The Bonnet Problem.
Is a surface characterized by its metric and curvatures?

Alexander Bobenko

Technische Universität Berlin

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new results on Bonnet pairs joint with T. Hoffmann and A. Sageman-Furnas

SFB TRR 109 Discretization in Geometry and Dynamics

Analytic description of surfaces

Conformally parametrized surfaces

$$f = (f_1, f_2, f_3) : \mathcal{R} \to \mathbb{R}^3,$$

$$< f_u, f_u >=< f_v, f_v >= e^{2h},\ < f_u, f_v >= 0,\ < n, n >= 1.$$

Complex coordinate $z = u + \mathrm{i}\, v$. First and second fundamental forms

$$\begin{aligned} < df, df > &= e^{2h} dz d\bar{z}, \\ - < df, dn > &= H e^{2h} dz d\bar{z} + Q dz^2 + \bar{Q} d\bar{z}^2 \end{aligned}$$

Mean curvature $H = \frac{1}{2}(k_1 + k_2)$, Gaussian curvature $K = k_1 k_2$.

$$< f_{z\bar{z}}, n >= \frac{1}{2} H e^{2h}.$$

Hopf differential

$$Q =< f_{zz}, n > .$$

Gauss-Codazzi equations. Bonnet theorem

$$\begin{array}{lrcl} \text{Gauss equation} & h_{z\bar{z}} + \frac{1}{4} H^2 e^{2h} - |Q|^2 e^{-2h} & = & 0, \\ \text{Codazzi equation} & Q_{\bar{z}} & = & \frac{1}{2} H_z e^{2h}. \end{array}$$

Theorem (Bonnet theorem)

Given a metric $e^{2h} dz d\bar{z}$, a quadratic differential $Q\,dz^2$, and a mean curvature function H on $\mathcal{R}$ satisfying the Gauss–Codazzi equations, there exists an immersion

$$f : \tilde{\mathcal{R}} \to \mathbb{R}^3$$

with the corresponding fundamental forms. Here $\tilde{\mathcal{R}}$ is the universal covering of $\mathcal{R}$. The immersion f is unique up to Euclidean motions in $\mathbb{R}^3$.

The Bonnet problem

- A natural question is whether some of the data e^{2h}, Q, H of the fundamental forms are superfluous.
- Note that the Gaussian curvature is determined by the metric
- Do the metric and the curvatures determine a surface?
- Generally yes.

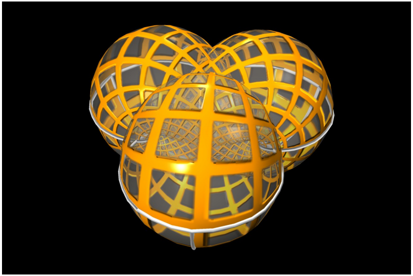
3 exceptional cases (known to Bonnet):

- cmc surfaces
- Bonnet pairs: two surfaces
- Bonnet families: one parameter family of non-cmc surfaces

Local and global theory

All 3 cases are described by integrable systems

Constant mean curvature surfaces



Wente torus:
one family of planar curvature lines
$\Delta u = \sinh u$

- Cmc surfaces: associated family $Q \to \lambda Q$, $|\lambda| = 1$.
- Modern global theory of cmc surfaces.
- Wente torus ['86], Abresch, Walter [87].
- all tori, description as integrable systems. Hitchin, Pinkall, Sterling ['89]
- explicit formulas in terms of RS (theta functions). Bobenko ['91]
- Higher genus: Heller [2010-]

Bonnet pairs

- A Bonnet pair is two non-congruent isometric surfaces $\mathcal{F}^+$ and $\mathcal{F}^-$ with the same mean curvature at corresponding points.
- Lawson-Tribuzy ['81], 3 immersions $\Rightarrow$ one-parameter family
- $\Rightarrow$ There exist at most 2 compact immersions
- Claims (retracted) that they do not exist. Sabitov ['12]
- Sufficient conditions for non-existence. Jensen -Musso-Nicolodi ['18]

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Bonnet Problem:
Do there exist compact Bonnet pairs?

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Compact Bonnet pairs?

A smooth Bonnet pair $\mathcal{F}^{\pm}$ is given by conformal immersions

$$f^{\pm} : \mathcal{R} \to \mathbb{R}^3$$

of the same Riemann surface, with common metric e^{2h}, common mean curvature function H and different Hopf differentials $Q^{+} \not\equiv Q^{-}$.

- Codacci equations $\Rightarrow Q_h := Q^{+} - Q^{-}$ is holomorphic
- There exists no Bonnet pairs of genus $g = 0$. Lawson-Tribuzy ['81], $Q_h = 0$ on a sphere.
- The set of umbilic points coincides with the zero divisor of Q_h. Bobenko ['08] , Sabitov ['12]

Bonnet pairs of genus $g = 1$?

No umbilic points on Bonnet tori:

$$Q^{\pm} = \frac{1}{2}(\alpha \pm \mathrm{i}\,),$$

where $\alpha : \mathcal{R} \to \mathbb{R}$ smooth.
The Gauss–Codazzi equations of Bonnet tori

$$\begin{aligned} 4h_{z\bar{z}} + H^2 e^{2h} - (1+\alpha^2)e^{-2h} &= 0, \\ \alpha_{\bar{z}} &= e^{2h} H_z. \end{aligned}$$

- solve the GC equations
- solve the frame equations
- find doubly periodic immersions

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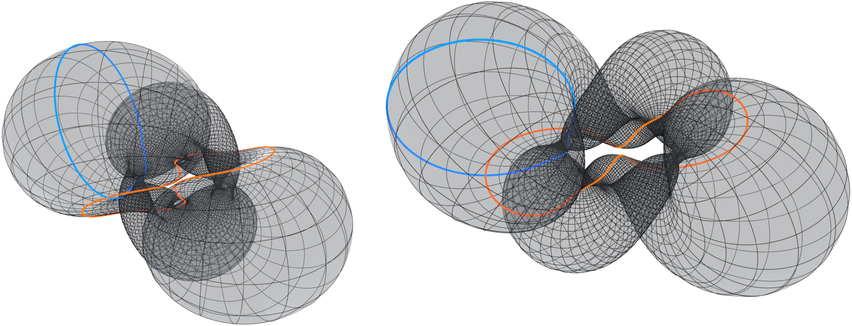
This is not the way to solve the problem!

Main result: Bonnet pairs of genus $g = 1$ do exist



A compact Bonnet pair of genus $g = 1$.
Corresponding generators are shown.
The orange generators are not congruent

Isothermic surfaces

Isothermic surfaces = conformal curvature line parametrization

$$f_{uv} \in \mathrm{span}\{f_u, f_v\}, \quad Q(z, \bar{z}) \in \mathbb{R}$$

Dual isothermic surface

$$df^* := e^{-2h}(f_u du - f_v dv).$$

The periodicity properties of $f : \mathcal{R} \to \mathbb{R}^3$ are not respected.
Quaternionic description of surfaces

$$(u, v) \to f(u, v) \in \mathrm{Im}\, \mathbb{H}$$

$$df^* = -(f_u)^{-1} du + (f_v)^{-1} dv$$

Bonnet pairs from isothermic surfaces

Description of Bonnet pairs (frames) in terms of isothermic surfaces in S^3 by Bianchi [1903].
Quaternionic description for simply connected $D \subset \mathbb{R}^2$ is crucial in our construction

Theorem (Kamberov-Pedit-Pinkall ('98))

The immersions $f^\pm : D \to \mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$ build a Bonnet pair if and only if there exists an isothermic surface $f : D \to \mathrm{Im}\,\mathbb{H}$ and a real number $\epsilon \in \mathbb{R}$ such that

$$df^\pm = (\pm\epsilon - f)df^*(\pm\epsilon + f),$$

where f^* is the dual isothermic surface.

Periodicity conditions

Torus $\mathcal{R} = \mathbb{C}/\mathcal{L}$. If γ is a cycle on $\mathcal{R}$ that is closed on the isothermic surface $f : \mathcal{R} \to \mathbb{R}^3$.
The corresponding curves are closed on the Bonnet pair iff

$$(\mathcal{A}-\text{periodicity condition}) \quad \int_\gamma -fdf^*f + \epsilon^2 df^* = 0,$$

$$(\mathcal{B}-\text{periodicity condition}) \quad \int_\gamma [df^*, f] = 2\int_\gamma \mathrm{Im}_{\mathbb{H}}(df^*f) = 0.$$

- Parameter ϵ is not essential
- Isothermic surfaces are Möbius invariant

Lemma

Let f be an isothermic torus such that f^* and $(f^{-1})^*$ are also tori. Then the $\mathcal{A}$-periodicity condition for Bonnet pairs is satisfied for any γ

$$\int -fdf^*f + \epsilon^2 df^* = (f^{-1})^* + \epsilon^2 f^*.$$

Discrete Bonnet pairs. Local theory

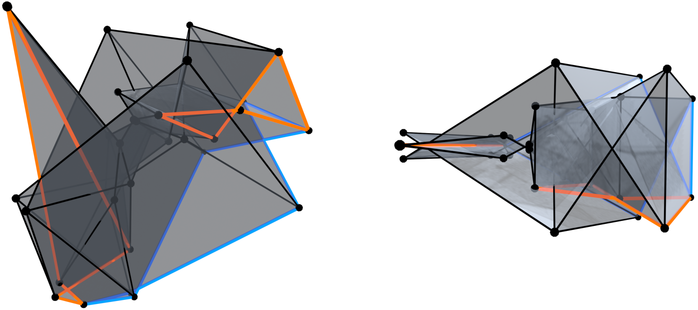
Discrete isothermic surfaces Bobenko-Pinkall ['96].
$f : \mathbb{Z}^2 \to \mathrm{Im}\mathbb{H} = \mathbb{R}^3$ is a *discrete isothermic net* if for each quad

$$(f_1 - f)(f_{12} - f_1)^{-1}(f_2 - f)(f_{12} - f_2)^{-1} = -1$$

Discretization of the Kamberov-Pedit-Pinkall formula by Hoffmann-Sageman-Furnas-Wardetzky ['17]
$\Rightarrow$ Discrete Bonnet pairs

$$f^{\pm}_{1,2} - f^{\pm} := \mathrm{Im}_{\mathbb{H}}\Big((\pm\epsilon - f)(f^*_{1,2} - f^*)(\pm\epsilon + f_{1,2})\Big)$$

Discrete compact Bonnet pairs. Numerical example



Two views of a discrete isothermic torus. Extremely coarse numerical example on a 5×7 lattice.

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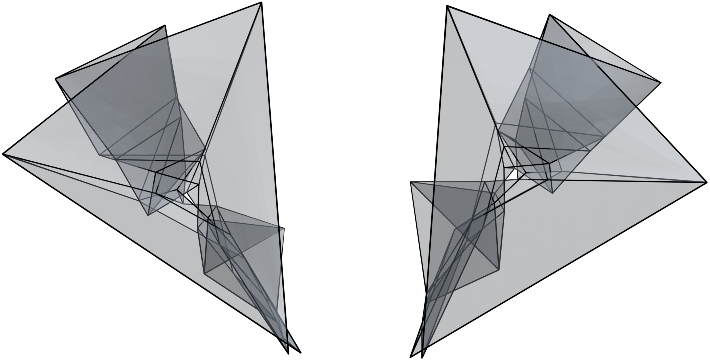
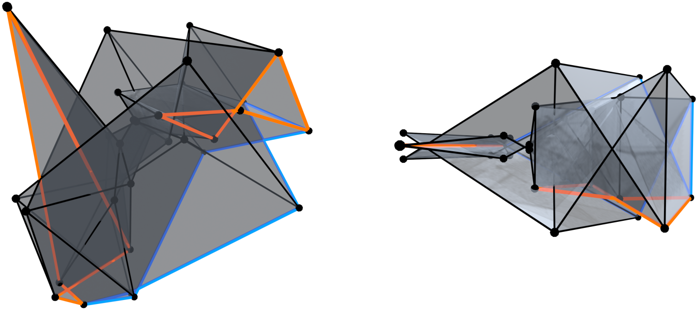


The corresponding discrete Bonnet pair

Discrete compact Bonnet pairs. Numerical example



On the discrete isothermic torus, the curvature lines with 5 vertices are planar (orange) and the curvature lines with 7 vertices are spherical (blue)

Isothermic surifaces with one family of planar curvature lines

Classification by Dabroux ['1883]. Involved and tricky computations

Theorem

Every isothermic surface $f(u,v)$ with one family (u-curves) of planar curvature lines has its planes tangent to a cone, and is given by

$$\begin{aligned} f(u,v) &= \Phi^{-1}(v)\gamma(u,\mathrm{w}(v))\mathbf{j}\Phi(v) \\ \Phi'(v)\Phi^{-1}(v) &= \sqrt{1-\mathrm{w}'(v)^2}\,W_1(\mathrm{w}(v))\mathbf{k}, \end{aligned}$$

where $\mathrm{w}(v)$ is a reparametrization function satisfying $|\mathrm{w}'(v)| \leq 1$, the lattice is rectangular, and $\omega \in \mathbb{R}$, and $\gamma(u,\mathrm{w})$ and $W_1(\mathrm{w})$ are given by explicit formulas in theta functions.

Formula for the curves γ

$$\gamma(u,\mathrm{w}) = -\mathrm{i}\,\frac{2\vartheta_4(\omega)^2}{\vartheta_1'(0)\vartheta_1(2\omega)}\,\frac{\vartheta_1\left(\frac{u+\mathrm{i}\,\mathrm{w}-3\omega}{2}\right)}{\vartheta_1\left(\frac{u+\mathrm{i}\,\mathrm{w}+\omega}{2}\right)}\,e^{(u+\mathrm{i}\,\mathrm{w})\frac{\vartheta_4'(\omega)}{\vartheta_4(\omega)}}.$$

- Problem: $\gamma(u,\mathrm{w})$ is never periodic in u
- Darboux classification contains no cylinders or tori. He considers elliptic functions on rectangular lattices only.

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- Problem: $\gamma(u, \mathrm{w})$ is never periodic in u
- Darboux classification contains no cylinders or tori. He considers elliptic functions on rectangular lattices only.
- Classification must be extended by including the elliptic functions on rhombic lattices
- Similar to the story with cmc tori, but more involved

Rectangular and rhombic lattices: the curves γ

Rectangular: $\tau \in \mathrm{i}\,\mathbb{R}$

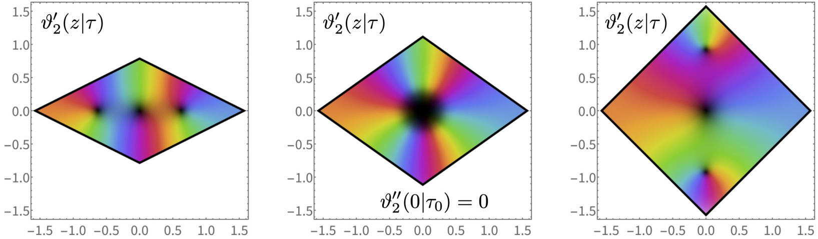
$$\gamma(u, \mathrm{w}) = -\mathrm{i}\,\frac{2\vartheta_4(\omega)^2}{\vartheta_1'(0)\vartheta_1(2\omega)}\,\frac{\vartheta_1\left(\frac{u+\mathrm{i}\,\mathrm{w}-3\omega}{2}\right)}{\vartheta_1\left(\frac{u+\mathrm{i}\,\mathrm{w}+\omega}{2}\right)}\,e^{(u+\mathrm{i}\,\mathrm{w})\frac{\vartheta_4'(\omega)}{\vartheta_4(\omega)}}.$$

Rhombic: $\tau \in \frac{1}{2} + \mathrm{i}\,\mathbb{R}$

$$\gamma(u, \mathrm{w}) = -\mathrm{i}\,\frac{2\vartheta_2(\omega)^2}{\vartheta_1'(0)\vartheta_1(2\omega)}\,\frac{\vartheta_1\left(\frac{u+\mathrm{i}\,\mathrm{w}-3\omega}{2}\right)}{\vartheta_1\left(\frac{u+\mathrm{i}\,\mathrm{w}+\omega}{2}\right)}\,e^{(u+\mathrm{i}\,\mathrm{w})\frac{\vartheta_2'(\omega)}{\vartheta_2(\omega)}}.$$

Zeros of ϑ_2' for rhombic lattices

real zeroes $0, \pm\omega,\ \omega > 0$ — **triple zero** 0 — **imaginary zeroes** $0, \pm\omega\mathrm{i},\ \omega > 0$

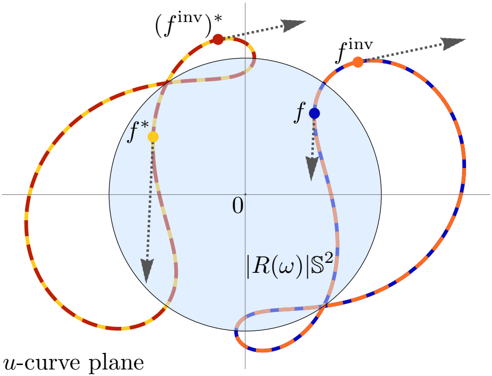


$\tau = \frac{1}{2} + \frac{1}{4}\mathrm{i}$ — $\tau = \tau_0 \approx \frac{1}{2} + \frac{1325085}{3735476}\mathrm{i}$ — $\tau = \frac{1}{2} + \frac{1}{2}\mathrm{i}$

$\vartheta_4'(\omega)$ in rectangular case never vanishes,
$\vartheta_2'(\omega)$ in rhombic case can vanish.

Bonnet periodicity conditions when f has a family of closed planar curvature lines

The Bonnet periodicity conditions (A and B) simplify



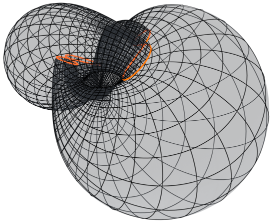
The spherical inversion and dualization operations map each planar curvature line, and therefore the entire surface, onto (minus) itself

Fundamental piece

The **fundamental piece** Π is the parametrized cylindrical patch

$$\Pi = \{f(u,v) \,\big|\, u \in [0,2\pi], v \in [0,\mathcal{V}]\}, \mathrm{w}(v+\mathcal{V}) = \mathrm{w}(v).$$

- f has an **axis** A and **generating rotation angle** $\theta \in [0,\pi]$ with $\Phi(0)^{-1}\Phi(\mathcal{V}) = \cos(\theta/2) + \sin(\theta/2)\mathrm{A}$.



Bonnet periodicity conditions when f has a family of closed planar curvature lines

Theorem

Let $f(u, v)$ be an isothermic cylinder with one family (u-curves) of closed planar curvature lines, and with periodic $\mathrm{w}(v + \mathcal{V}) = \mathrm{w}(v)$ that yields a fundamental piece with axis A and generating rotation angle $\theta \in [0, \pi]$. Denote its Gauss map by n and metric by e^{2h}.

Then the resulting Bonnet pair cylinders $f^{\pm}$ are tori if and only if

1. *(Rationality condition)*

$$k\theta \in 2\pi\mathbb{N} \text{ for some } k \in \mathbb{N},$$

2. *(Vanishing axial $\mathcal{B}$ part)*

$$\left\langle \mathrm{A}, \int_0^{\mathcal{V}} e^{-h(\omega,\mathrm{w}(v))} n(\omega, v) dv \right\rangle_{\mathbb{R}^3} = 0.$$

Second family of spherical curvature lines

- The primary challenge, is that the angle θ, axis A, and Gauss map n, depend on the frame $\Phi(v)$, which cannot be computed explicitly.
- The periodicity conditions can be refined when the second family of curvature lines is spherical.
- The key geometric insight is that the centers of the curvature line spheres are collinear and lie on the axis.
- Both periodicity conditions are given as Abelian integrals.

Periodicity conditions as Abelian integrals

Theorem

Let $f(u, v)$ be an isothermic cylinder of rhombic type, with spherical v-curvature lines. Then the arising Bonnet pair cylinders $f^{\pm}$ are tori if and only if there exist parameters ω, δ, s_1, s_2 such that

$$\textit{Rationality condition:} \quad \frac{\theta}{2} = \int_{s_1^-}^{s_1^+} \frac{Z_0}{\tilde{\mathcal{Q}}_2(s)} \frac{\mathcal{Q}_2(s)}{\sqrt{\mathcal{Q}(s)}} ds \in \pi\mathbb{Q}.$$

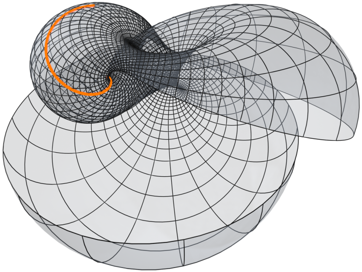
$$\textit{Vanishing axial } \mathcal{B}\textit{- part:} \quad \int_{s_1^-}^{s_1^+} \frac{\mathcal{Q}_2(s)}{\sqrt{\mathcal{Q}(s)}} ds = 0.$$

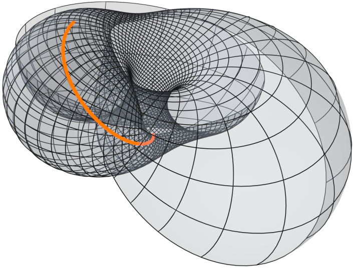
Here, s_1^-, s_1^+ are the two real zeroes of the v-elliptic curve $\mathcal{Q}(s) = -(s - s_1)^2(s - s_2)^2 + \delta^2 \mathcal{Q}_3(s)$, where the u-elliptic curve $\mathcal{Q}_3(s)$ and $\mathcal{Q}_2, \tilde{\mathcal{Q}}_2, Z_0$ are given by explicit formulas.

Proof of existence

- Proof of existence by asymptotic analysis $\delta \to 0$

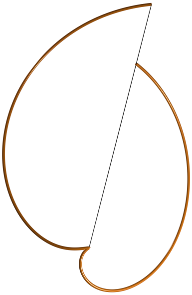
Compact Bonnet pairs from isothermic tori with planar and spherical curvature lines

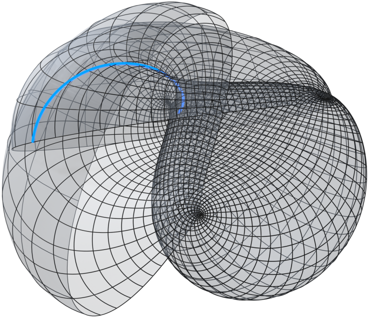




Fundamental pieces of an example with 3-fold symmetry

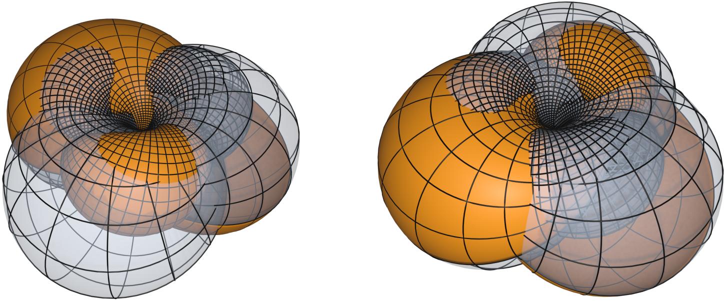
Compact Bonnet pairs from isothermic tori with planar and spherical curvature lines





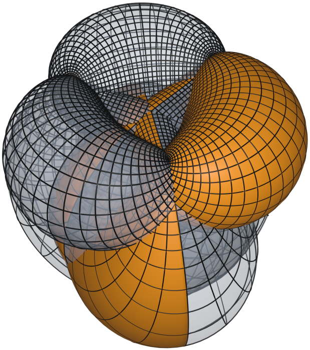
The pair of corresponding orange curves and the fundamental piece of the isothermic torus

Compact Bonnet pairs from isothermic tori with planar and spherical curvature lines



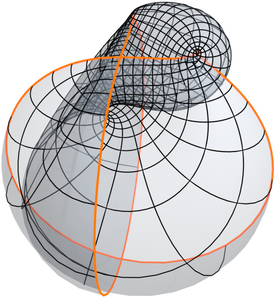
The example with 3-fold symmetry

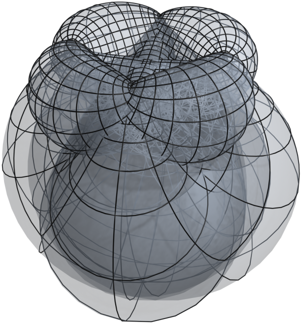
Compact Bonnet pairs from isothermic tori with planar and spherical curvature lines



The corresponding isothermic torus

Example with 4-fold symmetry

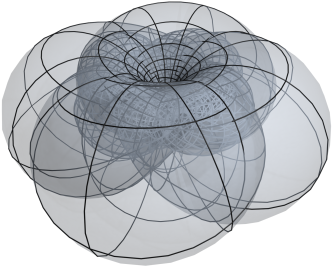


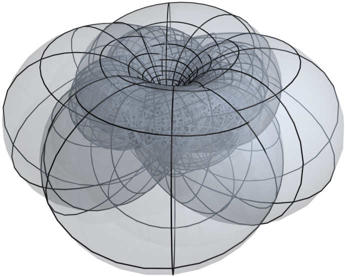


The fundamental piece and full isothermic surface

One surface!

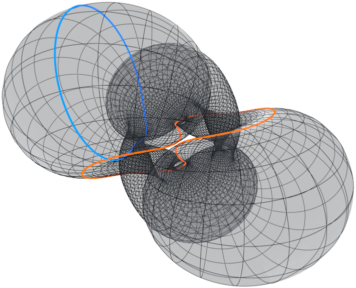
- The corresponding Bonnet tori f^+ and f^- are mirror images of each other. Note that the mirror symmetry mapping f^+ to f^- is not the mean curvature preserving isometry.

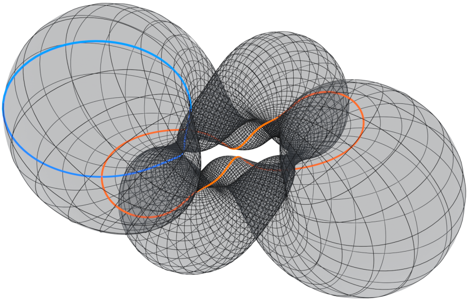




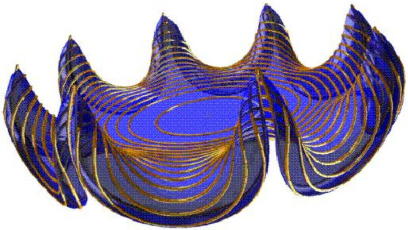
Compact Bonnet pairs with two different surfaces

Proof by a small perturbation.





Bonnet families



Bonnet family:
non-cmc surface

- One parameter families of non-cmc surfaces
- Umbilic free surface is a Bonnet surface iff [Graustein '24]:
  - (i) it is isothermic
  - (ii) $1/Q$ harmonic

Local classification

[E. Cartan '42] 3 types

$$A: \quad |Q|^2 = \frac{4}{sin^2 2t}$$
$$B: \quad |Q|^2 = \frac{4}{sinh^2 2t}$$
$$C: \quad |Q|^2 = \frac{1}{t^2}$$

Analytic treatment via ODE. Hazzidakis equation [1887]

$$\left(\left(\frac{H''}{H'}\right)' - H'\right)\frac{1}{|Q|^2} = 2 - \frac{H^2}{H'}.$$

$$1/Q = h + \bar{h},\ w = \int \frac{dz}{h'},\ t = w + \bar{w},\ H'(t) < 0,\ e^u = -\frac{2|Q|^2}{H'}$$

Intrinsic isometries by imaginary shifts $w \mapsto w + ia$

Description via monodromy problems

Bobenko, Eitner ['98]

$$x = e^{-4(w+\bar{w})}, \quad \lambda = e^{-4w}$$

Frame equations of Bonnet families B:

$$\begin{aligned} \frac{\partial \Phi}{\partial \lambda} \Phi^{-1} &= \frac{B_0(x)}{\lambda} + \frac{B_1(x)}{\lambda - 1} + \frac{B_x(x)}{\lambda - x}, \\ \frac{\partial \Phi}{\partial x} \Phi^{-1} &= -\frac{B_x(x)}{\lambda - x} + C(x), \end{aligned}$$

and Hazzidakis equation

$$4 \left(x \frac{\mathcal{H}''(x)}{\mathcal{H}'(x)} \right)' + \mathcal{H}'(x) = \frac{4}{(x-1)^2} \left(2 + \frac{\mathcal{H}^2(x)}{4\, x\, \mathcal{H}'(x)} \right),$$

Description via Painlevé equations

For any solution of the Painlevé VI

$$\frac{d^2 y}{d x^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right) y'^2 - \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right) y' + \frac{y\,(y-1)\,(y-x)}{2\,x^2\,(x-1)^2}\left(\theta^2\,\frac{(x-1)}{(y-1)^2} - \theta\,(\theta+2)\,\frac{x\,(x-1)}{(y-x)^2}\right)$$

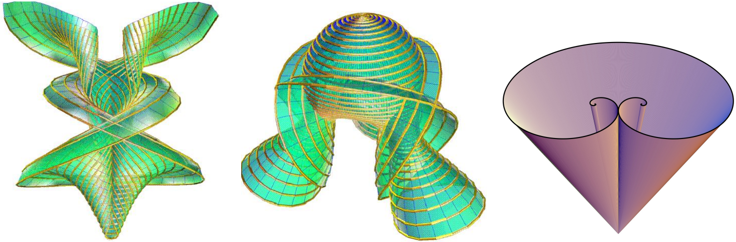
The function

$$\mathcal{H}(x) \equiv -2\,\frac{(x-1)\,(\theta^2\,y(x)^2 - x^2 y'^2(x))}{y(x)\,(y(x)-1)(y(x)-x)}$$

solves the Hazzidakis equation.
Similar formulas for Bonnet families C and Painlevé V.

Bonnet families. Global classification

Bonnet families as maximal immersions of open stripe, half-plane and disc. Also with critical points $dH = 0$ (isolated). No compact examples. Bobenko-Eitner ['00]



Maximal immersions of Bonnet families of types A, B and C.