The Riemann–Hilbert Problem in Higher Genus and Some Applications

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Abstract Painlevé transcendents as well as solutions of nonlinear waves are deeply linked to solutions of Riemann—Hilbert problems on the sphere. At their core, these problems define a (trivial) vector bundle on the sphere, and the poles of the transcendents correspond to non-trivial bundles where the partial indices of the associated problem become non-zero. In higher genus there are additional issues linked to the index; the role of degree—zero bundles is better played by degree ng (with n the rank and g the genus). The practical application of the theory of infinitesimal variations then requires a matrix version of the Cauchy kernel that contains as parameters the Turin data, namely the moduli of a reference bundle. While these notions seem closer to algebraic geometry than to Integrable Systems, I will indicate how they become necessary to address certain problems stemming from asymptotic analysis of Padé approximations on Riemann surfaces.

Introduction: genus and index for people in the RH community

On the plane, the prototype RHP is

size $n \times n$

$$Y(z_+) = Y(z_-)G(z), \quad z \in S^1, \qquad Y(\infty) = \mathbf{1}, \ Y(z), Y^{-1}(z) \text{anal. \& bdd. on } \mathbb{C} \backslash S^1$$

Obstruction!

$$\mathrm{ind}_{S^1} \det G \neq 0.$$

Let $\mathcal{C}$ be a smooth R.S. of genus g and γ the bdry of an embedded disk $\mathbb{D}$.

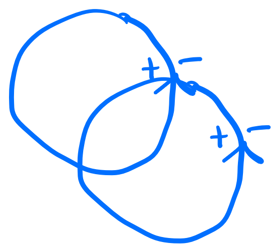
Question

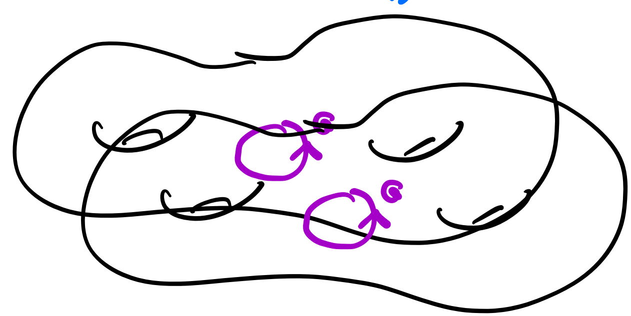
Can we repeat the problem above?

jump matrix.

$G : \gamma \xrightarrow{\text{jump}} GL_n(\mathbb{C})$

$G : \gamma \to GL_n(\mathbb{C})$





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Let $\mathcal{C}$ be a smooth R.S. of genus g and γ the bdry of an embedded disk $\mathbb{D}$.

Question

Can we repeat the problem above?

Answer: NO!

- If we insist on index 0 we must allow poles (try even for the scalar problem!).
- If we insist on holomorphicity of $Y(z)$ we must choose $\mathrm{ind}_\gamma \deg G = ng.$

Alg. geometers know this very well; this is a vector bundle $\mathscr{E}$ of degree ng and Riemann–Roch says

$$h^0(\mathscr{E}) = h^1(\mathscr{E}) + ng - n(g-1) = h^1(\mathscr{E}) + n.$$

Generically we have unique solution (modulo normalization); on the **non-Abelian Theta divisor**

$$h^1(\mathscr{E}) > 0.$$

$\underset{h_j}{\boxed{\;}}\, Y(p_j) = 0$

For a RHP of index ng then $\det Y$ will have ng zeros!

Example (Simple zeros)

$\mathrm{div}\,(\det Y) = \sum_{j=1}^{ng} p_j =: \mathscr{T}$. Then $Y(p_j)$ has co-rank 1; $\mathrm{Ker}_{\mathrm{row}}(Y(p_j)) = \mathbb{C}\{\mathbf{h}_j\}$.

Definition (Tyurin data)

The collection of $\mathscr{T}$ (divisor of degree ng) and $\mathbf{h}_j \in \mathbb{P}^{n-1}$ is called **Tyurin data**.
They classify the moduli space of vector bundles of degree ng (up to common GL_n action)

For higher multiplicity points the description is subtler, see loc.cit.

With an eye to small norm theorems and Deift–Zhou-type of problems we need to study "infinitesimal" deformations:

$$Y(z_+;\epsilon) = Y(z_-;\epsilon)\left(G(z) + \epsilon\delta G(z)\right), \quad z \in \gamma.$$

Then, like in genus 0:

$Y(\infty) = \mathbb{1}$

$$\dot Y(z_+) = \dot Y(z_-)G(z) + Y(z_-)\delta G(z)$$

Question

How to solve this **non-homogeneous** RHP? In genus zero

$$\dot Y(z) = \left(\oint_\gamma Y(w_-)\delta G(w) Y^{-1}(w_+)\frac{\mathbf{1}\,\mathrm{d}w}{(w-z)2i\pi}\right)Y(z)$$

The problem is what goes instead of $\frac{\mathbf{1}\,\mathrm{d}w}{w-z}$?

simple pole at ∞

simple pole $z=w$,

simple zero $z=\infty$

Simple Tyurin points (example): equation for Ξ

Consider $\mathscr{T} = \sum_{j=1}^{ng} p_j$

$$\textbf{Tyurin vectors}: \qquad \mathbf{h}_p^{t} Y(p) = 0 \quad \mathbf{h}_p \in \mathbb{P}^{n-1}$$

Let $\omega_\ell(z) \in H^0(\mathcal{K}_{\mathcal{C}})$ be the (normalized) holomorphic differentials:

$$\omega_\ell(\mathscr{T}) := \mathrm{diag}\Big(\omega_\ell(p_1), \dots, \omega_\ell(p_{ng})\Big) \in \mathrm{Mat}_{ng\times ng} \qquad \mathbb{H} := \begin{bmatrix} \mathbf{h}_1 \\ \hline \mathbf{h}_2 \\ \hline \\ \vdots \\ \hline \mathbf{h}_{ng} \end{bmatrix} \in \mathrm{Mat}_{ng\times n}$$

Theorem

Brill–Noether–Tyurin matrix

$$\mathbb{T} := \Big[\ \omega_1(\mathscr{T})\mathbb{H} \ \Big|\ \omega_2(\mathscr{T})\mathbb{H} \ \Big|\ \dots \ \Big|\ \omega_g(\mathscr{T})\mathbb{H}\ \Big] \in \mathrm{Mat}_{ng\times ng}$$

$$h^1(\mathscr{E}) = \boxed{\operatorname{corank} \mathbb{T}}.$$

0

The non-Abelian Theta divisor is thus a divisor: $\Xi = \{\det \mathbb{T} = 0\}$.

Cauchy kernel and affine connection

Definition

The Cauchy kernel $\mathbf{C}_\infty(w,z) = C_\infty(w,z)\,\mathrm{d}w$ is a differential/function matrix such that

1. $\mathbf{C}_\infty(w,z) = \left(\frac{\mathbf{1}}{w-z} + \mathbf{F}(z) + \mathcal{O}(w-z)\right)\mathrm{d}w$ i.e. $\operatorname*{res}_{w=q} \mathbf{C}_\infty = \mathbf{1}$;
2. as a differential in w/function in z:

$$\operatorname{div}\left(\mathbf{C}_\infty(w,z)\right)_w \geqslant -\infty - z \qquad \operatorname{div}\left(\mathbf{C}_\infty(w,z)\right)_z \geqslant -w + \infty - \mathscr{T}$$

3. for every $p_j \in \mathscr{T}$ (here version for simple Tyurin data only)

$$\mathbf{h}_j^t \mathbf{C}_\infty(w,z) = \mathcal{O}(\overset{1}{w} - p_j) \qquad \text{as } w \to p_j \tag{1}$$

$$\mathbf{C}_\infty(w,z) = \frac{\mathbf{h}_j^t \mathbf{v_j}}{z - p_j} + \text{regular} \tag{2}$$

It exists and is unique for $\mathscr{E} \notin \boldsymbol{\Xi}$.

Affine connection

The matrix $\mathbf{F}(z)$ is an affine connection under change of coordinates:

$$\mathbf{F}(\zeta) = \frac{\mathrm{d}\zeta}{\mathrm{d}z}\mathbf{F}(z) + \frac{\mathbf{1}}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)$$

Formula for $\mathbf{C}_\infty$

The kernel is not abstract:

$$\mathbf{C}_{ij}(q,p) = \frac{\det \left[\begin{array}{c|c|c|c} \omega_1(\mathscr{T})\mathbb{H} & \cdots & \omega_g(\mathscr{T})\mathbb{H} & \omega_{p,\infty}(\mathscr{T})\mathbb{H}\mathbf{e}_j \\ \hline \omega_1(q)\mathbf{e}_i^t & \cdots & \omega_g(q)\mathbf{e}_i^t & \omega_{p,\infty}(q)\delta_{ij} \end{array}\right]}{\det \mathbb{T}}, \quad i,j = 1,\ldots,n.$$

$$\omega_j(\mathscr{T}) = \begin{bmatrix} \omega_j(p_1) & 0 & 0 & \cdots & 0 \\ 0 & \omega_j(p_2) & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & & & \\ 0 & 0 & \cdots & & \omega_j(p_{ng}) \end{bmatrix} \in \mathrm{Mat}_{ng\times ng} \qquad \mathbb{H} := \begin{bmatrix} \mathbf{h}_1 \\ \hline \mathbf{h}_2 \\ \hline \vdots \\ \hline \mathbf{h}_{ng} \end{bmatrix} \in \mathrm{Mat}_{ng\times n}$$

Explicit formulas are essential for applications.

Applications: from symplectic geometry of moduli space to nonlinear steepest descent (Padé)

Logarithmic form of Liouville's tautological form

$$T^\star \mathcal{V} \overset{\mathbf{A}+\mathbf{F}_{\mathscr{D}}}{\longleftrightarrow} \widehat{\mathscr{A}}(\mathscr{D}) \xrightarrow{\mathcal{M}} \mathfrak{R}(\mathcal{C}, \mathrm{GL}_n)$$

Goldman sympl. struct.

- $\mathbf{F}_{\mathscr{D}}$ reference affine connection constructed directly from Cauchy kernel;
- $\mathcal{M}$ the monodromy map;
- $\mathcal{V}$ moduli space of (stable) vector bundles of degree ng
- $\widehat{\mathscr{A}}(\mathscr{D})$ connections (with fixed polar divisor).
- $\mathfrak{R}(\mathcal{C}, \mathrm{GL}_n)$ is the **character variety**

Theorem

If θ_{can} is tautological one form on $T^*\mathcal{V}$ then

- $\varphi := (\mathcal{M}^{-1})^* \theta_{can}$ *is a "potential" on* $\mathfrak{R}(\mathcal{C}, \mathrm{GL}_n)$ *for the Goldman symp. form:*

$$\mathrm{d}\varphi = \omega_G$$

- φ *is a* LOGARITHMIC FORM *on* $\mathfrak{R}(\mathcal{C}, \mathrm{GL}_n)$ *with pole along* $\mathcal{M}(\Xi)$ *and residue* $-h^1(\mathscr{E})$*:*

$$\varphi + \mathrm{d} \ln \det \mathbb{T} = \mathcal{O}(1). \quad \text{(Locally near } \Xi\text{)}$$

Informally: the (class of the) Goldman symplectic form is "Poincaré dual" to Ξ.

More practical application: Padé on Riemann surfaces

Given a measure the **Weyl-(Stiltjes)** function (or generating function of moments):

Markov-

$$W(z) := \int_{\mathbb{R}} \frac{e^{w(x)}\mathrm{d}x}{z-x} = \sum_{j\geqslant 0} \frac{\mu_j}{z^{j+1}}$$

The Padé approximation is a rational approximation scheme:

$$W(z) = \frac{Q_{n-1}(z)}{P_n(z)} + \mathcal{O}(z^{-2n+1}), \quad |z| \to \infty.$$

Fact:

The denominators are the orthogonal polynomials for the measure.

Can we merge these two worlds? (B)OPs on RSs?

Very little literature:

- Fasondini-Olver-Xu (2020) arXiv:2011.10884: Orthogonal "polynomials" on elliptic curves
- C. Charlier: spectral curves and matrix OPs *Trans. Math. Appl.* **5** (2021), no. 2, tnab004, 35 pp.

Two generalization directions

Generalizations: either via meromorphic functions or meromorphic half–differentials.

$$\int P_n(z)P_m(z)\mathrm{e}^{w(x)}\,\mathrm{d}x = \int \underbrace{P_n(z)\sqrt{\mathrm{d}x}}_{\varphi_n}\ \underbrace{P_m(z)\sqrt{\mathrm{d}x}}_{\varphi_m}\ \mathrm{e}^{w(x)} \tag{5}$$

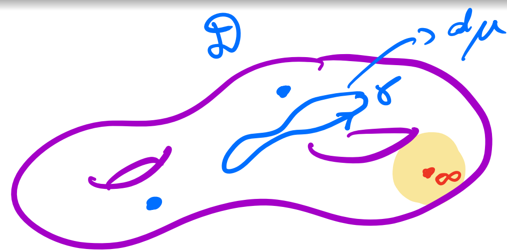
Meromorphic functions with pole at a given point.

I am going to describe only the second setting here. The first one is necessary for application to MOPs: also generalizes nicely multi–point Padé approximations.

Padé on Riemann surfaces

We need the following data:

- A smooth R.S. $\mathcal{C}$ of genus g;
- a (generic) divisor $\mathscr{D}$ of degree g;
- a fixed chosen point $\infty \in \mathcal{C}$;
- a local coordinate $z : \mathbb{D}_\infty \to \mathbb{C}$ such that $\frac{1}{z(\infty)} = 0$.
- a curve $\gamma \subset \mathcal{C}$;
- a density (measure) $\mathrm{d}\mu$ on γ.



The (scalar) Cauchy kernel

$\mathbf{C}_\infty(p,q)$ is a differential in p and function in q such that:

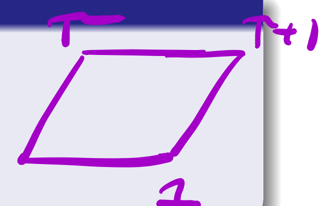
g parameters

1. as a differential w.r.t. p it has poles at q, ∞ and residues $+1, -1$; zeros at $p \in \mathscr{D}$;
2. as a function w.r.t. q it has poles at $p, \mathscr{D}$ and zero at ∞.

Such object exists and is unique.

Example (genus 1)

$$\mathbf{C}_\infty(z,w) = \left(\zeta(z-w) + \zeta\left(w-a\right) - \zeta(z) + \zeta(a)\right)\, \mathrm{d}z.$$



∞ is $z = 0$ and $\mathscr{D} = a$.

Definition (Weyl differential)

We define it by

$$\mathcal{W}(p) = \int_{q\in\gamma} \mathbf{C}_\infty(p,q)\,\mathrm{d}\mu(q)$$

disc. on γ

$\mathcal{W}(p_+) - \mathcal{W}(p_-) = 2\pi i\, d\mu(p)$

$p \in \gamma$

The space of polynomials of degree n is now replaced by the line bundle $\mathscr{L}(n\infty + \mathscr{D})$ (of dimension $n+1$ like the space of polynomials by Riemann–Roch).

Problem (Padé approximation problem)

Find $P_n \in \mathscr{L}(\mathscr{D} + n\infty)$ and $\mathfrak{Q}_{n-1} \in \mathcal{K}((n+1)\infty)$

$$\operatorname{div}\left(\frac{\mathfrak{Q}_{n-1}}{P_n} - W\right) \geqslant 2\mathscr{D} + (2n-1)\infty.$$

Theorem ("Orthogonality")

(nonhermitian)

$$\int_\gamma P_n(p)P_m(p)\,\mathrm{d}\mu(p) = h_n\delta_{nm}.$$

What survives?

We now use the local coordinate z and define a reference basis of **"monic"** meromorphic functions ;

$$\zeta_j(q) := \operatorname*{res}_{p=\infty} z(p)^j \mathbf{C}(p,q) = z^j + \mathcal{O}(z^{-1}).$$

"monomials".

1. **Pseudo–moments** $\mu_{j,k}$ (not Hankel!):

$$\mu_{j,k} = \oint_\gamma \zeta_j(p)\zeta_k(p)\,\mathrm{d}\mu(p) = -\operatorname*{res}_{q=\infty} \oint_{p\in\gamma} \zeta_j(q)\mathbf{C}(q,p)\zeta_k(p)\,\mathrm{d}\mu(p)$$

$$D_n := \det\Big[\mu_{j,k}\Big]_{j,k=0}^{n-1}$$

2. **Heine formula**

$$P_n(p) := \frac{1}{D_n}\int_{\gamma^n} \det\Big[\zeta_{a-1}(p_b)\Big]_{a,b=1}^{n+1} \det\Big[\zeta_{a-1}(p_b)\Big]_{a,b=1}^{n} \prod_{j=1}^n \mathrm{d}\mu(p_j), \qquad p_{n+1} = p.$$

3. **Riemann–Hilbert problem** (see next).

The departed

1. Three term recurrence relation; replaced by a $2g+3$ recurrence relation.

Fokas Its Kitaev for OP

Problem

Let Y_n be a 2×2 matrix with functions in the first column and differentials in the second column, meromorphic in $\mathcal{C} \backslash \gamma$

$$Y_n(p_+) = Y_n(p_-) \left[\begin{array}{cc} 1 & \mathrm{d}\mu(p) \\ 0 & 1 \end{array} \right], \qquad p \in \gamma.$$

In addition we require that the matrix is such that it has poles at $\mathscr{D}$ in the first column and zeros in the second column, and also the following growth condition at ∞:

$$Y_n(p) = \left[\begin{array}{cc} \mathcal{O}(\mathscr{D} + n\infty) & \mathcal{K}(-\mathscr{D} - (n-1)\infty) \\ \mathcal{O}(\mathscr{D} + (n-1)\infty) & \mathcal{K}(-\mathscr{D} - (n-2)\infty) \end{array} \right]. \tag{6}$$

$$Y_n(p) = \left(\mathbf{1} + \mathcal{O}(z(p)^{-1})\right) \left[\begin{array}{cc} z^n(p) & 0 \\ 0 & \frac{\mathrm{d}z(p)}{z^n(p)} \end{array} \right], \qquad p \to \infty. \tag{7}$$

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Problem inherently of index $2g$

$\det Y_n \in \mathcal{K}(2\infty)$; it has $2g$ zeros! How to prove uniqueness? Existence? Tyurin divisor....

Theorem

The solution of the RHP exists and is unique if and only if $D_n \neq 0$.

Note that it is different from genus 0; the solution if it exists is unique. Now it may exist and be not unique (if $D_n = 0$).

$$Y_n(p) = \begin{bmatrix} P_n(p) & \mathfrak{R}_n(p) \\ \widetilde{P}_{n-1}(p) & \widetilde{\mathfrak{R}}_{n-1}(p) \end{bmatrix}$$

$$\mathfrak{R}_n(p) := \int_\gamma \mathbf{C}(p,q) P_n(q)\, \mathrm{d}\mu(q) \qquad \widetilde{\mathfrak{R}}_{n-1}(p) := \int_\gamma \mathbf{C}(p,q) \widetilde{P}_{n-1}(q)\, \mathrm{d}\mu(q).$$

$$P_n(p) = \frac{1}{D_n} \det \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n,0} \\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n,1} \\ \vdots & & & \vdots \\ \zeta_0(p) & \zeta_1(p) & \cdots & \zeta_n(p) \end{bmatrix} \in \mathscr{L}(\mathscr{D} + n\infty)$$

$$\widetilde{P}_{n-1}(p) = \frac{1}{D_n} \det \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} \\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1} \\ \vdots & & & \vdots \\ \zeta_0(p) & \zeta_1(p) & \cdots & \zeta_{n-1}(p) \end{bmatrix} \in \mathscr{L}(\mathscr{D} + (n-1)\infty).$$

If $\mathcal{C}$ has antiholomorphic involution fixing γ and $\mathrm{d}\mu$ is a positive measure, then $D_n > 0$ (easy to show).
Genus 1. Elliptic curve $E_\tau = \mathbb{C}/2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$, In Weierstraß form the elliptic curve is

$$Y^2 = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$$

with $e_1 + e_2 + e_3 = 0$ and $e_1 < e_2 < e_3$.
Antiholomorphic involution $z \to \frac{\omega_1}{\omega_1}\overline{z} = \overline{z}$. We choose $\infty = \{0\}$ and $\mathscr{D} = \{a\}$, with $a \in (0, 2\omega_1)$.

$$\mathscr{L}(\mathscr{D} + n\infty) = \mathbb{C}\left\{1, \zeta(z) - \zeta(z-a) - \zeta(a), \wp(z), \wp'(z), \ldots, \wp^{(n-2)}(z)\right\}.$$

Real–analytic: $\overline{f(z)} = f(\overline{z})$.

Theorem

The orthogonal sections π_n exist and have $n+1$ zeros. These lie all on γ for $(n+1)$ even, while for $(n+1)$ odd one zero belongs to α.

Question

Interlacing?

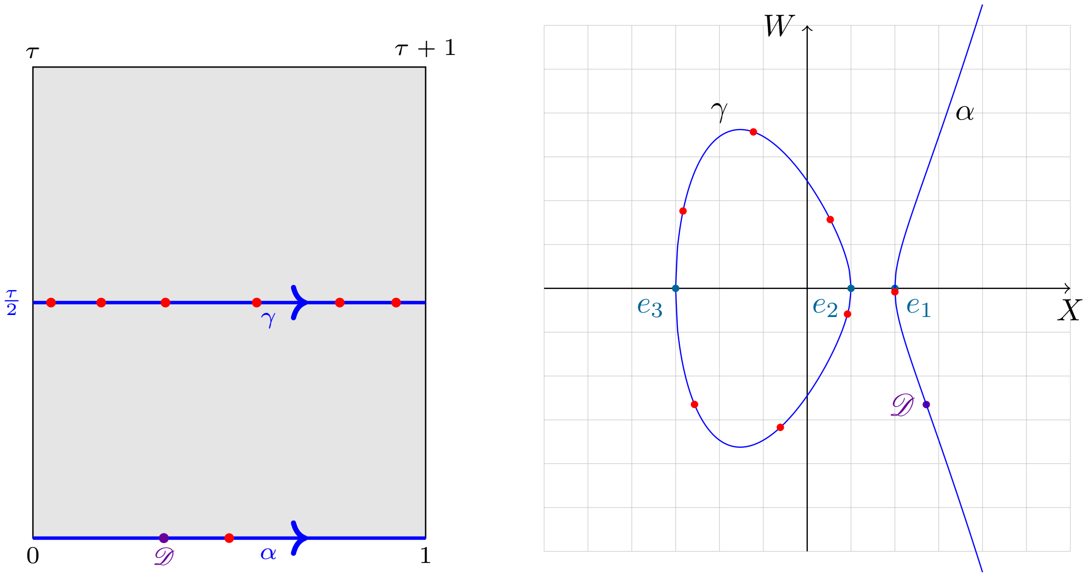


Figure: An example of real elliptic curve (specifically $W^2 = 4(X-1)(X-2)(X+3)$). On the left pane we have the "elliptic" parametrization as the quotient of $\mathbb{C}$ by the lattice Λ_τ. On the right the representation of the real section of $\mathcal{E}_\tau$ in the Weierstrass parametrization. The divisor $\mathscr{D}$ consists of a single point on the real oval of the α cycle (in this example $\mathscr{D} = 1/3$ in the elliptic parametrization), while the measure of orthogonality is defined on the cycle γ and it is given by an arbitrary smooth positive function $w(p)$ on γ times the holomorphic normalized differential $\mathrm{d}p = \frac{\mathrm{d}X}{2\omega_1 W}$. Also plotted are the zeros of the orthogonal section π_6 with respect to the "flat" measure with $w(p) \equiv 1$. Note that the zero on α is already (for $n = 6$) extremely close to e_1: this zero, for even n converges to e_1 exponentially fast.

We now present the asymptotic analysis under

Assumption

The function $w(p)$ is analytic in a strip containing γ and real on γ.

This is a non-scaling regime.

$$\int_\gamma P_n(p)P_m(p)\mathrm{e}^{w(p)}\,\mathrm{d}p = \delta_{nm}\mathrm{h}_n$$

Nonlinear Steepest descent analysis

In genus 1 no practical difference between functions/differentials.

Problem

Let $Y = Y_n(p)$ be the 2×2 matrix, meromorphic on $\mathcal{E}_\tau \backslash \gamma$ and with poles at $p = 0, \mathscr{D}$, such that

1. Near $p = 0 \equiv \Lambda_\tau$ we have the behaviour

$$Y(p) = (\mathbf{1} + \mathcal{O}(p)) \begin{bmatrix} p^{-n} & 0 \\ 0 & p^{n-2} \end{bmatrix}, \quad p \to 0 \mod \Lambda_\tau$$

2. Near $p = \mathscr{D} \mod \Lambda_\tau$ we have that

$$Y(p) = \begin{bmatrix} \mathcal{O}((p-\mathscr{D})^{-1}) & \mathcal{O}(p-\mathscr{D}) \\ \mathcal{O}((p-\mathscr{D})^{-1}) & \mathcal{O}(p-\mathscr{D}) \end{bmatrix}$$

3. The boundary values at $p \in \gamma$ are bounded and satisfy:

$$Y(p_+) = Y(p_-) \begin{bmatrix} 1 & \mathrm{e}^{w(p)} \\ 0 & 1 \end{bmatrix}$$

Note that $\det Y(p)$ has 2 **zeros**: usual argument for uniqueness fails. But the theorem earlier guarantees existence since (using Andreief) one sees $D_n > 0$.

A quick rundown of the DZ method and novelties I

1. The g–function is found explicitly and along similar lines;
2. the steps of (i) normalization (using the g–function) of the singularity and (ii) opening lenses is also without major surprises.
3. The "model problem" (aka "outer parametrix") is found explicitly $M(p)$; alas, its determinant has also 2 zeros $\mathrm{div}\,\det M = (1/4) + (3/4)$. These zeros and the corresponding kernel spaces are the **Tyurin data**.

A quick rundown of the DZ method and novelties II

1. The issue is in the error analysis: to see consider the prototype

$$Y_+(z) = Y_-(z)J(z), \quad |z| = 1, \quad Y(\infty) = \mathbf{1}.$$

$$Y(z) = \mathbf{1} + \frac{1}{2i\pi} \oint_{|w|=1} Y_-(w)\big(J(w) - \mathbf{1}\big)\frac{\mathrm{d}w}{w - z}.$$

The latter expression needs a **matrix Cauchy kernel** that is defined given the Tyurin data: $\mathbf{C}_0(p, q)\,\mathrm{d}p$ is a matrix–valued differential with respect to the variable p and meromorphic function with respect to the variable q satisfying the following properties

1. It has a simple pole for $p = q$ and $p = 0$ and no other poles with respect to p;
2. The residue matrix for $p = q$ is $\mathbf{1}$ (and hence at $p = 0$ is $-\mathbf{1}$)
3. It has a simple pole for $q = p$ and at the Tyurin divisor $\mathscr{T} = (1/4) + (3/4)$ and all entries vanish for $q = 0$.
4. The expression $M^{-1}(p)\mathbf{C}_0(p, q)M(q)$ is locally analytic with respect to q and p at $\mathscr{T}$.

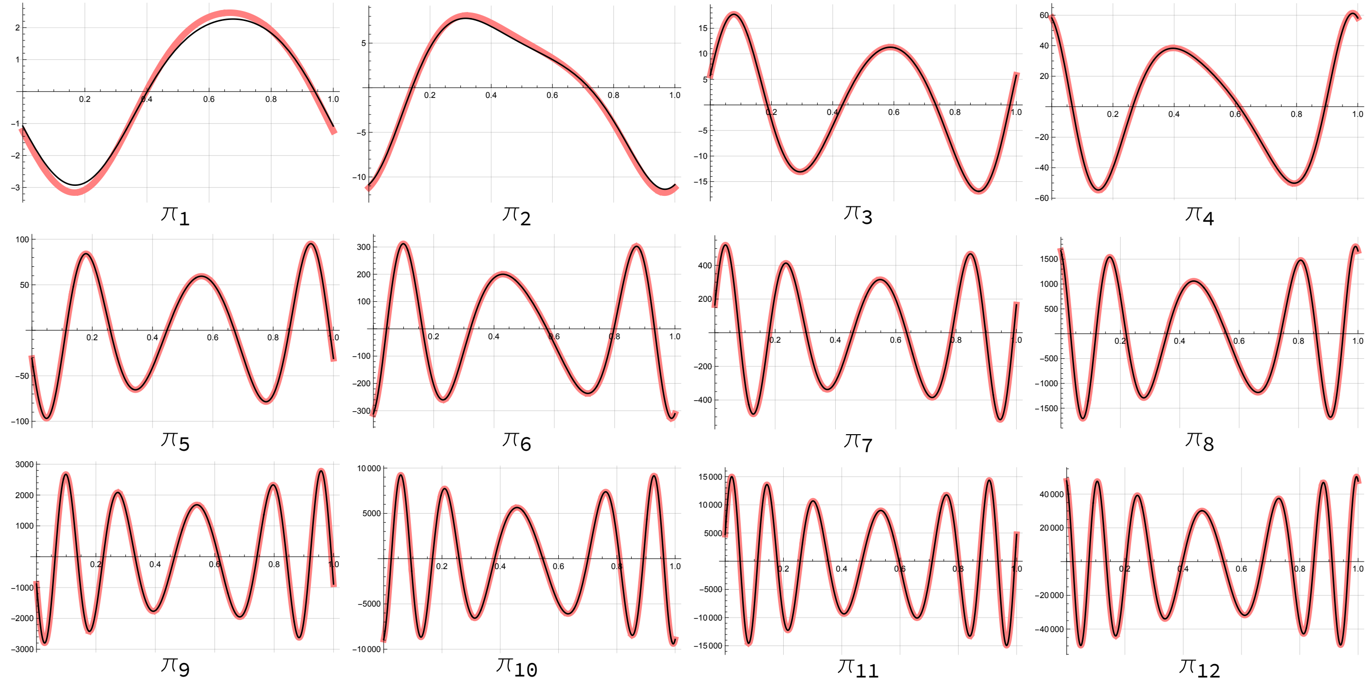


Figure: The first few monic orthogonal sections plotted as a function of $s \in [0, 1]$ via $p = \frac{\tau}{2} + s$; here $\pi_n(p) \in \mathscr{P}_n$ are the "monic" sections behaving like $\pi_n(p) = p^{-n}(1 + \mathcal{O}(p))$. The elliptic curve is $W^2 = 4X^3 - 19X + 15 = 4(X-1)(X-3/2)(X+5/2)$. Here $\tau \simeq 0.6563i$. We have set $\mathscr{D} = 1/3 \in \mathbb{R}$ and $\infty = 0$. The contour γ is the segment $[\tau/2, \tau/2 + 1]$ in $\mathcal{E}_\tau$; in the X–plane this is the segment $X \in [e_3, e_2]$ (on both sheets). The thick line is the plot of the orthogonal section obtained by computing explicitly the moments. The thin line is the approximation. Observe that the approximation is almost perfect starting from $n = 2$, confirming the exponential rate of convergence discussed in the text.

Asymptotic results I

1. For every compact subset of $\mathcal{E}_\tau \backslash \gamma$ we have

$$\pi_n(p) = \mathrm{e}^{-S_\infty} M_{11}(p) \mathrm{e}^{(n-1)g(p)+S(p)} \left(1 + \mathcal{O}(\mathrm{e}^{-nc_0})\right).$$

$$M_{11}(p) = \mathrm{e}^{-i\pi p} \frac{\theta_1(\mathscr{D}; 2\tau)\theta_1(p - \mathscr{D} - \tau; 2\tau)\theta_1'(0; 2\tau)\theta_{\{2,3\}}(p; 2\tau)}{\theta_1(\mathscr{D} + \tau; 2\tau)\theta_1(p - \mathscr{D}; 2\tau)\theta_1(p; 2\tau)\theta_{\{2,3\}}(0; 2\tau)}, \tag{8}$$

where the choice between θ_2, θ_3 is according to the parity of n. The function $S(p)$ is the "Szegö" function for the function $w(p)$ defined in terms of the Cauchy kernel and w.
The g–function is given by:

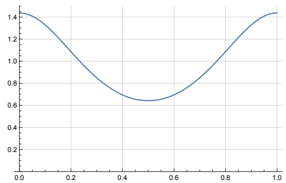
$$\mathrm{e}^{g(p)} = \mathrm{e}^{\ell} \begin{cases} \mathrm{e}^{i\pi(p - \frac{\tau}{2}) - \frac{i\pi}{2}} \dfrac{\theta_1(p; 2\tau)}{\theta_1(p - \tau; 2\tau)} & \Im \frac{\tau}{2} < \Im p < \Im \tau \\ \mathrm{e}^{-i\pi(p - \frac{\tau}{2}) + \frac{i\pi}{2}} \dfrac{\theta_1(p - \tau; 2\tau)}{\theta_1(p; 2\tau)} & 0 < \Im p < \frac{1}{2} \Im \tau. \end{cases}$$

$$\mathrm{e}^{\ell} = -i \frac{\theta_1'(0; 2\tau)}{\theta_1(\tau; 2\tau)} \mathrm{e}^{-i\pi \frac{\tau}{2}} > 0$$

2. For $p \in \gamma$ we have the modulated oscillatory behaviour for $p = s + \frac{\tau}{2} + i0$:

$$\pi_n(p) = 2\mathrm{e}^{(n-1)\ell - S_\infty} \Re\left(M_{11}(p_+)\mathrm{e}^{S(p_+)} \left(\mathrm{e}^{i\pi s - \frac{i\pi}{2}} \frac{\theta_1\left(s + \frac{\tau}{2}; 2\tau\right)}{\theta_1\left(s - \frac{\tau}{2}; 2\tau\right)} \right)^{n-1} \right) \left(1 + \mathcal{O}(\mathrm{e}^{-nc_0})\right)$$

3. For every continuous function ϕ defined on $\gamma \subset \mathcal{E}_\tau$



$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{2\lfloor \frac{n+1}{2} \rfloor} \phi(z_j^{(n)}) = \int_\gamma \phi(p)\sqrt{e_1 - \wp(p)}\,\frac{\mathrm{d}p}{2\pi}$$

Asymptotic results III

4. The extra zero of π_n for n even tends at exponential rate to $p = \frac{1}{2}$ (i.e. $X = e_1$).
5. The square of the norms of the monic orthogonal sections have the asymptotics

$$\|\pi_n\|^2 = 2\pi \mathrm{e}^{2(n-1)\ell - 2S_\infty} \mathrm{e}^{-i\pi\tau} \frac{\mathrm{e}^{-2i\pi\mathscr{D}}\theta_1^2(\mathscr{D};2\tau)}{\theta_1^2(\mathscr{D}+\tau;2\tau)} \frac{\theta_1'(0;2\tau)}{\theta_4(0;2\tau)} \left(\frac{\theta_3(0;2\tau)}{\theta_2(0;2\tau)}\right)^{\sharp_n} (1+\mathcal{O}(\mathrm{e}^{-nc_0}))$$

where $\sharp_n = 1$ for even n and -1 for odd n.

Matrix (Bi)Orthogonal Polynomials I

Matrix weight $W(z)$ on the real axis (or contour γ in $\mathbb{C}$) gives rise to matrix BOPs.

$$\int_\gamma P_n(z)W(z)P_m^\vee(z)\,\mathrm{d}z = \delta_{nm}\mathbf{H}_n,$$

Notable applications to the Aztec diamond (see Arno's talk).
Connection with scalar orthogonality on a Riemann surface already recognized by [Charlier '20] (implicitly in [Duits-Kuijlaars '17]).
It is sufficient that the eigenvectors of $W(z)$ live on an algebraic surface $\mathcal{C}$ (of genus g).

Example (arxiv:2107.12998)

$$Z:\mathcal{C}\to\mathbb{CP}^1,\quad \mathrm{div}\,(Z)\geqslant -r\infty$$

$$\int_\gamma \psi_n\psi_m^\vee \mathrm{e}^w,\quad \psi_n^{(\vee)}\in\sqrt{\mathcal{K}}((n+1)\infty)\otimes\mathcal{X}^{(\vee)}$$

$$\Psi_k^{(\vee)}(z) := \begin{bmatrix} \psi_{rk}^{(\vee)}(z^{(1)}) & \dots & \psi_{rk}^{(\vee)}(z^{(r)}) \\ \vdots & & \vdots \\ \psi_{rk+r-1}^{(\vee)}(z^{(1)}) & \dots & \psi_{rk+r-1}^{(\vee)}(z^{(r)}) \end{bmatrix}$$

Theorem

The matrices

$$P_n(z) := \Psi_k(z)\Psi_0^{-1}(z)\ , \qquad P_n^\vee(z) := (\Psi_0^\vee)^{-1}(z)\Psi_k^\vee(z)$$

are **polynomials** *and (bi)-orthogonal for the weight*

$$\mathbb{W}(z) = W(z)\,\mathrm{d}z := \Psi_0(z)\Lambda(z)\Psi_0^\vee(z), \qquad \Lambda(z) = \mathrm{diag}\Big(Y(z^{(1)}),\ldots,Y(z^{(r)})\Big).$$

Example

It works also if $\mathcal{C}$ is the sphere! $Z(t) = (t-c)^2 : \mathbb{CP}^1 \to \mathbb{CP}^1$

$$W_L(z)\,\mathrm{d}z = \begin{bmatrix} 1 & \alpha+1-c-\sqrt{z} \\ \alpha+1-c-\sqrt{z} & (\alpha+1-c-\sqrt{z})^2 \end{bmatrix} (c+\sqrt{z})^\alpha \frac{\mathrm{e}^{-c-\sqrt{z}}}{2\sqrt{z}}\,\mathrm{d}z.$$

$$P_j(z) = \begin{bmatrix} \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j}^\alpha(s)(s+1+\alpha-2c)\,\mathrm{d}s}{z-Z(s)} & \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j}^\alpha(s)\,\mathrm{d}s}{Z(s)-z} \\ \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j+1}^\alpha(s)(s+1+\alpha-2c)\,\mathrm{d}s}{z-Z(s)} & \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j+1}^\alpha(s)\,\mathrm{d}s}{Z(s)-z} \end{bmatrix}.$$

$$\int_{c^2}^\infty P_j(z)W_L(z)P_k^t(z)\,\mathrm{d}z = \delta_{jk}\begin{bmatrix} \frac{\Gamma(2j+\alpha+1)}{(2j)!} & 0 \\ 0 & \frac{\Gamma(2j+\alpha+2)}{(2j+1)!} \end{bmatrix}.$$

KP and 2–Toda: stitching together Krichever's tau functions

Tensor $\mathscr{L}$ by a zero-degree bundle with transition function $\mathrm{e}^{\sum t_\ell z^\ell(p)}$ near ∞.
A section of $\mathscr{L}_{\boldsymbol{t}}(n\infty + \mathscr{D})$ satisfies:

$$(\psi_n) \geqslant -\mathscr{D}, \quad \psi_n(p) = z^n \mathrm{e}^{\sum t_\ell z^\ell(p)}(1 + \mathcal{O}(z^{-1})).$$

Note:

For $n = 0$ it is the Baker–Akhiezer function of Krichever.

Take

$$\psi \in \widehat{\mathscr{L}_{\boldsymbol{t}}} := \bigoplus_{n \geqslant 0} \mathscr{L}_{\boldsymbol{t}}(n\infty + \mathscr{D})$$

$$\phi \in \widehat{\mathscr{L}_{\boldsymbol{s}}} := \bigoplus_{n \geqslant 0} \mathscr{L}_{\boldsymbol{s}}(n\infty + \mathscr{D})$$

Pairing:

$$\langle \phi, \psi \rangle_{\boldsymbol{t},\boldsymbol{s}} = \int_\gamma \phi(p)\psi(p)\,\mathrm{d}\mu(p)$$

We can construct biorthogonal sections $\{\psi_n, \phi_n\}_{n\in\mathbb{N}}$ (if non-degenerate!)
A basis is

$$\zeta_j(p; \boldsymbol{t}) = z^j \mathrm{e}^{\sum t_\ell z^\ell}(1 + \mathcal{O}(z^{-1})) \ \text{ (similarly for } \boldsymbol{s})$$

Tau function

Definition (The Tau function)

The Tau function is defined by

$$\begin{aligned}\tau_n(\boldsymbol{t},\boldsymbol{s}) :=& \frac{1}{n!}\Theta(F(\boldsymbol{t}))\Theta(\mathbb{F}(\boldsymbol{s}))\mathrm{e}^{Q(\boldsymbol{t})+Q(\boldsymbol{s})+nA(\boldsymbol{t})+nA(\boldsymbol{s})}\times\\ &\times\int_{\gamma^n}\det\left[\zeta_{a-1}(r_b;\boldsymbol{t})\right]_{a,b=1}^{n}\det\left[\zeta_{a-1}(r_b;\boldsymbol{s})\right]_{a,b=1}^{n}\prod_{j=1}^{n}\mathrm{d}\mu(r_j) =\\ =&\tau_{Kr}(\boldsymbol{t})\tau_{Kr}(\boldsymbol{s})\mathrm{e}^{nA(\boldsymbol{t})+nA(\boldsymbol{s})}\det\Big[\mu_{ab}(\boldsymbol{t},\boldsymbol{s})\Big]_{a,b=0}^{n-1}\end{aligned}$$

The expression $Q(\boldsymbol{t})$ is a quadratic form and $A(\boldsymbol{t})$ is a linear form in the times.

Theorem

The tau function

1. *Is a KP tau function w.r.t. both sets of times (satisfies HBI):*

$$\operatorname*{res}_{x=\infty} \tau_n(\boldsymbol{t} - [x], \boldsymbol{s})\tau_n(\tilde{\boldsymbol{t}} + [x], \boldsymbol{s})\mathrm{e}^{\xi(x;\boldsymbol{t})-\xi(x;\tilde{\boldsymbol{t}})}\,\mathrm{d}z(x) \equiv 0$$

2. *It is a tau function for 2–Toda Hierarchy (Adler-VanMoerbeke)*

$$\operatorname*{res}_{x=\infty} \tau_n(\boldsymbol{t} - [x]; \boldsymbol{s})\tau_{m+1}(\tilde{\boldsymbol{t}} + [x]; \tilde{\boldsymbol{s}})\frac{\mathrm{e}^{\xi(x;\boldsymbol{t})-\xi(x;\tilde{\boldsymbol{t}})+A(\tilde{\boldsymbol{t}}-\boldsymbol{t})}\,\mathrm{d}z(x)}{z(x)^{m-n+1}} =$$

$$= \operatorname*{res}_{x=\infty} \tau_{n+1}(\boldsymbol{t}; \boldsymbol{s} + [x])\tau_m(\tilde{\boldsymbol{t}}; \tilde{\boldsymbol{s}} - [x])\frac{\mathrm{e}^{\xi(x;\tilde{\boldsymbol{s}})-\xi(x;\boldsymbol{s})+A(\boldsymbol{s}-\tilde{\boldsymbol{s}})}\,\mathrm{d}z(x)}{z(x)^{n-m+1}}$$

3. *If $P_n(p; \boldsymbol{t}, \boldsymbol{s}), Q_n(p; \boldsymbol{t}, \boldsymbol{s})$ are the biorthogonal sections then the Baker and dual Baker functions are (up to prefactors) $P_n(x; \boldsymbol{t}, \boldsymbol{s})$ and*

$$\mathfrak{R}_n(x; \boldsymbol{t}, \boldsymbol{s}) := \int_{r\in\gamma} \mathbf{C}(x, r; \boldsymbol{t})Q_{n-1}(r; \boldsymbol{t}, \boldsymbol{s})\,\mathrm{d}\mu(r)$$

respectively (note that dual BA is a differential).

4. *$\tau_n(\boldsymbol{t}, \boldsymbol{s}) = 0$ if and only if $\tau_{Kr} = 0$ or the pairing is degenerate on*

$$\mathscr{L}_{\boldsymbol{t}}(n\infty + \mathscr{D}) \otimes \mathscr{L}_{\boldsymbol{s}}(n\infty + \mathscr{D})$$

Outlook

1. Varying weights: this requires study of equilibrium problem on RS: we need appropriate Green functions.
2. One can study DRPF: the projection operator (in the Harnack case) gives a TP kernel defined on the curve.
3. New integrable systems? Connection with Hitchin systems (higher genus generalization of Calogero–Moser types).
4. Interface with algebraic geometry of vector bundles.