Analytic Langlands correspondence over $\mathbb{C}$ and $\mathbb{R}$

(with E. Frenkel and D. Kazhdan)

G - split reductive group (usually $G = PGL_2$).

Langlands theory for G over $\mathbb{F}_q(X)$ = harmonic analysis on $L^2(Bun_G(X)(\mathbb{F}_q), \mu)$

irreducible smooth projective curve

$\mu(E) = \frac{1}{|Aut(E)|}$. Namely, we are interested in eigenfunctions of Hecke operators acting on this space, and the Langlands correspondence says (roughly) that such eigenfunctions correspond to homomorphisms from the Galois group of $\mathbb{F}_q(X)$ to $G^\vee$.

Analytic Langlands correspondence = the same for $Bun_G(X)(F)$ where F is a local field

$$F = \begin{cases} \text{archimedian } (\mathbb{R} \text{ or } \mathbb{C}) \\ \text{non-archimedian (finite extension of } \mathbb{Q}_p \text{ or } \mathbb{F}_p((x))\text{)}. \end{cases}$$

We will mostly focus on $F = \mathbb{R}, \mathbb{C}$.

One important difference of this setting from the usual Langlands: While in the usual Langlands theory we have to work with ALL bundles, even the most degenerate ones (as each bundle E has nonzero measure $\frac{1}{|\mathrm{Aut}\, E|}$), in analytic Langlands (if $g(X) \geq 2$) we may restrict to stable bundles, since other bundles form a "set of measure zero". This simplifies the story significantly, since the space of stable bundles is an analytic F-manifold (while $\mathrm{Bun}_G(X)$ is a complicated algebraic stack), but there is also a price to pay, since it leads to divergences of integrals defining Hecke operators.

More precisely, to do harmonic analysis on $Bun_G(X)(F)$, we first define the Hilbert space stable bundles

$\mathcal{H} = L^2(Bun_G^s(X)(F))$. Since we don't have a natural measure on $Bun_G^s(X)(F)$, we define it to be the space of L^2 half-densities (rather than functions); then

$\|f\|^2 := \int_{Bun_G^s(X)(F)} |f|^2$ is canonically defined.

We will define Hecke operators $H_x : \mathcal{H} \to \mathcal{H}$, $x \in X(F)$, which are pairwise commuting, and for $F = \mathbb{R}$ and $\mathbb{C}$ also commute with certain commuting differential operators called the quantum Hitchin hamiltonians, constructed by Beilinson and Drinfeld as a key ingredient in the geometric Langlands program. Then harmonic analysis on $\mathrm{Bun}_G(X)(F)$ reduces to describing joint eigenfunctions of these operators on $\mathcal{H}$.

The connection to geometric Langlands is that the D-module on $\mathrm{Bun}_G(X)$ generated by the eigenfunction ψ (for $F=\mathbb{R}$ or $\mathbb{C}$) is a Hecke eigensheaf in the sense of Beilinson and Drinfeld.

Let us now get into the details. We will consider bundles with parabolic structures. Let $t_0,\dots,t_{N-1}\in X(F)$ be distinct points.

Recall that a PGL_2-bundle on X is a GL_2-bundle (= rank 2 vectorbundle) E up to tensoring with line bundles.

A (quasi) parabolic structure on E at t_j is a choice $s_j \in \mathbb{P}E_{t_j}$.

A parabolic bundle is a bundle E equipped with s_j for all $j \in [0, N-1]$.

Slope of a parabolic bundle E is

$$\mu(E) = \frac{1}{2}\deg(E) + \frac{N}{4}$$

Slope of a line subbundle $L \subset E$ is

$$\mu(L) = \deg(L) + \frac{N_L}{2},$$

where N_L is the number of j s.t. $s_j = L_{t_j} \subset E_{t_j}$.

A parabolic bundle E is stable if $\forall L \subset E$, $\mu(L) < \mu(E)$, and semistable if $\forall L \subset E$, $\mu(L) \leq \mu(E)$.

So if N is odd, every semistable

bundle is stable.

Theorem. Stable bundles form a smooth quasiprojective variety Bun^s. Semistable bundles (mod S-equivalence) form a projective variety (not nec. smooth) Bun^{ss}, such that $Bun^s \subset Bun^{ss}$ is an open subset.

Thus if N is odd, $Bun^s = Bun^{ss}$ is a smooth projective variety.

Now we define the Hilbert space $\mathcal{H} = L^2(Bun^s(F))$, the space of square integrable half-densities f on $Bun^s(F)$: $\int_{Bun^s(F)} |f|^2 < \infty$.

Now we want to define Hecke operators on $\mathcal{H}$.

Hecke correspondence (stable part):

$$Z = \{(E, E', s, x) \mid E, E' \in Bun^s,\ s \in \mathbb{P}E_x$$
$$x \in X,\quad 0 \to E \to E' \to \delta_s \to 0\}.$$ Thus

$\Gamma(U, E')$ = sections of E over U

regular outside x and allowed to have a first order pole at x with residue in s.

$$\begin{tikzcd}
 & Z \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "q = p_3"] & \\
\mathrm{Bun}^s & \mathrm{Bun}^s & X
\end{tikzcd}$$

E' is called the Hecke modification of E at x along s; $E' = HM_{x,s}(E)$.

In ordinary Langlands, we define the Hecke operator H_x by

$$(H_x \psi)(E) = \sum_{(E, E', s, x) \in Z} \psi(E')$$

where the sum is taken over all Hecke modifications of E at x (a finite sum with $q+1$ terms).

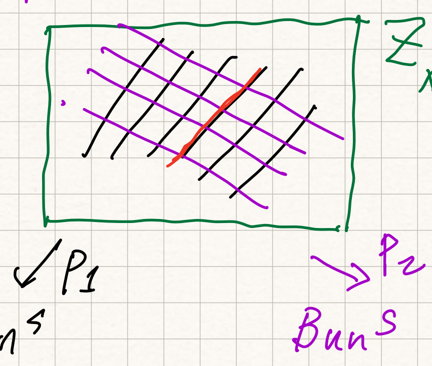
But over a local field instead of sum we have to take an integral, so there is a question which measure to use. Here comes handy the following theorem of Beilinson and Drinfeld.

Theorem. $\exists$ a canonical nonvanishing section

$$a \in \Gamma(p_1^* K_{Bun} \otimes p_2^* K_{Bun}^{-1} \otimes \omega^2 \otimes q^* K_X^{-1})$$

where ω is the cotangent bundle to the fiber $p_1 \times q : Z \to Bun^s \times X$.

Now we define the Hecke operators of analytic Langlands:



$\forall x \neq t_j, \quad j \in [0, N-1]$

$$(H_x \psi)(E) = \int_{(E, E', s) \in Z_x \subset \mathbb{P}^1(F) \cong \mathbb{P}E_x} \psi(E') \|a\|^{1/2}$$

Thus H_x is a $-\frac{1}{2}$-density with respect to x.

The integration is over a fiber of $p_1 \times q$ which is a $\mathbb{P}^1$ with some missing points (as we consider only stable bundles). So we have to answer questions:

1. Is the integral convergent?
2. Does it define a bounded operator $\mathcal{H} \longrightarrow \mathcal{H}$?

Def. A stable bundle E is said to be very stable if it does not admit a nonzero nilpotent Higgs field (a section ϕ of $\mathrm{ad}E \otimes K_X \otimes \bigotimes_{i=0}^{N-1} \mathcal{O}(t_i)$ such that ϕ preserves $s_i\ \forall i$ and $\phi^2 = 0$).

Very stable bundles form an open set $Bun^{vs} \subset Bun^{s}$.

Theorem. If ψ is a compactly supported smooth half-density on $Bun^{vs}(F)$ then the integral $H_x\psi$ converges absolutely and defines a smooth half-density on $\mathrm{Bun}^s(F)$.

Thus we obtain a densely defined operator H_x on $\mathcal{H}$.

Assume $X(F) \neq \emptyset$.

Compactness Conjecture. The operators H_x land in $\mathcal{H}$ and define compact, self-adjoint commuting operators $\mathcal{H} \to \mathcal{H}$. Also $\bigcap_x \mathrm{Ker}\, H_x = 0$.

Theorem. The compactness conjecture holds in genus 0 (with $N \geq 3$).

Corollary. The joint spectrum of $\{H_x\}$ is a countable set Σ and eigenspaces are finite dimensional. Thus

$$\mathcal{H} = \bigoplus_{\Lambda \in \Sigma} \mathcal{H}_\Lambda, \quad \dim \mathcal{H}_\Lambda < \infty.$$

Now consider the case $X = \mathbb{P}^1$ (genus zero) in more detail. Let $t_{N-1} = \infty$ and $m = N-2$.

We have two components
Bun^0 and Bun^1 of Bun^S,
bundles of degree 0 and 1,
and automorphisms $S_i : Bun^S \circlearrowleft$
$i \in [0, m+1]$ given by Hecke modifi-
cations at t_i along s_i.
We have $S_i S_j = S_j S_i$ and $S_i^2 = 1$,
so they define an action of
$(\mathbb{Z}/2)^{m+2}$ on Bun^S. Let us identify
Bun^0 and Bun^1 using S_{m+1} (at ∞),
so we may regard Hecke opera-
tors H_x as acting on $\mathcal{H} = L^2(Bun^0)$.
We also have an action of
$(\mathbb{Z}/2)^{m+1}$ on $\mathcal{H}$ by S_i, $i \in [0, m]$,
which commute with H_x.

A generic $E \subset Bun^0$ is trivial:
$E \cong \mathcal{O} \oplus \mathcal{O}$, so $s_i \in \mathbb{P}^1$.
Using the action of PGL_2, we
can bring the parabolic structures

of E to the following form:

$$\begin{array}{ccccc} t_0 & t_1 & \cdots & t_m & t_{m+1}=\infty \\ (1,0) & (1,y_1) & & (1,y_m) & (0,1) \\ \| & \| & & \| & \| \\ s_0 & s_1 & & s_m & s_{m+1}. \end{array}$$

Here $y=(y_1,\dots,y_m)$ is well defined up to scaling. We denote this bundle by E_y.

Proposition. The Hecke modification is given by the formula

$HM_{x,s}(E_y) = E_z$ where

$$z_i = \frac{xy_i - st_i}{y_i - s}$$

The space $\mathcal{H}$ can be realized as the space of functions $\psi(y_1,\dots,y_m)$ on F^m

$\|\lambda\| = |\lambda|$ for $\mathbb{R}$

$|\lambda|^2$ for $\mathbb{C}$

such that

$$\psi(\lambda y) = \|\lambda\|^{-\frac{m}{2}} \psi(y)$$

satisfying an appropriate L^2 condition, or

$\psi(y_0, y_1, \dots, y_m)$ on F^{m+1} satisfying the above condition and

$$\psi(y_0 + a, \dots, y_m + a) = \psi(y_0, \dots, y_m).$$

Theorem. In the last realization

$$(H_x\psi)(y) = \left\|\prod_{i=0}^{m}(x - t_i)\right\|^{-1/2} (\mathbb{H}_x\psi)(y)$$

$(\mathbb{H}_x\psi)(y)$ = modified Hecke operator =

$$\int_F \psi\left(\frac{t_0 - x}{s - y_0}, \dots, \frac{t_m - x}{s - y_m}\right) \left\| \frac{ds}{\prod_{i=0}^{m}(s - y_i)} \right\|.$$

Theorem. The spectrum Σ (for $\mathbb{C}$) is simple, i.e. $\forall \Lambda \in \Sigma$, the space $\mathcal{H}_\Lambda$ is 1-dimensional: $H_\Lambda = \mathbb{C}\,\psi_\Lambda$ for some joint eigenfunction ψ_Λ.

The function ψ_Λ is defined uniquely (up to sign) by the condition that $\|\psi_\Lambda\|_{L^2} = 1$ and ψ_Λ is real.

The main challenge is now to describe Σ. The description is in terms of local systems on X with values in $G^\vee = SL_2(\mathbb{C})$, which constitutes what we call the analytic Langlands correspondence (for $F = \mathbb{R}$ or $\mathbb{C}$). Namely, what arises is a special kind of local systems called opers.

To see the appearance of opers, consider the eigenvalues of H_x:

$$H_x \psi_\Lambda = \beta_\Lambda(x) \psi_\Lambda$$

where β_Λ is a $-\frac{1}{2}$-density on $X(F)$ defined away from the points t_i.

Theorem (the oper equation): The eigenvalues $\beta_\Lambda(x)$ satisfy the differential equation $L_\Lambda \beta_\Lambda = 0$ where

$$L_\Lambda = \partial_x^2 + \frac{1}{4}\sum_{i=0}^{m}\frac{1}{(x-t_i)^2} - \sum_{i=0}^{m}\frac{\mu_i(\Lambda)}{x-t_i}.$$

(for some $\mu_i(\Lambda) \in \mathbb{C}$)

Moreover, if $F = \mathbb{C}$ then we also have

$$\overline{L_\Lambda}\,\beta_\Lambda = 0.$$

For $F = \mathbb{C}$

Corollary. The monodromy of the differential equation $L_\Lambda \beta = 0$ can be conjugated to $SU(1,1) \simeq SL_2(\mathbb{R})$

Proof. Let f_1, f_2 be a basis of solutions of this equation (locally). Since β_Λ is real, we have

$$\beta_\Lambda(x) = \sum_{i,j=1}^{2} c_{ij} f_i(x) \overline{f_j(x)}$$

where c is a Hermitian matrix. Since β_Λ is single valued, c defines an invariant Hermitian form on the space of solutions, which is nondegenerate since the monodromy is irreducible. Also monodromy at t_i is unipotent, so the Hermitian form must be indefinite $\blacksquare$

Thus in some basis of solutions we have

$$\beta_\Lambda(x) = f_1(x)\overline{f_2(x)} + f_2(x)\overline{f_1(x)}.$$

Definition. An oper is a differential operator

$$\boxed{L = \partial_x^2 + \frac{1}{4}\sum_{i=0}^{m}\frac{1}{(x-t_i)^2} - \sum_{i=0}^{m}\frac{\mu_i}{x-t_i}},$$

$\mu_i \in \mathbb{C}$, $\sum \mu_i = 0$, $\sum t_i \mu_i = \frac{m}{4}$.

An oper L is real if the monodromy of the equation $L\beta = 0$ lands in $SU(1,1) \cong SL(2,\mathbb{R})$ up to conjugation.

Theorem (Faltings) Real opers form a discrete subset $Op_{\mathbb{R}}$ in the space of opers $Op \cong \mathbb{C}^{m-1}$

$\|x\| = |x|^2$ for $F = \mathbb{C}$

Lemma: $H_x \sim \|x\|^{1/2} \log \|x\|$, $x \to \infty$.

Corollary: $\beta_\Lambda(x) \sim \|x\|^{1/2} \log \|x\|$

This and the above differential equation determine $\beta_\Lambda(x)$

uniquely. Thus we obtain an inclusion $\Sigma \subset Op_{R}$.

We expect that in fact $\Sigma \cong Op_{R}$, and we have shown it for $N=4$ and 5.

Finally, let us explain the meaning of the constants $\mu_i(\Lambda)$. Recall that the operators H_x are supposed to commute with the quantum Hitchin system. In our case this system is the Gaudin system.

$$G_i = \sum_{\substack{0\le j\le m \\ j\ne i}} \frac{-(y_i-y_j)^2\partial_i\partial_j + (y_i-y_j)(\partial_i-\partial_j)+\frac{1}{2}}{t_i-t_j}$$

$$[G_i,G_j]=[G_i,H_x]=0.$$

Theorem. $G_i\psi_\Lambda = \mu_i(\Lambda)\psi_\Lambda$.

In particular, ψ_Λ are smooth on the open subset Bun^{vs}.

In fact, the operators G_i are naturally defined as unbounded commuting self-adjoint operators on $\mathcal{H}$, and the spectral decomposition of H_x is also one for G_i.

Moreover, in the case $F=\mathbb{C}$ we also have $\overline{G_i}\psi_\Lambda=\overline{\mu_i(\Lambda)}\psi_\Lambda$, and $\mu_i(\Lambda)$ are exactly the constants for which the monodromy of the system $G_i\psi=\mu_i\psi$ lands in $GL_{2^{N-3}}(\mathbb{R})$, up to conjugation.

Example. Since the (Schwarz) kernel of the operator H_x is positive, by the infinite analog of the Perron–Frobenius theorem, $\exists$ an eigenvalue $\beta_0(x) > 0$ for which $\psi_0 > 0$.

The corresponding oper L_0 is called the uniformization oper.

Indeed, in this case we get $f_1\overline{f_2} + f_2\overline{f_1} > 0$, so $\mathrm{Re}\left(\frac{f_2}{f_1}\right) > 0$.

Thus $h(z) = \frac{if_2}{f_1}(z)$ is a multivalued holomorphic function which lands in the upper half-plane H_+, and monodromy acts by elements $g \in SL_2(\mathbb{R})$

It is not difficult to show that this monodromy is a discrete Fuchsian group $F \subset SL_2(\mathbb{R})$ and $H_+/F = \mathbb{C}P^1 \setminus \{t_0, .., t_m\}$; the cover $H_+ \longrightarrow \mathbb{C}P^1 \setminus \{t_0, .., t_m\}$ is given by h^{-1}.

Moreover, as β_0 is a $-1/2$-density, β_0^{-2} is a density, and it is nothing but the Poincaré conformal metric on $\mathbb{C}P^1 \setminus \{t_0, .., t_m\}$ (curvature $= -1$).