Quantum Anomalies as Kinematic Properties of a Classical Fluid

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Celebrating the work of Igor Krichever

Dirac, Euler (3+1 dimensions)

Dirac Fermions $\mathscr{L} = \bar{\psi}\gamma^{\mu}(i\partial_{\mu} + A_{\mu})\psi$

Euler equation $m(\partial_t + \mathbf{v}\cdot\nabla)\,\mathbf{v} + \rho^{-1}\nabla P = \mathbf{E} + \mathbf{v}\times\mathbf{B}$

Fermions in Fluid

- Under certain conditions Dirac fermions (electrons or quarks) may form a fluid (He^3, quark-gluon plasma)

- Quantum Anomaly is a kinematic property of Dirac fermions largely insensitive to an interaction,

- Semiclassical flows of quantum fluid are governed by classical hydrodynamics (Euler equations)

- Question:

Does Euler equation reflects anomalies?

QUANTUM ANOMALIES AS KINEMATIC PROPERTIES OF A CLASSICAL FLUID

Based on works with A.G. Abanov

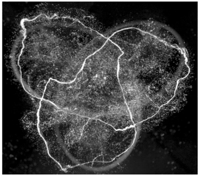
Important work in the subject: DT Son, P Surowka - Phys. Rev. Lett., 2009

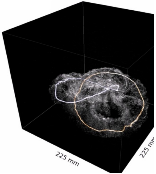
Quantization in perfect fluid: Kelvin, Moffat

$$\text{Helicity} \qquad \mathscr{H} = \int \mathbf{p}\cdot(\nabla\times\mathbf{p})\,d\mathbf{x} = \text{linking number}\times(\text{circulation})^2$$

$$\mathbf{p} = m\mathbf{v} - \text{momentum}$$

$$\text{Helicity is conserved} \quad \dot{\mathscr{H}} = 0 \quad \text{regardless of a Hamiltonian}$$





Helicity visualization

$$\boxed{\mathscr{H} \sim \int \mathbf{v} \cdot (\nabla \times \mathbf{v})\, d\mathbf{x}}$$

Arnold: stationary solutions of the Euler equation fall into classes of a given value of helicity

Foliation: Geometric interpretation of hydrodynamics

Vorticity surfaces are integral and form a foliation of space-time

Covariant form of the Euler equation

Carter-Lichnerowitz form of the Euler equation:

$$j^\mu \overbrace{(\partial_\mu p_\nu - \partial_\nu p_\mu)}^{\text{vorticity}} = 0, \qquad \partial_\mu j^\mu = 0,$$

$$\iota_j \Omega = 0, \quad dj = 0, \qquad \Omega = dp$$

j^μ — conserved current, p_μ — momentum :

Example : $j_\mu = (\rho, \rho \mathbf{v})$, $p_\mu = (\Phi, m\mathbf{v})$

Frobenius condition: $\iota_j \Omega = 0$,

Euler's flows form a foliation of space-time by vorticity integral surfaces

Helicity current

Conservation of helicity (a global characteristic) yields a locally conserved current

$$j_A^\mu = \epsilon^{\mu\nu\lambda\rho} p_\nu \partial_\lambda p_\rho$$

$$j_A = p \wedge dp$$

$$d j_A = dp \wedge dp = \Omega \wedge \Omega = 0$$

Conservation of helicity current follows from the Carter-Lichnerowitz form before specifying a relation between current and momentum.

Axial current is not a Noether current

Axial Current Anomaly: Adler 1969; Bell and Jakiw 1969

Dirac Fermions $\psi = (\psi_{L,\sigma}, \psi_{R,\sigma})$ coupled with electromagnetic (Abelian gauge) field

$$\mathscr{L} = \bar{\psi}\gamma^{\mu} i\partial_{\mu}\psi + A_{\mu} j^{\mu} = \psi_L^{\dagger} i(D_t - \sigma\cdot\nabla)\psi_L + \psi_R^{\dagger} i(D_t + \sigma\cdot\nabla)\psi_R$$

- Vector current $\quad j^{\mu} = \bar{\psi}\gamma^{\mu}\psi = \psi_L^{\dagger}\sigma^{\mu}\psi_L + \psi_R^{\dagger}\sigma^{\mu}\psi_R = j_L + j_R$
- Axial current $\quad j_A^{\mu} = \bar{\psi}\gamma_5\gamma^{\mu}\psi = \psi_L^{\dagger}\sigma^{\mu}\psi_L - \psi_R^{\dagger}\sigma^{\mu}\psi_R = j_L - j_R$

$$\psi_L \to e^{i\alpha_L}\psi_L, \quad \psi_R \to e^{i\alpha_R}\psi_R$$

Adler 1969; Bell and Jakiw 1969

$$Q = \int j^0 dx, \quad Q_A = \int j_A^0 dx$$

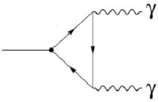
$$\tfrac{d}{dt} Q = 0, \quad \tfrac{d}{dt} Q_A = 2 \int \mathbf{E} \cdot \mathbf{B} \, dx$$

$$\begin{cases} \partial_\mu j^\mu = 0, \\ \partial_\mu j_A^\mu = \frac{k}{4} \, {}^\star F \cdot F = k \, \mathbf{E} \cdot \mathbf{B}, \quad \text{QED}: \; k = 2\,. \end{cases}$$

Triangle Diagram

$$dj_A = \frac{k}{4} F \wedge F$$

2=Triangle Diagram



In units of the Planck constant h the constant k is integer-valued, and $k = 2$ is the number of Weyl fermions in the Dirac multiplet.

$$\mathrm{Tr}\, \gamma_5 = \mathrm{Tr}\left[\mathrm{diag}(\mathbb{1}, -\mathbb{1}) e^{-\epsilon D^2}\right]$$

Axial-current Anomaly=Linking number

$$\frac{d}{dt}Q_A = 2\int \mathbf{E}\cdot\mathbf{B}\,dx$$

$$Q_A|_{t=\infty} - Q_A|_{t=0} = 2\int \mathbf{A}\cdot\mathbf{B}\,d^3x\Big|_{t=0}^{t=\infty}$$

Change of chirality = Twice the change of the linking number of magnetic loops

Axial external field (chiral imbalance)

$$\mathscr{L} = \bar{\psi}\gamma^{\mu} i\partial_{\mu}\psi + A_{\mu} j^{\mu} + A^{\mathrm{A}}_{\mu} j^{\mu}_{\mathrm{A}}$$

Vector current	$j = j_L + j_R$
Axial current	$j_A = j_L - j_R$

$$\partial_{\beta} T^{\beta}_{\alpha} = F_{\alpha\beta} j^{\beta} + F^{\mathrm{A}}_{\alpha\beta} j^{\beta}_{\mathrm{A}} ,$$

$$dj = F \wedge F^{\mathrm{A}}$$

$$dj_{\mathrm{A}} = \frac{1}{2} F \wedge F$$

EULER

$$H = \int \left(\tfrac{1}{2} m\rho \mathbf{v}^2 + \varepsilon[\rho]\right) d\mathbf{x}$$

$$\begin{cases} \dot{\rho} + \nabla(\rho \mathbf{v}) = 0, \\ m\rho(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} + \nabla P = 0 \end{cases}$$

Two conserved currents:

Vector (mass) current: $j = (\rho, \rho\mathbf{v})$,

Axial (*helicity*) current: $j_A^\mu = \epsilon^{\mu\nu\lambda\rho} p_\nu \partial_\lambda p_\rho = p \wedge dp$

$$\partial \cdot j = 0$$

$$\partial \cdot j_A = 0$$

Coupling to the gauge (electromagnetic) field

$$H \to H - \int A \cdot j \, d\mathbf{x}$$

$$m(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} + \rho^{-1}\nabla P = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

Coupling to the gauge field amounts to replacing momentum with canonical momentum.

Carter-Lichnerowitz equation

$$p \to \pi = p + A, \quad \Omega = d\pi, \quad j^\alpha \Omega_{\alpha\beta} = 0$$

Conservation of canonical helicity

$$p \wedge dp \to \pi \wedge d\pi, \quad d[\pi \wedge d\pi] = 0$$

FLUID CHIRALITY

In the gauge field the *fluid helicity* current $\pi \wedge d\pi = (p+A) \wedge d(p+A)$

is conserved, but not gauge invariant, despite the helicity $\mathscr{H} = \int (\pi \wedge d\pi)$ is.

Instead, we must consider

the *fluid chirality*

$$j_{\rm A} = p \wedge dp + F \wedge p$$

It is a gauge invariant but is not conserved.

The Euler equation yields the axial-current anomaly

$$dj_{\rm A} = \tfrac{1}{2} F \wedge F$$

Axial anomaly in fluid mechanics

$$j_{\text{A}} = p \wedge dp + F \wedge p \qquad \longleftrightarrow \qquad \psi_L^\dagger \sigma^\mu \psi_L - \psi_R^\dagger \sigma^\mu \psi_R$$

$$\boxed{dj_{\text{A}} = \tfrac{1}{2} F \wedge F}$$

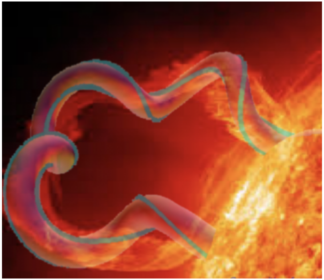
$$Q_A = \int m\mathbf{v} \cdot (m\nabla \times \mathbf{v} + 2\mathbf{B})\, d\mathbf{x},$$

$$\frac{d}{dt} Q_A = 2 \int \mathbf{E} \cdot \mathbf{B}\, d\mathbf{x}$$

$$\boxed{Q_A = 2\mathrm{Link}[\omega] + 2\mathrm{Link}[\omega, B]}$$

2π twist of a vortex changes the action of the fluid by π as does the Weyl fermion

A link between two vortex lines is a Dirac fermion changes the action of the fluid by 2π as does Dirac fermion



CME: Corona mass expulsion

Fluid coupled with axial external field

$$H \to H - \int [A \cdot j + A_{\mathrm{A}} \cdot j_{\mathrm{A}}] d\mathbf{x}$$

$$\begin{cases} \partial_\beta T_\alpha^\beta = F_{\alpha\beta} j^\beta + F^{\mathrm{A}}_{\alpha\beta} j^\beta_{\mathrm{A}}\,, \\ dj = F \wedge F^{\mathrm{A}} \\ dj_{\mathrm{A}} = \frac{1}{2} F \wedge F \end{cases}$$

$$\begin{cases} j = \rho u + F_{\mathrm{A}} \wedge p & \longleftrightarrow & \psi_L^\dagger \sigma \psi_L + \psi_R^\dagger \sigma \psi_R \\ j_{\mathrm{A}} = p \wedge dp + F \wedge p & \longleftrightarrow & \psi_L^\dagger \sigma \psi_L - \psi_R^\dagger \sigma \psi_R \end{cases}$$

Beltrami flow: Chiral Imbalance

What are stationary solutions of the Euler equation? $\longleftrightarrow$ eigenstates of the quantized fluid

$$\text{Beltrami flows}: \quad \mathbf{v} = \mu_{\mathrm{A}} \nabla \times \mathbf{v}$$

$$\text{Energy=Helicity}: \quad \frac{1}{2} m \mathbf{v}^2 = \frac{\mu_{\mathrm{A}}}{2m} \mathbf{v} \cdot (\nabla \times \mathbf{v})$$

Beltrami flow becomes the ground state under the axial external field.

Beltrami flow is known to be chaotic: onset of Lagrangian turbulence

Stationary electric current on the Beltrami flow

$$\boxed{\bar{\mathbf{j}} = 2\mu_{\mathrm{A}} (\mathbf{B} + \nabla \times \mathbf{v})}$$