FROM KOORNWINDER THEORY TO CLUSTER ALGEBRA:

PROOF OF THE MACDONALD-Q-SYSTEM CONJECTURE

[P. Di Francesco & R. Kedem]

1. Koornwinder operators / polynomials
2. Macdonald Specializations (BCD + twisted D, A).
3. Duality : Pieri rules and Hamiltonians
4. q Whittaker limit ($t \to 0$): Whittaker operators / functions
5. Time translation (from $SL_2(\mathbb{Z})$ symmetry of DAHA)
6. Main Theorems : Q-systems and raising operators
7. Proof universal solutions and Fourier transform
8. Conclusion

IDPEIS22

0. Discrete Integrable Dynamical Systems

- cluster algebras = discrete dyn. sys.
  times: periodic mutation sequences
- bonus = come with a natural quantization
- question = integrable directions on cluster Algebra?

This talk = answer for q-cluster algebras based on affine (twisted) Algebras A, B, C, D (Q-systems)
= theory of Koornwinder q-difference operators
[Van Diejen, Sahi, Cherednik, Stokman...]

1. KOORNWINDER OPERATORS

$$K_1^{(abcd;q,t)}(x) = \sum_{i=1}^{N} \sum_{\varepsilon=\pm 1} \phi^{(abcd;q,t)}(x_i^{\varepsilon}) \prod_{j\neq i} \frac{t x_i^{\varepsilon} - x_j}{x_i^{\varepsilon} - x_j} \frac{t x_i^{\varepsilon} x_j - 1}{x_i^{\varepsilon} x_j - 1} \left(\Gamma_i^{\varepsilon} - 1\right)$$

$$\phi(x) = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}$$

$$\Gamma_i^{\pm 1} f = f\big|_{x_i \to q^{\pm 1} x_i}$$

1. acts on functions $f(x_1, x_2, \dots, x_N)$
2. leaves symmetric Laurent polynomials invariant
   $\quad$ " Weyl-invariant (BC) $\quad \mathfrak{S}_N \ltimes \mathbb{Z}_2$.

KOORNWINDER POLYNOMIALS

- Eigenfunctions of K_1, monic Laurent $P_\lambda^{(a,b,c,d)} = x^\lambda + \text{lower}$

- Define $\mathcal{D}_1^{abcd} = K_1^{abcd}(x) + \frac{1-t^N}{1-t}\left(1 + \frac{abcd}{q} t^{N-1}\right)$

$$\boxed{\mathcal{D}_1^{abcd} P_\lambda^{abcd}(x) = \sigma t^{N-1}\, \hat{e}_1(s) \cdot P_\lambda^{abcd}(x)}$$

$\sigma = \sqrt{\frac{abcd}{q}}$ $\qquad$ $\lambda = (\lambda_1, \dots, \lambda_N)$

$s = (s_1, \dots, s_N)$; $\boxed{s_i = \sigma t^{N-i} q^{\lambda_i}}$

$\hat{e}_1(s) = \boxed{\sum_{i=1}^{N} s_i + s_i^{-1}} =$ first elementary Weyl-inv^t function

HIGHER KOORNWINDER OPERATORS I

[van Diejen 95]

- family of commuting operators $\left\{ \begin{array}{l} V_\alpha^{(abcd)} \quad \alpha = 1, 2, \dots, N \\ V_1 = K_1 \end{array} \right.$
- "higher order" in $\Gamma_i^{\pm 1}$

Definition A.1. *The van Diejen operator of order α is defined as:*

"order α"

$$\mathcal{V}_\alpha^{(a,b,c,d)} := \sum_{\substack{J\subset[1,N],\ |J|=\alpha \\ \epsilon_j=\pm1,\ j\in J}} \sum_{s=1}^{\alpha} (-1)^{s-1} \sum_{\emptyset\subsetneq J_1\subsetneq\cdots\subsetneq J_s=J} \prod_{r=1}^{s} V^{(a,b,c,d)}_{\{x\},\{\epsilon\};J_r\setminus J_{r-1};K_r} \Big(\prod_{j\in J_1} \Gamma_j^{\epsilon_j} - 1\Big) \tag{A.5}$$

where we set $J_0 = \emptyset$, $K_r = J_r^c$ and

$$\phi^{abcd}(x_i^{\varepsilon_i})$$

$$V^{(a,b,c,d)}_{\{x\},\{\epsilon\};J;K} := \prod_{i\in J} \frac{(1-ax_i^{\epsilon_i})(1-bx_i^{\epsilon_i})(1-cx_i^{\epsilon_i})(1-dx_i^{\epsilon_i})}{(1-x_i^{2\epsilon_i})(1-qx_i^{2\epsilon_i})}$$

$$\times \prod_{i<j\in J} \frac{1-tx_i^{\epsilon_i}x_j^{\epsilon_j}}{1-x_i^{\epsilon_i}x_j^{\epsilon_j}} \frac{1-qtx_i^{\epsilon_i}x_j^{\epsilon_j}}{1-qx_i^{\epsilon_i}x_j^{\epsilon_j}} \prod_{\substack{i\in J\\ j\in K}} \frac{1-tx_i^{\epsilon_i}x_j}{1-x_i^{\epsilon_i}x_j} \frac{tx_i^{\epsilon_i}-x_j}{x_i^{\epsilon_i}-x_j} \tag{A.6}$$

where $J, K \subset [1,N]$, $J\cap K = \emptyset$.

HIGHER KOORNWINDER OPERATORS II

[van Diejen 95, Rains 16]

Rains operators:

"order $\frac{N}{2}$"

$$\mathcal{R}_N^{(u,v)}(x;q,t) = \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_N=\pm 1} \prod_{i=1}^{N} \frac{(1-ux_i^{\epsilon_i})(1-vx_i^{\epsilon_i})}{1-x_i^{2\epsilon_i}} \prod_{1\le i<j\le n} \frac{1-tx_i^{\epsilon_i}x_j^{\epsilon_j}}{1-x_i^{\epsilon_i}x_j^{\epsilon_j}} \prod_{i=1}^{N} \Gamma_i^{\epsilon_i/2}$$

$$R_N^{\left(\frac{a}{\sqrt{q}},\frac{b}{\sqrt{q}}\right)}(x;q,t)\; P_\lambda^{(a,b,c,d)}(x) = \frac{1}{q^{|\lambda|}} \prod_{i=1}^{N} (1-abq^{\lambda_i-1}t^{N-i})\; P_\lambda^{\left(\frac{a}{\sqrt{q}},\frac{b}{\sqrt{q}},c\sqrt{q},d\sqrt{q}\right)}(x)$$

$$\boxed{\hat{\mathcal{D}}_N^{(abcd)}(x;q,t) = R_N^{(a,b)}\; R_N^{\left(\frac{c}{\sqrt{q}},\frac{d}{\sqrt{q}}\right)}}$$

commutes with the previous family!

2. MACDONALD SPECIALIZATIONS

- Koornwinder Polynomial P_λ^{abcd} $\rightarrow$ Macdonald (B, C, D, twisted)

$\mathfrak{g}$	a	b	c	d	σ	$\mathfrak{g}^*$
$D_N^{(1)}$	1	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	1	$D_N^{(1)}$
$B_N^{(1)}$	t	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	$t^{1/2}$	$C_N^{(1)}$
$C_N^{(1)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	$-t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	t	$B_N^{(1)}$
$A_{2N-1}^{(2)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	$t^{1/2}$	$A_{2N-1}^{(2)}$
$D_{N+1}^{(2)}$	t	-1	$t\,q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	t	$D_{N+1}^{(2)}$
$A_{2N}^{(2)}$	t	-1	$t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	$-t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	t	$A_{2N}^{(2)}$

KOORNWINDER - MACDONALD OPERATORS I

DEF

suitable $\mathbb{C}$-linear combination

$$\mathcal{D}_\alpha = \sum_{j=0}^{\alpha} d_j \, V_{\alpha-j}$$

[PDF + R. Kedem 21]

THM [DF + Kedem 21]

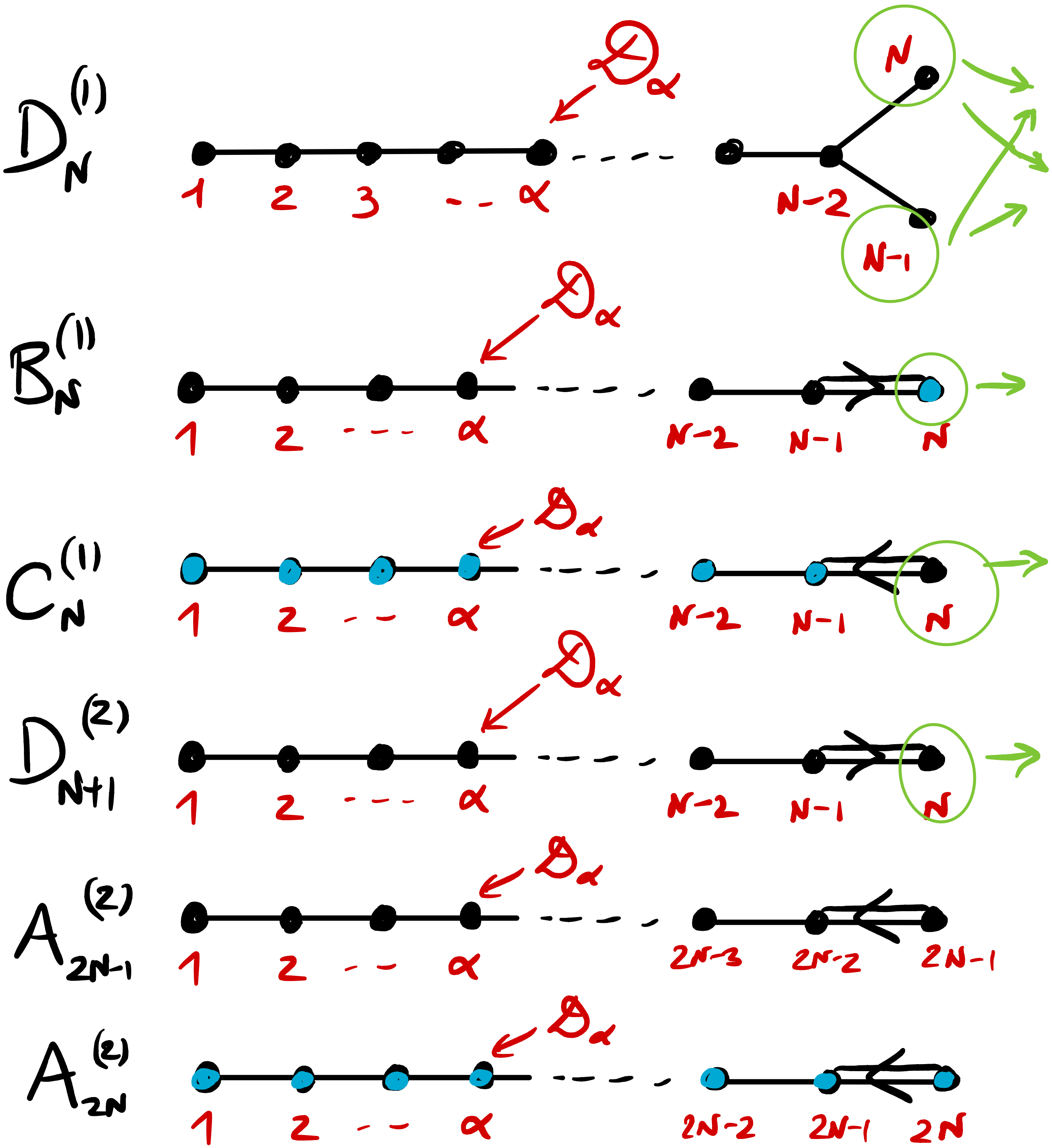
$$\boxed{\mathcal{D}_\alpha \, P_{\lambda}(x) = \sigma^\alpha t^{\alpha(N-\frac{\alpha+1}{2})} \hat{e}_\alpha(s) \cdot P_\lambda(x)}$$

$\hat{e}_\alpha(s)$ = higher elementary W-symmetric functions

$$\prod_{\varepsilon=\pm 1} \prod_{i=1}^{N} (1 + u s_i^\varepsilon) = \sum_{\alpha=0}^{2N} u^\alpha \hat{e}_\alpha(s) \quad ; \quad \hat{e}_{2N-\alpha} = \hat{e}_\alpha$$

$$\hat{e}_\alpha(s_1, s_2, \dots, s_N) = e_\alpha(s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_N, s_N^{-1})$$

KOORNWINDER-MACDONALD OPERATORS II



Sum $= R_N^{(1,-1)}$ (Rains)

product $= \mathcal{D}_N$(v.D:ej)

$= \hat{\mathcal{D}}_N^{(t,-1,\sqrt{q},-\sqrt{q})}$ (Rains)

$= R_N^{(\sqrt{t},-\sqrt{t})}$ (Rains)

$= R_N^{(t,-1)}$ (Rains)

3. KOORNWINDER DUALITY

$$\boxed{\frac{P_\lambda^{(abcd)}(\sigma^* t^\rho q^\mu)}{P_\lambda^{(abcd)}(\sigma^* t^\rho)} = \frac{P_\mu^{(a^*b^*c^*d^*)}(\sigma t^\rho q^\lambda)}{P_\mu^{(a^*b^*c^*d^*)}(\sigma t^\rho)}}$$

[Sahi + van Diejen]

$$a^* = \sqrt{\frac{abcd}{q}} = \sigma \;;\; b^* = -\sqrt{\frac{qab}{cd}} \;;\; c^* = \sqrt{\frac{qac}{bd}} \;;\; d^* = -\sqrt{q\frac{ad}{bc}}$$

$$\sigma^* = a \qquad (* = \text{involution})$$

- upon specialization $B_N^{(1)} \overset{\text{duality}}{\underset{*}{\longleftrightarrow}} C_N^{(1)}$ and all other types are self-dual

- Duality relates eigenvector eqns to Pieri rules

$$\boxed{\mathcal{D}_\alpha \, P_\lambda(x) = \theta_\alpha \, \hat{e}_\alpha(s) \, P_\lambda(x)}$$

- use duality $\quad x = \sigma^* t^\rho q^\mu \quad$ (recall $s = \sigma t^\rho q^\lambda$).
- Γ_i^ε acts by $\quad \Gamma_i^\varepsilon = e^{\varepsilon \frac{\partial}{\partial \mu_i}}$ (shifts μ_i by ε)

$$\boxed{\frac{P_\lambda^{(abcd)}(\sigma^* t^{\rho^*} q^\mu)}{P_\lambda^{(abcd)}(\sigma^* t^\rho)} = \frac{P_\mu^{(a^*b^*c^*d^*)}(\sigma t^\rho q^\lambda)}{P_\mu^{(a^*b^*c^*d^*)}(\sigma t^\rho)}}$$

- Duality relates eigenvector eqns to Pieri rules

$$\boxed{\mathcal{D}_\alpha \; P_\lambda(x) = \theta_\alpha \hat{e}_\alpha(s) \; P_\lambda(x)}$$

- use duality $\quad x = \sigma^* t^\rho q^\mu \quad$ (recall $s = \sigma t^\rho q^\lambda$).
- Γ_i^ε acts by $\quad \Gamma_i^\varepsilon = e^{\varepsilon \frac{\partial}{\partial \mu_i}}$ (shifts μ_i by ε)

$$\mathcal{D}_\alpha \; P_\lambda(\sigma^* t^\rho) \frac{P_\lambda(\sigma^* t^\rho q^\mu)}{P_\lambda(\sigma^* t^\rho)} = \mathcal{D}_\alpha \; \cancel{P_\lambda(\sigma^* t^\rho)} \frac{P^*_\mu(\sigma t^\rho q^\lambda)}{P^*_\mu(\sigma t^\rho)}$$

$$= \theta_\alpha \hat{e}_\alpha(\underbrace{\sigma t^\rho q^\lambda}_{s}) \; \cancel{P_\lambda(\sigma^* t^\rho)} \frac{P^*_\mu(\sigma t^\rho q^\lambda)}{P^*_\mu(\sigma t^\rho)}$$

$$\boxed{\hat{e}_\alpha(s) \; P^*_\mu(s) = \frac{1}{\theta_\alpha} \left\{ P^*_\mu(\sigma t^\rho) \, \mathcal{D}_\alpha \, P^*_\mu(\sigma t^\rho)^{-1} \right\} P^*_\mu(s)}$$

$x \leftrightarrow s$!

KOORNWINDER PIERI RULES & HAMILTONIANS

$$\boxed{\begin{gathered}\hat{e}_\alpha(x)\, P^*_\mu(x) = H^{(a^\bullet b^\bullet c^\bullet d^\bullet)}_\alpha P^*_\mu(x) \\ H^{(a^\bullet b^\bullet c^\bullet d^\bullet)}_\alpha = \frac{1}{\Theta_\alpha} P^*_\mu(\sigma t^\rho)\, \mathcal{D}^{(a,b,c,d)}_\alpha P^*_\mu(\sigma t^\rho)^{-1}\end{gathered}}$$

Koornwinder duality relates $\mathcal{D}^{(abcd)}$ to $H^{(a^\bullet b^\bullet c^\bullet d^\bullet)}$

- commuting difference operators in variable $s = \sigma t^\rho q^\mu$
- "Hamiltonians" ?

Upon specialization:

Q^* Pieri rules are dual to Q Macdo eigenvalue eqns.

4. SPECIALIZATION + qWHITTAKER LIMIT

Whittaker limit : $t \to \infty$. Macdonald polynomials become qWhittaker functions = eigenfunctions of qToda Hamiltonians.

$$\text{specialization}\left(H_\alpha^{abcd}\right)\Big|_{t\to\infty} \sim \underbrace{H_\alpha^{(g)}}_{\text{qToda Hamiltonian}}$$

TODA EIGENVALUE EQN.

$$\boxed{H_\alpha \Pi_\lambda(x) = \hat{e}_\alpha(x)\Pi_\lambda(x)}$$

$$\begin{cases} x = q^{\rho+\mu} & \leftarrow \text{reps.} \\ \Lambda = q^\lambda & \leftarrow \text{original} \end{cases}$$

acts as diff op in $\Lambda = q^\lambda$

$\lim_{t\to\infty} \text{spec}(P_\lambda(x))$

q WHITTAKER LIMIT of MACDONALD OPERATORS

$(t \to \infty)$

$$D_1^{(\mathfrak{g})}(x;q) = 1 + \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^{N} \phi_{i,\epsilon}^{(\mathfrak{g})}(x)(\Gamma_i^\epsilon - 1),$$

$$\phi_{i,\epsilon}^{(\mathfrak{g})}(x) = \prod_{j\neq i} \frac{x_i^\epsilon}{x_i^\epsilon - x_j} \frac{x_i^\epsilon x_j}{x_i^\epsilon x_j - 1} \times \begin{cases} 1 & (D_N^{(1)}); \\ \frac{x_i^\epsilon}{x_i^\epsilon - 1}, & (B_N^{(1)}), \\ \frac{x_i^{2\epsilon}}{x_i^{2\epsilon}-1} \frac{q x_i^{2\epsilon}}{q x_i^{2\epsilon} - 1}, & (C_N^{(1)}), \\ \frac{x_i^{2\epsilon}}{x_i^{2\epsilon}-1}, & (A_{2N-1}^{(2)}), \\ \frac{x_i^\epsilon}{x_i^\epsilon - 1} \frac{q^{\frac{1}{2}} x_i^\epsilon}{q^{\frac{1}{2}} x_i^\epsilon - 1}, & (D_{N+1}^{(2)}), \\ \frac{x_i^\epsilon}{x_i^\epsilon - 1} \frac{q x_i^{2\epsilon}}{q x_i^{2\epsilon} - 1}, & (A_{2N}^{(2)}). \end{cases}$$

TODA HAMILTONIANS from spec/dual/whitt. limit

$$H_1^{(\mathfrak{g})}(\Lambda;q)=\sum_{i=1}^{N}(1-\Lambda^{-\alpha^*_{i-1}})T_i+\sum_{i=1}^{N_{R*}}(1-\Lambda^{-\alpha^*_i})T_i^{-1}+M^{(\mathfrak{g})}(\Lambda;q),$$

where:

$$M^{(\mathfrak{g})}(\Lambda;q)=\begin{cases}(1-\Lambda^{-\alpha^*_N})(T_N^{-1}+(1-\Lambda^{-\alpha^*_{N-1}})T_{N-1}^{-1}), & \mathfrak{g}=D_N^{(1)},\\ (1-\Lambda^{-\alpha^*_N})(1-q\Lambda^{-\alpha^*_N})T_N^{-1}+\Lambda^{-\alpha^*_N}(q^{-1}\Lambda^{-\alpha^*_{N-1}}-(1+q^{-1})), & \mathfrak{g}=B_N^{(1)},\\ 0, & \mathfrak{g}=C_N^{(1)},A_{2N-1}^{(2)}\\ (1-\Lambda^{-\alpha^*_N})(1-q^{\frac{1}{2}}\Lambda^{-\alpha^*_N})T_N^{-1}, & \mathfrak{g}=D_{N+1}^{(2)},\\ -\Lambda^{-\alpha^*_N}, & \mathfrak{g}=A_{2N}^{(2)}.\end{cases}$$

R = finite root lattice of $\mathfrak{g}$ roots α

R^* = " " " $\mathfrak{g}^*$ roots α^*

N_{R*} = $N-2$ ($D_N^{(1)}$), $N-1$ ($B_N^{(1)}$), N otherwise.

$T_i \Lambda_j = q^{\delta_{ij}} \Lambda_j T_i$ (quantum torus; $\Lambda_i = q^{\lambda_i}$; $T_i = q^{\lambda_i \frac{\partial}{\partial \lambda_i}}$)

$\boxed{T_i \Lambda_j = q^{\delta_{ij}} \Lambda_j T_i}$

$$H_1^{(D_N^{(1)})}(\Lambda) = T_1 + \sum_{i=2}^{N}\left(1-\frac{\Lambda_i}{\Lambda_{i-1}}\right)T_i + \sum_{i=1}^{N-2}\left(1-\frac{\Lambda_{i+1}}{\Lambda_i}\right)T_i^{-1}$$
$$+\left(1-\frac{\Lambda_N}{\Lambda_{N-1}}\right)\left(1-\frac{1}{\Lambda_{N-1}\Lambda_N}\right)T_{N-1}^{-1} + \left(1-\frac{1}{\Lambda_{N-1}\Lambda_N}\right)T_N^{-1},$$

$$H_1^{(B_N^{(1)})}(\Lambda) = T_1 + \sum_{i=2}^{N}\left(1-\frac{\Lambda_i}{\Lambda_{i-1}}\right)T_i + \sum_{i=1}^{N-1}\left(1-\frac{\Lambda_{i+1}}{\Lambda_i}\right)T_i^{-1}$$
$$+\left(1-\frac{1}{\Lambda_N^2}\right)\left(1-\frac{q}{\Lambda_N^2}\right)T_N^{-1} + \frac{q^{-1}}{\Lambda_{N-1}\Lambda_N} - \frac{1+q^{-1}}{\Lambda_N^2},$$

$$H_1^{(C_N^{(1)})}(\Lambda) = T_1 + \sum_{i=2}^{N}\left(1-\frac{\Lambda_i}{\Lambda_{i-1}}\right)T_i + \sum_{i=1}^{N-1}\left(1-\frac{\Lambda_{i+1}}{\Lambda_i}\right)T_i^{-1} + \left(1-\frac{1}{\Lambda_N}\right)T_N^{-1},$$

$$H_1^{(A_{2N-1}^{(2)})}(\Lambda) = T_1 + \sum_{i=2}^{N}\left(1-\frac{\Lambda_i}{\Lambda_{i-1}}\right)T_i + \sum_{i=1}^{N-1}\left(1-\frac{\Lambda_{i+1}}{\Lambda_i}\right)T_i^{-1} + \left(1-\frac{1}{\Lambda_N^2}\right)T_N^{-1},$$

$$H_1^{(D_{N+1}^{(2)})}(\Lambda) = T_1 + \sum_{i=2}^{N}\left(1-\frac{\Lambda_i}{\Lambda_{i-1}}\right)T_i + \sum_{i=1}^{N-1}\left(1-\frac{\Lambda_{i+1}}{\Lambda_i}\right)T_i^{-1}$$
$$+\left(1-\frac{1}{\Lambda_N}\right)\left(1-\frac{q^{\frac{1}{2}}}{\Lambda_N}\right)T_N^{-1} - \frac{1+q^{-\frac{1}{2}}}{\Lambda_N},$$

$$H_1^{(A_{2N}^{(2)})}(\Lambda) = T_1 + \sum_{i=2}^{N}\left(1-\frac{\Lambda_i}{\Lambda_{i-1}}\right)T_i + \sum_{i=1}^{N-1}\left(1-\frac{\Lambda_{i+1}}{\Lambda_i}\right)T_i^{-1} + \left(1-\frac{1}{\Lambda_N}\right)T_N^{-1} - \frac{1}{\Lambda_N}.$$

5. TIME TRANSLATION OF q-WHITTAKER OPERATORS

- Double Affine Hecke Algebra [Cherednik 95]

→ generators $\langle T_i, X_i, Y_i \rangle_{i=1}^{N}$ + relations (solid torus reps).

→ Koornwinder–Macdonald $\mathcal{D}_\alpha \leftrightarrow \rho(\hat{e}_\alpha(Y_1, \dots, Y_N))$
= polynomial reps ρ [Noumi]

→ $SL_2(\mathbb{Z})$ symmetry acts on $\mathcal{D}_\alpha$ $\quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

in the polynomial reps: $T = \mathrm{Ad}_{\gamma^{-1}}$, where:

$$\boxed{\gamma = e^{\frac{\sum_1^N (\log x_i)^2}{2 \log q}}}$$ Cherednik's "Gaussian".

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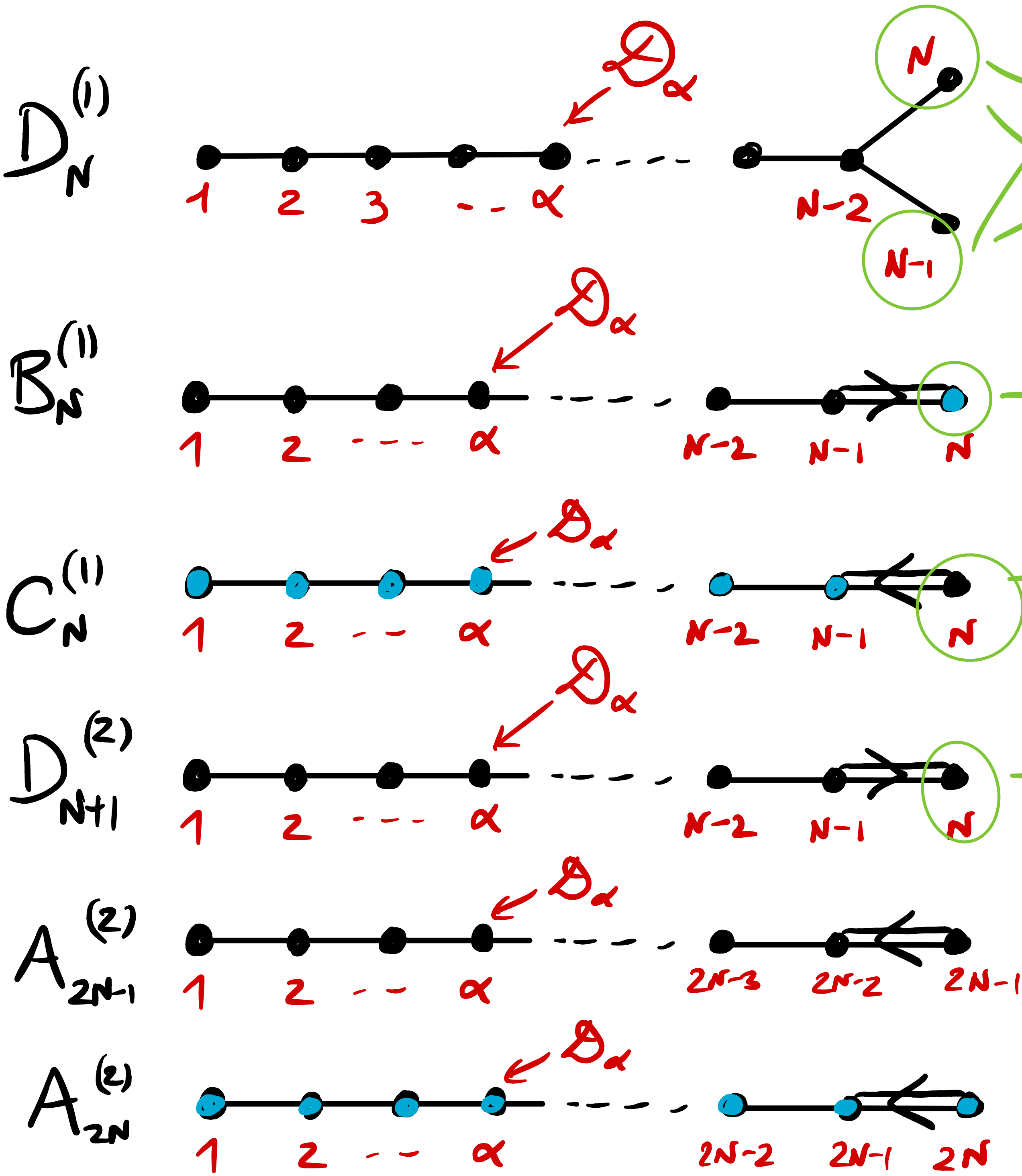
"Gaussian".

- Define (after $t \to \infty$)

$$\boxed{\begin{aligned} D_{\alpha,n} &= q^{-\frac{n\alpha}{2}} \gamma^{-n} D_\alpha \gamma^n \quad \text{for long labels} \\ D_{\alpha,2n+\varepsilon} &= q^{-n\alpha} \gamma^{-n} D_{\alpha,\varepsilon} \gamma^n, \ \varepsilon = 0, -1 \quad \text{for short} \end{aligned}}$$

KOORNWINDER-MACDONALD OPERATORS II



Sum $= R_N^{(1,-1)}$ (Rains)

product $= \mathcal{D}_N(\text{v.D.}\, e_j)$

$= \hat{\mathcal{D}}_N^{(t,-1,\sqrt{q},-\sqrt{q})}$ (Rains)

$= R_N^{(\sqrt{t},-\sqrt{t})}$ (Rains)

$= R_N^{(t,-1)}$ (Rains)

● long labels

● short labels

with $D_{\alpha,0} = D_\alpha$, and for short labels •

- $$\boxed{D^{(g)}_{1,-1} := \sum_{i,\varepsilon} \Phi^{(g)}_{i,\varepsilon}(x) \frac{1}{x_i^{\varepsilon}} (\Gamma_i^{\varepsilon} - 1) \qquad (C_N^{(1)}, A_{2N}^{(2)}) \\ D^{(g)}_{a,-1} := \frac{(-1)^a}{q-1} [D^{(g)}_{1,-a}, D^{(g)}_{a-1,0}]_{q^a} \leftarrow q^a\text{-commutator} \;\Big|\; [A,B]_{q^a} = AB - q^a BA}$$

- $$\boxed{D_{N,-1} := R^{(1)}_{N,-1} \, R^{(0)}_N \qquad (B_N^{(1)})}$$

$$\begin{cases} R_N^{(0)} \sim \lim_{t\to\infty} \mathcal{R}_N^{(1,-1)} \; ; \; R_N^{(1)} \sim \lim_{t\to\infty} \mathcal{R}_N^{(t,-1)} \text{ (Rains)}. \\ R^{(1)}_{N,-1} = q^{N/2} \gamma R_N^{(1)} \gamma^{-1} \end{cases}$$

composite of time -1 and time 0.

6. MAIN THEOREMS

Thm 1 The time-translated Whittaker operators obey the quantum Q-system relations for all affine & twisted cases.

Thm 2 The time ± 1 operators are raising/lowering operators for the q-Whittaker functions

wts of R^*

wts of R

$$D_{a,1}\,\Pi_\lambda = q^{(\omega_a^*,\lambda)}\,\Pi_{\lambda+\omega_a}$$

$$D_{a,-1}\,\Pi_\lambda = q^{(\omega_a^*,\lambda)}\left(1-q^{-(\alpha_a^*,\lambda)}\right)\Pi_{\lambda-\omega_a}$$

roots of R^*

QUANTUM Q-SYSTEMS

- $\Lambda^{(\mathfrak{g})}_{ab} = \omega_a^* \cdot \omega_b$
- $t_a = 1$ long roots, $t_a = 2$ short

•Commutations:
$$\mathcal{Q}_{a;t_a k+i}\, \mathcal{Q}_{b;t_b k+j} = q^{\Lambda^{(\mathfrak{g})}_{a,b}\, j - \Lambda^{(\mathfrak{g})}_{b,a}\, i}\, \mathcal{Q}_{b;t_b k+j}\, \mathcal{Q}_{a;t_a k+i},$$

•mutations:
$$q^a\, \mathcal{Q}_{a;n+1}\, \mathcal{Q}_{a;n-1} = \mathcal{Q}_{a;n}^2 - \mathcal{Q}_{a+1;n}\, \mathcal{Q}_{a-1;n}, \qquad a \in [1, \bar{N}_{\mathfrak{g}}], \text{ all } \mathfrak{g}.$$

$$\begin{aligned}
D_N^{(1)}: \quad q^{N-2}\, \mathcal{Q}_{N-2;n+1}\, \mathcal{Q}_{N-2;n-1} &= \mathcal{Q}_{N-2;n}^2 - q^{-\frac{(N-2)n}{4}}\, \mathcal{Q}_{N-3;n}\, \mathcal{Q}_{N-1;n}\, \mathcal{Q}_{N;n},\\
q^{\frac{N}{4}}\, \mathcal{Q}_{N-1;n+1}\, \mathcal{Q}_{N-1;n-1} &= \mathcal{Q}_{N-1;n}^2 - q^{\frac{(N-4)n}{4}}\, \mathcal{Q}_{N-2;n},\\
q^{\frac{N}{4}}\, \mathcal{Q}_{N;n+1}\, \mathcal{Q}_{N;n-1} &= \mathcal{Q}_{N;n}^2 - q^{\frac{(N-4)n}{4}}\, \mathcal{Q}_{N-2;n},
\end{aligned}$$

$$\begin{aligned}
B_N^{(1)}: \quad q^{N-1}\, \mathcal{Q}_{N-1;n+1}\, \mathcal{Q}_{N-1;n-1} &= \mathcal{Q}_{N-1;n}^2 - \mathcal{Q}_{N-2;n}\mathcal{Q}_{N;2n},\\
q^{\frac{N}{2}}\, \mathcal{Q}_{N;2n+1}\, \mathcal{Q}_{N;2n-1} &= \mathcal{Q}_{N;2n}^2 - q^{-n}\, \mathcal{Q}_{N-1;n}^2,\\
q^{\frac{N}{2}}\, \mathcal{Q}_{N;2n+2}\, \mathcal{Q}_{N;2n} &= \mathcal{Q}_{N;2n+1}^2 - q^{\frac{N}{2}-n-1}\, \mathcal{Q}_{N-1;n+1}\mathcal{Q}_{N-1;n},
\end{aligned}$$

$$\begin{aligned}
C_N^{(1)}: \quad q^{N-1}\, \mathcal{Q}_{N-1;2n+1}\, \mathcal{Q}_{N-1;2n-1} &= \mathcal{Q}_{N-1;2n}^2 - q^{-\frac{Nn}{2}}\, \mathcal{Q}_{N-2;2n}\mathcal{Q}_{N;n}^2,\\
q^{N-1}\, \mathcal{Q}_{N-1;2n+2}\, \mathcal{Q}_{N-1;2n} &= \mathcal{Q}_{N-1;2n+1}^2 - q^{-\frac{Nn}{2}}\, \mathcal{Q}_{N-2;2n+1}\mathcal{Q}_{N;n+1}\mathcal{Q}_{N;n},\\
q^{\frac{N}{2}}\, \mathcal{Q}_{N;n+1}\, \mathcal{Q}_{N;n-1} &= \mathcal{Q}_{N;n}^2 - q^{\frac{(N-2)n}{2}}\, \mathcal{Q}_{N-1;2n},
\end{aligned}$$

$$\begin{aligned}
D_{N+1}^{(2)}: \quad q^{N-1}\, \mathcal{Q}_{N-1;n+1}\, \mathcal{Q}_{N-1;n-1} &= \mathcal{Q}_{N-1;n}^2 - q^{-\frac{Nn}{4}}\, \mathcal{Q}_{N-2;n}\mathcal{Q}_{N;n}^2,\\
q^{\frac{N}{4}}\, \mathcal{Q}_{N;n+1}\, \mathcal{Q}_{N;n-1} &= \mathcal{Q}_{N;n}^2 - q^{\frac{(N-2)n}{4}}\, \mathcal{Q}_{N-1;n},
\end{aligned}$$

$$A_{2N-1}^{(2)}: \quad q^{N}\, \mathcal{Q}_{N;n+1}\, \mathcal{Q}_{N;n-1} = \mathcal{Q}_{N;n}^2 - q^{-n}\, \mathcal{Q}_{N-1;n}^2,$$

$$\begin{aligned}
A_{2N}^{(2)}: \quad q^{N}\, \mathcal{Q}_{N;2n+1}\, \mathcal{Q}_{N;2n-1} &= \mathcal{Q}_{N;2n}^2 - q^{-n}\mathcal{Q}_{N-1;2n}\mathcal{Q}_{N;2n},\\
q^{N}\, \mathcal{Q}_{N,2n+2}\, \mathcal{Q}_{N,2n} &= \mathcal{Q}_{N;2n+1}^2 - q^{-n}\, \mathcal{Q}_{N-1;2n+1}\mathcal{Q}_{N;2n+1}.
\end{aligned}$$

Remark 1 except for $A_{2N}^{(2)}$, all Q-systems are mutations of a quantum cluster algebra

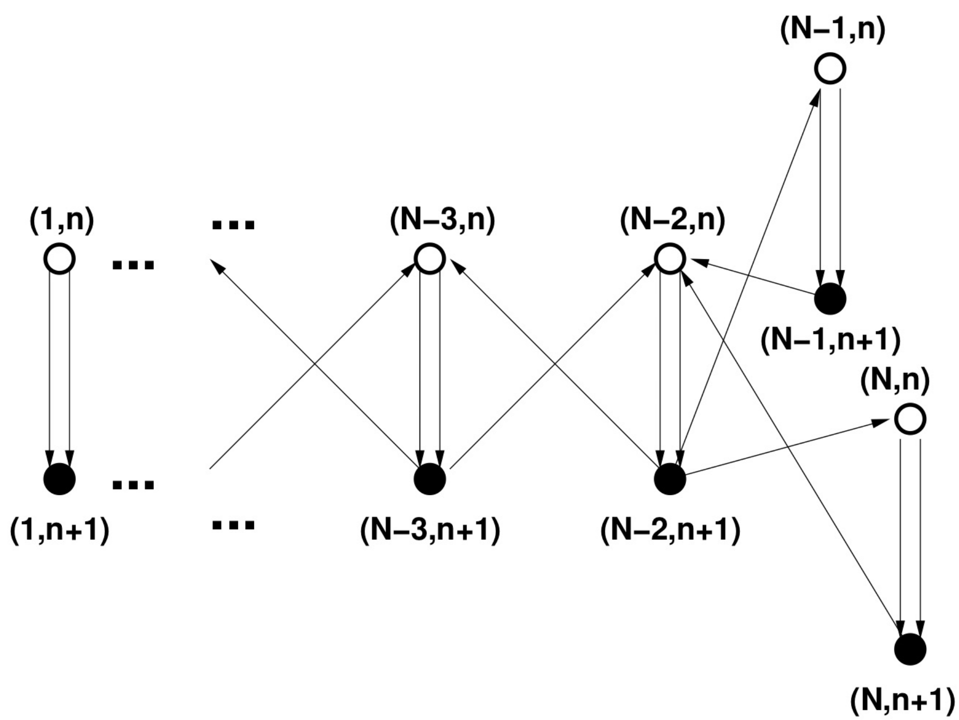
Exchange matrix: $B = \begin{pmatrix} C^t - C & -C^t \\ C & 0 \end{pmatrix}$ (untwisted)

(C = Cartan) skew-symmetric

$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$ (twisted)

skew-symmetrizable

Ex: $D_N^{(1)}$:



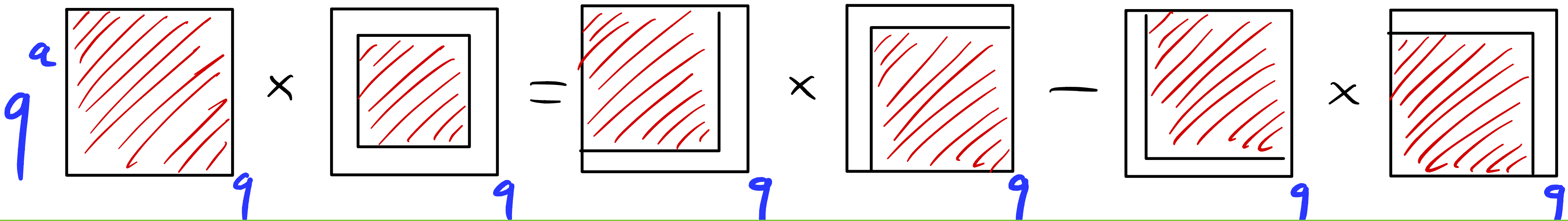
Remark 2

Van Diejen ops + Macdo combination is equivalent to the quantum Wronskian solution of:

$$\boxed{Q_{a,n}\,Q_{b,n+1} = q^{\mathrm{Min}(a,b)}\,Q_{b,n+1}\,Q_{a,n}}$$

[q-commutation relations]

$$q^{a}\,Q_{a,n+1}\,Q_{a,n-1} = Q_{a,n}^{2} - Q_{a+1,n}\,Q_{a-1,n}$$



[q-Desnanot-Jacobi & q-Lewis Carroll]

or q-Hirota

$$\boxed{Q_{a,n} = \det_q\left(Q_{1,n+i-j}\right)_{1\le i,j\le a}}$$

inversions #(-1)

$$\det_q(Q) = \sum_{A\in ASM_a} (-q)^{I(A)-N(A)} \prod_{i=1}^{a} Q_{1,n+a-i-m_i(A)}$$

$(A\rho)_i$

statistics on ASMs

7. PROOF OF THE MAIN THEOREMS

Key ingredients

1. universal solutions

2. duality

3. Fourier transform

UNIVERSAL SOLUTIONS [Shiraishi-Noumi; Stokman].

Remark: $\mathcal{D}_1^{(abcd)}$ has a series exp^n in $x^{-\alpha_i}$

Thm there exists a unique "universal" solution to the Koornwinder eigenvalue equation:

$$\mathcal{D}_1^{abcd}(x)\, P^{abcd}(x;s) = \sigma t^{N-1} \hat{e}_1(s)\, P^{abcd}(x;s)$$

$$P^{abcd}(x;s) = q^{\lambda\cdot\mu} \sum_{\beta\in R_+} c_\beta(s)\, x^{-\beta}$$

$$x = q^{\mu} t^{\rho} \sigma^{*} \; ; \; s = q^{\lambda} t^{\rho} \sigma \; ; \; c_\beta(0) = 1$$

λ, μ complex!

Proof: triangular non-singular system for c_β.

Thm There exists a unique universal solution to the Koornwinder Pieri equation:

$$H_1^{abcd}(s)\, Q^{abcd}(s;x) = \hat{e}_1(x)\, Q^{abcd}(s;x)$$

$$Q^{abcd}(s;x) = q^{\lambda\cdot\mu} \sum_{\beta\in R_+} \tilde{c}_\beta(x)\, s^{-\beta} \;;\quad \tilde{c}_\beta(0)=1$$

Proof: same as before as H_1 series of $s^{-\alpha}$

THM (duality): the solutions P, Q are series in both x, s and are related via:

$$P^{abcd}(x;s) = \Delta^{abcd}(x)\; Q^{abcd}(s;x)$$

$$\Delta^{a^*b^*c^*d^*}(s)\, P^{abcd}(x;s) = \Delta^{abcd}(x)\, P^{a^*b^*c^*d^*}(s;x)$$

$$\Delta^{(a,b,c,d)}(x) \;:=\; \prod_{i=1}^{N} \frac{\left(\frac{q}{x_i^2};q\right)_\infty}{\left(\frac{q}{ax_i};q\right)_\infty\left(\frac{q}{bx_i};q\right)_\infty\left(\frac{q}{cx_i};q\right)_\infty\left(\frac{q}{dx_i};q\right)_\infty} \prod_{1\le i<j\le N}\;\prod_{\epsilon=\pm1} \frac{\left(\frac{qx_j^\epsilon}{x_i};q\right)_\infty}{\left(\frac{qx_j^\epsilon}{tx_i};q\right)_\infty}.$$

$(x;q)_\infty = \prod_{n=0}^{\infty}(1-q^n x)$ q-dilogarithm

- Reduces to Koornwinder polynomial duality
- Equivalently

$$\boxed{Q^{abcd}(s;x) = Q^{a^*b^*c^*d^*}(x;s)}$$

FOURIER TRANSFORM

$$\mathcal{F}: \quad \mathrm{Diff}(\underbrace{x,\Gamma}_{\text{quantum torus}}) \longrightarrow \mathrm{Diff}(\underbrace{s,T}_{\text{quantum torus}})$$

$$\Gamma_i x_j = q^{\delta_{ij}} x_j \Gamma_i \qquad T_i s_j = q^{\delta_{ij}} s_j T_i$$

$$f(x) \longmapsto \hat{f}(s)$$

Such that:

$$\boxed{f(x)\, P(x;s) = \hat{f}(s)\, P(x;s)}$$

Main example:

$$\mathcal{D}_1(x)\, P(x;s) = \alpha t^{N-1} \widehat{e}_1(s)\, P(x,s)$$

$$\text{operator} \xrightarrow{\mathcal{F}} \text{eigenvalue}$$

BACK TO THE PROOF

$(t\to\infty \quad s \rightsquigarrow q^{\lambda})$

- Start from $\mathcal{F}(D_a) \sim \lim_{t\to\infty} \vec{e}_a(s) \sim q^{\omega_a^*\cdot\lambda} = \hat{D}_a$
- construct a reps of the opposite q-Q system $\hat{D}_{a,n}$

5 STEPS

1. Solve the commutation relations

$$\hat{D}_{a,0}\hat{D}_{b,1} = q^{-\omega_a^*\cdot\omega_b}\hat{D}_{b,1}\hat{D}_{a,0} \Rightarrow \boxed{\hat{D}_{a,1} = q^{\omega_a^*\cdot\lambda}\, T_{\omega_a}}$$

2. Use Q^{op}-system as recursive def^{n} for $\hat{D}_{a,n}$

$$\hat{D}_{b,n} = \text{L.P.}(\hat{D}_{a,0}, \hat{D}_{a,1}) \in \mathbb{C}(q^{\lambda_i})[T_j^{\pm}]$$

3. Construct "time translation operator" g such that

$$\hat{D}_{a,n} = q^{-\frac{na}{2}} g^{n} \hat{D}_{a} g^{-n}$$ long labels

$$\hat{D}_{a,2n+\varepsilon} = q^{-na} g^{n} \hat{D}_{a,\varepsilon} g^{-n}$$ short labels

4. Show that $\boxed{g = \hat{\gamma}}$ (Fourier transform)

5. Conclude that $\boxed{D_{a,n}(x)\, P(x;s) = \hat{D}_{a,n} P(x;s)}$

$\Rightarrow$ D satisfies opposite relations to $\hat{D}$ $\Rightarrow$ qQ-system!

$\Rightarrow$ raising/lowering for $n = \pm 1$

qed

Step 3

$$g_T := q^{\frac{\sum (\log T_i)^2}{2\log q}} = \gamma(T)\ ;\ g_\Lambda := \prod_{i=1}^{N-1} \left(\frac{\Lambda_{i+1}}{\Lambda_i}; q\right)_\infty^{-1}$$

(Gaussian of T_i)

$$g^{(\mathfrak{g})} = \begin{cases} g_T\, g_\Lambda \left(\frac{1}{\Lambda_{N-1}\Lambda_N}; q\right)_\infty^{-1}, & \mathfrak{g} = D_N^{(1)}, \\ \left(g_T^{1/2} \left(\frac{1}{\Lambda_N^2}; q\right)_\infty^{-1}\right)^2 g_\Lambda, & \mathfrak{g} = B_N^{(1)}, \\ (g_T\, g_\Lambda)^2 \left(\frac{1}{\Lambda_N}; q\right)_\infty^{-1}, & \mathfrak{g} = C_N^{(1)}, \\ g_T\, g_\Lambda \left(\frac{1}{\Lambda_N^2}; q^2\right)_\infty^{-1}, & \mathfrak{g} = A_{2N-1}^{(2)}, \\ g_T\, g_\Lambda \prod_{n=0}^{\infty} \left(\frac{1}{\Lambda_N}; q^{\frac{1}{2}}\right)_\infty^{-1}, & \mathfrak{g} = D_{N+1}^{(2)}, \\ g_T\, g_\Lambda \left(q^{\frac{1}{2}}\frac{1}{\Lambda_N}; q\right)_\infty^{-1} g_T\, g_\Lambda \left(\frac{1}{\Lambda_N}; q\right)_\infty^{-1}, & \mathfrak{g} = A_{2N}^{(2)}, \end{cases}$$

Step 4

$$\boxed{g(s)P(x;s) = \gamma(x)P(x;s)}$$

1. Lemma : $[g, H_1] = 0$

2. deduce $g\,P(x;s)$ proportional to $P(x;s)$ by function of x (uniqueness of universal Pieri sol$^{\underline{n}}$).

3. calculate eigenvalue by $g \cdot q^{\lambda\mu} \sim q^{\frac{\sum \mu_i^2}{2}} q^{\lambda\mu} g + O(s^{-\alpha})$

$(x = q^{\mu})$.

Remark

$$\gamma(x) = e^{\frac{\sum_i (\log x_i)^2}{2\log q}} = \prod_{n=0}^{\infty} \prod_{i=1}^{N} \frac{(1+q^{n+\frac{1}{2}})^2}{(1+q^{n+1/2}x_i)(1+q^{n+1/2}x_i^{-1})}$$

$$= \prod_{n=0}^{\infty} \frac{(1+q^{n+\frac{1}{2}})^{2N}}{\sum_{a=0}^{2N} q^{(n+\frac{1}{2})a} \hat{e}_a(x)}$$

$\Rightarrow$ Fourier transform $g = \hat{\gamma}$ is a g.f of Hamiltonians

(Pieri $= H_a P = \hat{e}_a(x) P$)

$$g = \prod_{n=0}^{\infty} \frac{(1+q^{n+\frac{1}{2}})^{2N}}{\sum_{a=0}^{2N} q^{(n+\frac{1}{2})a} H_a} \qquad (H_{2N-a} = H_a).$$

= "Baxter Q-operator"
for BCD+twisted types

[Schrader, Shapiro]
for type A

Step 5

$$\boxed{D_{a,n}\, P(x;s) = \widehat{D}_{a,n}\, P(x;s)}$$

- easy for long labels ● (reduces to eigenvalue eqn for D_a)

$$D_a P = \widehat{D}_a P$$

$$(\gamma^{-n} D_a \gamma^n) P = (\gamma^{-n} D_a) g^n P = g^n \gamma^{-n} (D_a P)$$

$$= g^n \gamma^{-n} (\widehat{D}_a P) = g^n \widehat{D}_a \gamma^{-n} P = (g^n \widehat{D}_a g^{-n}) P \quad \text{qed}$$

(cf $A^{(1)}_{N-1}$ case)

- short labels ●: must prove for $n=-1$ first.

case $B^{(1)}_N$ most subtle: one must use the Rains operators:

$$D_{N,-1} = R^{(1)}_{N,-1}\, R^{(0)}_{N,0}$$

sends P to P'

sends P' to P

and compute the Fourier transform.

8. CONCLUSION

- Proof of the Macdonald – Clusteralgebra conjecture
  Q-system
  - other variables?
- Duality / Universal solutions / Fourier Transform
  cf. [Cherednik; Shiraishi–Noumi, Stokman]
- Next steps: – clusteralgebra for finite t Macdonald theory? [DF+K+Shapiro +G. Schrader]
  $SL_2(\mathbb{Z})$ action?
  – clusteralgebra for Koornwinder (or DAHA?)
- Elliptic Hall Algebra for A, B, C, D?

Thank You!

References:

- P. Di Francesco and R. Kedem, "Macdonald operators and Quantum Q-systems for classical types" arXiv 1908.00806[math-ph] Progress in Math, Springer 2021.

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