TT^*- equations of Cecotti and Vafa

Riemann-Hilbert method

and

Iwasawa factorization

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$\boxed{\text{AGMPS - 2022} \quad \text{Celebrating the work of Igor Krichever}}$

The talk is based on the joint work

with

Martin Guest and Chang-Shou Lin,

and

it represents an ongoing joint project

with

Martin Guest and Igor Krichever

tt^* - equations

$$2(w_j)_{t\bar{t}} = -e^{2(w_{j+1}-w_j)} + e^{2(w_j - w_{j-1})}$$

$$w_j : \mathbb{C}^* \longrightarrow \mathbb{R}$$

$$j \in \mathbb{Z}$$

$w_j = w_{j+n+1}$ (periodicity)

$w_j + w_{n-j} = 0$ (anti symmetry)

$w_j = w_j(|t|)$ (radial symmetry)

1991, Cecotti & Vafa, study of topological - anti - topological fusion.

More on history.

1. Mikhailov - Olshanetsky - Perelomov classification (1981)

   - A_n^2 - Toda equation

2. 1992: Dubrovin proved the isomonodromic integrability of the general tt^* - equations and discussed the Riemann - Hilbert setting for their global solutions

3. Lax-pair integrability of the hyperbolic version - 1980, Mikhailov

4. The long time asymptotics of the solution of the Cauchy problem - 1984, Novokshenov. via the RH approach.

5. $n = 1$, $n = 2$ (P_{III}) asymptotic connection formula: Novokshenov (1984-85) Kitaev (1989)

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Lax pair (Mikhailov)

$$\Psi_t = \left(w_t + \frac{1}{\lambda} W\right)\Psi$$

$$\Psi_{\bar{t}} = \left(-w_{\bar{t}} + \lambda W^T\right)\Psi$$

$$w = \mathrm{diag}\,(w_0, w_1 \ldots w_n)$$

$$W = \begin{pmatrix} 0 & e^{w_1 - w_0} & 0 & 0 \\ 0 & & e^{w_2 - w_1} & 0 \\ & & & 0 \\ & & \ddots & e^{w_n - w_{n-1}} \\ e^{w_0 - w_n} & 0 \cdots & 0 & \end{pmatrix} \quad (n+1)\times(n+1)$$

$\boxed{n = 2}$ $\quad w_0 = -w_2$

$w_1 = 0$

$$W = \begin{pmatrix} 0 & e^{-w_0} & 0 \\ 0 & 0 & e^{-w_0} \\ e^{2w_0} & 0 & 0 \end{pmatrix} \qquad w = \begin{pmatrix} w_0 & & \\ & 0 & \\ & & -w_0 \end{pmatrix}$$

$$2 w_{0\,t\bar{t}} = e^{4w_0} - e^{-2w_0}$$

$$w_0 \equiv w_0(x), \quad x = |t|$$

$$W_{0xx} + \frac{1}{x} W_{0x} = 2e^{4W_0} - 2e^{-2W_0}$$

(Bullough - Dodd - Mikhailov)

$$\Psi(t, \bar{t}, \lambda) \equiv \Psi(x, \lambda/t)$$

$$\Psi(x, \xi) \quad , \quad \xi = \lambda/t$$

$$\frac{\partial \Psi}{\partial \xi} = \left(- \frac{1}{\xi^2} W - \frac{1}{\xi} x W_x + x^2 W^T \right) \Psi \equiv A(\xi) \Psi$$

$$\frac{\partial \Psi}{\partial x} = \left(- W_x + 2x \xi W^T \right) \Psi$$

Cauchy data:

$$w_0 = \gamma \ln x + \delta + o(1)$$

$$x \to 0, \qquad -\frac{1}{2} < \gamma < 1$$

$$\boxed{w_0 \equiv w_0(x; \gamma, \delta)}$$

$$\left(\exists!\ \ w_0(x) \in C^{\infty}(\mathbb{R}_+ \setminus \{0\}),\ \ \underset{x \to \infty}{w_0(x) \to 0} \right)$$

$$\delta = \delta(\gamma)\ !\ ?$$

Asymptotic data:

Spec. case: $\quad w_0(x) \cong \varkappa\, x^{-1/2} e^{-2\sqrt{3}\,x}$

$$x \to +\infty$$

general case: $\quad \boxed{w_0 \equiv w_0(x; \tau, \varkappa)}$

Monodromy data:

$$\Psi_f^{(\infty)}(\xi) = P_\infty \left(I + O(1/\xi)\right) e^{x^2 \xi d_3}$$

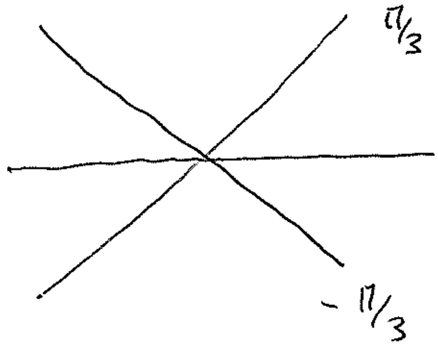
$$\Psi_f^{(0)}(\xi) = P_0 \left(I + O(\xi)\right) e^{\frac{1}{\xi} d_3}$$

$$d_3 = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} \qquad \omega = e^{2\pi i/3}$$

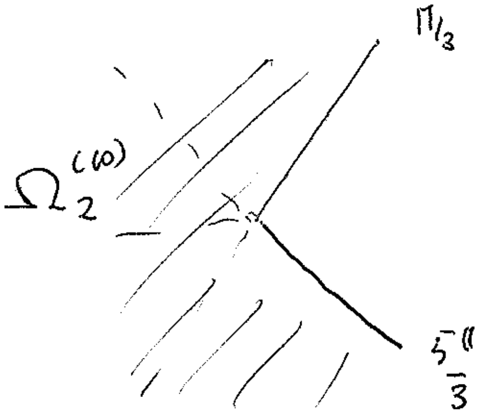
$$P_\infty = e^{W} \Omega^{-1}, \quad P_0 = e^{-W} \Omega$$

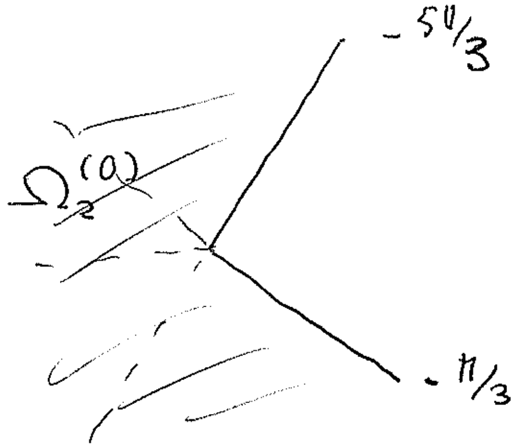
$$\Omega = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

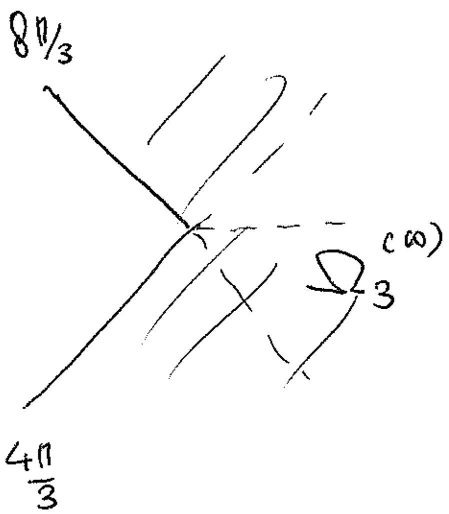
Stokes rays:

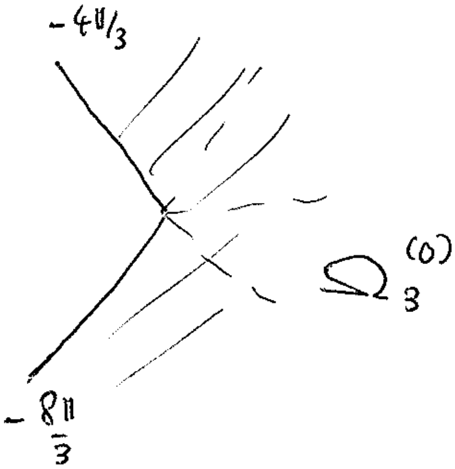


Stokes sectors:









$$\Psi_k^{(p,0)}(\xi) \simeq \Psi_f^{(\infty,0)}(\xi)$$

$$\xi \to \infty, \quad \xi \in \Omega_k^{(p,0)}$$

$$k = 1, 2, 3.$$

$$S_k^{(p,0)} = \left[\Psi_k^{(p,0)}\right]^{-1} \Psi_{k+1}^{(p,0)} \qquad k = 1, 2$$

$$E_1 = \left[\Psi_1^{(0)}\right]^{-1} \Psi_1^{(0)}.$$

Symmetrie:

$$\Delta A^T(-\xi)\Delta = A(\xi) \qquad \Delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\omega d_3^{-1} A(\omega\xi) d_3 = A(\xi)$$

$$\Delta A\left(\frac{1}{x^2\xi}\right)\Delta = A(\xi)$$

$$\overline{A(\overline{\xi})} = A(\xi)$$

$$\Downarrow$$

$$\left\{ S_k^{(0,\infty)}, E_1 \right\} = \{s\} \qquad m = (a, e)$$

$$S_1^{(0)} = \begin{pmatrix} 1 & 0 & a \\ a & 1 & a^2 - \omega a \\ -a\omega & a & 1 \end{pmatrix} \qquad a = \omega^{1/2} s, \quad \overline{s} = s.$$

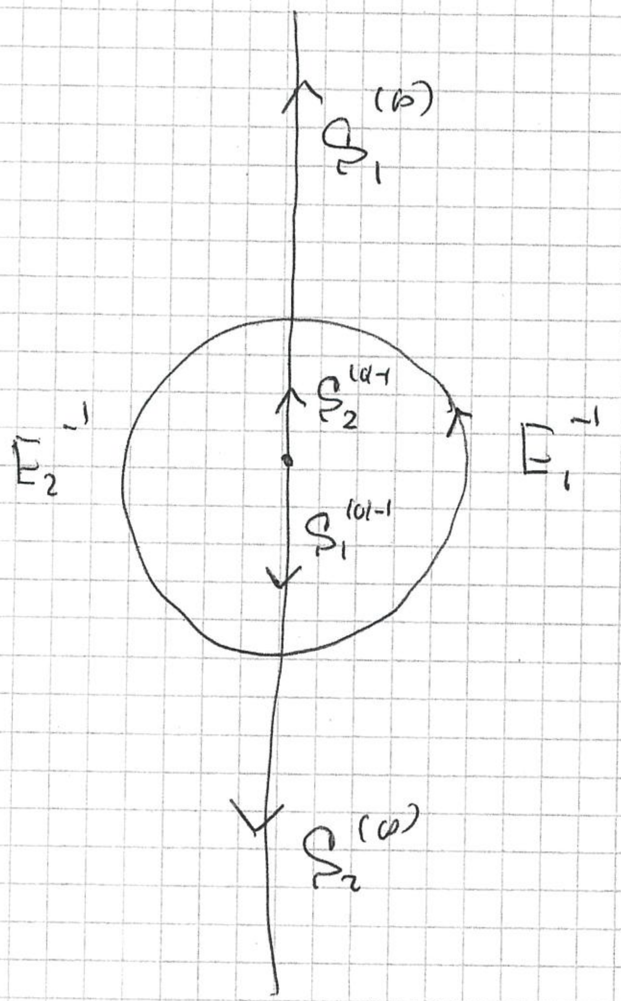
$$E_1^{-1} = \begin{pmatrix} A & B & C \\ B - a\omega C & C & A + aC \\ -aB + C & A + aB & B \end{pmatrix} \qquad A + \omega^2 B + \omega C = 3$$

$$AC + \omega AB + (a\omega + \omega^2) BC = 0$$

$$A = \overline{A}$$

$$C = \overline{B}$$

RHP:



$$\Psi_+ = \Psi_- S.$$

$$\Psi(\xi) \simeq P_\infty \left(I + O\left(\tfrac{1}{\xi}\right)\right) e^{x^2 \xi d_3}$$

$$\Psi(\xi) \simeq P_0 \left(I + O(\xi)\right) e^{\frac{1}{\xi} d_3}$$

$$E_2 = S_1^{(0)-1} E_1 S_2^{(\infty)-1}$$

$$\boxed{m \xmapsto[\text{RH}]{} w_0(x; m)}$$

The scheme for global analysis:

!

$$
\begin{tikzcd}
 & (m) \arrow[dl, leftrightarrow, "\text{Iwasawa}"'] \arrow[dd, "\text{RHP}"] \arrow[dr, leftrightarrow, "\text{NSD}"] & \\
(\gamma, s) & & (r, \varphi) \\
2\gamma \ln x + s & W_0(x) \arrow[l, "x \to 0"'] \arrow[r, "x \to \infty"'] & \sim r\, e^{-2\sqrt{3}x} x^{-1/2}
\end{tikzcd}
$$

NSD for special global solutions

$\downarrow$

$$r = \frac{3^{1/4}}{4\sqrt{\pi}}\, s \,.$$

Special global case:

$$E_1 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left|\frac{s+1}{2}\right| < 1$$

2. Iwasawa factorization.

(Bobenko, Its, 1997
Dorfmeister – Pedit – Wu
1993 theory of harmonic maps.)

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Holomorphic data.

$$C(c_j, k_j), \quad c_j > 0, \quad k_j > -1 \quad j = 0, 1$$

$$\downarrow$$

$$\eta = \begin{pmatrix} 0 & 0 & p_0 \\ p_1 & 0 & 0 \\ 0 & p_1 & 0 \end{pmatrix} \qquad p_j(z) = c_j z^{k_j}$$

$$\downarrow$$

$$\frac{dL}{dz} = \frac{1}{\lambda} L \eta, \quad L\Big|_{z=0} = I$$

$$L = \sum_{j=0}^{\infty} \lambda^{-j} C_j, \quad C_0 = I$$

$$\cap$$

$$H(\mathbb{C}^*)$$

also: $L \in \Lambda SL^{\tau}(3, \mathbb{C}) = \left\{ C = \sum_{j=-\infty}^{\infty} c_j \lambda^{-j}, \det C = 1, \right.$

$$\Delta C^{T-1}(-\lambda) \Delta = C(\lambda)$$

$$\Delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{Pressley \& Segal:}}$

$$L = L_R L_+ \quad , \quad L_R, L_+ \in \Lambda SL(3, \mathbb{C})$$

$$\Delta L_R(z, \bar{z}, \tfrac{1}{\lambda}) \Delta = L_R(\lambda)$$

$$L_+ = \sum_{j=0}^{\infty} \lambda^j L_j \quad ; \quad L_0 = \begin{pmatrix} b_0 & & \\ & 1 & \\ & & b_0^{-1} \end{pmatrix}$$

$$b_0 > 0$$

$$L_+\big|_{z=0} = L_R\big|_{z=0} = I$$

$z \longmapsto L_R, L_+$ – C^∞-maps.

<u>Key point:</u> (cf. Krichever, 1980, nonlinear analog of the d'Alembert's formula) also, Bobenko, I (1995)

$$\cancel{\frac{dL_R}{dz} = \ldots}$$

$$L_R^{-1} \frac{dL_R}{dz} = L_+ \frac{1}{\lambda} \eta L_+^{-1} - \frac{dL_+}{dz} L_+^{-1} \qquad L_R = L L_+^{-1}$$

$$= \frac{A^T}{\lambda} + B_0 + B_1 \lambda + \cdots \tag{9.1}$$

$$A^T = L_0 \eta L_0^{-1}$$

$$L_R^{-1} \frac{dL_R}{d\bar{z}} = -\frac{dL_+}{d\bar{z}} L_+^{-1} \simeq -\frac{dL_0}{d\bar{z}} L_0^{-1} + O(\lambda)$$

$$= -\operatorname{diag}\left(\frac{d}{d\bar{z}} \ln b_0 \;\; 0 \;\; -\frac{d}{d\bar{z}} \ln b_0\right) + O(\lambda).$$

at the same time,

$$L_R^{-1} \frac{dL_R}{dz} = \triangle\, \overline{L_R^{-1}\left(\frac{1}{\bar{\lambda}}\right)} \, \frac{d \overline{L_R\left(\frac{1}{\bar{\lambda}}\right)}}{dz} \triangle$$

$$= \triangle \overline{\left(L_R^{-1}\left(\frac{1}{\bar{\lambda}}\right) \frac{dL_R\left(\frac{1}{\bar{\lambda}}\right)}{d\bar{z}}\right)} \triangle$$

$$= -\operatorname{diag}\left(-\frac{d}{dz} \ln b_0 \;\; 0 \;\; +\frac{d}{dz} \ln b_0\right) + O\left(\frac{1}{\lambda}\right). \quad (9.2)$$

(9.1), (9.2)

$\Downarrow$

$$L_R^{-1} \frac{dL_R}{dz} = a + \frac{A^T}{\lambda}$$

$$a = \operatorname{diag}\left(\frac{d}{dz} \ln b_0 \;\; 0 \;\; -\frac{d}{dz} \ln b_0\right).$$

Conclusion:

$$\frac{dL}{dz} = \frac{1}{\lambda} L \eta$$

$$\Downarrow$$

$$\frac{dL_R}{dz} = L_R\left(a + \frac{1}{\lambda} A^T\right)$$

$$\frac{dL_R}{d\bar{z}} = L_R\left(-\bar{a} + \lambda \bar{A}\right) \qquad \left(\Leftarrow \ \frac{d}{d\bar{z}} L_R = \Delta\left(\frac{d}{dz}\overline{L_R\left(\tfrac{1}{\lambda}\right)}\right)\Delta \cdots \right)$$

$$a = \begin{pmatrix} \frac{d}{dz}\ln b_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{dz}\ln b_0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 0 & A_0 \\ A_1 & 0 & 0 \\ 0 & A_1 & 0 \end{pmatrix} \qquad \begin{aligned} A_0 &= \rho_0 b_0^2 \\ A_1 &= \rho_1 b_0^{-1} \end{aligned}$$

Now,

$$G(z,\bar{z}) := \begin{pmatrix} |h_0|/h_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h_0/|h_0| \end{pmatrix}$$

$$h_0 = \rho_1^{1/3} \rho_0^{-1/3}$$

(12)

$$\frac{d}{dz}(L_R G_1) = L_R G_1 \left(w_z + \frac{1}{\lambda} \nu W^T \right)$$

$$w_0 = \ln \frac{h_0}{|h_0|} \quad (12.1)$$

$$\frac{d}{d\bar{z}}(L_R G_1) = L_R G_1 \left(-w_{\bar{z}} + \lambda \bar{\nu} W \right)$$

$$\nu = \rho_0^{1/3} \rho_1^{2/3}$$

$$z \longmapsto t : \quad \frac{dt}{dz} = \nu , \quad t = (c_0 c_1^2)^{1/3} z^{N/3} \frac{3}{N}$$

$$N = k_0 + 2k_1 + 3$$

$$\Psi(t, \bar{t}, \lambda) := (L_R G_1)^T$$

$$\frac{d\Psi}{dt} = \left(w_t + \frac{1}{\lambda} W \right) \Psi$$

$$\frac{d\Psi}{d\bar{t}} = \left(-w_{\bar{t}} + \lambda W^T \right) \Psi \Rightarrow$$

w_0 from (12.1) satisfies tt^*-equation.

and:

$$w_0 = -\ln|h_0| + o(1) = \gamma \ln x + \rho + o(1)$$

$$x \to 0$$

$$\gamma = \frac{k_0 - k_1}{N} = \frac{\alpha_0 - \alpha_1}{\alpha_0 + 2\alpha_1} \qquad \alpha_j = k_j + 1$$

$$N = \alpha_0 + 2\alpha_1 = k_0 + 2k_1 + 3$$

$$m_0 = -\gamma/2$$

$$\rho = \ln\left[\left(\frac{N}{3}\right)^{\gamma} (c_0 c_1^2)^{-\gamma/3} \left(\frac{c_0}{c_1}\right)^{1/3}\right]$$

$$= \ln\left[c_0^{1/2} (c_0 c_1^2)^{\frac{4m_0 - 1}{6}} \left(\frac{N}{3}\right)^{-2m_0}\right].$$

Important: Dependence on $x = |t|$ only.

$$L(z,\lambda) = \begin{pmatrix} e^{-i\varphi_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\varphi_1} \end{pmatrix} L(ze^{i\varphi}, \lambda e^{i\beta\varphi}) \begin{pmatrix} e^{i\varphi_1} & & \\ & 1 & \\ & & e^{-i\varphi_1} \end{pmatrix}$$

$$\varphi_1 = \frac{k_0 - k_1}{3}\varphi, \quad \beta = \frac{N}{3}$$ (the same for L_R, L_+)

$$\Downarrow$$

$$L_0 = L_0(|z|) \Rightarrow W_0 = \frac{\ln b_0}{|b_0|} \equiv W_0(|z|)$$
$$\equiv W_0(x).$$

<u>moreover:</u>

$$\lambda \frac{\partial L}{\partial \lambda} + \frac{z}{\beta} \frac{\partial L}{\partial z} = - [m, L]$$

$$m = \begin{pmatrix} -\gamma & & \\ & 0 & \\ & & \gamma \end{pmatrix}$$

$$\cancel{\left(\lambda \frac{\partial}{\partial \lambda} + z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}\right) g}$$

$$\lambda \frac{\partial L_R}{\partial \lambda} + \frac{z}{\beta} \frac{\partial L_R}{\partial z} - \frac{\bar{z}}{\beta} \frac{\partial L_R}{\partial \bar{z}} = - [m, L_R]$$

put $\quad g(\lambda) := \lambda^m$

then:

$$\left(\lambda \frac{\partial}{\partial \lambda} + \frac{z}{\beta} \frac{\partial}{\partial z}\right) g L = g L m$$

$$\left(\lambda \frac{\partial}{\partial \lambda} + z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}\right) (g L_R G) = 0$$

$$\Phi(\lambda) := (g(\lambda) L)^T$$

$$\Psi(\lambda) := (g(\lambda) L_R G_e)^T$$

$$\frac{d\Phi}{d\lambda} = \left[-\frac{z}{\beta\lambda^2} \eta^T + \frac{1}{\lambda} m \right] \Phi$$

$$\frac{d\Psi}{d\lambda} = \left[-\frac{t}{\lambda^2} W - \frac{x w_x}{\lambda} + \bar{t}\, W^T \right] \Psi$$

also

$$\mathring{\Phi}(\xi) = t^{-m}\, \Phi(\lambda)\Big|_{\lambda = \xi t}$$

$$\Psi(\xi) = (g L_R G_e)\Big|_{\lambda = \xi t}$$

Then:

$$\overset{\circ}{\Psi}(\xi) = \overset{\circ}{\Psi}(\xi).$$

$$\Psi(\xi) = \Psi(x, \xi).$$

$$\frac{d\Psi}{d\xi} = \left[-\frac{1}{\xi^2}W - \frac{1}{\xi}xW_x + x^2 W^T\right]\Psi.$$

$$\frac{d\Psi}{dx} = \left[-W_x + 2x\xi W^T\right]\Psi$$

$$\frac{d\overset{\circ}{\Psi}}{d\xi} = \left[-\frac{1}{\xi^2}\begin{pmatrix} 0 & \varkappa & 0 \\ 0 & 0 & \varkappa \\ \varkappa^{-2} & 0 & 0 \end{pmatrix} + \frac{1}{\xi}m\right]\overset{\circ}{\Psi} \quad (16.1)$$

$$m = \begin{pmatrix} -\gamma & & \\ & 0 & \\ & & \gamma \end{pmatrix}$$

$$\varkappa = \left(\frac{3}{N}\right)^{\gamma} c_1^{\frac{2\gamma+1}{3}} c_0^{\frac{\gamma-1}{3}} = e^{-\rho}.$$

$$\left(w_0 = \gamma \ln x + \rho + o(1), \; x \to 0 \right)$$

Relations between Ψ and $\mathring{\Phi}$ monodromy data.

Ψ-data: $$\Psi_k^{(0,\infty)} = P_{0,\infty}\left(I + O(\xi, 1/\xi)\right)e^{\frac{1}{\xi}d_3,\ x^2\xi d_3}$$

$$\xi \to 0, \infty, \quad \xi \in \Omega_k^{(0,\infty)}$$

$$k = 1, 2, 3$$

$$\boxed{\begin{aligned} S_k^{(0,\infty)} &= \left[\Psi_k^{(0,\infty)}\right]^{-1}\Psi_{k+1}^{(0,\infty)}, \quad k = 1, 2 \\ E_1 &= \left[\Psi_1^{(0)}\right]^{-1}\Psi_1^{(\infty)} \end{aligned}}$$

$\mathring{\Phi}$-data: $$\mathring{\Phi}_k^{(0)} = \mathring{O}_0\left(I + O(\xi)\right)e^{\frac{1}{\xi}d_3}, \quad \xi \to 0, \ \xi \in \Omega_k^{(0)}$$

$$k = 1, 2, 3$$

$$\mathring{O}_0 = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{pmatrix}\Omega, \quad \Omega = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

$$\mathring{\Phi}^{(\infty)} = \left(I + O(1/\xi)\right)\xi^m, \quad \xi \to \infty$$

$$\boxed{\begin{aligned} R_k &= \left[\mathring{\Phi}_k^{(0)}\right]^{-1}\mathring{\Phi}_{k+1}^{(0)}, \quad k = 1, 2 \\ D_k &= \left[\mathring{\Phi}_k^{(0)}\right]^{-1}\mathring{\Phi}^{(\infty)} \end{aligned}}$$

Claim 1: $\boxed{S_k^{(0)} = R_k.}$

General reason:

$$\Psi(\xi) = (g L_R G)^T$$

$$= (g L L_+^{-1} G)^T = (L_+^{-1} G)^T (gL)^T$$

$$= \underbrace{(L_+^{-1} G)^T L^m}_{\substack{\text{holom. at} \\ \xi = 0}} \mathring{\Phi}(\xi)$$

$$\boxed{\text{Details: p. 13-16, Lecture 13.}}$$

Claim 2:

$$E_1 = D_2 \Delta \overline{D_2^{-1}} d_3^{-1} E_1^{(global)}.$$

$$E_1^{(global)} = \frac{1}{3} C.$$

$$\boxed{\text{Details: p. 17-21, Lecture 13}}$$

$$\Delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The last observation:

(16.1) can be solved explicitly !

$$\Downarrow$$

$$a = \omega^{-1/2} s \qquad s = \bar{s}$$

$$s = -1 + 2\cos\left(\frac{2}{3}\pi\gamma + \frac{1}{3}\pi\right) \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_1 = R_1 K^{-1} \begin{pmatrix} e & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} K R_1^{-1} \frac{1}{3} C$$

$$R_1 = \begin{pmatrix} 1 & 0 & \omega^{1/2} s \\ \omega^{1/2} s & 1 & \omega s^2 - \omega^{-1/2} s \\ 0 & 0 & 1 \end{pmatrix} \left(= S_1^{(0)} \right)$$

$$K = \begin{pmatrix} 1 & \omega^{\gamma} & \omega^{-\gamma} \\ 1 & \omega & \omega^{-1} \\ 1 & \omega^{2-\gamma} & \omega^{\gamma-2} \end{pmatrix} \qquad \omega = e^{\frac{2\pi i}{3}}$$

$$e = \frac{2(\gamma-1)^3}{x^2} \, 3^{2\gamma-2} \, \frac{\Gamma\left(\frac{\gamma-1}{3}\right)\Gamma\left(\frac{2\gamma-2}{3}\right)}{\Gamma\left(\frac{2-2\gamma}{3}\right)\Gamma\left(\frac{1-\gamma}{3}\right)}$$

$$\left(w_0 = \gamma \ln x + \varsigma + o(1), \ x \to 0, \ x = e^{-\varsigma} \right)$$

3 "Internal" version.

Theorem 1. (Iwasawa) Let $L(z,\lambda)$ be the element of $\Lambda SL^{\tau}(3,\mathbb{C})$ defined on p. 8. Then there is a unique factorization of $L(z,\lambda)$ in the form

$$L(z,\lambda) = L_R(z,\bar{z},\lambda)\, L_+(z,\bar{z},\lambda)$$

where $L_R, L_+ \in \Lambda SL^{\tau}(3,\mathbb{C})$ and

- $$\Delta\, \overline{L_R(z,\bar{z}, 1/\bar{\lambda})}\, \Delta = L_R(z,\bar{z},\lambda)$$
$$\Delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- $$L_+ = \sum_{j=0}^{\infty} L_j \lambda^j, \quad L_0 = \mathrm{diag}(b_0, 1, b_0^{-1}), \quad b_0 > 0$$

- $z \mapsto L_R$, $z \to L_+$ are C^∞ maps of a small neighborhood of 0 into $\Lambda SL^{\tau}(3,\mathbb{C})$ and
$$L_R\big|_{z=0} = L_+\big|_{z=0} = I$$

First proof. (Krichever's direct problem)

$$(c_0, c_1; k_0, k_1) \to L$$

$$\downarrow$$

$$\gamma := \frac{k_0 - k_1}{N}, \qquad N = k_0 + 2k_1 + 3$$

$$\left(\text{Note:} \quad -\tfrac{1}{2} < \gamma < 1 \Leftarrow k_{0,1} > -1 \right)$$

$$\varrho := \ln \left[\left(\frac{N}{3} \right)^{\gamma} (c_0 c_1^2)^{-\gamma/3} \left(\frac{c_0}{c_1} \right)^{1/3} \right]$$

$$t := (c_0 c_1^2)^{1/3} \frac{3}{N} z^{N/3}$$

$$x = |t|$$

Let $w_0(x)$ be a unique solution of

$$w_{0xx} + \frac{1}{x} w_{0x} = 2e^{4w_0} - 2e^{-w_0} \qquad (21.1)$$

$$w_0 = \gamma \ln x + \delta + o(1), \quad x \to 0$$

$$0 < x < x_0$$

~~Notation~~

$$p_j = c_j z^{k_j} \quad j = 0, 1$$

Define also

$$b_0 := |h_0| e^{w_0} \qquad h_0 = \left(p_1 / p_0\right)^{1/3}$$

Note that

$$b_0 = 1 + o(1), \quad x \to 0$$

Now, put

$$a = \mathrm{diag}\left\{ \frac{d}{dz} \ln b_0 \quad 0 \quad -\frac{d}{dz} \ln b_0 \right\}$$

$$A = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_1 \\ A_0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} A_0 &= p_0 b_0^{2} \\ A_1 &= p_1 b_0^{-1} \end{aligned}$$

<u>Claim 1.</u> Put

$$U(\lambda) := a + \frac{1}{\lambda} A^T$$

$$V(\lambda) := -\bar{a} + \lambda \bar{A}$$

then, (21.1) $\Leftrightarrow$ $U_{\bar{z}} - V_z = [U, V]$

$$\Downarrow$$

$\exists\ L_R(z, \bar{z}, \lambda)$:

$$\partial_z L_R = L_R U(\lambda) \qquad L_R\big|_{z=0} = I$$

$$\partial_{\bar{z}} L_R = L_R V(\lambda)$$

Moreover: $L_R(\lambda) \in \Lambda SL(3, \mathbb{C})$ and

$$\Delta L_R^{T-1}(-\lambda)\Delta = L_R(\lambda)$$

$$\Delta \overline{L_R\left(\frac{1}{\bar{\lambda}}\right)} \Delta = L_R(\lambda)$$

$$\Delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Now, the Key step:

locally in z_0:

Birkhoff - Grothendieck:

$$L(\lambda) = L_{\rho}(\lambda)\, L_{+}(\lambda)$$

where:

$$L_{+}(\lambda) = \sum_{j=0}^{\infty} L_j \lambda^{j}$$

$$L(\lambda) = \sum_{j=0}^{\infty} C_j \lambda^{-j}$$

$$L_0 = \begin{pmatrix} b_0 & & \\ & 1 & \\ & & b_0^{-1} \end{pmatrix} \quad \text{where } b_0 = |b_0| e^{w_0}$$

We also note that

$$\det L_{+} = \det L = 1$$

$$L_{+}\big|_{z=0} = L\big|_{z=0} = I$$

The theorem will be proven if we show that

$$\frac{dL}{d\bar{z}} = 0, \quad \frac{dL}{dz} = \frac{1}{\lambda} L \eta.$$

We have:

$$\underline{L^{-1}\frac{dL}{d\bar z}} = L_+^{-1} L_B^{-1}\left(\frac{dL_B}{d\bar z} L_+ + L_B \frac{dL_+}{d\bar z}\right)$$

$$= L_+^{-1} \cancel{L_B}(-\bar a + \lambda \bar A) L_+ + L_+^{-1}\frac{dL_+}{d\bar z}$$

$$= -\cancel{\bar a} + \cancel{\bar a} + O(\lambda)$$

$$L^{-1}\frac{dL}{d\bar z} = O\left(\frac{1}{\lambda}\right) \quad \Rightarrow \quad \boxed{L^{-1}\frac{dL}{d\bar z} = 0}$$

$$\underline{L^{-1}\frac{dL}{dz}} = L_+^{-1}\left(a + \frac{1}{\lambda} A^T\right) L_+ + L_+^{-1}\frac{dL_+}{dz}$$

$$= 2a + \frac{1}{\lambda}\eta + [\eta, B_1] + O(\lambda)$$

$$\Downarrow$$

$$L^{-1}\frac{dL}{dz} = \frac{1}{\lambda}\eta + R$$

$$R = 2a + [\eta, B_1]$$

using symmetries

$$\Delta L_+^{-T}(-\lambda)\Delta = L_+(\lambda)$$

$$\Delta L^{-T}(-\lambda)\Delta = L(\lambda) \quad (24.7)$$

$$d_3 L_R(\omega\lambda) d_3^{-1} = L_R(\lambda) \quad (25.1)$$

$$\Downarrow$$

$$d_3 L_+(\omega\lambda) d_3^{-1} = L_+(\lambda) \quad (25.2)$$

$$d_3 L(\omega\lambda) d_3^{-1} = L(\lambda) \quad (25.3).$$

$$(25.2) \Rightarrow B_1 = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & c \\ d & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$R = \begin{pmatrix} \tau_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_0 \end{pmatrix} \qquad \tau_0 = 2\frac{d}{dz}\ln b_0 + p_0 d - c p_1$$

The last thing to take into account – homogeneity.

Indeed, from diff eqn (22.1) ~~[illegible]~~ which determine L_R, it follows:

$$\begin{pmatrix} e^{-i\varphi_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\varphi_1} \end{pmatrix} L_R\left(e^{i\varphi}z, e^{-i\varphi}\bar z, \lambda e^{i\varphi\rho}\right) \begin{pmatrix} e^{i\varphi_1} & & \\ & 1 & \\ & & e^{-i\varphi_1} \end{pmatrix}$$

$$= L_R(z, \bar z, \lambda)$$

$$\varphi_1 = \frac{k_0 - k_1}{3}, \qquad \rho = N/3$$

$$\Downarrow$$

$$e^{i\varphi} R(z e^{i\varphi}) = R(z)$$

$$\Downarrow$$

$$e^{i\varphi}\, r_0(z e^{i\varphi}) = r_0(z)$$

$$\left.\frac{d}{d\varphi}\right|_{\varphi=0} \qquad \left(i e^{i\varphi} r_0(z e^{i\varphi}) + i e^{2i\varphi} \frac{dr_0}{dz} \right.$$

$$\Downarrow$$

$$r_0 + z\frac{dr_0}{dz} = 0 \quad \Rightarrow \quad r_0(z) = \frac{\kappa}{z}$$

$$\frac{dL}{dz} = L\left(\frac{1}{\lambda}\xi + \frac{\kappa}{z}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right)$$

$$L = C_0 + \frac{1}{\lambda} C_1 + \cdots$$

$$\frac{dC_0}{dz} = C_0 \frac{1}{z}\begin{pmatrix} \kappa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\kappa \end{pmatrix} \Rightarrow C_0 = C z^{\begin{pmatrix} \kappa & & \\ & 0 & \\ & & -\kappa \end{pmatrix}}$$

contradict $L|_{z=0} = I$ unless $\kappa = 0$

Theorem 1 is proven.

~~(details: [illegible] 10–21, I.K.3)~~

Second Proof (Krichever's inverse problem)

We start with L given.

- Put: $M(\bar z, \lambda) := \Delta\, \overline{L\left(z, \frac{1}{\bar\lambda}\right)}\, \Delta$

$$= I + \sum_{j\geq 1}^{\infty} \lambda^j \tilde C_j(\bar z) \quad \text{- entire in } \lambda.$$

$$\tilde C_j = \Delta \bar C_j \Delta$$

and

$$\frac{dM}{d\bar z} = \lambda M \bar\eta^T$$

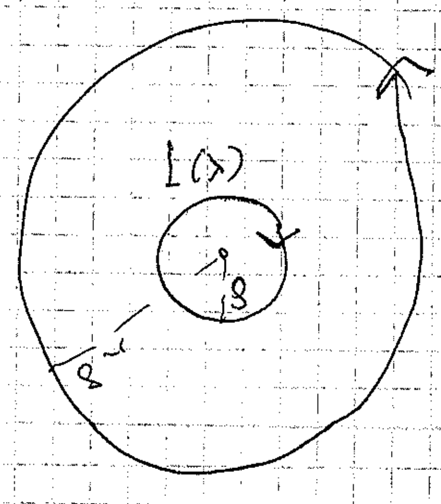
Define:

$$C_\rho = \{|z| = \rho\}, \quad C_{\rho^{-1}} = \{|z| = \rho^{-1}\}.$$

$$0 < \rho < 1$$

$$\Gamma = C_\rho + C_{\rho^{-1}}$$

$G(\lambda)$:



and Set the following RHP:

- $Y(\lambda) \in H(\mathbb{C} \setminus \Gamma)$

- $Y_+(\lambda) = G(\lambda) Y_-(\lambda) \qquad \lambda \in \Gamma$.

- $Y(0) = I$

Since $G(\lambda)\Big|_{z=0} = I \Rightarrow$ problem is locally

and smoothly in z solvable

$$Y(\lambda) = Y(z, \bar{z}, \lambda)$$

and $Y(\lambda)\Big|_{z=0} = I$.

$$\sigma \overline{G\left(\frac{1}{\bar{\lambda}}\right)} \sigma = G(\lambda) \leftarrow \text{by construction.}$$

$$\Downarrow$$

$$\sigma \overline{Y\left(\frac{1}{\bar{\lambda}}\right)} \sigma = Y(\lambda) D^{-1}$$

$$D = Y(\infty).$$

Note:

$$\sigma \bar{D} \sigma = D^{-1} \quad (28.1)$$

Also, one can check:

$$\Delta G^{-T}(-\lambda)\Delta = G(\lambda)$$

$$d_3 \, G(\omega\lambda)\, d_3^{-1} = G(\lambda)$$

$$\Downarrow$$

In addition to (28.1), we will have

$$D = \begin{pmatrix} \delta & & \\ & s & \\ & & \delta^{-1} \end{pmatrix}$$

$\delta = \bar{\delta}$, in fact $\delta > 0$. (locally)

~~(Details: [illegible])~~

Now, we are ready to produce L_R, L_+:

$$L_R(\lambda) = Y(\lambda)\, D^{-1/2} \qquad \varrho < |\lambda| < \varrho^{-1}$$

$$L_R(\lambda) = L\,Y(\lambda)\, D^{-1/2} \qquad 0 < |\lambda| < \varrho.$$

$$L_R(\lambda) = M\,Y(\lambda)\, D^{-1/2} \qquad |\lambda| > \varrho^{-1}$$

and:

$$L_+(\lambda) = D^{1/2}\, Y^{-1}(\lambda) \qquad |\lambda| < \varrho$$

$$L_+(\lambda) = D^{1/2}\, Y^{-1}(\lambda) L(\lambda) \,, \qquad \varrho < |\lambda| < \varrho^{-1}$$

$$L_+(\lambda) = D^{1/2}\, Y^{-1}(\lambda) M^{-1}(\lambda) L(\lambda) \qquad |\lambda| > \varrho^{-1}$$

$$\left(\begin{array}{ll} L_R L_+ = L & |\lambda| < \varrho \\ L_R L_+ = L & \varrho < |\lambda| < \varrho^{-1} \\ L_R L_+ = L & |\lambda| > \varrho^{-1} \end{array}\right)$$

Theorem 1 is proven.

~~(details are on pp. 34-37, I.K.-2)~~

<u>Remark</u>. ~~This,~~ This, second proof can be extended to the general element

$$L(\lambda) \in \Lambda SL(3, \mathbb{C})\,.$$

$$L(\lambda) = \sum_{j=-\infty}^{\infty} c_j \lambda^j$$

$$\det L = 1\,, \quad \Delta L^{-T}(-\lambda)\Delta = L(\lambda)$$

$$d_3 L(\omega\lambda) d_3^{-1} = L(\lambda)$$

~~(details on p 38-39, I.K.2)~~

Third Proof. (D-P-W?)

The idea: Suppose, we have Iwasawa:

$$L = L_R L_+ \qquad \hat{C}(L) = \sigma \overline{L\left(\frac{1}{\bar{\lambda}}\right)} \sigma$$

or

$$\hat{C}(L) = \left[L^*\left(\frac{1}{\bar{\lambda}}\right) \right]^{-1}$$

$\Downarrow$

$$\hat{C}(L) = \hat{C}(L_R)\,\hat{C}(L_+) = L_R\,\hat{C}(L_+).$$

$$J = \hat{C}^{-1}(L)L = \underbrace{\hat{C}^{-1}(L_+)}_{-\text{ function!}} L_+$$

$\Downarrow$

if we define $Y(\lambda) := L_+^{-1}(0) \begin{cases} L_+(\lambda) & |\lambda|<1 \\ \hat{C}(L_+)(\lambda) & |\lambda|>1 \end{cases}$

We should have the solution of the following RHP:

- $Y(\lambda) \in H(\overline{\mathbb{C}} \setminus \{|\lambda|=1\})$
- $Y_+(\lambda) = Y_-(\lambda) J(\lambda)$, $|\lambda| = 1$. (32.1)
- $Y(0) = I$.

Suppose now that we have solution of (32.1). Let us try to produce the Invariance from it:

First we notice.

$$\hat{C}(J) = J^{-1}(\lambda).$$

From this we get that

$$\hat{C}(Y)(\lambda) = D^{-1} Y(\lambda)$$

$$D = Y(\infty).$$

and

$$\sigma \bar{D} \sigma = D^{-1}$$

Put now:

$$\hat{Y}(\lambda) = D^{-1/2} Y(\lambda)$$

(probably again need $\lambda \to \omega\lambda$!)

$$\hat{C}(\hat{Y})(\lambda) = \hat{Y}(\lambda)$$

and put:

$$L_+(\lambda) = \begin{cases} \hat{Y}(\lambda) & |\lambda| < 1 \\ \hat{Y}(\lambda) J(\lambda) & |\lambda| > 1 \end{cases}$$

$\Downarrow$ $L_+(\lambda) \in H(\mathbb{C})$

and define:

$$\boxed{L_R(\lambda) := L L_+^{-1}}$$

We have that $L_R(\lambda) \in H(\mathbb{C} \setminus \{0\})$.

and

Let us check $\hat{C}(L_R)(\lambda) = L_R(\lambda)$: (34)

$$\hat{Y}(\lambda)\Big|_{|\lambda|>1} = \hat{C}\left(\hat{Y}\big|_{|\lambda|<1}\right)$$

$$L_+(\lambda) = \hat{Y}(\lambda)\Big|_{|\lambda|<1} = \hat{Y}(\lambda)\Big|_{|\lambda|>1} J(\lambda)$$

$$= \hat{C}(L_+)\,J(\lambda) = \hat{C}(L_+)\,\hat{C}^{-1}(L)\,L$$

$$\not= \qquad \Downarrow$$

$$\hat{C}(L)\,\hat{C}^{-1}(L_+) = L\,L_+^{-1} = L_R(\lambda)$$

$$\parallel$$

$$\hat{C}(L_R) \qquad \text{O.K.}$$

<u>Remark.</u> This ~~both~~ reduction to B-G, allows to use Vanishing Lemma to prove global existence of Ivanama for compact $\hat{C}$.

$$\vec{Y}_+ = \vec{Y}_- J$$

$$\vec{Y} \to \vec{0} \quad \lambda \to 0$$

$$0 = \oint \vec{Y}_+(\lambda)\, \vec{Y}_+^*\!\left(\tfrac{1}{\lambda}\right) \frac{d\lambda}{2\pi i \lambda} = \oint \vec{Y}_-(\lambda)\, J(\lambda)\, \vec{Y}_-^*\!\left(\tfrac{1}{\lambda}\right) \frac{d\lambda}{2\pi i \lambda}$$

$$= \oint \vec{Y}_-(\lambda) \left(\cancel{J(\lambda)} \vec{Y}_-(\lambda) J \right)^* \frac{d\lambda}{2\pi i \lambda}$$

$$= \oint \langle \vec{Y}_-(\lambda), J \vec{Y}_-(\lambda) \rangle \frac{d\lambda}{2\pi i \lambda} .$$

Final Remark.

$$\frac{d\Psi}{d\lambda} = \left[\frac{Z}{N\lambda^2} \eta^T + \frac{1}{\lambda} m \right] \Psi$$

$$|||$$

$$\frac{d\Psi^{(CP^2)}}{d\lambda} = \left[\frac{1}{\lambda^2} U + \frac{1}{\lambda} V \right] \Psi^{(CP^2)} \Big|_{(O z^2 c)}$$

$$\begin{tikzcd} tt^* \arrow[rr, leftrightarrow, "!?"'] & & \overline{P_{VI}} \end{tikzcd}$$