The tropical Prym variety

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Algebraic Geometry, Mathematical Physics, and Solitons

Columbia University

Celebrating the work of Igor Krichever

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- The kernel $\mathrm{Ker}\,\mathrm{Nm}$ has two connected components.
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The even connected component of $\mathrm{Ker}\,\mathrm{Nm}$ is a principally polarized abelian variety of dimension $g(X) - 1$ called the **Prym variety** $\mathrm{Prym}(\widetilde{X}/X)$ of the double cover $\pi : \widetilde{X} \to X$.

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The dimension of the moduli space of Pryms of dimension g is $\dim\mathcal{R}_g=3g$, which is more than the dimension $\dim\mathcal{M}_g=3g-3$ of the moduli space of Jacobians.

Characterization of Prym varieties

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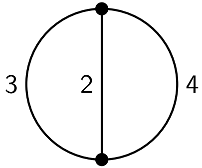
Theorem (Grushevsky–Krichever, 2010)

Prym varieties are characterized by the existence of a symmetric pair of quadrisecant planes of the associated Kummer variety.

Tropicalization: algebraic curves to metric graphs

Algebraic curve



Metric graph

There is a **tropicalization** procedure for algebraic curves over a non-Archimedean field K, e.g. $K = \mathbb{C}((t))$:

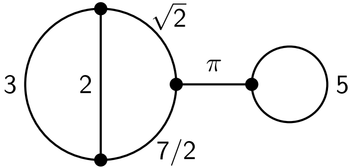
$$X \text{ algebraic curve over } K \quad \to \quad \Gamma_X \text{ metric graph}$$

The graph Γ_X records the degeneration behavior of X.

Metric graphs

Metric graphs are the tropical analogues of algebraic curves.

A metric graph Γ may have loops and multi-edges, and has a **length function** $\ell : E(\Gamma) \to \mathbb{R}_{>0}$ on the edges:



The **genus** $g(\Gamma)$ of a graph Γ is its first Betti number:

$$g(\Gamma) = b_1(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1.$$

Jacobian of a metric graph (Mikhalkin–Zharkov, 2008)

For a smooth algebraic curve X over $\mathbb{C}$, the Jacobian is

$$\mathrm{Jac}(X)=\frac{H_0(X,\Omega^1_X)^\vee}{H_1(X,\mathbb{Z})},$$

with $H_1(X,\mathbb{Z})$ embedded in $H_0(X,\Omega^1_X)^\vee$ by the integration pairing.

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- $H_1(\Gamma, \mathbb{Z})$ is the simplicial homology group:

$$H_1(\Gamma, \mathbb{Z}) = \left\{ \sum_{e \in E(\Gamma)} a_e \cdot e \,\middle|\, a_e \in \mathbb{Z}, \sum_{e \text{ into } v} a_e = \sum_{e \text{ out of } v} a_e \right\}.$$

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- The group of **harmonic** 1**-forms** is also $H_1(\Gamma,\mathbb{Z})$:

$$H_1(\Gamma,\mathbb{Z}) = H_0(\Gamma,\Omega^1_\Gamma), \quad \gamma = \sum_{e\in E(\Gamma)} a_e \cdot e \quad \leftrightarrow \quad \sum_{e\in E(\Gamma)} a_e\, de.$$

The edge length pairing

We introduce a (symmetric, positive definite) pairing on $H_1(\Gamma,\mathbb{Z})$ (think of the first $H_1(\Gamma,\mathbb{Z})$ as $H_0(\Gamma,\Omega^1_\Gamma)$):

$$[\cdot,\cdot]: H_1(\Gamma,\mathbb{Z})\times H_1(\Gamma,\mathbb{Z})\to\mathbb{R},\quad [\gamma_1,\gamma_2]=\int_{\gamma_2}\gamma_1,$$

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We embed $H_1(\Gamma,\mathbb{Z})$ into $H_1(\Gamma,\mathbb{Z})^\vee = \mathrm{Hom}(H_1(\Gamma,\mathbb{Z}),\mathbb{R})$ via the edge length pairing. The quotient torus is the **Jacobian variety** of Γ:

$$\gamma \mapsto [\cdot,\gamma] \in H_1(\Gamma,\mathbb{Z})^\vee, \quad \mathrm{Jac}(\Gamma) = \frac{H_1(\Gamma,\mathbb{Z})^\vee}{H_1(\Gamma,\mathbb{Z})} \simeq \mathbb{R}^g/\mathbb{Z}^g.$$

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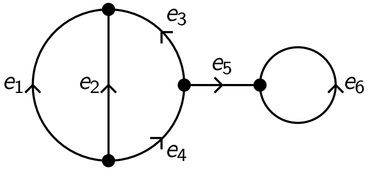
The isomorphism $H_1(\Gamma,\mathbb{Z})=H_0(\Gamma,\Omega^1_\Gamma)$ is the statement that $\mathrm{Jac}(\Gamma)$ is a **principally polarized tropical abelian variety**.

The intersection matrix and Jacobian variety: example



For this graph,

$$H_1(\Gamma,\mathbb{Z})=\mathbb{Z}\gamma_1\oplus\mathbb{Z}\gamma_2\oplus\mathbb{Z}\gamma_3,$$

$$\gamma_1=e_1-e_2,\quad \gamma_2=e_2-e_3-e_4,\quad \gamma_3=e_6,\quad \ell(e_i)=x_i.$$

	γ_1	γ_2	γ_3
γ_1	x_1+x_2	$-x_2$	0
γ_2	$-x_2$	$x_2+x_3+x_4$	0
γ_3	0	0	x_6

The Jacobian variety $\mathrm{Jac}(\Gamma)$ is the quotient of $\mathbb{R}^3$ by the lattice spanned by the columns of the intersection matrix.

Divisor theory on metric graphs

Mikhalkin–Zharkov (2008), Baker–Norine (2007)

- $\mathrm{Div}(\Gamma)$ is the free abelian group on the points of Γ.
- A rational function $f : \Gamma \to \mathbb{R}$ is continuous, **piecewise-linear** with **integer slopes**.
- The **divisor** $\mathrm{div}\, f$ of a rational function is

$$\mathrm{div}\, f = \sum_{x\in\Gamma} (\text{sum of incoming slopes at } x) \cdot x.$$

- The Jacobian variety is isomorphic to the set of linear equivalence classes of degree zero divisors:

$$\mathrm{Jac}(\Gamma) \simeq \mathrm{Pic}^0(\Gamma) = \frac{\mathrm{Div}^0(\Gamma)}{\mathrm{Prin}(\Gamma)}, \quad \mathrm{Prin}(\Gamma) = \{\mathrm{div}\, f | f \in \mathrm{Rat}(\Gamma)\}$$

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Theorem (Baker–Rabinoff, 2015)

If the metric graph Γ_X is the tropicalization of an algebraic curve X, then the Jacobian $\mathrm{Jac}(\Gamma_X)$ is the tropicalization of the Jacobian $\mathrm{Jac}(X)$.

The volume of the Jacobian variety of a metric graph

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Idea. Since $\dim \mathrm{Jac}(\Gamma) = g$, we should have $\mathrm{Vol}(\mathrm{Jac}(\Gamma)) = (\cdots) \cdot \mathrm{cm}^g$, so there should be a formula

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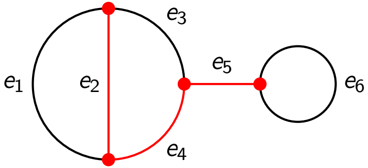
Observation: the genus $g(\Gamma)$ of a graph is the number of edges in the complement of any spanning tree:



$$g(\Gamma) = 3, \quad \Gamma \backslash T = \{e_1, e_3, e_6\}, \quad \ell(e_1)\ell(e_3)\ell(e_6) = x_1 x_3 x_6 \cdot \mathrm{cm}^3.$$

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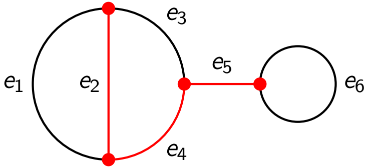

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Kirchhoff's theorem for metric graphs

Theorem (An–Baker–Kuperberg–Shokrieh, 2014)

The volume of the Jacobian of a metric graph Γ is given by

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Example. If Γ is a loop e with $\ell(e) = L$, then

$$H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}e, \quad ||e|| = \sqrt{[e,e]} = \sqrt{L},$$

hence

$$\mathrm{Jac}(\text{circle of length } L) = \text{circle of length } \sqrt{L}.$$

Geometric interpretation of the volume formula

We rearrange the volume formula as follows:

$$\mathrm{Vol}(\mathrm{Jac}(\Gamma)) = \frac{1}{\mathrm{Vol}(\mathrm{Jac}(\Gamma))} \sum_{\substack{F \subset E(\Gamma) \\ \Gamma \backslash F \text{ spanning tree}}} \mathrm{Vol}(F), \quad \mathrm{Vol}(F) = \prod_{e \in F} \ell(e).$$

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Fix a point $q \in \Gamma$, and consider the Abel–Jacobi map:

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The Abel–Jacobi map extends to the symmetric product

$$\Phi^g : \mathrm{Sym}^g(\Gamma) \to \mathrm{Jac}(\Gamma), \quad \Phi^g(p_1 + \cdots + p_g) = \Phi(p_1) + \cdots + \Phi(p_g).$$

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The symmetric product $\mathrm{Sym}^g(\Gamma)$ has a cellular decomposition:

$$\mathrm{Sym}^g(\Gamma) = \bigcup_{F\in \mathrm{Sym}^g(E(\Gamma))} C_F,$$

where the cells are indexed by g-tuples of edges of Γ:

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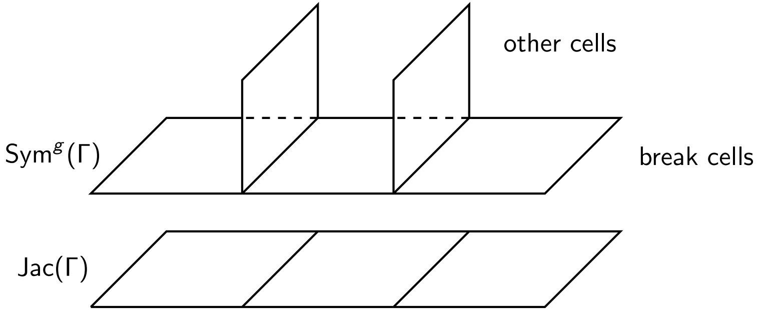
We say that C_F is a **break cell** if all e_i are distinct, and if $\Gamma\backslash F$ is a tree.

ABKS: structure of the tropical Abel–Jacobi map

The Abel–Jacobi map $\Phi^g : \mathrm{Sym}^g(\Gamma) \to \mathrm{Jac}(\Gamma)$ is affine linear on each cell C_F, and the cells fit together as follows:



In other words, Φ^g contracts all cells except the break cells, and the images of the break cells form a **tiling** of $\mathrm{Jac}(\Gamma)$.

ABKS decomposition of the Jacobian

Tropical Jacobi inversion (Mikhalkin–Zharkov 2008, ABKS 2014)

The Abel–Jacobi map

$$\Phi^g : \mathrm{Sym}^g(\Gamma) \to \mathrm{Jac}(\Gamma)$$

has a unique continuous section, whose image is the union of the break cells. Furthermore, for any cell $C_F \subset \mathrm{Sym}^g(\Gamma)$:

1. If C_F is a break cell, then

$$\mathrm{Vol}(\Phi^g(C_F)) = \frac{1}{\mathrm{Vol}(\mathrm{Jac}(\Gamma))}\mathrm{Vol}(F) = \frac{1}{\mathrm{Vol}(\mathrm{Jac}(\Gamma))}\prod_{e\in F}\ell(e).$$

2. Otherwise, $\mathrm{Vol}(\Phi^g(C_F)) = 0$.

Summing over the break cells, we get

$$\mathrm{Vol}(\mathrm{Jac}(\Gamma)) = \frac{1}{\mathrm{Vol}(\mathrm{Jac}(\Gamma))}\sum_{\substack{F\subset E(\Gamma)\\ \Gamma\backslash F \text{ spanning tree}}}\mathrm{Vol}(F).$$

Double covers of metric graphs

A **double cover** of metric graphs $\pi : \widetilde{\Gamma} \to \Gamma$ is a topological covering of degree two that also preserves edge lengths.

It is easy to see that

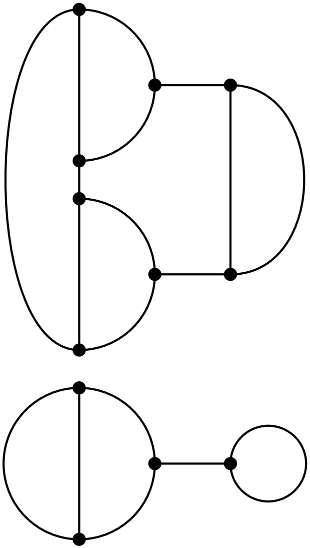
$$g(\widetilde{\Gamma}) = 2g(\Gamma) - 1.$$

There is an induced map on the homology groups and a surjective norm map on the Jacobians:

$$\pi_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z}),$$

$$\sum_{\tilde{e} \in E(\widetilde{\Gamma})} a_{\tilde{e}} \cdot \tilde{e} \mapsto \sum_{\tilde{e} \in E(\widetilde{\Gamma})} a_{\tilde{e}} \cdot \pi(\tilde{e}),$$

$$\mathrm{Nm} : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma).$$



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The even connected component is the **Prym variety** of the double cover:

$$\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) = \frac{(\mathrm{Ker}\,\pi_*)^\vee}{\mathrm{Ker}\,\pi_*}, \quad \dim \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) = g(\Gamma) - 1.$$

where $\mathrm{Ker}\,\pi_*$ is embedded in its dual by the integration pairing:

$$\gamma \in \mathrm{Ker}\,\pi_*, \quad \gamma \mapsto \frac{1}{2}[\cdot, \gamma] \in (\mathrm{Ker}\,\pi_*)^\vee.$$

The Prym variety of a double cover of metric graphs

Theorem (Jensen–Len, 2018)

Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a double cover of metric graphs.

1. The kernel of the norm map $\mathrm{Nm} : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma)$ has two connected components.
2. The even connected component carries a principal polarization that is $\frac{1}{2}$ of the polarization induced from $\mathrm{Jac}(\widetilde{\Gamma})$.

The even connected component is the **Prym variety** of the double cover:

$$\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) = \frac{(\mathrm{Ker}\,\pi_*)^\vee}{\mathrm{Ker}\,\pi_*}, \quad \dim \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) = g(\Gamma) - 1.$$

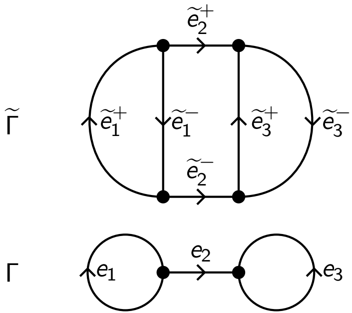
where $\mathrm{Ker}\,\pi_*$ is embedded in its dual by the integration pairing:

$$\gamma \in \mathrm{Ker}\,\pi_*, \quad \gamma \mapsto \frac{1}{2}[\cdot, \gamma] \in (\mathrm{Ker}\,\pi_*)^\vee.$$

Theorem (Len–Ulirsch, 2020)

If the double cover $\widetilde{\Gamma} \to \Gamma$ is the tropicalization of $\widetilde{X} \to X$, then $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ is the tropicalization of $\mathrm{Prym}(\widetilde{X}/X)$.

Prym variety of a double cover: example



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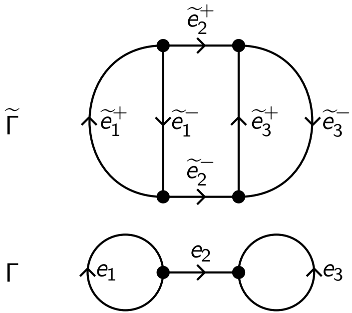

The kernel of $\pi_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ is generated by the cycle

$$\gamma = (\widetilde{e}_1^+ - \widetilde{e}_1^-) + 2(\widetilde{e}_2^+ - \widetilde{e}_2^-) - (\widetilde{e}_3^+ - \widetilde{e}_3^-),$$

$$\frac{1}{2}[\gamma, \gamma] = x_1 + 4x_2 + x_3, \quad x_i = \ell(e_i) = \ell(\widetilde{e}_i^\pm).$$

Hence $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ is a circle of circumference $\sqrt{x_1 + 4x_2 + x_3}$.

The cographic matroid

In analogy with the tropical Jacobian, there should be a formula

$$\mathrm{Vol}^2(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)) = \sum_{F\subset E(\Gamma)} A(F)\,\mathrm{Vol}(F), \quad \mathrm{Vol}(F) = \prod_{e\in F} \ell(e).$$

- The sum is taken over certain $(g-1)$-element subsets of $E(\Gamma)$.
- $A(F)$ are certain coefficients (powers of two?).

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Let Γ be a graph. We denote

$$\mathcal{M}^*(\Gamma) = \{F \subset E(\Gamma) | \Gamma \backslash F \text{ is connected}\}.$$

Then F is the complement of a spanning tree if and only if F is a maximal element $\mathcal{M}^*(\Gamma)$.

The sets $\mathcal{M}^*(\Gamma)$ are the independent sets of the **cographic matroid** of Γ.

Matroid of a double cover (Zaslavsky, 1982)

Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a double cover. We say that an edge set $F \subset E(\Gamma)$ is **independent** if

every connected component of $\Gamma \backslash F$ has connected preimage in $\widetilde{\Gamma}$.

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$$\begin{array}{cccccc} g(\Gamma_i) = 0 & \to & \pi^{-1}(\Gamma_i) \text{ disconnected} & \to & F \text{ not independent} \\ g(\Gamma_i) \geq 2 & \to & \text{can remove another edge} & \to & F \text{ not maximal} \end{array}$$

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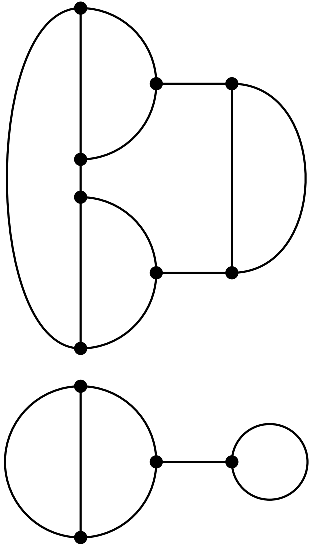
Hence F is a maximal independent set if

- $g(\Gamma_i) = 1$ for all i.
- Each $\pi^{-1}(\Gamma_i)$ is connected.

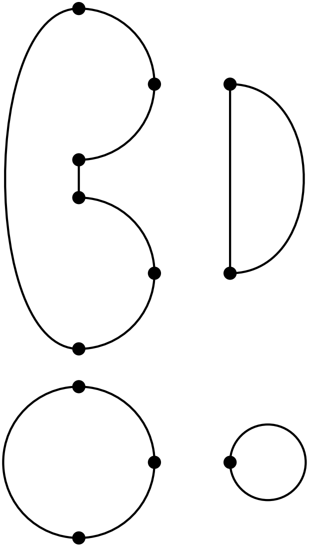
We call such a set $F \subset E(\Gamma)$ an **odd genus one decomposition (ogod)**. The number k of connected components of $G \backslash F$ is the **rank** $r(F)$ of F.

Example of a double cover

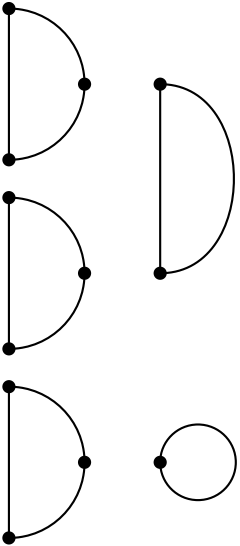


$g(\widetilde{\Gamma}) = 5$, $g(\Gamma) = 3$, ogods consist of $3 - 1 = 2$ edges.

Odd genus one decompositions: example



ogod of rank 2



not an ogod

The volume of the tropical Prym variety

Theorem (Len-Z)

The volume of the tropical Prym variety of a double cover $\pi : \widetilde{\Gamma} \to \Gamma$ is

$$\mathrm{Vol}^2(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)) = \sum_{F \subset E(\Gamma) \text{ ogods}} 4^{r(F)-1}\,\mathrm{Vol}(F),$$

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For discrete graphs, an analogous result was proved by Zaslavsky (1982) and Reiner–Tseng (2004).

Geometrization of the volume formula and Abel–Prym map

The Abel–Prym map associated to a double cover $\pi : \widetilde{\Gamma} \to \Gamma$ with associated involution $\iota : \widetilde{\Gamma} \to \widetilde{\Gamma}$ is

$$\Psi : \widetilde{\Gamma} \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma), \quad p \mapsto p - \iota(p)$$

It extends to symmetric powers:

$$\Psi : \mathrm{Sym}^{g-1}(\widetilde{\Gamma}) \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma), \quad \Psi(p_1 + \cdots + p_{g-1}) = \Psi(p_1) + \cdots + \Psi(p_{g-1}).$$

Our main result regarding Ψ is the following:

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Theorem (Len–Z.)

The tropical Abel–Prym map

$$\Psi : \mathrm{Sym}^{g-1}(\widetilde{\Gamma}) \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$$

is a harmonic morphism of polyhedral complexes of degree 2^{g-1}.

The action of Ψ on the cells of $\mathrm{Sym}^{g-1}(\widetilde{\Gamma})$

We consider the natural cellular decomposition for $\mathrm{Sym}^{g-1}(\widetilde{\Gamma})$:

$$\mathrm{Sym}^{g-1}(\widetilde{\Gamma}) = \bigcup_{F \in \mathrm{Sym}^{g-1}(E(\widetilde{\Gamma}))} C_F.$$

We say that C_F is an **ogod cell of degree** 2^{k-1} if $\pi(F) \subset E(\Gamma)$ is an ogod of rank $k = r(\pi(F))$.

Theorem (Len–Z)

Let $\Psi : \mathrm{Sym}^{g-1}(\widetilde{\Gamma}) \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ be the Abel–Prym map.

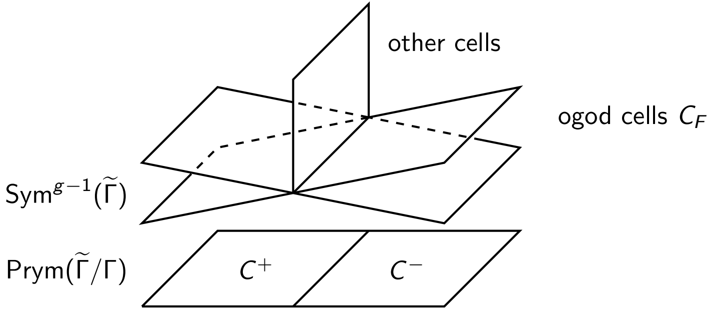
1. If $C_F \subset \mathrm{Sym}^{g-1}(\widetilde{\Gamma})$ is an ogod cell of degree 2^{k-1}

$$\mathrm{Vol}(\Psi(C_F)) = 2^{k-1}\frac{\mathrm{Vol}(F)}{\mathrm{Vol}(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma))}.$$

2. Otherwise, $\mathrm{Vol}(\Psi(C_F)) = 0$ (Ψ contracts C_F).

The ogod cells form a **multivalued tiling** of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. A generic point lies in 2^{g-1} tiles (counted with degree).

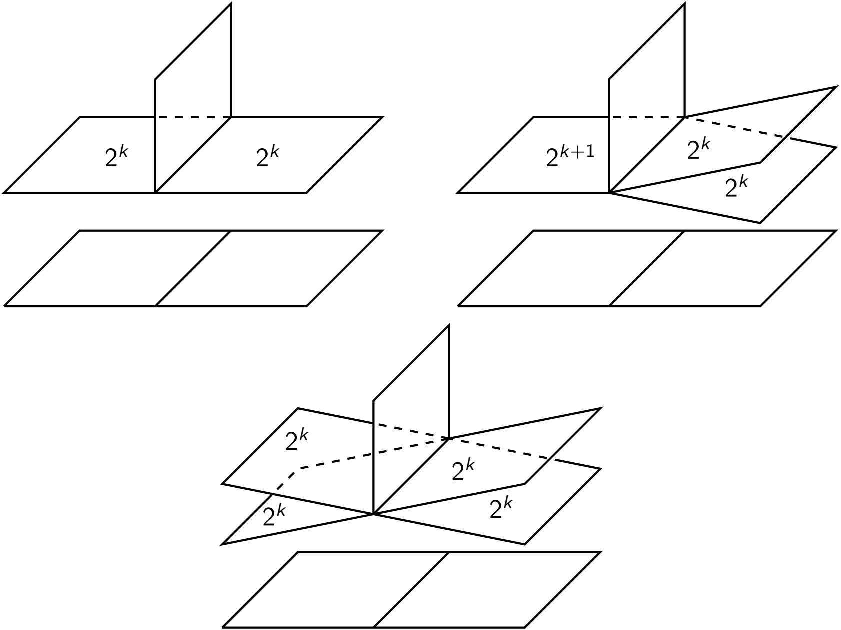
Harmonicity of the Abel–Prym map



Harmonicity of the Abel–Prym map

Let C^+ and C^- be two cells of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ of dimension $g-1$ with common boundary of dimension $g-2$. The total degrees over all cells $C(F)$ mapping to C^+ and to C^- are equal:

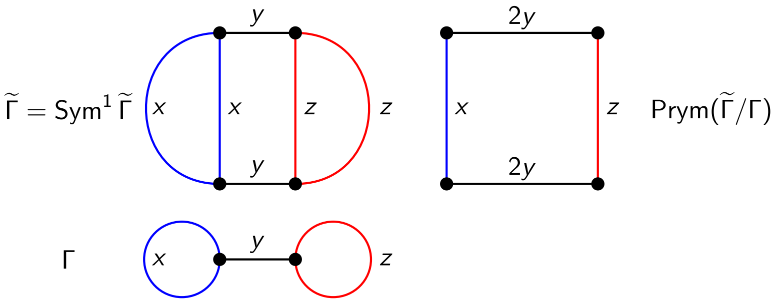
$$\sum_{F:\Psi(C_F)=C^+} 2^{r(\varphi(F))-1} = \sum_{F:\Psi(C_F)=C^-} 2^{r(\varphi(F))-1}.$$


2^k
2^k
2^{k+1}
2^k
2^k
2^k
2^k
2^k
2^k

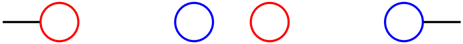
The Abel–Prym map: examples for $g = 2$

We consider the double cover $\pi : \widetilde{\Gamma} \to \Gamma$ and Abel–Prym map

$$\widetilde{\Gamma} = \mathrm{Sym}^1(\widetilde{\Gamma}) \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma).$$



The odd genus one decompositions are



$r = 1, \mathrm{Vol} = x$ $\quad r = 2, \mathrm{Vol} = y$ $\quad r = 1, \mathrm{Vol} = z$

Hence the volume of the Prym is $\mathrm{Vol}^2(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)) = x + 4y + z$.

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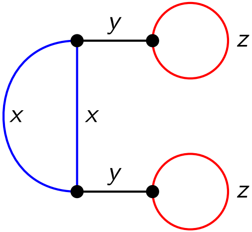
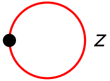
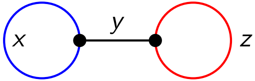


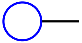


$\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$

Γ



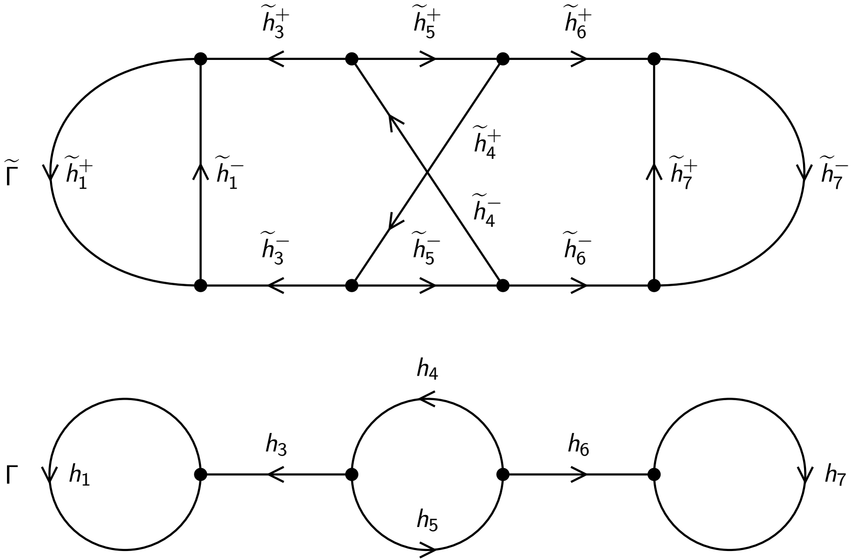
The only odd genus one decomposition is

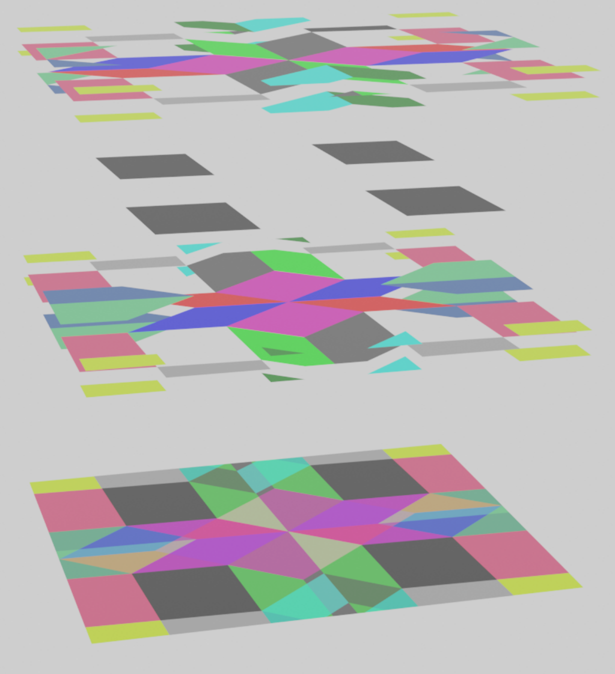


$r = 1, \mathrm{Vol} = z$

$$\mathrm{Vol}^2(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)) = z.$$

The Abel–Prym map: example for $g = 3$





The trigonal construction

Theorem (Recillas, 1974)

There is a bijection between étale double covers of trigonal curves $\widetilde{X} \to X \to \mathbb{P}^1$ and generic tetragonal curves $Y \to \mathbb{P}^1$ such that

$$\mathrm{Prym}(\widetilde{X}/X) \simeq \mathrm{Jac}(Y).$$

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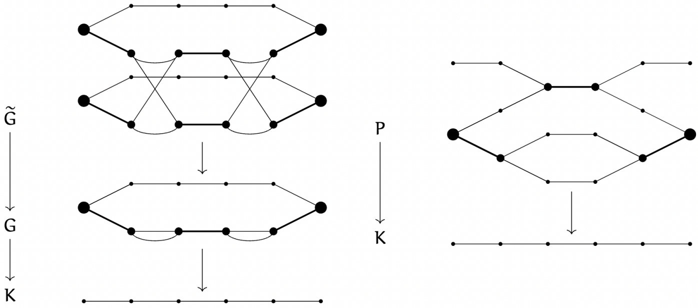
Our main technique is **tropical homology theory** (Itenberg–Katsarkov–Mikhalkin–Zharkov, Gross–Shokrieh), and we are able to closely model our proof on the algebraic case.

Example of the tropical trigonal construction



A double cover of a trigonal tropical curve $\widetilde{G} \to G \to K$ and a tetragonal tropical curve $P \to K$. Thickness indicates dilation factor.

$$\mathrm{Prym}(\widetilde{G}/G) \simeq \mathrm{Jac}(P).$$

The end

THANK YOU!