On algebraic de Rham theorem

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"Algebraic Geometry, Mathematical Physics, and Solitons"
Celebrating the work of Igor Krichever
October 7-9, 2022
Columbia University, New York

Introduction

Let M be an algebraic variety. According to Atiyah and Hodge, closed meromorphic p-form φ on M is called *differential of a second kind*, if it has zero residues on open subsets $M \setminus D$ for sufficiently large divisors D.
The quotient groups

$$\frac{\{p\text{-forms of the second kind}\}}{\{\text{exact forms}\}}$$

have an interpretation in terms of spectral sequences for certain complex of sheaves of meromorphic forms on M. In particular, one gets the statement

$$H^1_{\mathrm{dR}}(M, \mathbb{C}) \simeq \frac{\{1\text{-forms of the second kind}\}}{\{\text{exact forms}\}}.$$

1. Curves

Let X be compact Riemann surface of genus g, $\mathcal{O}_X$ — the sheaf of holomorphic functions on X, $\mathcal{M}_X$ — the sheaf of meromorphic functions, and $\mathcal{M}$ — the vector space of meromorphic functions on X. Let d be the exterior derivative on X. The sheaf $\mathrm{d}\mathcal{M}_X$ is a sheaf of differentials of the 2nd kind and $\Omega^{(2\mathrm{nd})} = H^0(X, \mathrm{d}\mathcal{M}_X)$ is the vector space of the differentials of the 2nd kind.
Algebraic de Rham theorem is the statement

$$H^1_{\mathrm{dR}}(X, \mathbb{C}) = H^{1,0}(X, \mathbb{C}) \oplus H^{0,1}(X, \mathbb{C}) \simeq \Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}, \qquad (1)$$

which is easily proved using a sheaf-theoretic de Rham isomorphism

$$H^1_{\mathrm{dR}}(X, \mathbb{C}) \simeq H^1(X, \underline{\mathbb{C}}).$$

Namely, consider the short exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{\ i\ } \mathcal{M}_X \xrightarrow{\ \mathrm{d}\ } \mathrm{d}\mathcal{M}_X \longrightarrow 0,$$

where $\underline{\mathbb{C}}$ is the constant sheaf. Since $H^1(X, \mathcal{M}_X) = \{0\}$, the corresponding short exact sequence in the cohomology gives (1).

The infinite-dimensional vector space $\Omega^{(2\mathrm{nd})}$ has a natural skew-symmetric bilinear form

$$\omega_X^{(1)}(\theta_1,\theta_2)=\sum_{P\in X}\mathrm{Res}_P(\mathrm{d}^{-1}\theta_1\theta_2),\quad \theta_1,\theta_2\in\Omega^{(2\mathrm{nd})}.$$

Theorem 1 (i) The restriction of $\omega_X^{(1)}$ to $\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}$ is non-degenerate and

$$\dim_{\mathbb{C}}\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}=2g.$$

(ii) For every choice of non-special divisor $D=P_1+\cdots+P_g$,

$$\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}\simeq\Omega^{(2\mathrm{nd})}\cap H^0(X,K_X+2D).$$

(iii) For every choice of local coordinates $z_i=z(P_i)$ at P_i, $\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}$ has a symplectic basis $\{\vartheta_i,\tau_i\}_{i=1}^g$, uniquely characterized by

$$\vartheta_i=(\delta_{ij}+O(z-z_j))\,dz\ \text{ and }\ \tau_i=\left(\frac{\delta_{ij}}{(z-z_j)^2}+O(z-z_j)\right)dz.$$

2. Remarks

(1) Put

$$\Omega^{(2\mathrm{nd})}(2D) = \mathbb{C}\tau_1 \oplus \cdots \oplus \mathbb{C}\tau_g.$$

The vector space $\Omega^{(2\mathrm{nd})}(2D)$ is dual to $H^0(X, K_X)$ with respect to the pairing given by the symplectic form $\omega_X^{(1)}$.

(2) The choice of a non-special effective divisor D with g distinct points P_i and local coordinates is as an algebraic analogue of the choice of a-cycles. The differentials ϑ_i are analogues of differentials of the first kind with normalized a-periods, and the differentials τ_i are analogues of differentials of the second kind with second-order poles, zero a-periods and normalized b-periods. The symplectic property of the basis $\{\vartheta_i, \tau_i\}_{i=1}^{g}$ is an analogue of the reciprocity laws for differentials of the first kind and the second kind.

(3) Every choice choice of non-special effective divisor D of degree g defines the isomorphism

$$H^{0,1}(X, \mathbb{C}) \simeq \Omega^{(2\mathrm{nd})}(2D).$$

(4) By Dolbeault isomorphism,

$$\operatorname{Pic}^0(X) = H^{0,1}(X, \mathbb{C})/H^1(X, \mathbb{Z}),$$

so the choice of a non-special effective divisor D of degree g allows to identify holomorphic tangent space to $\operatorname{Jac}(X)$ with $\Omega^{(2\mathrm{nd})}(2D)$; the holomorphic cotangent space is naturally identified with $H^{1,0}(X, \mathbb{C})$, with the pairing given by $\omega_X^{(1)}$.
(5) Fix a non-special divisor $D_0 = Q_1 + \cdots + Q_g$ and consider the map

$$X^{(g)} \ni D \to \mu^{(g)}(D) \in \operatorname{Jac}(X),$$

where $\mu^{(g)}$ is the Abel-Jacobi sum: for $D = P_1 + \cdots + P_g$

$$\mu^{(g)}(D) = \left(\sum_{i=1}^{g} \int_{Q_i}^{P_i} \vartheta_1, \ldots, \sum_{i=1}^{g} \int_{Q_i}^{P_i} \vartheta_g \right), \qquad (2)$$

and $\{\vartheta_i\}_{i=1}^{g}$ is the basis of $H^0(X, K_X)$ from Theorem 1, specialized to the divisor D_0. The 1-forms dz_i at the base point $\mu^g(D_0)$ correspond to the differentials of the first kind ϑ_i, and the vector fields $\dfrac{\partial}{\partial z_i}$ — to the differentials of the second kind τ_i.

(6) If divisor D is also non-special, then it follows from the group law on the Jacobian and Theorem 1 that dz_i and $\dfrac{\partial}{\partial z_i}$ at a point $\mu^{(g)}(D)$ are given by the symplectic basis of $\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}$.

(7) The vector fields $\dfrac{\partial}{\partial z_i}$ on $\mathrm{Jac}(X)$ can be described using the formalism of Lax equations on algebraic curves, developed by **Igor Krichever** (Commun. Math. Phys. **229**, 2002, and Mosc. Math. J., **2**:4, 2002).

(8) Namely, Igor's meromorphic 1-forms $L(z)dz$ are holomorphic in case $r = 1$ and become differentials of the first kind ϑ, while the analogues of rational functions $M(z)$ are defined as follows. Consider the vector space

$$\mathcal{L}_{D+D_0} = \{f \in \mathcal{M} : (f) + D + D_0 \geq 0\}, \quad \dim \mathcal{L}_{D+D_0} = g + 1.$$

For any fixed choice of principal parts of f at D_0, not all of them zero, there is a unique $f \in \mathcal{L}_{D+D_0}$, at all points of D satisfying

$$f(z) = \frac{\alpha_i}{z - z_i} + O(1), \quad z_i = z(P_i). \tag{3}$$

Functions f play the role of rational functions $M(z)$ in case $r = 1$.

(9) We have

$$df = \tau - \tau_0,$$

where $\tau \in \Omega^{(2\mathrm{nd})}(2D)$ and $(\tau_0) + 2D_0 \geq 0$. By the residue theorem,

$$-\sum_{i=1}^{g} \mathrm{Res}_{P_i}(f\vartheta) = \omega_X^{(1)}(\vartheta, \tau) = \omega_X^{(1)}(\vartheta, \tau_0), \quad \vartheta \in H^0(X, K_X),$$

so the pairing (2.22) in Igor's papers coincides with the pairing given by the symplectic form $\omega_X^{(1)}$. Choosing the symplectic basis of $\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}$, we see that there is a correspondence

$$f \mapsto \mathscr{L}_f = -\sum_{i=1}^{g} \alpha_i \frac{\partial}{\partial z_i}$$

between rational functions $f \in \mathcal{L}_{D+D_0}$ and vector fields on $\mathrm{Jac}(X)$.
(10) Along an integral curve $D(t)$, where $D(0) = D$, we have

$$\dot{z}_i(t) = -\alpha_i(t), \quad i = 1, \ldots, g, \tag{4}$$

where the dot stands for the t-derivative. In case when X is a hyperelliptic curve, these are classical Dubrovin equations, arising in the theory of finite-gap integration for the KdV equation.

(11) Using Dubrovin equations, we see that along the integral curve equations (3) take the form

$$f_t(z) = -\frac{\dot{z}_i(t)}{z - z_i(t)} + O(1), \quad i = 1, \ldots, g. \tag{5}$$

Thus introducing

$$\Psi(z) = \exp\left\{\int_0^T f_t(z)dt\right\}$$

we see from (5) that Ψ is a meromorphic function on $X \setminus D_0$ having simple poles only at D, simple zeros only at $D(T)$, and essential singularities at the points of D_0. The function Ψ is nothing but the celebrated *Baker-Akhiezer function*, introduced by **Igor Krichever** in 1977!

2. Quadratic differentials

In order to formulate an analog of algebraic de Rham theorem for higher order differentials, one needs to fix a projective structure on X (or to choose a uniformizer at each $P \in X$). One can assume that a projective structure is given by the Fuchsian uniformization $X \simeq \Gamma\backslash\mathbb{H}$ (or by quasi-Fuchsian uniformization for holomorphic families).

2.1 Quadratic differentials of the second kind. We have

$$H^0(X, \mathcal{M}(K_X^2)) \simeq \mathscr{M}_4(\mathbb{H}, \Gamma),$$

the space of weight 4 meromorphic automorphic forms for Γ and

$$H^0(X, K_X^2) \simeq \mathscr{H}_4(\mathbb{H}, \Gamma),$$

the subspace of holomorphic automorphic forms of weight 4. Correspondingly, for the space $\mathscr{V}$ of meromorphic vector fields on X

$$\mathscr{V} = H^0(X, \mathcal{M}(K_X^{-1})) \simeq \mathscr{M}_{-2}(\mathbb{H}, \Gamma).$$

It is a classical result

$$\mathscr{M}_{-2}(\mathbb{H},\Gamma) \ni v \mapsto q = v''' \in \mathscr{M}_4(\mathbb{H},\Gamma),$$

which allows (given a choice of a projective atlas) to consider the sheaf $\mathrm{d}^3\mathcal{M}(K_X^{-1})$ as a subsheaf of $\mathcal{M}(K_X^2)$.
The infinite-dimensional vector space $\Omega^{(2\mathrm{nd})} = H^0(X, \mathrm{d}^3\mathcal{M}(K_X^{-1}))$ — the space of *quadratic differentials of the second kind* — is the subspace of meromorphic automorphic forms of weight 4 whose singular series at the poles do not contain terms of orders $-3, -2$ and -1.
Explicitly,

$$q(z) = \sum_{n=N}^{\infty} (n^3 - n) a_n (z - z_0)^{n-2} \qquad (6)$$

near each pole $z_0 \in \mathbb{H}$, where coefficients a_{-1}, a_0, a_1 can set to be 0. (Though this condition is not well defined for arbitrary choice of a local parameter, it is stable under the fractional-linear transformations.)

2.2 Algebraic de Rham theorem I.

Let $\mathscr{P}_2(K_X^{-1})$ be the sheaf of local holomorphic sections of K_X^{-1}, which are polynomials of degree ≤ 2 (in a given projective structure on X).

The formal analogue of the algebraic de Rham theorem for quadratic differentials is the isomorphism

$$H^1(X, \mathscr{P}_2(K_X)) \simeq \Omega^{(2\mathrm{nd})}/\mathrm{d}^3\mathscr{V}, \tag{7}$$

which follows from the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{P}_2(K_X^{-1}) \xrightarrow{\ i\ } \mathcal{M}(K_X^{-1}) \xrightarrow{\ \mathrm{d}^3\ } \mathrm{d}^3\mathcal{M}(K_X^{-1}) \longrightarrow 0$$

as before, since $H^1(X, \mathcal{M}(K_X^{-1})) = 0$.

2.3 Symplectic form. The infinite-dimensional vector space $\Omega^{(2\mathrm{nd})}$ has a natural skew-symmetric bilinear form

$$\omega_X^{(2)}(q_1, q_2) = \sum_{z \in \Gamma\backslash\mathbb{H}} \mathrm{Res}_z(d^{-3} q_1 q_2),$$

It follows from the residue theorem that

$$\omega_X^{(2)}(q_1, q_2) = 0 \quad q_1 \in \Omega^{(2\mathrm{nd})}, \quad q_2 \in \mathrm{d}^3\mathscr{V},$$

so $\omega_X^{(2)}$ can be restricted to the quotient space $\Omega^{(2\mathrm{nd})}/\mathrm{d}^3\mathscr{V}$.

Theorem 2 (i) The restriction of $\omega_X^{(2)}$ to $\Omega^{(2\mathrm{nd})}/\mathrm{d}^3\mathscr{V}$ is non-degenerate and

$$\dim_{\mathbb{C}} \Omega^{(2\mathrm{nd})}/\mathrm{d}^3\mathscr{V} = 6g - 6.$$

(ii) For every choice of degree g non-special effective divisor D,

$$\Omega^{(2\mathrm{nd})}/\mathrm{d}^3\mathscr{V} \simeq \Omega^{(2\mathrm{nd})} \cap H^0(2K_X + 4D).$$

(iii) Let $K_X + D = \sum_{i=1}^{3g-3} P_i$ be an effective divisor of degree $3g-3$ with distinct points, The vector space $\Omega^{(2\mathrm{nd})}/\mathrm{d}^3\mathscr{V}$ has a symplectic basis $\{q_i, r_i\}_{i=1}^{3g-3}$ uniquely characterized by

$$q_i = (\delta_{ij} + O(z - z_j))dz^2 \quad r_i = \left(6\frac{\delta_{ij}}{(z - z_j)^4} + O(z - z_j)\right)dz^2.$$

(iv) The subspace $\Omega^{(2\mathrm{nd})}(4D) = \mathbb{C}\cdot r_1 \oplus \cdots \oplus \mathbb{C}\cdot r_{3g-3}$ is a complementary isotropic subspace to $H^0(X, K_X^2) \oplus \mathrm{d}^3\mathscr{E}$ in $\Omega^{(2\mathrm{nd})}$.

2.4 Eichler integrals and Eichler cohomology.

For $q \in \Omega^{(2\mathrm{nd})}$ we have

$$q(z) = \mathscr{E}'''(z),$$

where $\mathscr{E}$ is an Eichler integral of weight -1, a meromorphic function on $\mathbb{H}$ which has an expansion

$$\mathscr{E}(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^{n+1}$$

near each pole z_0 of q and for every $\gamma \in \Gamma$ satisfies

$$\mathscr{E}(z) - \frac{\mathscr{E}(\gamma z)}{\gamma'(z)} = \chi(\gamma^{-1})(z), \tag{8}$$

where $\chi(\gamma)(z) \in \mathscr{P}_2$, the vector space of polynomials of degree ≤ 2. The Eichler integral $\mathscr{E} = \mathrm{d}^{-3} q$ is defined up to the addition of a quadratic polynomial in z.
The mapping $\chi : \Gamma \to \mathscr{P}_2$ satisfies

$$\chi(\gamma_1 \gamma_2) = \chi(\gamma_1) + \gamma_1 \cdot \chi(\gamma_2), \quad (g \cdot P_2)(z) = \frac{P_2(g^{-1} z)}{(g^{-1})'(z)}$$

where $g \in \mathrm{PSL}(2, \mathbb{C}),\ P_2 \in \mathscr{P}_2$.

We have $\chi \in Z^1(\Gamma, \mathscr{P}_2)$, the space of 1-cocycles for the group Γ with coefficients in the Γ-module $\mathscr{P}_2$. Corresponding coboundaries $B^1(\Gamma, \mathscr{P}_2)$ are $\chi(\gamma) = \gamma \cdot P_2 - P_2$ for some $P_2 \in \mathscr{P}_2$, and

$$H^1(\Gamma, \mathscr{P}_2) = Z^1(\Gamma, \mathscr{P}_2)/B^1(\Gamma, \mathscr{P}_2)$$

is the first Eichler cohomology group of Γ (group cohomology with coefficients in the Γ-module $\mathscr{P}_2$).
We have a standard isomorphism

$$H^1(\Gamma, \mathscr{P}_2) \simeq H^1(X, \mathscr{P}_2(K_X^{-1})). \tag{9}$$

2.5 Eichler integrals and Eichler and Bers cocycles.
The solution of the equation $\mathscr{E}''' = q$ is

$$\mathscr{E}(z) = \frac{1}{2}\int_{z_0}^{z} (z-u)^2 q(u)du, \tag{10}$$

and corresponding cocycle χ is given by explicit formula

$$\chi(\gamma)(z) = \frac{1}{2}\int_{z_0}^{\gamma z_0} (z-u)^2 q(u)du. \tag{11}$$

In particular, we have a $\mathbb{C}$-linear mapping

$$H^0(X, K_X^2) \ni q \mapsto \imath_E(q) = [\chi] \in H^1(\Gamma, \mathscr{P}_2),$$

where χ is given by (11) with holomorphic q.
We call such χ *Eichler cocycles* and denote by $H_E^1(\Gamma, \mathscr{P}_2)$ the image of $H^0(X, K_X^2)$. This map is injective, so that

$$\dim_{\mathbb{C}} H_E^1(\Gamma, \mathscr{P}_2) = 3g - 3.$$

Another $\mathbb{C}$-antilinear mapping $\imath_B : H^0(X, K_X^2) \to H^1(\Gamma, \mathscr{P}_2)$: for $q \in H^0(X, K_X^2)$ put $\mu = y^2\bar{q}$ and consider the following $\bar{\partial}$-problem

$$F_{\bar{z}} = \mu \quad \text{and} \quad F = o(|z|^2) \ \text{ as } \ |z| \to \infty.$$

The function F is a *Bers potential* of the harmonic Beltrami differential $\mu = y^2\bar{q}$. We have

$$F(z) = -\frac{1}{4}\overline{\int_{z_0}^{z} (\bar{z} - u)^2 q(u) du} = -\frac{1}{4}\int_{z_0}^{z} (z - \bar{u})^2 \overline{q(u)} d\bar{u},$$

and for $\gamma \in \Gamma$,

$$F(z) - \frac{F(\gamma z)}{\gamma'(z)} = \sigma(\gamma^{-1})(z) \in \mathscr{P}_2,$$

where $\sigma \in Z^1(\Gamma, \mathscr{P}_2)$ is a *Bers cocycle*,

$$\sigma(\gamma)(z) = -\frac{1}{4}\int_{z_0}^{\gamma z_0} (z - \bar{u})^2 \overline{q(u)} d\bar{u}. \tag{12}$$

Now the mapping $\imath_B$ is defined by

$$H^0(X, K_X^2) \ni q \mapsto \imath_B(q) = [\sigma] \in H^1(\Gamma, \mathscr{P}_2),$$

and we denote by $H_B^1(\Gamma, \mathscr{P}_2)$ the image of $H^0(X, K_X^2)$. Comparing formula (12) for the Bers cocycle for q with formula (11) for the Eichler cocycle for q, we get

$$\sigma(\gamma)(z) = -\frac{1}{2}\overline{\chi(\bar{z})}. \tag{13}$$

The injectivity of the map $\imath_B$ follows from the injectivity of $\imath_E$ and

$$\dim_{\mathbb{C}} H^1_B(\Gamma, \mathscr{P}_2) = 3g - 3.$$

Lemma $H^1_E(\Gamma, \mathscr{P}_2) \cap H^1_B(\Gamma, \mathscr{P}_2) = \{0\}$.

2.6 Eichler-Shimura periods and bilinear relations.

$$(q_1, q_2) = \frac{\sqrt{-1}}{2}\,\omega_{\mathrm{G}}(\chi_1, \bar{\chi}_2), \quad \chi_1 = \imath_E(q_1),\ \chi_2 = \imath_E(q_2)$$

and

$$\omega_X^{(2)}(q_1, q_2) = -\frac{1}{\pi}\,\omega_{\mathrm{G}}(\chi_1, \chi_2), \quad \chi_1 = \imath_E(q_1),\ \chi_2 = \imath_E(q_2),$$

where $\ ,\ )$ stands for the Petersson inner product, and ω_{G} is the Goldman symplectic form on $\mathrm{PSL}(2, \mathbb{C})$ character variety.

2.7 Algebraic de Rham theorem II. Every choice of a non-special effective divisor D of degree g gives an isomorphism

$$H^1(X, \mathscr{P}_2(K_X^{-1})) \simeq H^0(X, K_X^2) \oplus \Omega^{(2\mathrm{nd})}(4D).$$

Remarks

- Let T_g be the Teichmüller space of compact Riemann surfaces of genus $g > 1$. Then its holomorphic tangent space $T_{[X]}T_g$, for a choice of a non-special effective divisor D of degree g, can be identified with the subspace $\Omega^{(2\mathrm{nd})}(4D)$ of special meromorphic quadratic differentials on X, and the holomorphic cotangent space $T^*_{[X]}T_g$ — with the subspace $H^0(X, K_X^2)$ of holomorphic quadratic differentials. The pairing between $T_{[X]}T_g$ and $T^*_{[X]}T_g$ is given by the symplectic form $\omega_X^{(2)}$.

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- Relation with work of Krichever-Phong, and its elaboration by Grushevsky-Krichever on the algebro-geometric description of the vector fields on the moduli space of curves (with extra data)?

Рис.: Наташа и Игорь (и Таня). Lake Mohonk, NY, 1998

Рис.: Наташа и Игорь (и Леон). Lake Mohonk, NY, 1998

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