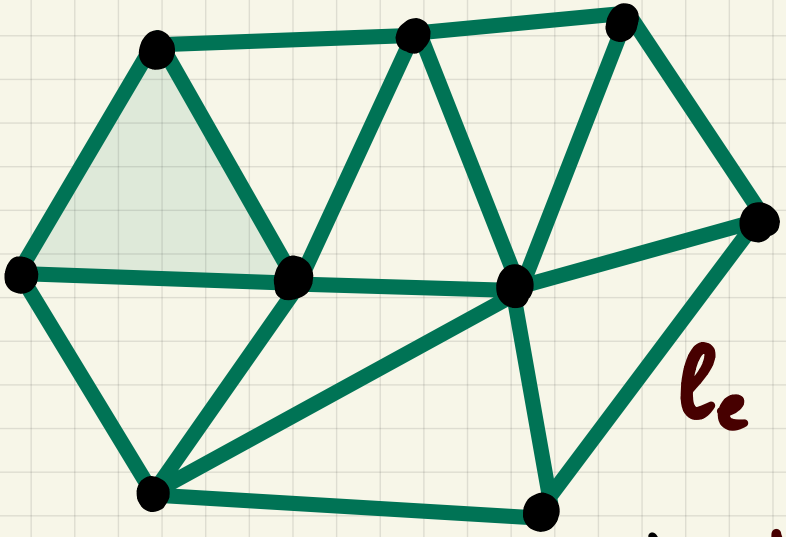
Integrability in gauge theory and sigma models

SIGMA MODELS

Fluctuating maps $\phi : \Sigma \longrightarrow X$

$$\int \mathcal{D}\phi \; e^{-S(\phi)}$$

Finite dimensional approximations



Γ - metric graph

$\varphi(v) \in X$

heat kernel

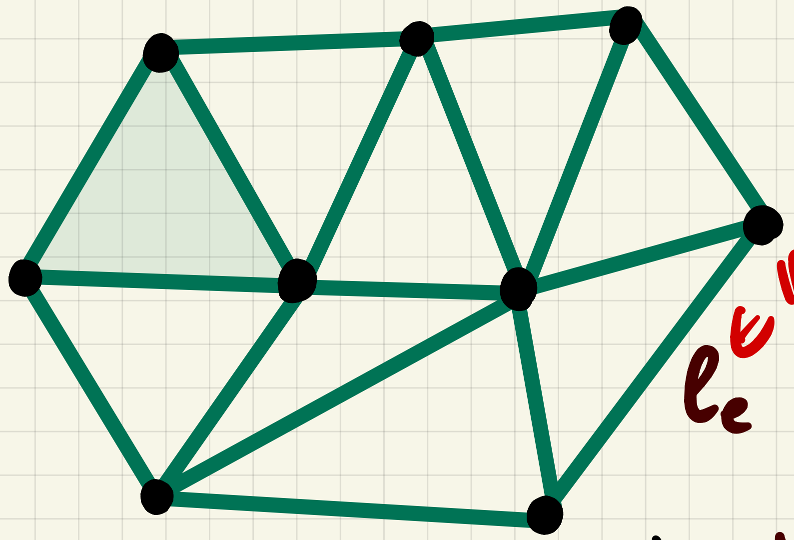
$M = X^{\Gamma^0}$

$\varphi \in M$

naive

$$\mu_\varphi = \prod_{\substack{e \\ \| \\ \langle vw \rangle}} K_{\ell_e}(\varphi(v), \varphi(w))$$

Finite dimensional approximations



Γ - metric graph

$\ell_e \in \mathbb{R}^+$, $m_e \in$ set of multiple geodesics on X

$\varphi(v) \in X$

$M = X^{\Gamma^0}$

$\varphi \in M$

heat kernel, in Selberg form
sum over geodesics

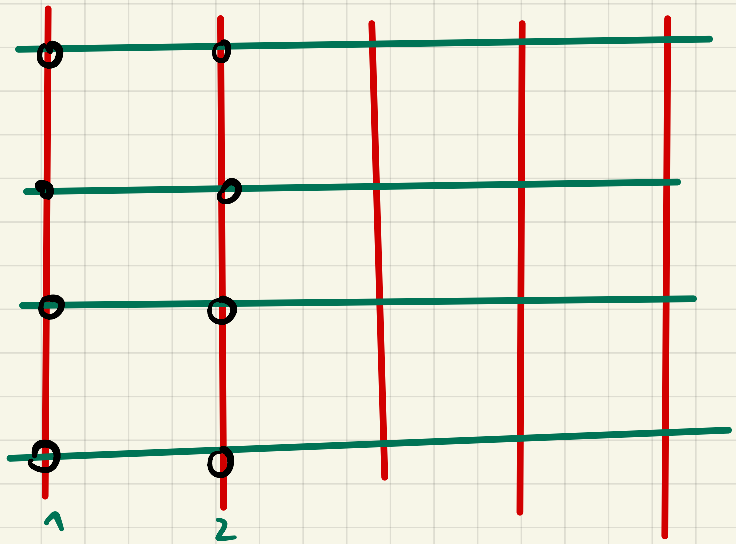
$$\mu_\varphi = \prod_{\substack{e \\ \| \\ \langle vw \rangle}} K^{m_e}_{\ell_e}(\varphi(v), \varphi(w)) \prod_F \delta_{\sum_{e \in \partial F} m_e}$$

to kill vortices

Finite dimensional approximations

Naive measure corresponds to transfer matrix

"Spin chain"



Integrable ?

Seems to depend on

X

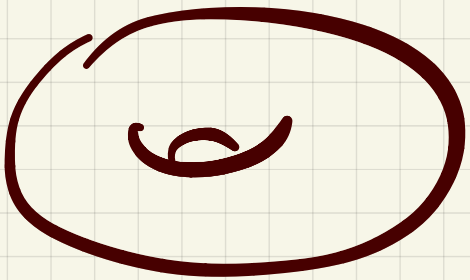
$$\int \Psi(X_{\text{column}_1}) \times \prod K(x_{c_1}, x_{c_2}) \, dx_{\text{column}_1} = \tilde{\Psi}(x_{\text{column}_2})$$

Lore : S^N_Y , $\mathbb{CP}^{N-1}_N$

Lefschetz thimbles

$$\int \mathcal{D}\phi \, e^{-S(\phi)} = \sum_c n_c \, e^{-S(\phi_c)} \times (1 + O(\hbar))$$

$\delta S|_{\phi_c} = 0$ complex critical points

$\Sigma =$ 

Turn on background gauge field B for $G \subset \mathrm{Isom}(X) \to G_{\mathbb{C}}$

$$X = S^{N-1} \qquad G = O(N)$$

$$\delta S = 0 \iff (-\Delta + u)\, q = 0\,, \quad q \in \mathbb{C}^N$$

double-periodic

Laplacian on T^2 (also complexified, $\tau, \bar{\tau} \neq \tau^*$)

Constraint $(q, q)_{\mathbb{C}^N} = 1$

Twist $T_x q = g_x q\,, \quad T_y q = g_y q$

$$g_x g_y = g_y g_x \qquad g_x, g_y \in O(N)$$

Fermi curve C_u = normalization of Bloch set

$$\left\{ (w_x, w_y) \;\middle|\; \text{s.t. } \exists\, \psi, \quad (-\Delta + u)\psi = 0, \quad \begin{array}{l} \psi(x+1,y) = w_x \psi(x,y) \\ \psi(x, y+1) = w_y \psi(x,y) \end{array} \right\}$$

Main claim (IK + NN)

C_u corresponding to $O(N)$ solutions is algebraic

has involution σ, and a meromorphic function E

two marked points P_+, P_- $\quad C_u^\sigma = \{P_+, P_-\}$

ψ – BA function 1° $\psi \sim e^{k_\pm \overset{z}{\bar z}} \sum_n \xi_n^\pm (z, \bar z) k_\pm^{-n}$ near $P_\pm$

Fermi curve C_u = normalization of Bloch set

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2° outside P_+, P_- is meromorphic with $(z, \bar z)$ independent divisor D of poles

$$D + D^\sigma = \underbrace{K}_{} + P_+ + P_-$$

K → divisor of zeroes of a holomorphic differential

Periodicity

$\exists$ Unique meromorphic differentials dp_x, dp_y s.t.

$$dp_x = dk_\pm \left(1 + O(k_\pm^{-2})\right), \quad dp_y = \begin{pmatrix} \tau \\ \bar{\tau} \end{pmatrix} dk_\pm \left(1 + O(k_\pm^{-2})\right)$$

near P_+, P_-

$$\mathrm{Re} \oint_{\text{all cycles}} dp_{x,y} = 0$$

Spectral curves $\qquad \oint dp_{x,y} \in 2\pi i \mathbb{Z}$

Function E

$\exists$ meromorphic function on C_u

σ-invariant

poles of E

$$r_i \, \psi(z, \bar{z}, q_{\pm}^{(i)}) = \psi_i^{\pm}$$

$$r_i^2 = \frac{1}{2(E_+ - E_-)} \operatorname{res}_{q_{\pm}^{(i)}} (E \, d\Omega)$$

$$\sum \psi_i^+ \psi_i^- = 1$$

$d\Omega$ - meromorphic on C_u/σ

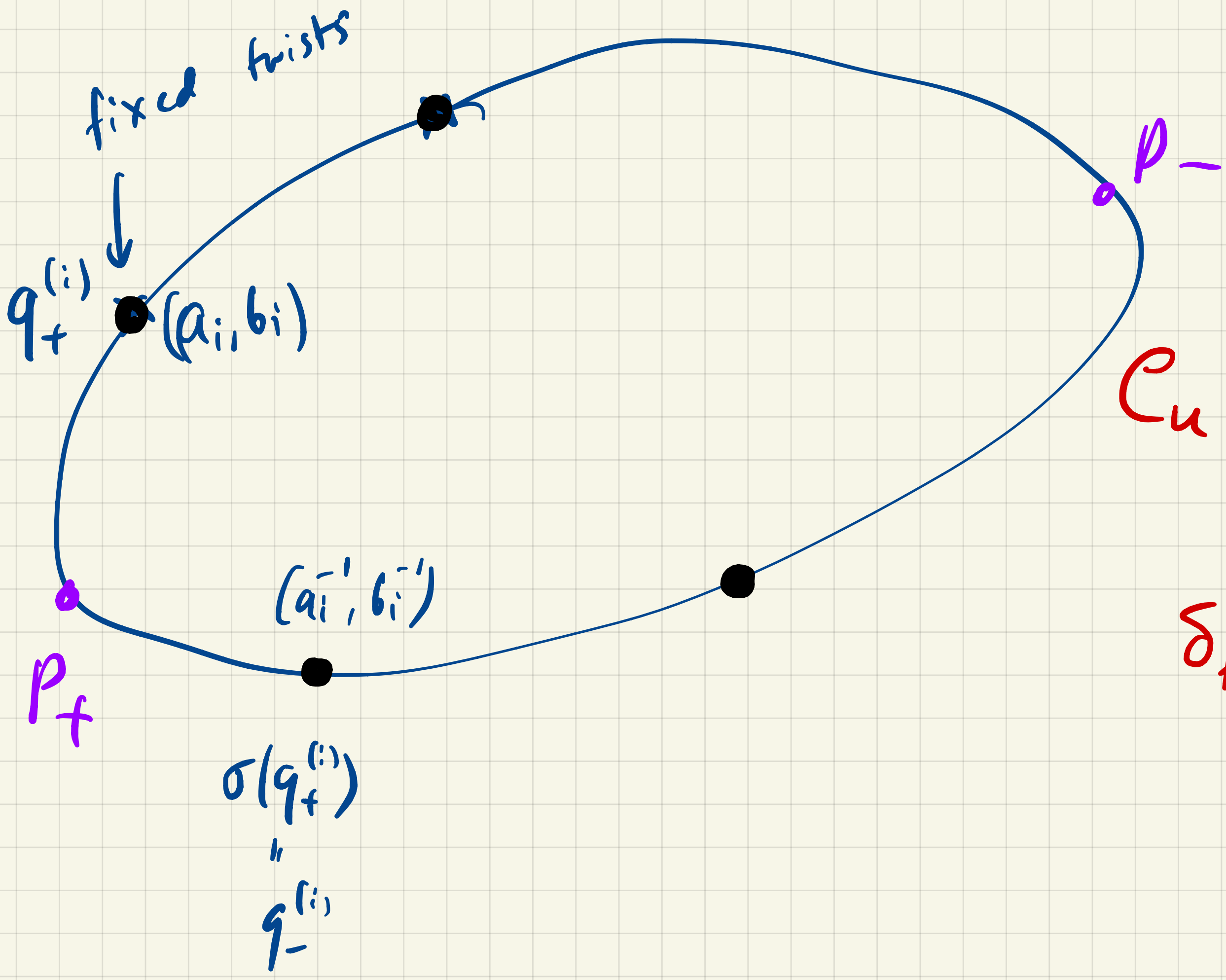
poles at $P_\pm$ only

$$\sigma(q_{\pm}^{(i)}) = q_{\mp}^{(i)}$$

can add more fixed points

$$\left(\sum \psi_i^+ \psi_i^- + x_i^2\right)$$

Why E exists



$\mathcal{U}_{a,b} = \{u \mid C_u$ passes through $(a_i^{\pm 1}, b_i^{\pm 1})\}$

Take any variation along $\mathcal{U}_{a,b}$ at u_*

$\updownarrow$

$\delta p_x dp_y$ is holomorphic on C_u/σ for critical u_* vanishing at $q^{(i)}$

$$q^{(i)} = (q_+^{(i)}, q_-^{(i)}) \in C_u/\sigma$$

$$g_0 + 1 - m = \#\{\delta p_x dp_y\}$$

$$\shortparallel$$

$$\dim S^{g_0}_{a,b} \times \quad h^0(q_+^{(1)} + \dots + q^{(m)}) = 2$$

$$m + 1 - g_0 + \cancel{=}$$

Novikov – Veselov

(for $\mathbb{CP}^{N-1}$ Dubrovin–Krichever–Novikov hierarchy)

$H\Psi = 0$ $\quad H = -\Delta + u$ $\quad H = -\Delta_A + u$

$(\partial_{t_n} - L_n)\Psi = 0$ $\quad (\partial_{t_n} - \bar{L}_n)\Psi = 0$

$$L_n = \partial_z^{2n+1} + \sum_{i=1}^{2n-1} w_{i,n}\, \partial_z^i$$

Manakov triple

$$\partial_{t_n} H = [L_n, H] + B_n H$$

Nontrivial fact : the S^{N-1} constraint is preserved
$\mathbb{CP}^{N-1}$ (generalization with gauge field)

Emergent

Alternative finite-dimensional approximation

– FINITE GAP (potentials, ...)

 → Fourier space → "trig polynomials"

4d gauge theory

Alternative finite dimensional approximation

$\longrightarrow$ equivariant localization

We are going to talk about a class of
$\mathcal{N}=2$ $d=4$ superconformal gauge theories

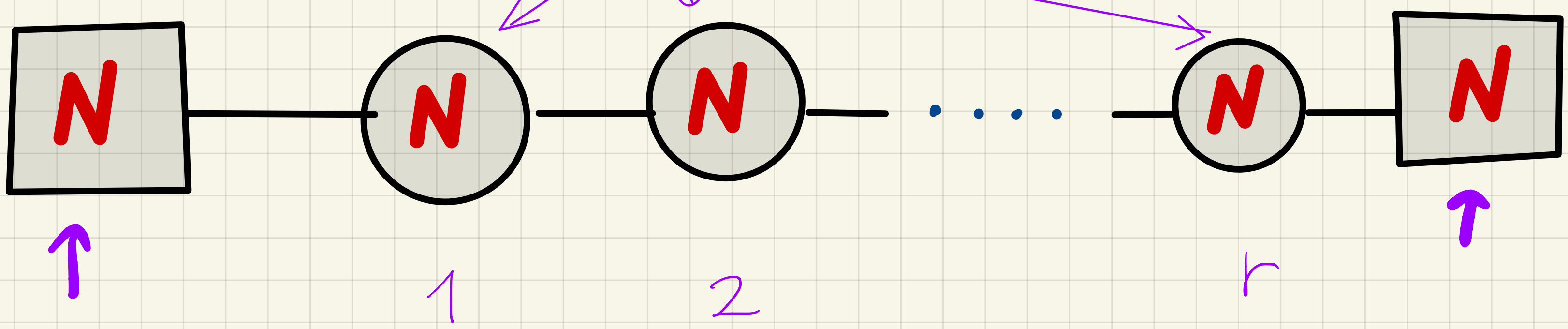
AA-type

$SU(N)^r$

quiver type

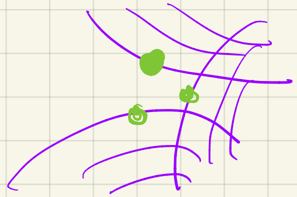
gauge group

nodes / gauge groups $SU(N)$



Susy Master formula for partition function

$$Z = \sum_{\Lambda} \left(\text{fugacities}\right)^{Ch\,\Lambda} \times \text{measure}(\Lambda)$$



dominating

$\longleftrightarrow$ instanton configurations in our gauge theory

fixed pts labels, e.g.

node, color $= 1, \dots, N$

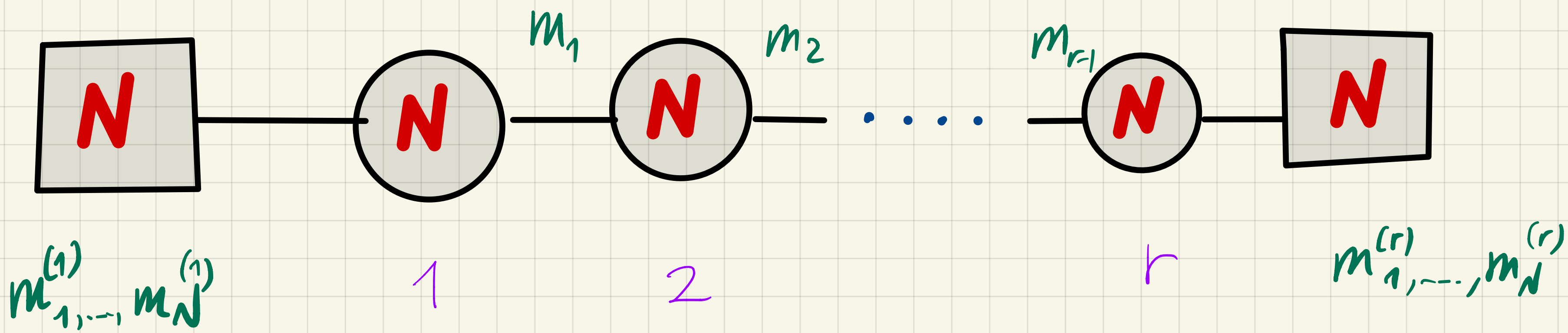
$$\Lambda = \left(\lambda^{(i,\alpha)} = \lambda^{(i,\alpha)}_1 \geq \lambda^{(i,\alpha)}_2 \geq \dots \geq 0\right)$$

non-increasing sequence of non-negative integers

Parameters of the model I : Bulk (4d)

$\vec{a}, \vec{m}, \vec{\tau}, \varepsilon_1, \varepsilon_2$

Masses of fundamental / Bi-fundamental hypers



Parameters of the model I : Bulk (4d)

$(\mathrm{Lie}\, SU(N)_i) \otimes \mathbb{C}$

$\cup$

$\vec{a}, \vec{m}, \vec{\tau}, \varepsilon_1, \varepsilon_2$

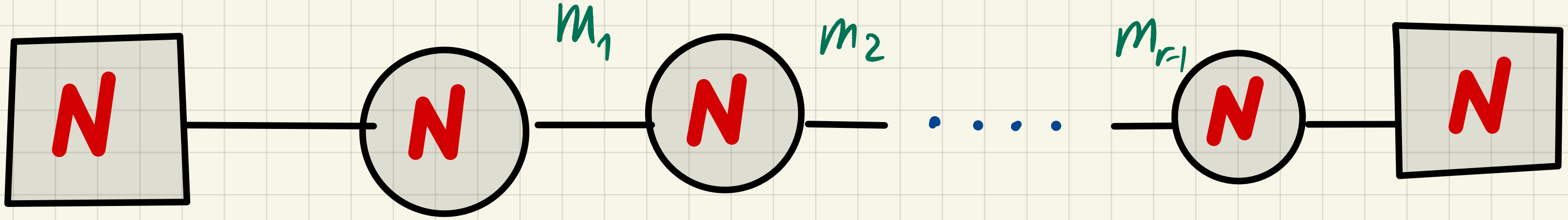
$\phi_i \longrightarrow \left(a^{(i,\alpha)} \in \mathbb{C} \right)_{i=1,\dots r}^{\alpha=1,\dots N}$

Coulomb parameters

$0 = \sum_{\alpha=1}^{N} a^{(i,\alpha)}$

$\circlearrowleft$ $SU(N)$

$\underline{a^{(1,\alpha)}}$ $\underline{a^{(2,\alpha)}}$ $\underline{a^{(r,\alpha)}}$



$m^{(1,f)}$ 1 2 r $m^{(r,f)}$

$f = 1, \dots, N$ $f = 1, \dots, N$

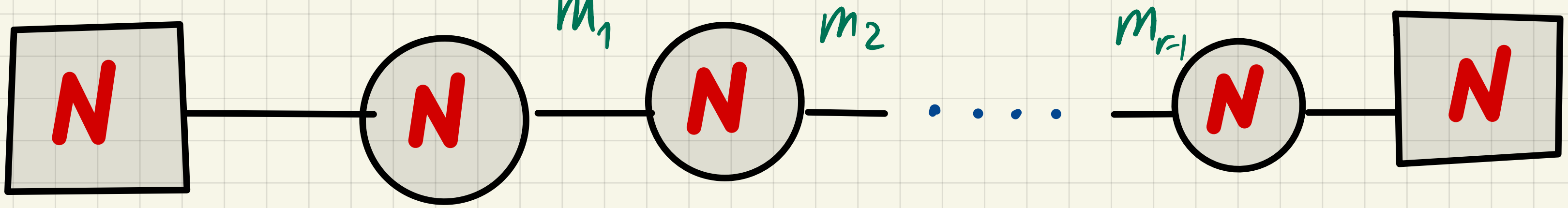
Parameters of the model I: Bulk (4d)

$\vec{a}, \vec{m}, \vec{\tau}, \varepsilon_1, \varepsilon_2$

$\left(a^{(i,\alpha)} \in \mathbb{C}\right)_{i=1,\dots,r}^{\alpha=1,\dots,N}$

Masses + Coulomb parameters

$0 = \sum_{\alpha=1}^{N} a^{(i,\alpha)}$ ↔ $SU(N)$



$$\mathbb{C}^{2N + r(N-1) + r-1} = \mathbb{C}^{(r+2)N} / \mathbb{C}$$

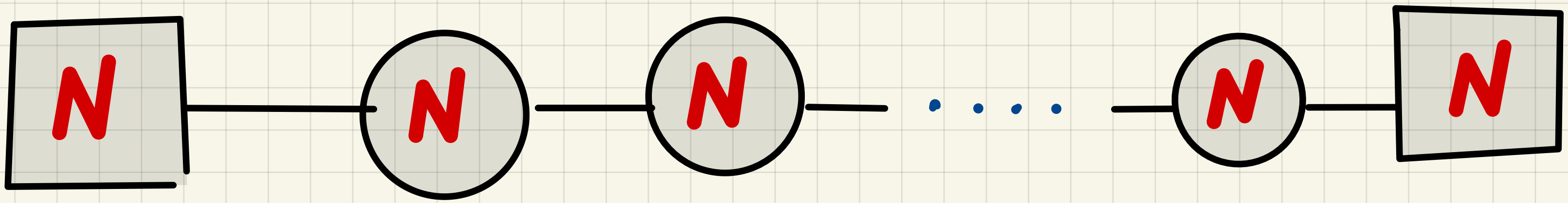
Parameters of the model I : Bulk (4d)

$\vec{a}, \vec{m}, \vec{\tau}, \varepsilon_1, \varepsilon_2$

$\mathrm{Tr}\,\phi_i - \mathrm{Tr}\,\phi_{i+1}$

$\sim m_{i,i+1}$

Masses + Coulomb parameters



Trade

$SU(N) \to U(N)$ $\quad \left(a^{(I,\alpha)} \in \mathbb{C} \right)_{\alpha = 1, \dots, N}^{I = 0, 1, \dots, r, r+1}$

$m^{(i,\alpha)}$

$a^{(i,\alpha)}$ obeying $\sum_\alpha a^{(i,\alpha)} = 0$

$\sim$ overall shift

m_e

$$\mathbb{C}^{2N + r(N-1) + r-1} = \mathbb{C}^{(r+2)N} / \mathbb{C}$$

Parameters of the model I : Bulk (4d)

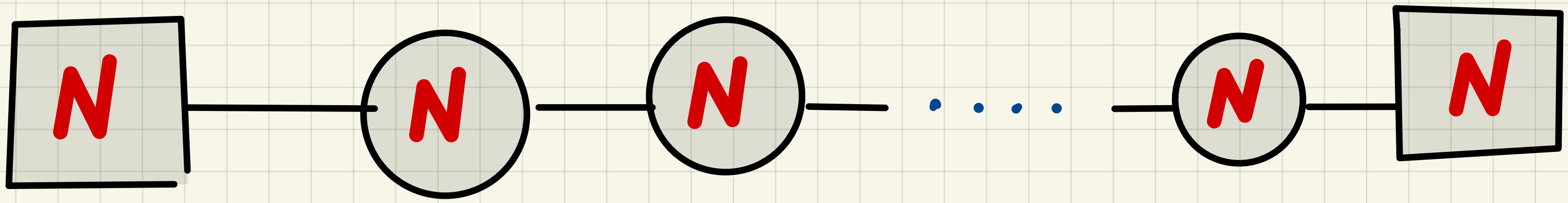
$$\left(a^{(I,\alpha)} \in \mathbb{C}\right)_{\alpha=1,\dots,N}^{I=0,1,\dots,r,r+1}$$

$\sim$ overall shift

$$\mathbb{C}^{(r+2)N}/\mathbb{C}$$

$$\vec{a}, \vec{m}, \vec{\tau}, \varepsilon_1, \varepsilon_2$$

Coupling constants



$q_{r+1}=0$

$q_0=0$ $\quad q_1 \quad q_2 \quad q_r$

$$\tau_i = \frac{\theta_i}{2\pi} + \frac{4\pi i}{g_i^2}$$

$$q_i = e^{2\pi\sqrt{-1}\,\tau_i} \in \mathbb{C}, \quad |q_i| < 1$$

convenient to extend to $(q_I)_{I=0,1,\dots,r,r+1}$

$q_0=0$

$q_{r+1}=0$

Parameters of the model I : Bulk (4d)

$$\left(a^{(I,\alpha)} \in \mathbb{C}\right)_{\alpha=1,\dots,N}^{I=0,1,\dots,r,r+1}$$

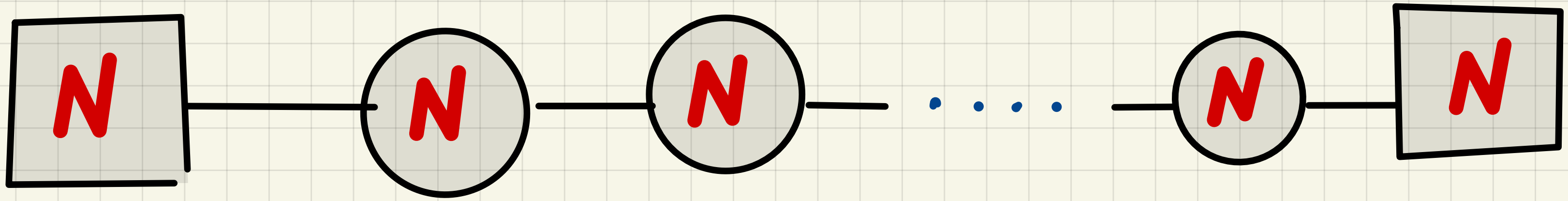
$\sim$ overall shift

$$\mathbb{C}^{(r+2)N}/\mathbb{C}$$

$$\vec{a}\,,\ \vec{m}\,,\ \vec{\tau}\,,\ \varepsilon_1, \varepsilon_2$$

$$(q_I)_{I=0,1,\dots,r,r+1}$$

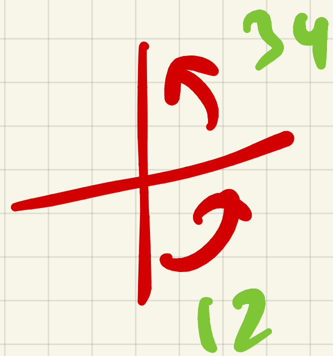
$$\mathbb{C}^r_{<1}$$



Equivariant parameters $\underbrace{\varepsilon_1, \varepsilon_2}$ $\in \mathbb{C}^2 = \mathrm{Lie}(\mathbb{C}^\times \times \mathbb{C}^\times)$

$\cap$

$Spin(4,\mathbb{C})$



$\mathbb{R}^4$

Weyl group

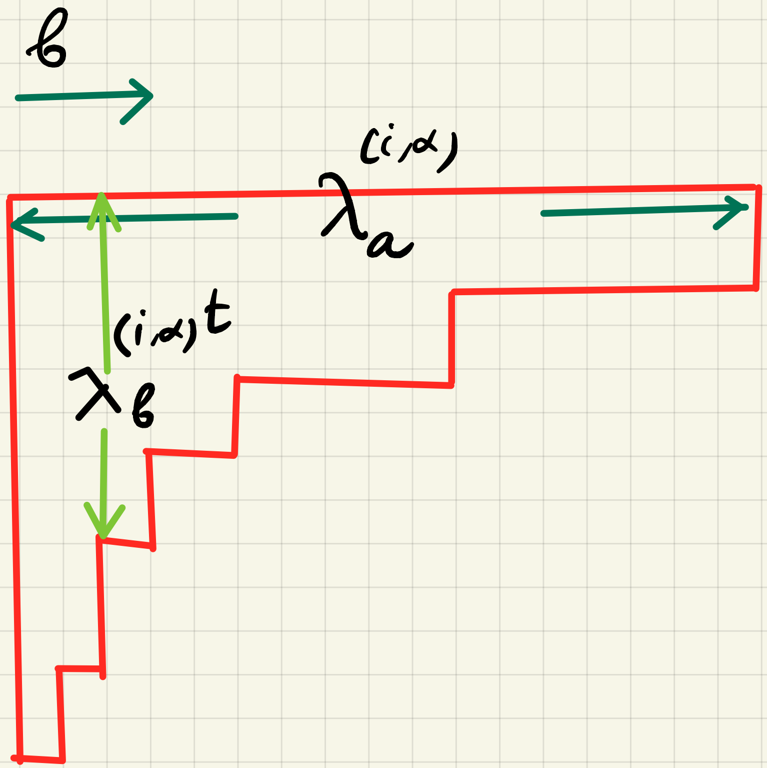
$\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\mathcal{Z} = \sum_{\Lambda} \left(\text{fugacities}\right) \times \text{measure}(\Lambda)$$

Ch Λ

$$\prod_{i=1}^{r}\prod_{\alpha=1}^{N} q_i^{-\frac{{a^{(i,\alpha)}}^2}{2\varepsilon_1\varepsilon_2} + |\lambda^{(i,\alpha)}|} \qquad |\lambda^{(i,\alpha)}| = \sum_a \lambda_a^{(i,\alpha)}$$

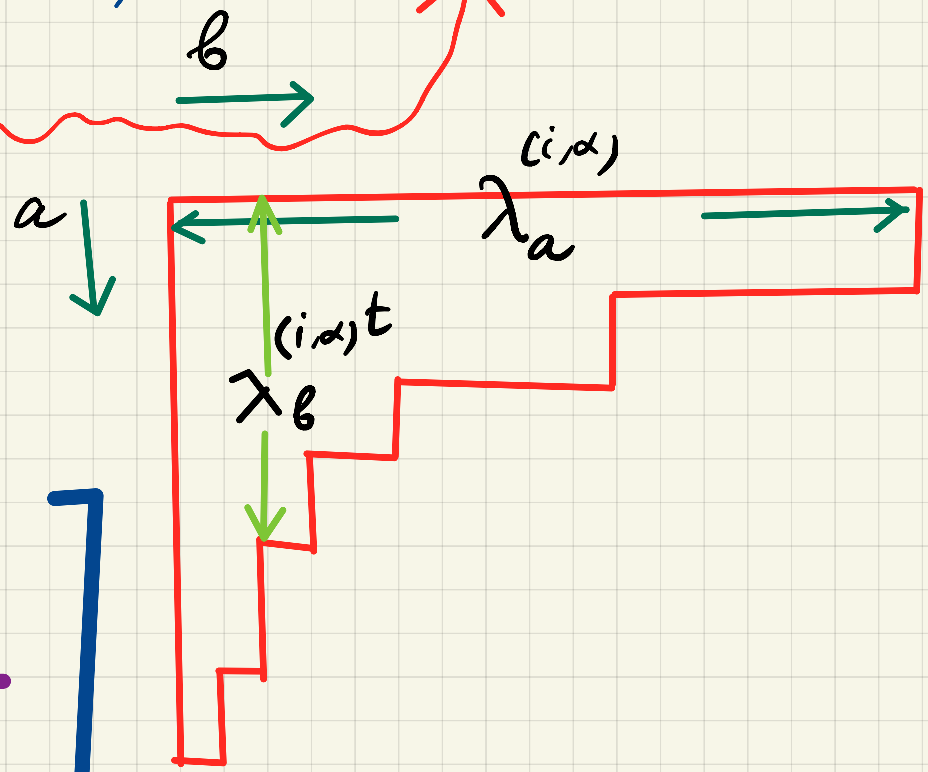
$$-\int_{\mathbb{C}^2}^{\mathbb{C}^\times \times \mathbb{C}^\times} ch_2(\mathcal{E}_i)$$



Young diagram of partition $\lambda^{(i,\alpha)}$

$\mathrm{Ch}\,\Lambda$

$$\mathbb{Z} = \sum_{\Lambda} \left(\text{fugacities}\right) \times \text{measure}(\Lambda)$$



$$E\left[\sum_{i=0}^{r+1} \frac{-S_i S_i^* + S_i S_{i+1}^*}{P_{12}^*}\right]$$

high temperature limit of a plethystic exponent

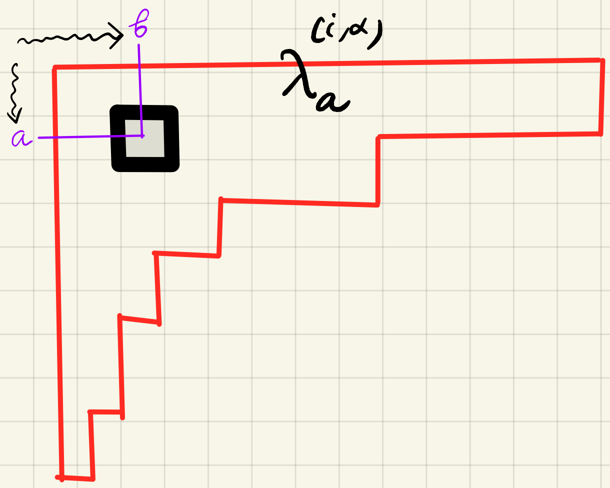
$$P_{12} = (1-e^{\beta\varepsilon_1})(1-e^{\beta\varepsilon_2})$$

$$E[a+b] = E[a]E[b]$$

To measure Λ need to build some character(s)

Virtual character for $i=1,\dots r$

$$S_i = N_i - P_{12} K_i$$



$$N_i = \sum_{\alpha=1}^{N} e^{\beta a^{(i,\alpha)}}$$

$$P_{12} = \left(1 - e^{\beta \varepsilon_1}\right)\left(1 - e^{\beta \varepsilon_2}\right)$$

$$K_i = \sum_{\alpha=1}^{N} \sum_{(a,b)\in\lambda^{(i,\alpha)}} e^{\beta\left(a^{(i,\alpha)} + \varepsilon_1(a-1) + \varepsilon_2(b-1)\right)}$$

Plethysm

$$E\left[e^{\beta x} - e^{\beta y}\right] = \frac{y}{x}$$

for finite virtual characters

Barnes integrals for infinite ones ....

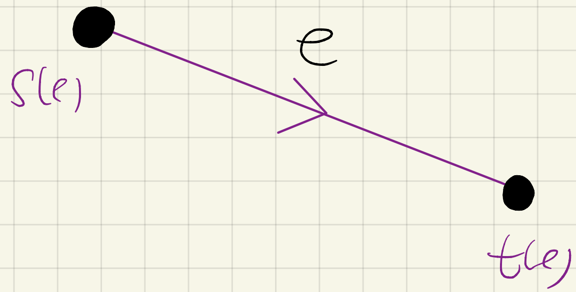
$$X_a^{(i,d)} = \{ a_{i,\alpha} + \varepsilon_1 (a-1-\lambda_a^{(i,d)}) \} \quad a=1\ldots\infty$$

Analogy with multi-matrix models

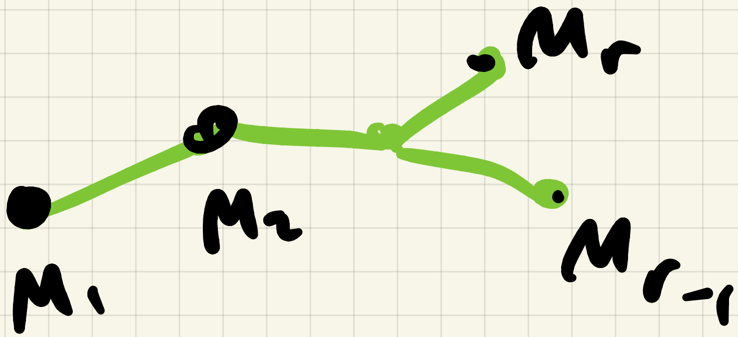
$$\int \prod_i dM_i\, e^{-V_i(M_i)} \prod_e d\Psi_e^\dagger d\Psi_e\; e^{-i\Psi_e^\dagger M_{s(e)}\Psi_e}\, e^{i\Psi_e M_{t(e)}\Psi_e^\dagger}$$

Single-trace potential with N critical points

$$\sim$$



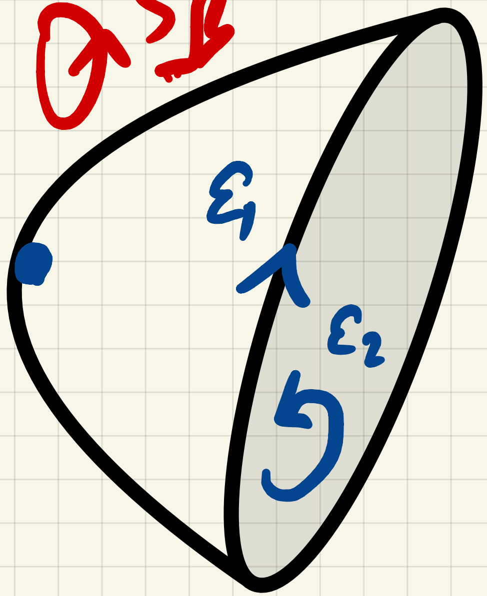
$$E\left[\sum_{i=0}^{r+1} \frac{-S_i S_i^* + S_i S_{i+1}^*}{P_{12}^*}\right]$$



$$\varepsilon_1 = -\varepsilon_2$$

Z — Supersymmetric partition function $\sim \lim_{\beta \to 0}$ of Witten index of 4+1 susy theory

$\mathbb{R}^4$ $\odot S^1_\beta$



S^3_∞

$a \longleftrightarrow \langle \phi \rangle$ ← scalars in vector multiplets

Local observables

$$\int dM_i \ldots ??$$

$$\left\langle \mathrm{Tr}\, \frac{1}{x-M_i} \right\rangle$$

$$Y_i(x) \sim x^N \exp \sum_{\ell=1}^{\infty} \frac{1}{\ell x^\ell} \mathrm{Tr}\, \phi_i^\ell$$

quiver node

auxiliary variable

Scalar in the vector multiplet of $SU(N)_i$

gauge for $i=1\ldots r$

flavor for $i=0, r+1$

Translating to the ensemble of $\Lambda = (\lambda^{(i,\alpha)})$'s

$$Y_i(x)\Big|_{\Lambda} = \prod_{\alpha=1}^{N} \left((x - a^{(i,\alpha)}) \prod_{(a,b)\in\lambda^{(i,\alpha)}} S\left(x - a^{(i,\alpha)} - \varepsilon_1(a-1) - \varepsilon_2(b-1)\right) \right)$$

$$S(x) = \frac{(x-\varepsilon_1)(x-\varepsilon_2)}{x(x-\varepsilon_1-\varepsilon_2)} \underset{x\to\infty}{\approx} 1 + \frac{\varepsilon_1\varepsilon_2}{x^2}$$

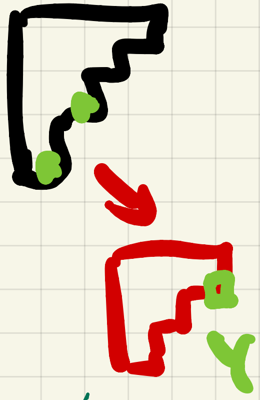
Translating to the ensemble of $\Lambda = (\lambda^{(i,\alpha)})$'s

$$Y_i(x)\Big|_{\Lambda} = \prod_{\alpha=1}^{N} (x - a^{(i,\alpha)}) \times \left(1 + O(x^{-2})\right)$$

functions of Λ

$$= x^N - \sum_{\alpha=1}^{N} a^{(i,\alpha)} x^{N-1} + \left(\sum aa + \varepsilon_1\varepsilon_2 k_i\right) x^{N-2} \left(1 + \frac{\varepsilon_1\varepsilon_2 \#}{x^2}\right)$$

$x \to \infty$



Measure is characterized by

Dyson-Schwinger relations

Analyticity of qq-characters expectation values

observable of gauge!

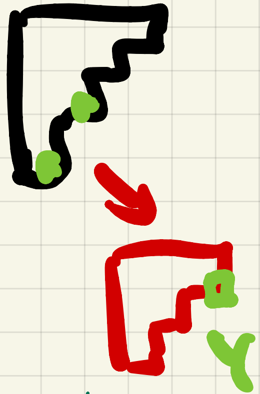
$$\left\langle \left(Y_i(x+\underbrace{\varepsilon_1+\varepsilon_2}_{\text{pole}}) + q_i \frac{\prod_{j\to i} Y_j(x) \prod_{i\to j} Y_j(x+\varepsilon)}{Y_i(x)} \right) + \dots \right\rangle$$

$\varepsilon = \varepsilon_1+\varepsilon_2$

pole

zero

has no poles in x !



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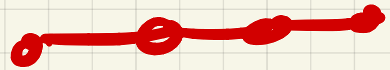
$$\left\langle \left(Y_i(x+\overbrace{\varepsilon_1+\varepsilon_2}^{\varepsilon}) + q_i \, \frac{\prod_{j\to i} Y_j(x) \prod_{i\to j} Y_j(x+\varepsilon)}{Y_i(x)} + \dots \right) \right\rangle$$

pole

zero

has no poles in x !

For $\varepsilon_1, \varepsilon_2 \to 0$ $\quad q_\alpha = e^{\beta \varepsilon_\alpha}$



$Q = A_r$

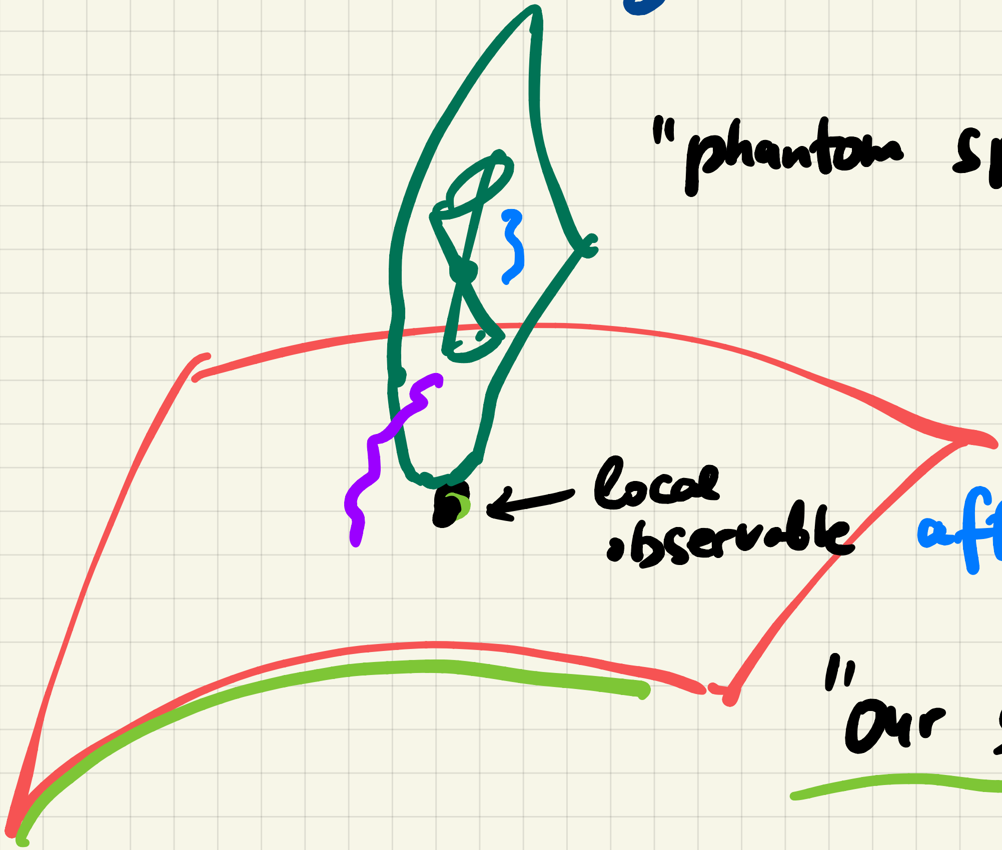
$\mathfrak{g}_Q = \mathfrak{sl}(r+1)$

$\underbrace{\text{qq- character}}_{5d}$ = Character of i'th fundamental representation of $\mathfrak{g}_Q$ $\quad \chi_i(x) =$

$$\mathrm{Tr}_{V_i}\left(\prod_{j=1}^{r}\left(Y_j(x)^{\alpha_j^\vee}\left(\underset{\uparrow q}{q_j} P_j(x)\right)^{\lambda_j^\vee}\right)\right) \sim g(x) \in G_Q \quad \forall x$$

For general quivers qq-characters are given by integrals over generalised Nakajima varieties

"phantom spacetime $\mathbb{R}^4/\Gamma$"



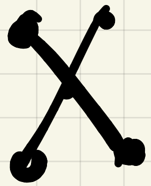
after phantom d.o.f. are integrated out

"our spacetime $\mathbb{R}^4$"

For A_r quiver - explicit formula for fundamental

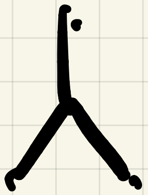
qq-characters

$$\chi_i(x) \qquad i = 1, \ldots, r$$



In the $\varepsilon_1, \varepsilon_2 \to 0$ limit

Y_0, Y_{r+1} fund hypers



$$\boxed{\text{qq-character} \longrightarrow \text{character}}$$

$$Y_i + \mathfrak{q}_i \frac{Y_{i+1} Y_{i-1}}{Y_i} + \cdots$$

$$\chi_i(x) \xrightarrow[\varepsilon_1, \varepsilon_2 \to 0]{} \frac{Y_0(x)}{z_1 \cdots z_i} \mathrm{Tr}_{\Lambda^i \mathbb{C}^{r+1}} \, g(x)$$

$$\binom{r+1}{i}$$

$$g(x) = \mathrm{diag}\left(z_i \frac{Y_i(x)}{Y_{i-1}(x)} \right)_{i=1}^{r+1}$$

$$\mathfrak{q}_i = \frac{z_{i+1}}{z_i}$$

$i = 1, \ldots, r$

Packaging Dyson-Schwinger eqs.

In the $\varepsilon_1, \varepsilon_2 \longrightarrow 0$ limit

polynomials in x of degree N

camera l affine curve in $\mathbb{C}^r \times \mathbb{CP}^1_x$

$$\boxed{\frac{T_i(x)}{z_1 \cdots z_i} \, \mathrm{Tr}_{\Lambda^i \mathbb{C}^{r+1}} g_\infty = \frac{Y_0(x)}{z_1 \cdots z_i} \, \mathrm{Tr}_{\Lambda^i \mathbb{C}^{r+1}} g(x)}$$

Seiberg-Witten curve(s)

$$g(x) = \mathrm{diag}\left(z_i \frac{\langle Y_i(x) \rangle}{\langle Y_{i-1}(x) \rangle} \right)_{i=1,\dots,r}$$

$$\boxed{q_i = \frac{z_{i+1}}{z_i}}$$

$$g_\infty = \mathrm{diag}\left(z_1, \dots, z_{r+1} \right)$$

$\langle Y_i(x) \rangle = Y_i(x$

"Simplicity" of A_r case

$$
\begin{tikzcd}
S(r+1) \curvearrowright C^{\text{cameral}} \arrow[rr, "r!:1"] \arrow[dd, "(r+1)!:1"'] & & C^{\text{Spectral}} \in \mathbb{CP}^1_x \times \mathbb{C}_z \arrow[ddll, "r+1:1"] \\
& & \\
\mathbb{CP}^1_x & &
\end{tikzcd}
$$

$$a^{(0,f)} = m^{(1,f)}$$

$$T_0(x) \equiv Y_0(x) = \prod_f \left(x - m^{(1,f)}\right)$$

$$T_{r+1}(x) \equiv Y_{r+1}(x) = \prod_f \left(x - m^{(r,f)}\right)$$

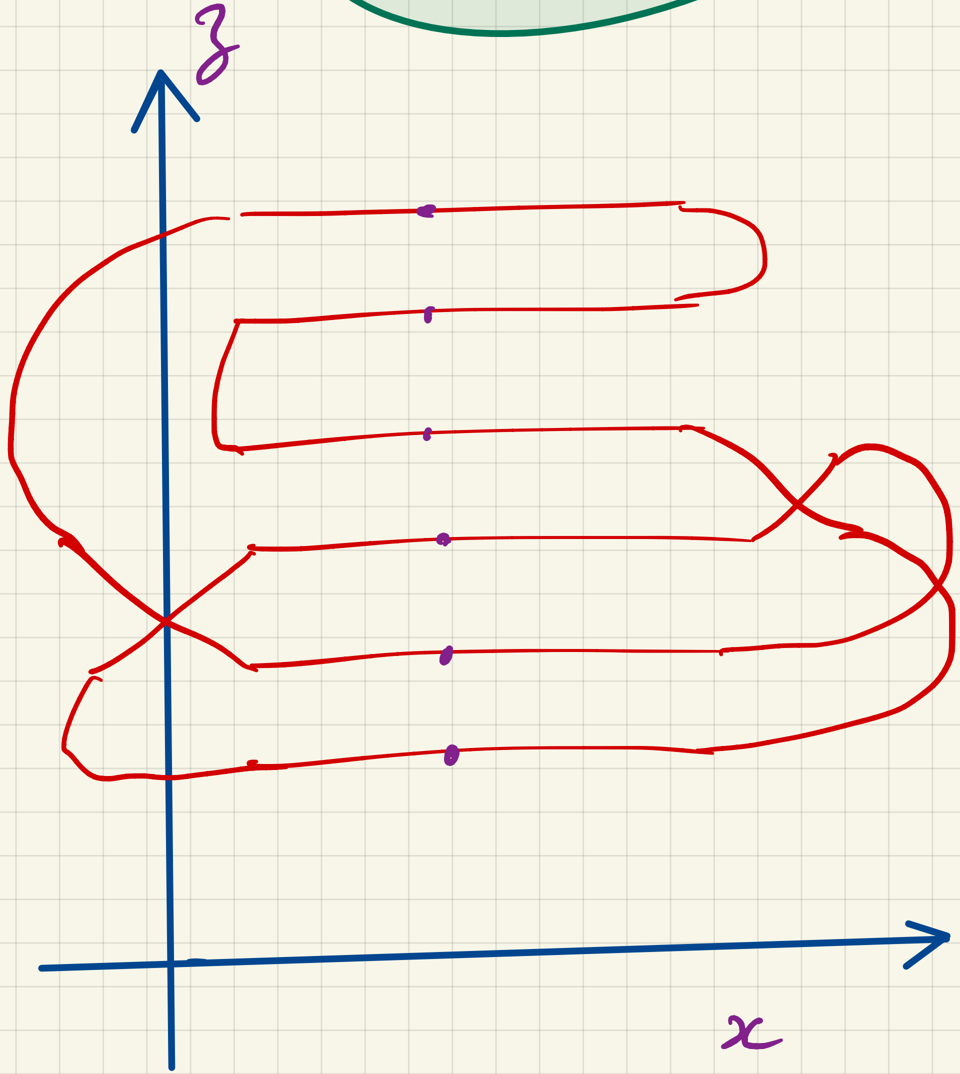
$$m^{(r,f)} = a^{(r+1,f)}$$

$$Y_0(x)\det\left(1 - \frac{g(x)}{z}\right) = 0 = \sum_{i=0}^{r+1} (-)^i z^{-i}\, T_i(x)\, \sigma_i(z_1, \dots, z_{r+1})$$

C^{spectral}

parametrizes the spectrum of a $GL(r+1)$-valued function over

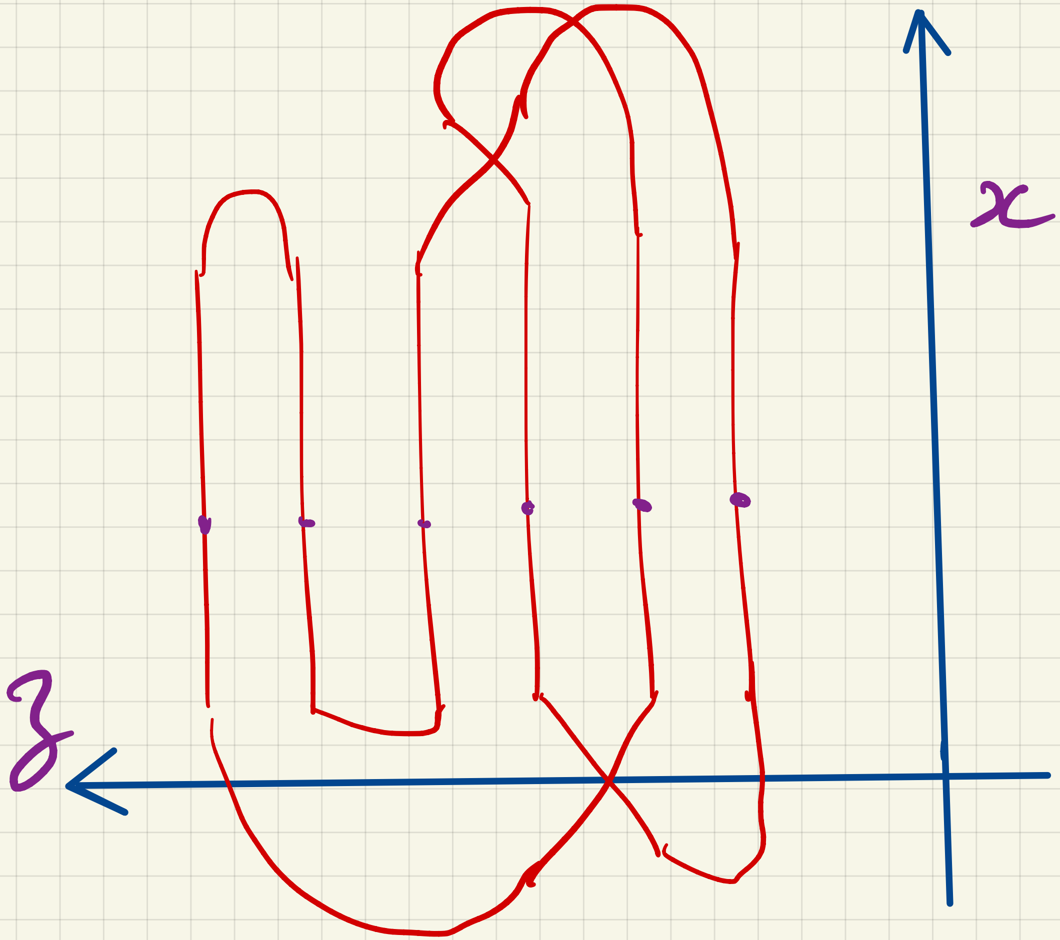
x



tropical limit

C^{spectral}

$$\frac{Y_0(x)\,\det_{r+1}\left(1-\frac{g(x)}{z}\right)}{\det_{r+1}\left(1-\frac{g_\infty}{z}\right)} = x^N + \dots = \mathrm{Det}_N\left(x - L(z)\right)$$

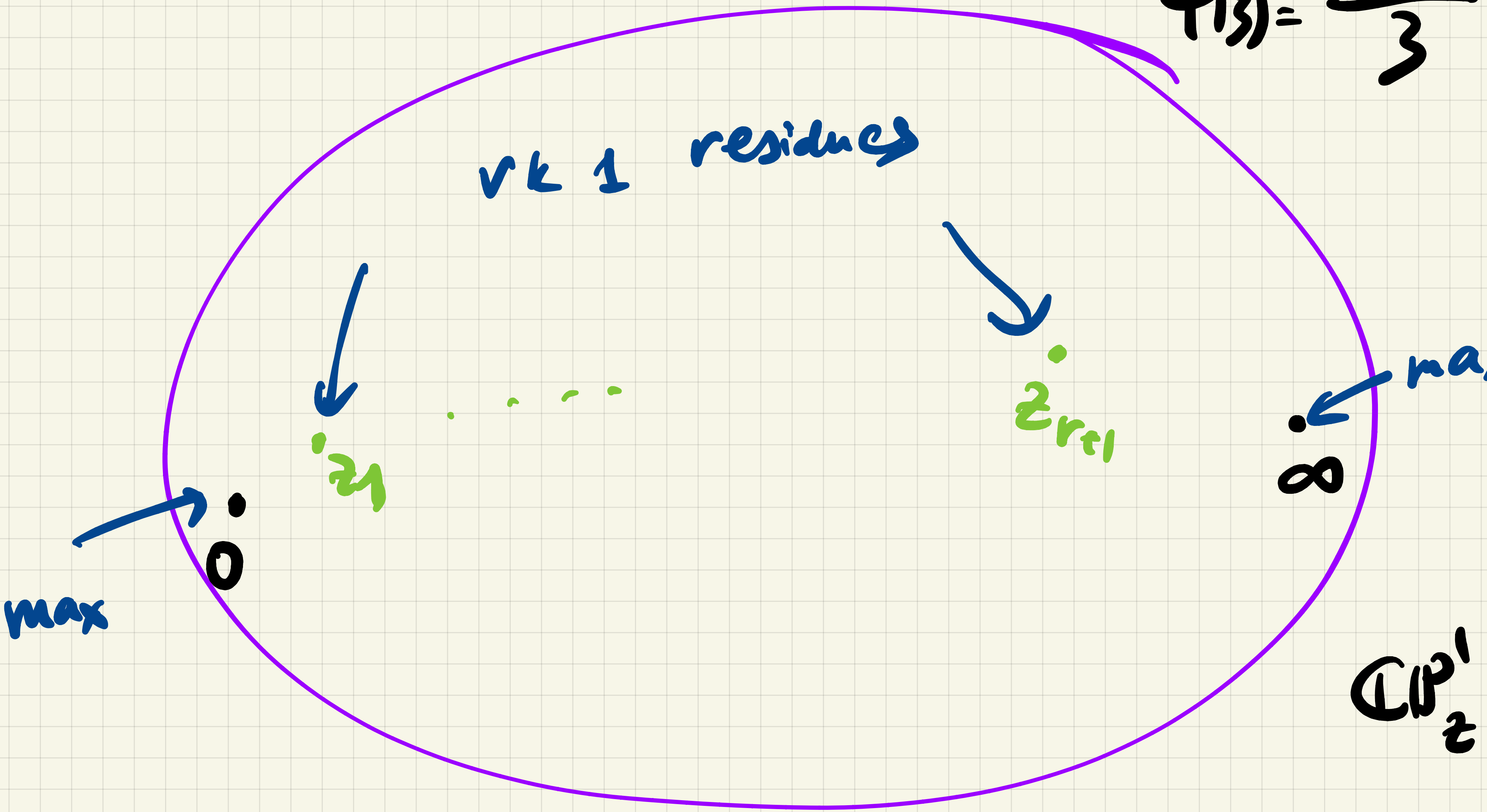


parametrizes the spectrum of a SL_N - valued Higgs field

$$\boxed{L(z) = L_0 + \sum_{i=1}^{r+1} \frac{z\, L_i}{z - z_i}}$$

rk 1

$$\Phi(z) = \frac{L(z)}{z}$$



What is $L(\mathfrak{z})$?

$$g(x) = P\exp \oint (A_3 + i\phi)$$

$\mathbb{R}^2 \times S^1$
$\shortparallel$
$\mathbb{C}_x$

High road

Fourier-Mukai
Nahm

Stringy ....

Low road

Gauge theory

Surface observables

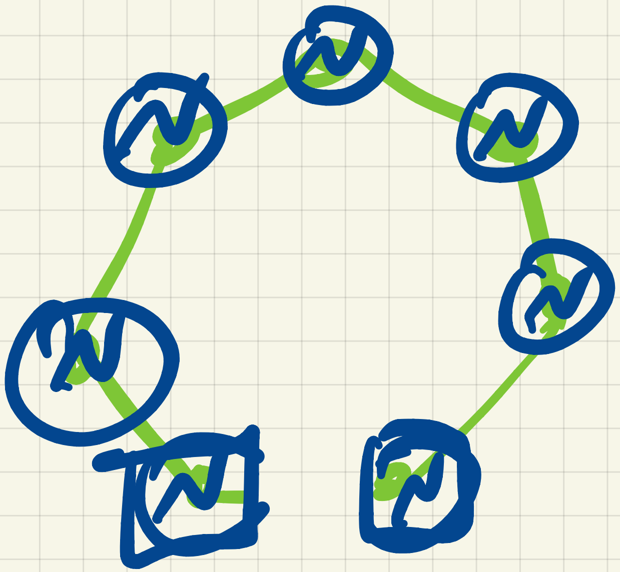
Philosophy

$\mathbb{C}^1 \times \mathbb{C}^2/\mathbb{Z}_{r+2}$

D3

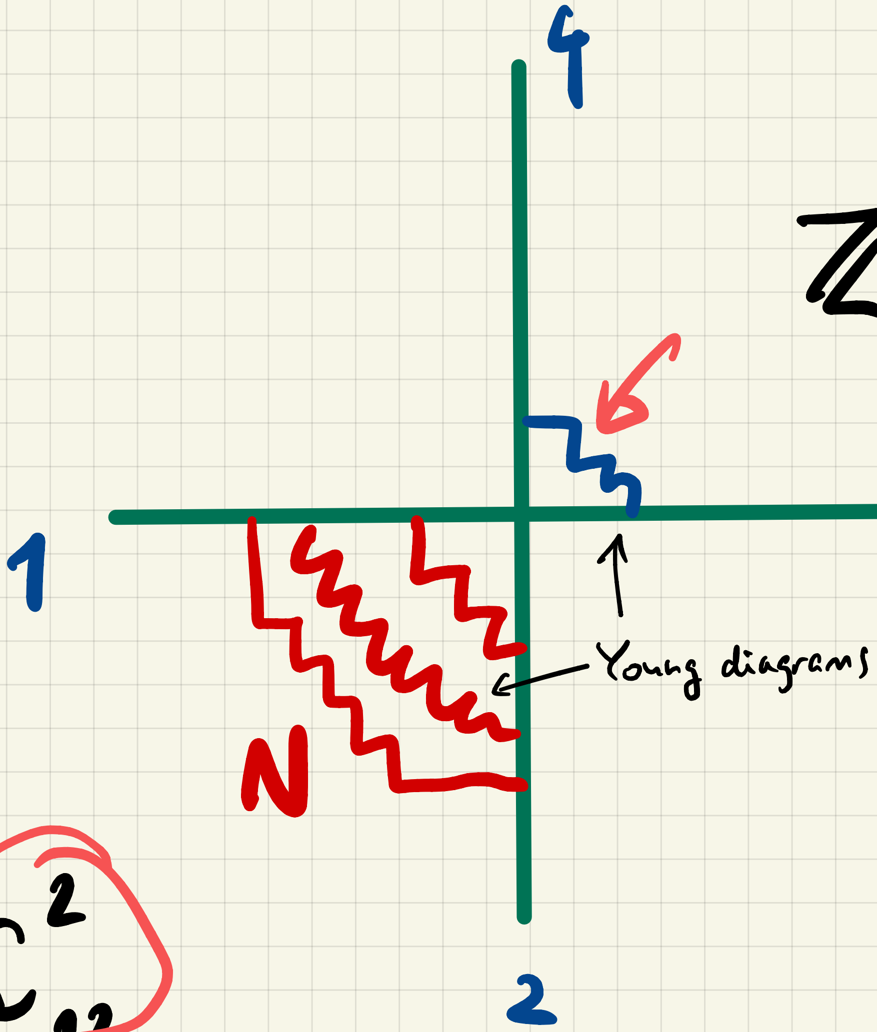
A_r quiver theory $=$

$\mathbb{Z}_{r+2}$ orbifold of $\mathcal{N}=4$ SYM



$q_0, q_{r+1} = 0$

$$\mathbb{C}^2_{12} \subset \mathbb{C}^4_{1234} \times \mathbb{C}^1$$



$\mathbb{Z}_{r+2}$ acts on $\mathbb{C}^2_{34}$

$$(\Omega z_3, \Omega^{-1} z_4)$$

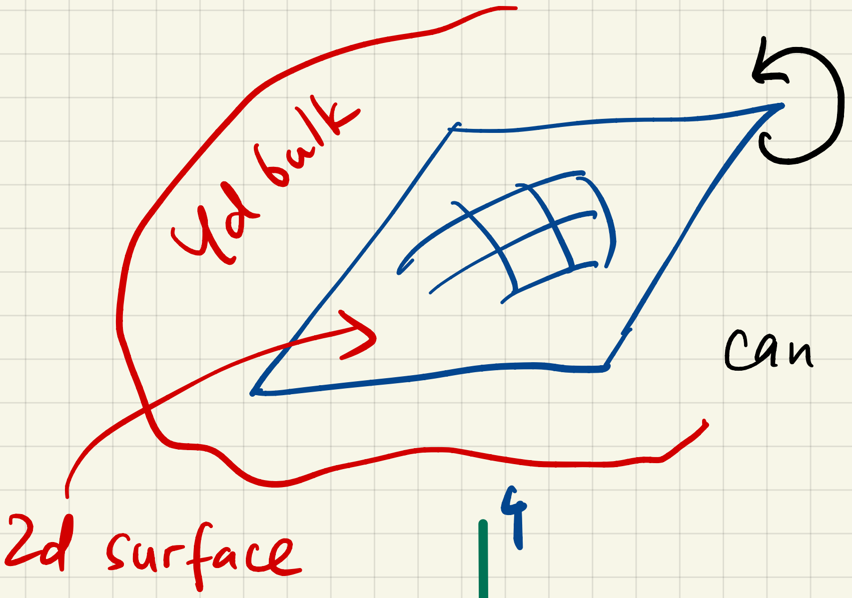
qq- observables = add gauge theory on $\mathbb{C}^2_{34}$

$\mathbb{C}^2_{12}$

physical space-time

Regular surface defect

$(G_{gauge} \to T_{gauge} \subset G_{gauge})$



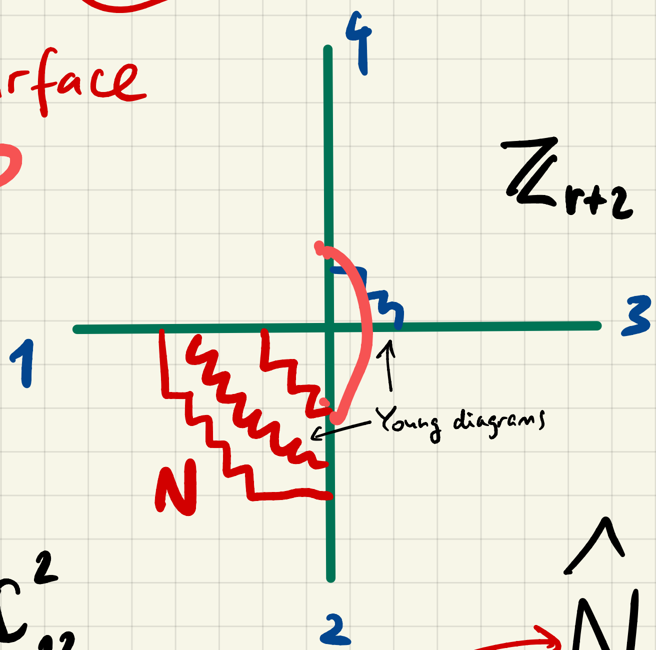
F_A curvature $\sim J \, \delta^{(2)}_{surface}$

can also be engineered using an $\mathbb{Z}_N$ orbifold

SU(3)

$\mathbb{C}^4_{1234}$

$z_2 = 0$

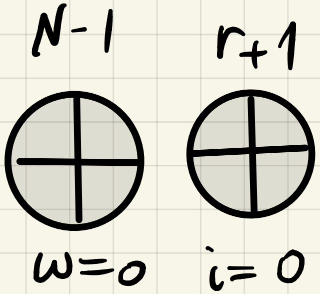


$\mathbb{Z}_{r+2}$ acts on $\mathbb{C}^2_{34}$ (and Chan-Paton spaces)

$\mathbb{Z}_N$ acts on $\mathbb{C}^2_{24}$ (and Chan-Paton spaces)

$\mathbb{C}^2_{12}$

$\hat{N}_{12} =$



Chan-Paton

$\mathbb{C} = \mathbb{C}[\mathbb{Z}_{r+2} \times \mathbb{Z}_N]$

Fractionalization (term borrowed from D-branes at orbifolds)

zeromodes bundles (sheaves)

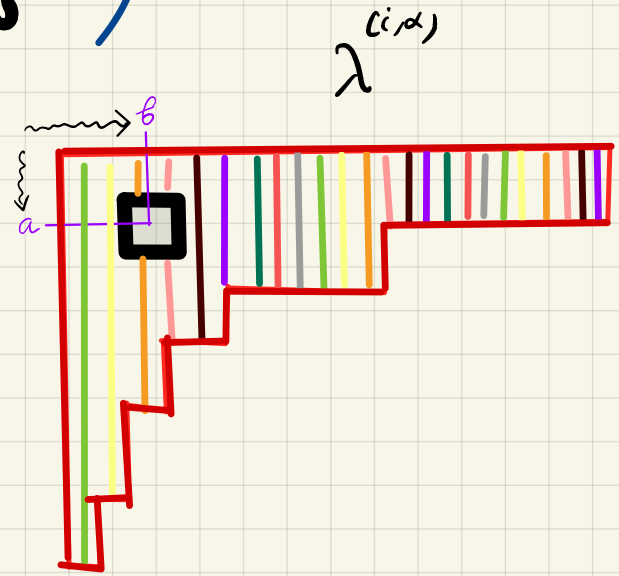
$\mathcal{O}_i \longrightarrow (\mathcal{E}_{i,\omega})$

$\omega = 0, \dots, N-1 \longleftrightarrow$ irreps of $\mathbb{Z}_N$

$$Y_i(x) \longrightarrow \hat{Y}_i(x) = \mathrm{diag}\left(Y_{i,\omega}(x)\right)$$

$\mathbb{Z}_{r+2}$ irreps

encode finer structure of Λ



N color of the box $\square_{a,b} \in \lambda^{(i,\alpha)}$

$$= \alpha - 1 + b - 1 \mod N$$

$$H^2(G/T) = \mathbb{Z}^{N-1}$$

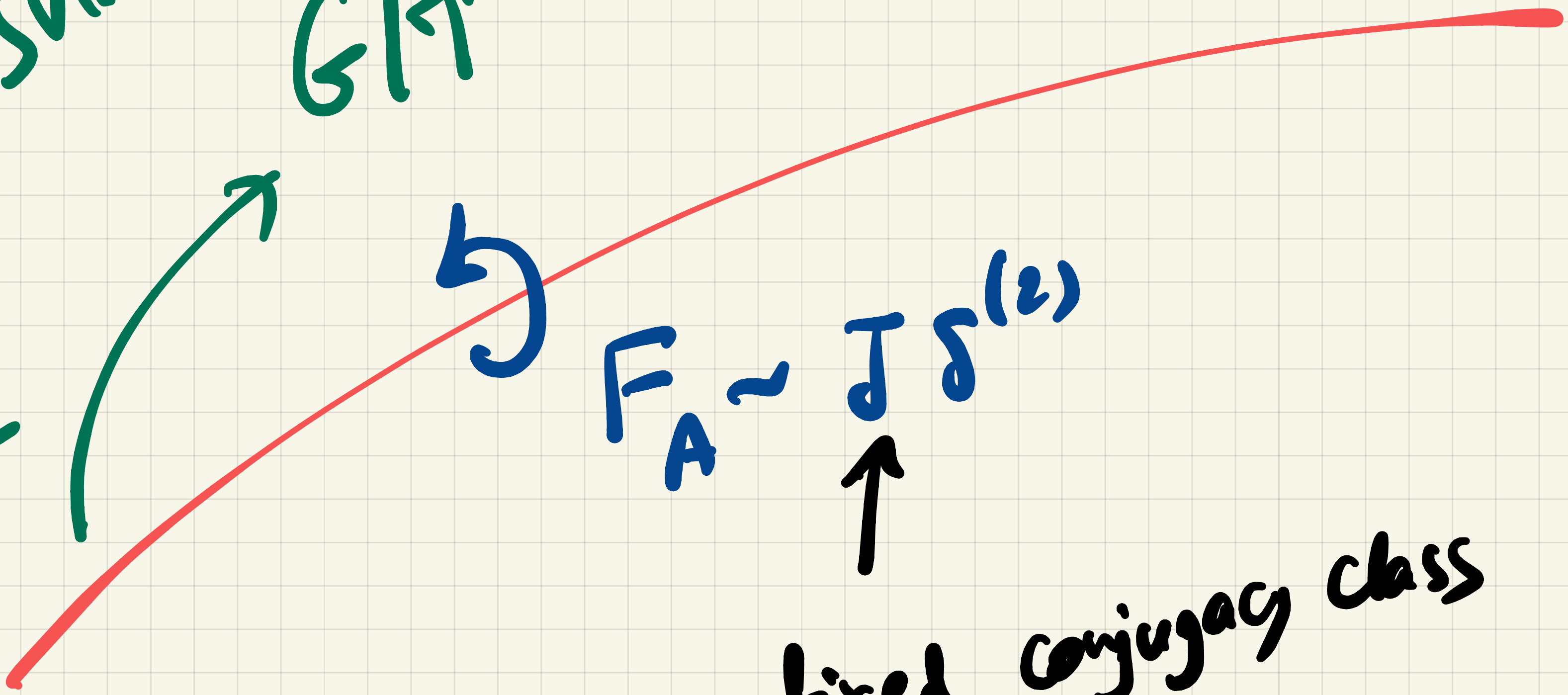
$G = SU(N)$, $T = U(1)^{N-1}$

G/T

$q_i \rightarrow q_{i,w}$

$w = 0, \ldots, N-1$

$\mathbb{R}^2 \subset \mathbb{R}^4$



$\mathbb{R}^2$

$F_A \sim J\,\delta^{(2)}$

fixed conjugacy class

Fractionalization (term borrowed from D-branes at orbifolds)

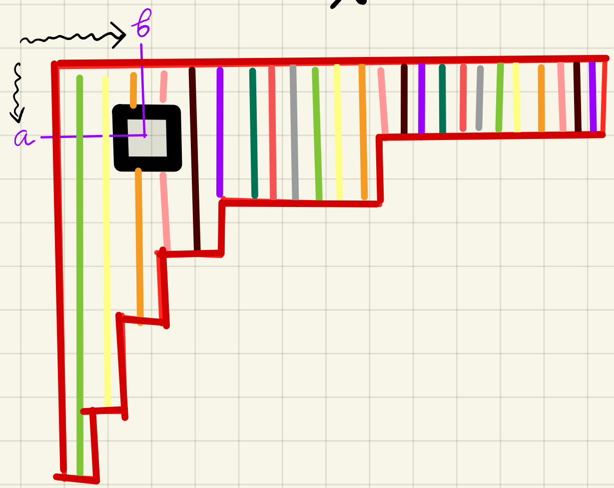
couplings

$$q_i \longrightarrow \left(q_{i,\omega} \right)$$

$\omega = 0, \dots, N-1 \longleftrightarrow$ irreps of $\mathbb{Z}_N$

$$z_i \longrightarrow \hat{Z}_i = \mathrm{diag}\left(z_{i,\omega} \right)$$

$\lambda^{(i,\alpha)}$



N color of the box

$\square_{a,b} \in \lambda^{(i,\alpha)}$

$= \alpha - 1 + b - 1 \mod N$

Why matrices? So far, $\hat{Z}, \hat{Y}$ diagonal...

Non-diagonal is also used:

$$\hat{C}_{\zeta} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ (-1)^N \zeta^{-1} & & & & 0 \end{pmatrix}$$

cyclic permutation with a twist

(all matrices below are $N \times N$)

Given $r+1$ diag matrices $\hat{Z}_1, \dots, \hat{Z}_{r+1} \in (\mathbb{C}^\times)^N$

and z diag matrices $\hat{M}_1, \hat{M}_r \in \mathbb{C}^N$

find $r+1$ diag matrix-valued multivalued functions $\hat{g}_1(x), \dots, \hat{g}_{r+1}(x)$ s.t.

$$\hat{g}_i = \frac{\hat{Y}_{i-1}}{\hat{Y}_i}$$

$\hat{g}_i(x) \to 1$, $x \to \infty$ (valued in $\mathbb{C}^\times$ on the physical sheet)

Such that

$$\prod_{i=1}^{r+1} \hat{g}_i(x) = \frac{x - \hat{M}_1}{x - \hat{M}_r} \equiv \frac{\hat{Y}_0(x)}{\hat{Y}_{r+1}(x)}$$

and

$$\overleftarrow{\prod_{i=1}^{r+1}} \left(1 + \hat{C}_z \hat{Z}_i\right)^{-1} \overrightarrow{\prod_{i=1}^{r+1}} \left(\hat{g}_i(x) + \hat{C}_z \hat{Z}_i\right) \hat{Y}_{r+1}(x)$$

Dyson-Schwinger in the presence of surface defect

has no singularities in x for any value of z !

$$\prod_{i=1}^{\overleftarrow{r+1}} \left(1+\hat{C}_z\hat{Z}_i\right)^{-1} \prod_{i=1}^{\overrightarrow{r+1}} \left(\hat{g}_i(x)+\hat{C}_z\hat{Z}_i\right) \hat{Y}_{r+1}(x)$$

has no singularities in x for any value of z

$$\underbrace{\prod_{i=1}^{\overleftarrow{r+1}} \left(1+\hat{C}_z\hat{Z}_i\right)}_{\displaystyle x\cdot 1_N - \underline{\hat{L}(z)}}$$

Here is the Lax

$$\hat{L}(z) = \hat{M}_{r+1} + \sum_{i=1}^{r+1} \frac{\hat{L}_i}{1 - z_i/z}$$

rank 1 residues, computable from $\hat{Z}_i, \hat{M}_1, \hat{M}_r$ and $\hat{P}_i$, $i=1\dots r+1$ with

$$\sum_{i=1}^{r+1} \hat{P}_i = \hat{M}_r - \hat{M}_1$$

$$\hat{g}_i(x) = 1 + \frac{\hat{P}_i}{x} + \cdots,$$

$$\mathrm{Det}\left(\overleftarrow{\prod_{i=1}^{r+1}} \left(1+\hat{C}_z \hat{Z}_i\right)^{-1} \underbrace{\overrightarrow{\prod_{i=1}^{r+1}} \left(\hat{g}_i(x)+\hat{C}_z \hat{Z}_i\right) \hat{Y}_{r+1}(x)} \right) =$$

$$= \prod_{i=1}^{r+1} \frac{z - z^{(i)}(x)}{z - z_i} \, Y_0(x) = R(x,z)$$

polynomial in x

$$z^{(i)}(x) = z_i \frac{Y_i(x)}{Y_{i-1}(x)}$$

branches of the spectral curve

$$\hat{g}_i(x) = \frac{\hat{Y}_{i-1}(x)}{\hat{Y}_i(x)}$$

diagonal matrices

$$Y_i(x) = \mathrm{Det}\, \hat{Y}_i(x)$$

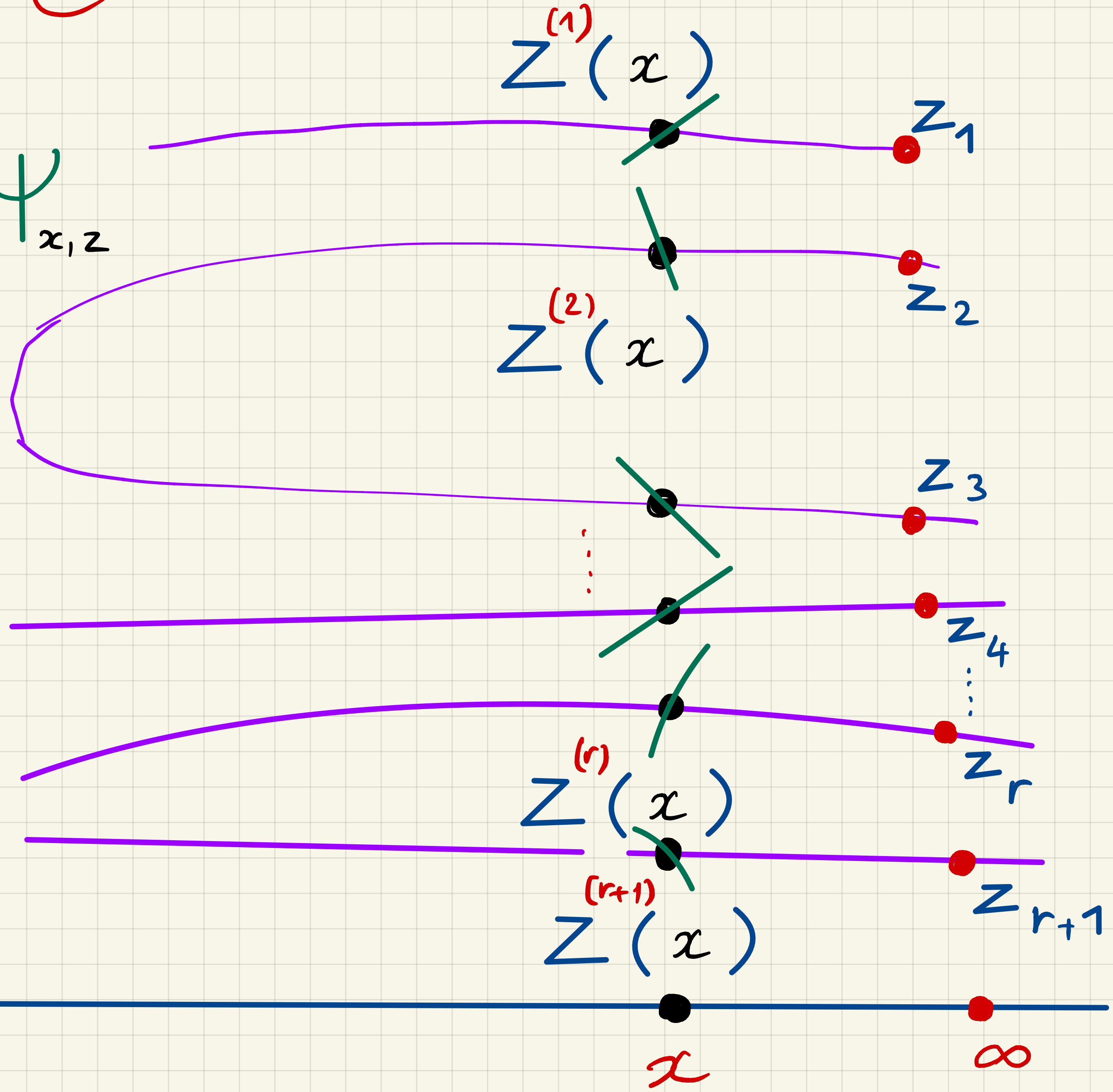
spectral curve $\mathcal{C}^{\text{spectral}}$

$$\hat{L}(z)\,\Psi_{x,z} = x\,\Psi_{x,z}$$

(x, z)

$$\psi^{(i)}_x = \psi_{x,\, z^{(i)}(x)}$$

well-defined outside the branching locus



$$\overleftarrow{\prod_{i=1}^{r+1}} \left(1+\hat{c}_z \hat{Z}_i\right)^{-1} \overrightarrow{\prod_{i=1}^{r+1}} \left(\hat{g}_i(x) + \hat{c}_z \hat{Z}_i\right) \widehat{Y}_{r+1}(x) =$$

$$= \overleftarrow{\prod_{i=1}^{r+1}} \hat{W}_i^{-1}\left(1+\hat{c}_{z/z_i}\right)^{-1}\hat{W}_i \, \widehat{Y}_0(x) \overrightarrow{\prod_{i=1}^{r+1}} \hat{V}_i^{-1}\left(1+\hat{c}_{z/z^{(i)}(x)}\right)\hat{V}_i$$

$$\frac{w_{i,\omega}}{w_{i,\omega-1}} = z_{i,\omega} \qquad \frac{v_{i,\omega}(x)}{v_{i,\omega-1}(x)} = \frac{w_{i,\omega}\, Y_{i,\omega}(x)}{w_{i,\omega-1}\, Y_{i-1,\omega}(x)}$$

$$w_{i,\omega+N} = w_{i,\omega} Z_i \qquad v_{i,\omega+N} = v_{i,\omega} Z^{(i)}(x)$$

QQ-System in disguise!

$$\Psi^{(r+1)}_x = \hat{V}^{-1}_{r+1}(x)\, e \implies \hat{V}_{r+1}(x) \text{ is computable (up to } \mathbb{C}^\times)$$

assuming components of Ψ don't vanish

$$\Psi^{(r)}_x = \hat{V}^{-1}_{r+1}(x)\left(1 + \hat{C}^{r+1}_r\right)^{-1} \hat{V}_{r+1}(x)\, \hat{V}^{-1}_r(x)\, e$$

$$\boxed{\hat{C}^j_i = \hat{C}_{z^{(i)}(x)/z^{(j)}(x)}}$$

$\vdots$

$$\Psi^{(i)} = \hat{V}^{-1}_{r+1}(x)\left(1+\hat{C}^{r+1}_i\right)^{-1} \hat{V}_{r+1}(x)\, \hat{V}_r(x) \cdots\cdots \left(1+\hat{C}^{i+1}_i\right)^{-1} \hat{V}_{i+1}(x)\, \hat{V}^{-1}_i(x)\, e$$

$\vdots$

$$\Psi^{(1)} = \hat{V}^{-1}_{r+1}(x)\left(1 + \hat{C}^{r+1}_1\right)^{-1} \hat{V}_{r+1} \hat{V}^{-1}_r \cdots \left(1+ \hat{C}^2_1\right)^{-1} \hat{V}_2(x)\, \hat{V}^{-1}_1(x)\, e$$

Microscopic parameters

$$q_{i,\omega} = \frac{Z_{i+1,\omega}}{Z_{i,\omega}} = \frac{w_{i+1,\omega+1}\, w_{i,\omega}}{w_{i+1,\omega}\quad w_{i,\omega+1}}$$

$$\wr\wr$$

$$q_{i,\omega+N}$$

$$\hat{P}_i = \mathrm{diag}\,(p_{i,\omega})$$

$$w_{i,\omega} \frac{\partial \mathcal{S}}{\partial w_{i,\omega}} = p_{i,\omega}$$

Free energy

partition function $Z_{\text{bulk}} \sim e^{\frac{1}{\varepsilon_1\varepsilon_2}\mathcal{F}}$

surface defect $\Psi_{\text{surface}} \sim e^{\frac{1}{\varepsilon_1\varepsilon_2}\mathcal{F} + \frac{1}{\varepsilon_1}\mathcal{S}}$

Hamiltonians

"Action" variables

$$\prod_{i=1}^{\overleftarrow{r+1}} \widehat{W}_i^{-1}\left(1+\widehat{\mathcal{C}}_{z/z_i}\right)^{-1}\widehat{W}_i \cdot \widehat{Y}_0(x)\cdot \prod_{i=1}^{\overrightarrow{r+1}} \widehat{V}_i^{-1}\left(1+\widehat{\mathcal{C}}_{z/z^{(i)}(x)}\right)\widehat{V}_i$$

$$= x - \widehat{L}(z)$$

fixed

fixed over the curve

but we can vary $\widehat{W}_i$, $\widehat{V}_i$

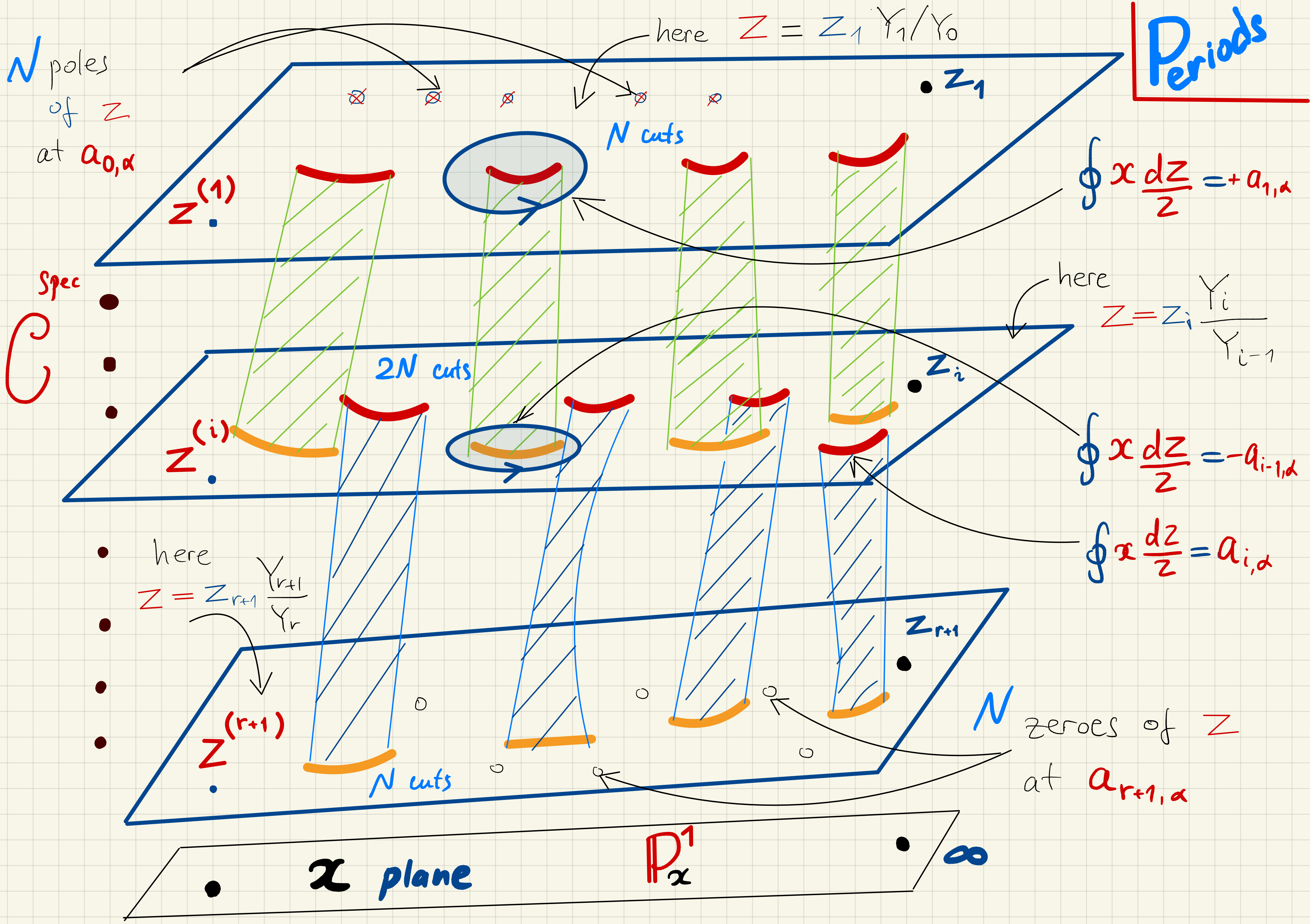
$\approx$ similar to the collection of rk residues $\widehat{L}_i$

$(a_{i,\alpha}) \sim$ periods of $x\, d\log Y_i$

not well-defined on $\mathcal{C}^{\text{spectral}}$

(not $x\, d\log Y_{i,w}$ though)

however the set of periods can be recovered from those of $x\, d\log z$


Periods
N poles of Z at $a_{0,\alpha}$
here $Z = Z_1 Y_1/Y_0$
z_1
N cuts
$Z^{(1)}$
$\oint x \frac{dz}{z} = +a_{1,\alpha}$
Spec
C
here $Z = Z_i \frac{Y_i}{Y_{i-1}}$
z_i
2N cuts
$Z^{(i)}$
$\oint x \frac{dz}{z} = -a_{i-1,\alpha}$
$\oint x \frac{dz}{z} = a_{i,\alpha}$
here $Z = Z_{r+1} \frac{Y_{r+1}}{Y_r}$
z_{r+1}
$Z^{(r+1)}$
N cuts
N zeroes of Z at $a_{r+1,\alpha}$
x plane
$\mathbb{P}^1_x$
∞

Turning on $\underline{\underline{\varepsilon}}$ (one out of two)

Modified "character" equations $\longrightarrow$ q-characters

Now we are looking for (meromorphic!) functions

$$Y_{i,w}(x)$$

obeying non-perturbative Dyson-Schwinger

One ε parameter turns characters into

q-characters

$i = 1, \dots, r+1$

$$g_i(x) = z_i \frac{Y_i(x+\varepsilon)}{Y_{i-1}(x)}$$

$$Y_0(x) \sum_{1 \le i_1 < i_2 < \dots < i_\ell \le r+1} g_{i_1}(x)\, g_{i_2}(x+\varepsilon)\, g_{i_3}(x+2\varepsilon) \cdots g_{i_\ell}(x+(\ell-1)\varepsilon)$$

$$= \sigma_\ell(z_1 \dots z_{r+1})\, T_\ell(x)$$

deg N monic polynomial

Packing supplies

Instead of the generating function of

X_i's $\longrightarrow$ spectral determinant $R(x, z)$

We form a **Generating operator**

(q-oper)

$$\hat{R} := Y_0(x)\left(1 - g_1(x)e^{\varepsilon\partial_x}\right)\left(1 - g_2(x)e^{\varepsilon\partial_x}\right)\cdots\left(1 - g_{r+1}(x)e^{\varepsilon\partial_x}\right)$$

Generating operator
(q-oper)

$Y_i(x)$

$$Y_0(x)\left(1-g_1(x)e^{\varepsilon\partial_x}\right)\left(1-g_2(x)e^{\varepsilon\partial_x}\right)\cdots\left(1-g_{r+1}(x)e^{\varepsilon\partial_x}\right)$$

$$= Y_0(x) - \sigma_1 T_1(x)e^{\varepsilon\partial_x} + \sigma_2 T_2(x)e^{2\varepsilon\partial_x} + \cdots$$

$$\cdots + (-)^{r+1}\sigma_{r+1}Y_{r+1}\left(x+(r+1)\varepsilon\right)e^{(r+1)\varepsilon\partial_x}$$

So, recovering $\langle Y_i(x)\rangle$'s, in the $\boxed{\varepsilon_2\to 0}$ limit $\Longleftrightarrow$ Miura

Solutions to the q-oper

$$\tilde{Q}^{(i)}(x) \sim z_{(i)}^{-\frac{x}{\varepsilon}}$$

$$\underline{\hat{\mathcal{R}}}\ \underline{\tilde{Q}} = 0 \quad \left(\text{dual solutions} \quad \hat{\mathcal{R}}^{*} \underset{\sim}{Q} = 0\right)$$

e.g. $$\underbrace{\left(1 - g_{r+1}(x)\, e^{\varepsilon \partial_x}\right)}\tilde{Q}^{(r+1)}(x) = 0$$

$$\frac{\tilde{Q}^{(r+1)}(x)}{\tilde{Q}^{(r+1)}(x+\varepsilon)} = z_{r+1} \frac{Y_{r+1}(x+\varepsilon)}{Y_r(x)}$$

Known polyn

complicated

Solutions to the q-oper / dual solutions

$$\tilde{Q}^{(i)}(x) \sim z_{(i)}^{-\frac{x}{\varepsilon}}$$

$$\hat{R}\,\tilde{Q} = 0 \quad \left(\text{dual solutions} \quad \hat{R}^{*}\,\underset{\sim}{Q} = 0\right)$$

e.g.

$$\left(1 - e^{-\varepsilon\partial_x} g_1(x)\right)\left(Y_0(x)\,\underset{\sim}{Q}^{(1)}(x)\right) = 0$$

$$z_1 Y_1(x)\,\underset{\sim}{Q}^{(1)}(x-\varepsilon) = g_1(x-\varepsilon)\,\underset{\sim}{Q}^{(1)}(x-\varepsilon)Y_0(x-\varepsilon) - \underset{\sim}{Q}^{(1)}(x)\,Y_0(x)$$

$$Q_1^{(1)}(x) \overset{\Downarrow}{=\!=} z_1^{+x/\varepsilon}\,\underset{\sim}{Q}^{(1)}(x)\,\frac{1}{\prod_{\alpha=1}^{N}\Gamma\left(1+\frac{x-a^{(0,\alpha)}}{\varepsilon}\right)}$$

Solutions to the q-oper /dual

$$\tilde{Q}^{(i)}(x) \sim z_i^{-\frac{x}{\varepsilon}}$$

$$\left(1 - g_{r+1}(x)\, e^{\varepsilon \partial_x}\right) \tilde{Q}^{(r+1)}(x) = 0$$

$$\Rightarrow \quad Q_r^{(1)}(x) = \frac{\tilde{Q}^{(r+1)}(x+\varepsilon)\, z_{r+1}^{x/\varepsilon}}{\prod_{\alpha=1}^{N} (-\varepsilon)^{\frac{x}{\varepsilon}}\, \Gamma\left(-\frac{x - a^{(r+1,\alpha)}}{\varepsilon}\right)}$$

Nesting....

QQ-system

$$\left(1-g_r(x)e^{\varepsilon\partial_x}\right)\left(1-g_{r+1}(x)e^{\varepsilon\partial_x}\right)\tilde{Q}^{(r)}(x)=0$$

etc.

Nesting....

QQ-system

$$\left(1-g_r(x)e^{\varepsilon\partial_x}\right)\left(1-g_{r+1}(x)e^{\varepsilon\partial_x}\right)\tilde{Q}^{(r)}(x)=0$$

$$\left(1-g_r(x)e^{\varepsilon\partial_x}\right)\left(\tilde{Q}^{(r)}(x)-\tilde{Q}^{(r)}(x+\varepsilon)\,\frac{\tilde{Q}^{(r+1)}(x)}{\tilde{Q}^{(r+1)}(x+\varepsilon)}\right)$$

$$\parallel$$

$$\left(1-g_r(x)e^{\varepsilon\partial_x}\right)\frac{W_2\left(\tilde{Q}^{(r)},\tilde{Q}^{(r+1)}\right)}{\tilde{Q}^{(r+1)}(x+\varepsilon)}$$

Nesting....

QQ-system

$$\left(1-g_r(x)e^{\varepsilon\partial_x}\right)\frac{W_2\left(\tilde{Q}^{(r)},\tilde{Q}^{(r+1)}\right)}{\tilde{Q}^{(r+1)}(x+\varepsilon)}=0$$

$$g_r(x)=z_r\frac{Y_r(x+\varepsilon)}{Y_{r-1}(x)}=z_r\frac{Q_r^{(1)}(x+\varepsilon)}{Q_r^{(1)}(x)}\frac{1}{Y_{r-1}(x)}$$

$$Y_{r-1}(x)=z_{r+1}z_r\frac{Y_{r+1}(x+2\varepsilon)\,W_2\left(\tilde{Q}^{(r)}\,\tilde{Q}^{(r+1)}\right)(x+\varepsilon)}{W_2\left(\tilde{Q}^{(r)},\tilde{Q}^{(r+1)}\right)(x)}$$

$$\parallel$$

$$\frac{Q_{r-1}^{(1)}(x)}{Q_{r-1}^{(1)}(x-\varepsilon)}$$

Nesting....

QQ-system

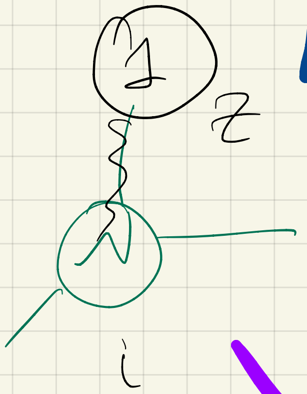
$$Q^{(1)}_{r-1}(x) = \frac{\left(z_r z_{r+1}\right)^{\frac{x}{\varepsilon}} W_2\left(\tilde{Q}^{(r)}, \tilde{Q}^{(r+1)}\right)(x+\varepsilon)}{\prod_{\alpha=r}^{N} (-\varepsilon)^{\frac{x}{\varepsilon}} \Gamma\left(-1-\frac{x-a^{(r+1,\alpha)}}{\varepsilon}\right)}$$

$$Y_{r-1}(x) = z_{r+1} z_r \frac{Y_{r+1}(x+2\varepsilon)\, W_2\left(\tilde{Q}^{(r)}, \tilde{Q}^{(r+1)}\right)(x+\varepsilon)}{W_2\left(\tilde{Q}^{(r)}, \tilde{Q}^{(r+1)}\right)(x)}$$

$$\Vert$$

$$\frac{Q^{(1)}_{r-1}(x)}{Q^{(1)}_{r-1}(x-\varepsilon)}$$

Microscopically



entire function of x

$$Y_i(x) = \frac{Q_i^{(1)}(x)}{Q_i^{(1)}(x-\varepsilon_1)} = \frac{Q_i^{(2)}(x)}{Q_i^{(2)}(x-\varepsilon_2)}$$

"minimal" surface defects $U(N)_i \to U(N-1)_i$

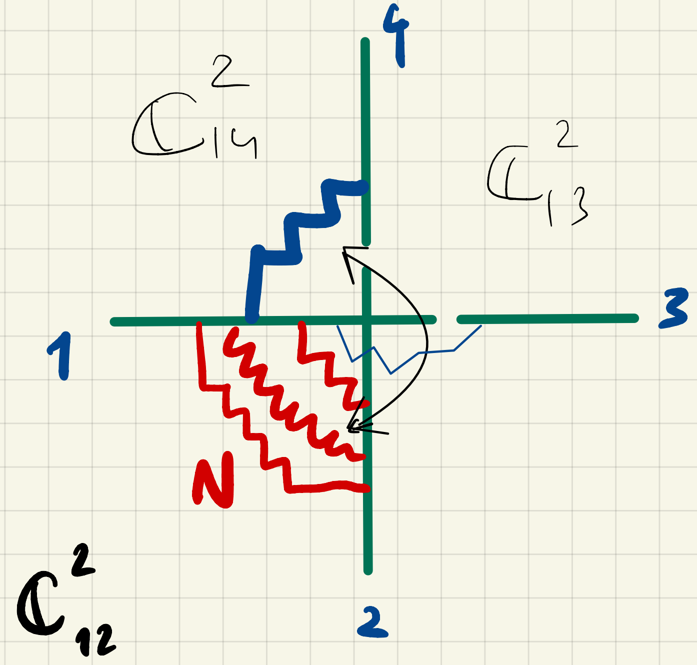
$$Q_i^{(1)}(x) = E\left[-e^{x}\hat{R}_i\left(\frac{\hat{S}_{12}^{*}}{P_1}\right)\right]^{\mathbb{Z}_{r+2}}$$

canonical

$$\Psi(z) = \sum_{x \in \text{lattice}} Q^{(1)}(x)\, z^{-x/\varepsilon_1}$$

More sophisticated surface defects

gauge origami



$\mathbb{C}^4_{1234}$

two types of surface defects at $z_1=0$

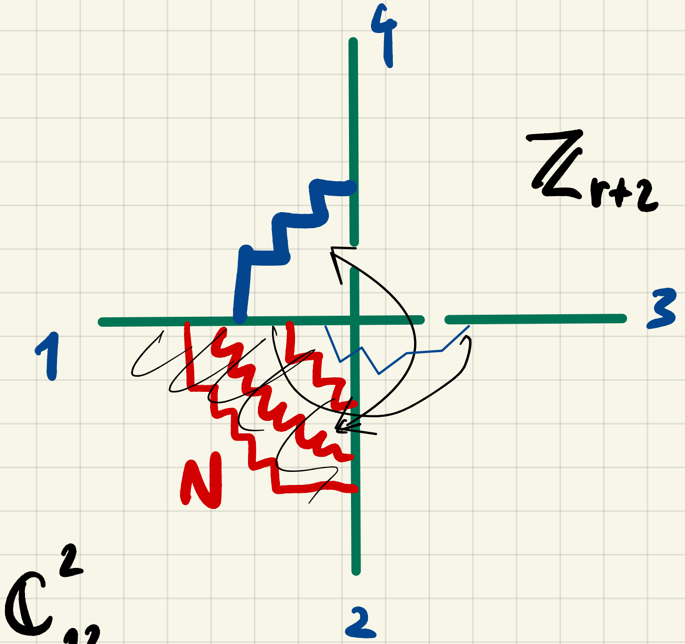
add

S_{14} or S_{13} and/or S_{24} or S_{23}

two types of surface defects at $z_2=0$

More sophisticated surface defects

gauge origami



$\mathbb{C}^2_{12}$

$\mathbb{C}^4_{1234}$

$\mathbb{Z}_{r+2}$ acts on $\mathbb{C}^2_{34}$

quiver gauge theory

two types of surface defects at $z_1 = 0$

add

$\mathcal{O}_{S_{14}}$

$\mathcal{O}_{S_{13}}$

$\mathcal{O}_{S_{24}}$

$\mathcal{O}_{S_{23}}$

two types of surface defects at $z_2 = 0$

$\varepsilon_2/\varepsilon_1 = k + N$

In this way we identify all the ingredients of the classical (Lax operator and its eigenvectors) and quantum (q-oper and its solutions) SW geometry with 4d $\mathcal{N}=2$ theory observables

One can go further, and find the place of the Lax evolution

$$\frac{\partial}{\partial t_{i,k}} \hat{L} = \left[\hat{L} , \hat{A}_{i,k} \right]$$

Spectral duality (Gaudin - XXX, ...)

CFT / CS $(\mathcal{N}=4\ d=4)$

outside RCFT / 3d CS

Hitchin / monopoles

$g=0,1$

beyond ?

My collaborators in these endeavours

- S. Jeong , N. Lee
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THANK YOU