Isomonodromic Deformations, Painlevé Equations & Integrable Systems: June 30, 2022

Integrable delay-differential equations

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Overview

- Delay (or differential-delay) equations arise in many important applications.
- Several differential-delay equations have been obtained as similarity reductions of integrable equations.
- In 1992, Quispel, Capel and Sahadevan obtained the equation

$$w(z)[w(z+1) - w(z-1)] = aw(z) + bw'(z).$$

- Delay equations from Bäcklund transformations of Painlevé equations.
- How can we find more examples of such equations?
- Is there an analogue of the QRT system?

Some delay Painlevé equations in the literature

- Quispel, Capel and Sahadevan (1992) obtained the equation

$$au(z) + bu'(z) = u(z)\left[u(z+1) - u(z-1)\right]$$

  as a symmetry reduction of the Manakov/Kac-van Moebeke equation.

- Grammaticos, Ramani and Moreira (1993) studied singularity confinement for delay equations of *bi-Riccati-type*.

- Levi and Winternitz (2007) considered symmetry reductions of Toda.

- Joshi (2009) used a direct method to find the following reduction of Toda:

$$au(z) + u'(z) = u(z)\left[v(z+1) - v(z)\right],$$

$$av(z) + v'(z) = 2\left[u(z)^2 - v(z-1)^2\right] + c,$$

- Fedorov, Gordoa and Pickering (2014) considered hierarchies of delay equations.

Two types of delay equations

- Equations for which we can solve for the highest/lowest shift, e.g.,

$$A(u,u')u(z+1)u(z-1)+B(u,u')u(z+1)+C(u,u')u(z-1)+D(u,u')=0,$$

  where $u \equiv u(z)$.

- Standard confinement-type arguments can be applied to these equations.

- Bi-Riccati:

$$\mathbf{U}(z)^t\mathbf{A}\mathbf{U}(z+1)=0,$$

  where $\mathbf{U}=(u' \;\; u^2 \;\; u \;\; 1)^t$ and $\mathbf{A}$ is a constant 4×4 matrix.

Analogues of constants and periodic functions

Consider the constant coefficient homogeneous differential-delay equation

$$u(z+1) - u(z) + ku'(z) = 0. \tag{1}$$

We see that $u(z) = \mathrm{e}^{\lambda z}$ is a solution if and only if $\mathrm{e}^{\lambda} - 1 + k\lambda = 0$. This characteristic equation has a countable infinity of solutions λ_n, $n \in \mathbb{Z}$. So for suitable constants c_n, equation (1) has the solution

$$u(z) = \sum_{n=-\infty}^{\infty} c_n \exp(\lambda_n z). \tag{2}$$

Note that when $k = 0$, we can index the λ_n's such that $\lambda_n = 2\pi i n$ and the expansion (2) becomes the Fourier series of the period one function u.

Discrete Painlevé equations as Bäcklund transformations

- This example is from Fokas, Grammaticos and Ramani.
- The third Painlevé equation with $\gamma = 0, \delta = -\alpha = 1$ is

$$w'' = \frac{w'^2}{w} - \frac{w'}{x} - \frac{1}{x}(w^2 - \beta) - \frac{1}{w}, \tag{3}$$

  where β is a parameter.
- If $w \equiv w(x, \beta)$ is a solution of equation (3) then

$$w(x;\beta+2) = \frac{x(1 + w'(x;\beta))}{w(x;\beta)^2} - \frac{\beta+1}{w(x;\beta)}, \text{ and} \tag{4}$$

$$w(x;\beta-2) = \frac{x(1 - w'(x;\beta))}{w(x;\beta)^2} - \frac{\beta-1}{w(x;\beta)}, \tag{5}$$

  are also solutions with β replaced by $\beta + 2$ and $\beta - 2$ respectively.
- Adding equations (4) and (5) gives

$$w(x;\beta+2) + w(x;\beta-2) = \frac{2x}{w(x;\beta)^2} - \frac{2\beta}{w(x;\beta)}.$$

Delay Painlevé equations from Bäcklund transformations

- Two BTs for $P_{\rm III}$ (with $\gamma = 0, \delta = -\alpha = 1$) are

$$w(x;\beta+2) = \frac{x(1+w_x(x;\beta))}{w(x;\beta)^2} - \frac{\beta+1}{w(x;\beta)}, \quad \text{and} \tag{6}$$

$$w(x;\beta-2) = \frac{x(1-w_x(x;\beta))}{w(x;\beta)^2} - \frac{\beta-1}{w(x;\beta)}. \tag{7}$$

- Taking the difference (6)−(7) gives

$$w(x;\beta+2) - w(x;\beta-2) = \frac{2xw_x(x;\beta)}{w(x;\beta)^2} - \frac{2}{w(x;\beta)}.$$

- The reduction $w(x;\beta) = u(z)$, where $z = (\beta/2) + \ln x$ gives

$$u(z+1) - u(z-1) = \frac{2\{u'(z) - u(z)\}}{u(z)^2},$$

which has a continuum limit to $P_{\rm I}$.

Bi-Riccati delay Painlevé equations from BTs

- Shifting $\beta \mapsto \beta + 2$ in the second BTs gives the pair

$$w(x;\beta)w(x;\beta+2) = \frac{x(1+w_x(x;\beta))}{w(x;\beta)} - (\beta+1), \text{ and} \tag{8}$$

$$w(x;\beta+2)w(x;\beta) = \frac{x(1-w_x(x;\beta+2))}{w(x;\beta+2)} - (\beta+1). \tag{9}$$

- Taking the difference (8)−(9) gives

$$[w(x;\beta)w(x;\beta+2)]_x + w(x;\beta+2) - w(x;\beta) = 0.$$

- The reduction $w(x;\beta) = \frac{b}{2}xu(z)$, where $z = \frac{\beta h}{2} + \frac{2a}{b}\ln x$ gives

$$a[u(z)u(z+h)]' + bu(z)u(z+h) + u(z+h) - u(z) = 0.$$

- Writing $u(z) = \frac{h^3}{2a}v(z) - \frac{h}{2a}$ and $b/a = \frac{3}{2}h^4 + O(h^5)$, then in the limit $h \to 0$ we recover P_I.

Another bi-Riccati delay Painlevé equation from BTs

- For P_{III} with $\gamma = -\delta = 1$, we have the BTs

$$\begin{aligned} w(x;-\alpha,-\beta) &= -w(x;\alpha,\beta), \\ w(x;-\beta,-\alpha) &= w(x;\alpha,\beta)^{-1}, \\ w(x;-\beta-2,-\alpha-2) &= w(x;\alpha,\beta)\left[1+\frac{2+\alpha+\beta}{x\left(\frac{w_x}{w}+w+\frac{1}{w}\right)-1-\beta}\right]. \end{aligned}$$

- Using these transformations we obtain the equation

$$\frac{w_x(x;\alpha+2,\beta+2)}{w(x;\alpha+2,\beta+2)}+\frac{w_x(x;\alpha,\beta)}{w(x;\alpha,\beta)}$$
$$=\left(w(x;\alpha+2,\beta+2)+\frac{1}{w(x;\alpha+2,\beta+2)}\right)-\left(w(x;\alpha,\beta)+\frac{1}{w(x;\alpha,\beta)}\right).$$

- The reduction $w(x;\beta) = u(z)$, where $z = \frac{(\alpha+\beta)h}{4} - kx$ gives

$$k[u(z)u(z+h)]' + [u(z+h)u(z)-1][u(z+h)-u(z)] = 0.$$

Addition laws for elliptic functions

Recall that the Weierstrass $\wp$ function satisfies

$$\wp'(z; g_2, g_3) = 4\wp(z; g_2, g_3)^3 - g_2\wp(z; g_2, g_3) - g_3,$$

(where "$'$" denotes the derivative with respect to the first argument) and the addition law

$$\wp(z \pm h; g_2, g_3) = \frac{1}{4}\left\{\frac{\wp'(z; g_2, g_3) \mp \wp'(h; g_2, g_3)}{\wp(z; g_2, g_3) - \wp(h; g_2, g_3)}\right\}^2 - \wp(z; g_2, g_3) - \wp(h; g_2, g_3).$$

It is straightforward to verify that

$$u(z) = \sqrt{\frac{\wp(h; g_2, g_3)}{h}}\,\{\wp(hz + c; g_2, g_3) - \wp(h; g_2, g_3)\}$$

satisfies

$$u(z)^2\,\{u(z+1) - u(z-1)\} = u'(z) \tag{10}$$

for arbitrary c, h, g_2 and g_3.

The symmetric QRT map

The symmetric Quispel-Roberts-Thompson map is

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_2(x_n) - x_{n-1}f_3(x_n)},$$

where

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = (\mathbf{A}_0\mathbf{X}_n) \times (\mathbf{A}_1\mathbf{X}_n), \quad \mathbf{X}_n = \begin{pmatrix} x_n^2 \\ x_n \\ 1 \end{pmatrix}, \quad \mathbf{A}_j = \begin{pmatrix} \alpha_j, & \beta_j, & \gamma_j \\ \beta_j & \epsilon_j & \zeta_j \\ \gamma_j & \zeta_j & \mu_j \end{pmatrix}, \quad j = 0, 1.$$

This system has the conserved quantity $K = \dfrac{\mathbf{X}_n^T\mathbf{A}_0\mathbf{X}_{n+1}}{\mathbf{X}_n^T\mathbf{A}_1\mathbf{X}_{n+1}}$.

Analogues of QRT mappings (with Bjorn Berntson)

An analogue of symmetric QRT mappings are those differential-delay equations of the form

$$A(u,u')u(z+1)u(z-1)+B(u,u')u(z+1)+C(u,u')u(z-1)+D(u,u')=0,$$

where $u \equiv u(z)$, that possess at least a two-parameter solution given in terms of elliptic functions. For example

$$\left(1-u(z)^2\right)\left\{u(z+1)-u(z-1)\right\}=bu'(z),$$

where b is a constant. This has a two-parameter family of solutions in terms of the Jacobi sn function because of the identity

$$\mathrm{sn}\,(z\pm h;k)=\frac{\mathrm{sn}\,(z;k)\mathrm{sn}\,'(h;k)\pm\mathrm{sn}\,'(z;k)\,\mathrm{sn}\,(h;k)}{1-k^2\mathrm{sn}\,(z;k)^2\,\mathrm{sn}\,(h;k)^2},$$

where "$'$" denotes the derivative with respect to the first argument.

Bi-Riccati analogues of QRT mappings

- We wish to identify equations of the form

$$\mathbf{W}^{\mathrm{T}}(z+h)K\mathbf{W}(z) = 0, \quad \text{where} \quad \mathbf{W}(z) = \begin{pmatrix} 1 \\ w(z) \\ w^2(z) \\ w'(z) \end{pmatrix},$$

admitting a two-parameter (ϵ and z_0) family of solutions of the form

$$w(z) = \frac{\alpha(\epsilon)\mathrm{sn}(\Omega(\epsilon)[z-z_0];k(\epsilon)) + \beta(\epsilon)}{\gamma(\epsilon)\,\mathrm{sn}(\Omega(\epsilon)[z-z_0];k(\epsilon)) + \delta(\epsilon)}.$$

- Ignoring the z_0 dependence, we look for solutions of the form

$$w(z) = \frac{\alpha\,\mathrm{sn}(\Omega z;k) + \beta}{\gamma\,\mathrm{sn}(\Omega z;k) + \delta}.$$

A simpler problem

The vector

$$\mathbf{U}(z) = \begin{pmatrix} 1 \\ u(z) \\ u^2(z) \\ u'(z) \end{pmatrix},$$

where

$$u(z) = \mathrm{sn}(\Omega z; k),$$

solves the equation

$$\mathbf{U}(z+h)^{\mathrm{T}} \mathbf{X} \mathbf{U}(z) = 0$$

if and only if $\mathbf{X}$ has the form

$$\mathbf{X} = \sum_{j=0}^{7} \lambda_j X_j,$$

where $\lambda_0, \ldots, \lambda_7$ are constants and the X_js have specific forms.

$$X_0 = s^{-1}\Omega \begin{pmatrix} s^2 & 0 & -1 & 0 \\ 0 & 2cd & 0 & 0 \\ -1 & 0 & k^2s^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_1 = s^{-1} \begin{pmatrix} 0 & \Omega cd & 0 & s \\ -\Omega & 0 & \Omega k^2 s^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = s^{-1} \begin{pmatrix} -\Omega s^2 & 0 & \Omega & 0 \\ 0 & -\Omega cd & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = s^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\Omega s^2 & 0 & \Omega & 0 \\ 0 & -\Omega cd & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = s^{-2} \begin{pmatrix} 0 & 0 & -\Omega^2 cd & 0 \\ 0 & \Omega^2(c^2+d^2) & 0 & 0 \\ -\Omega^2 cd & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2 \end{pmatrix}, \quad X_5 = s^{-1} \begin{pmatrix} 0 & \Omega & 0 & 0 \\ -\Omega cd & 0 & 0 & 0 \\ 0 & -\Omega k^2 s^2 & 0 & 0 \\ s & 0 & 0 & 0 \end{pmatrix},$$

$$X_6 = s^{-1} \begin{pmatrix} \Omega s^2 & 0 & 0 & 0 \\ 0 & \Omega cd & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & s & 0 & 0 \end{pmatrix}, \qquad X_7 = s^{-1} \begin{pmatrix} 0 & \Omega s^2 & 0 & 0 \\ 0 & 0 & \Omega cd & 0 \\ 0 & -\Omega & 0 & 0 \\ 0 & 0 & s & 0 \end{pmatrix}.$$

where $s = \mathrm{sn}\,(\Omega h; k)$, $c = \mathrm{cn}\,(\Omega h; k)$ etc.

Bi-Riccati QRT-type equations

- Consider the equation $\mathbf{U}(z+h)^{\mathrm{T}}\mathbf{X}\mathbf{U}(z) = 0$ with $u(z) = \mathrm{sn}(\Omega z; k)$ and

$$\mathbf{X} = \sum_{j=0}^{7} \lambda_j X_j.$$

- The transformation

$$w(z) = \frac{\alpha u(z) + \beta}{\gamma u(z) + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0$$

induces the transformation

$$\hat{X} = M^{\mathrm{T}} X M,$$

where

$$M = \begin{pmatrix} \delta^2 & 2\gamma\delta & \gamma^2 & 0 \\ \beta\delta & \alpha\delta + \beta\gamma & \alpha\gamma & 0 \\ \beta^2 & 2\alpha\beta & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha\delta - \beta\gamma \end{pmatrix}.$$

- In this way we obtain a number of delay-differential equations with multi-parameter families of elliptic function solutions. No geometric picture yet like QRT.

First-Order Difference Equations

- Consider the difference equation

$$y(z+1) = R(y(z)). \tag{11}$$

- If R is rational then equation (11) admits a non-constant meromorphic solution.
- If R is polynomial then equation (11) admits a non-constant entire solution.
- An immediate consequence of this theorem is that the Logistic map,

$$y(z+1) = \alpha y(z)(1 - y(z)),$$

  has a non-constant entire solution, $y(z) = w(z)$.
- The logistic map has a family of entire solutions:

$$y(z) = w(z - p(z)), \quad \text{where } p \text{ is periodic.}$$

- Nevanlinna theory provides a concept of "nice" meromorphic functions: *functions of finite order*.

Nevanlinna Theory

- Nevanlinna characteristic $T(r, f)$.
- For an entire function f,

$$T(r, f) \sim \log M(r, f), \quad M(r, f) = \max_{|z|=r} |f(z)|.$$

- More generally, for a meromorphic function f,

$$T(r, f) = m(r, f) + N(r, f),$$

  where $m(r, f)$ is a measure of how large f is on $|z| = r$ and $N(r, f)$ is a measure of how many poles f has in $D_r := \{z : |z| \leq r\}$.
- The order of f is $\rho(f) = \limsup_{r\to\infty} \frac{\log(T(r, f))}{\log r}$.
- The hyper-order of f is $\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log(T(r, f))}{\log r}$.
- Examples of finite-order meromorphic functions: e^z, $\cos z$, $\tan z$, $\wp(z)$.
- Infinite-order: $\exp(\exp z)$, $\exp(\cos(\sqrt{z}\,))$.

Difference equations of Painlevé type

- (Ablowitz, H, Herbst) An analogue of the Painlevé property for difference equations is the existence of sufficiently many finite-order meromorphic solutions.
- **Theorem** (Yanagihara) If the difference equation

$$y(z+1) = R(z, y(z)),$$

where

$$R(z,y) = \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q},$$

admits a finite-order non-rational meromorphic solution, then $\max(p,q) \leq 1$.

- This gives the difference Riccati equation

$$y(z+1) = \frac{\alpha(z)y(z) + \beta(z)}{\gamma(z)y(z) + \delta(z)},$$

which is linearized by

$$y(z) = \frac{\alpha(z-1)}{\gamma(z-1)} \left[\frac{w(z) - w(z-1)}{w(z)} \right].$$

Theorem (H. and Korhonen, 2007)

If the equation $\overline{w} + \underline{w} = R(z, w),$ (†)

has an admissible meromorphic solution of finite order, then either w satisfies the discrete Riccati eqn $\overline{w} = (\overline{p}\, w + q)/(w + p)$, or (†) can be transformed by a linear change of variables to one of the following equations:

$$\overline{w} + w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + \kappa_1$$

$$\overline{w} - w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + (-1)^z \kappa_1$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \pi_3}{w} + \pi_2$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \kappa_1}{w} + \frac{\pi_2}{w^2}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1)w + \pi_2}{(-1)^{-z} - w^2}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1)w + \pi_2}{1 - w^2}$$

$$\overline{w}w + w\underline{w} = p$$

$$\overline{w} + \underline{w} = p\, w + q$$

where p, q, π_k, κ_k are "small" functions and π_k and κ_k are periodic with period k.

Delay equations admitting meromorphic solutions with $\rho_2(w) < 1$ (with Risto Korhonen 2017)

- Let w be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z)),$$

  where R is rational in both its arguments and a is a rational function of z, such that $\rho_2(w) < 1$.
  Then $\mathrm{Deg}_w R(z,w) \leq 4$.

- Suppose furthermore that $R(z,w) = P(z,w)/Q(z,w)$, where $Q(z,0) \not\equiv 0$.
  Then either

  1. $\mathrm{Deg}_w P(z,w) = 1 + \mathrm{Deg}_w Q(z,w) \leq 3$

  or

  2. $\dfrac{P(z,w)}{Q(z,w)} = \dfrac{\alpha(z)w(z) + \beta(z)}{\gamma(z)w + \delta(z)}$.

Delay equations with meromorphic solutions with hyper-order < 1:

$\deg_w R(z,w) = 0$

- Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = b(z),$$

where $a(z) \not\equiv 0$ and $b(z)$ are rational. If the hyper-order of $w(z)$ is less than one and w has "a lot of simple zeros" then the coefficients $a(z)$ and $b(z)$ are both constants.

- Note that for any rational $a(z)$, if $b(z) \equiv p\pi i a(z)$, where $p \in \mathbb{N}$, then

$$w(z) = C \exp(p\pi i z), \qquad C \neq 0,$$

is a zero-free entire transcendental finite-order solution.

- Here "a lot of simple zeros" means that for any $\epsilon > 0$,

$$\overline{N}\left(r, \frac{1}{w}\right) \geq \left(\frac{3}{4} + \epsilon\right) T(r,w) + S(r,w).$$

A non-autonomous equation

Theorem

- Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) = \frac{a(z)w'(z) + b(z)w(z)}{w(z)^2} + c(z),$$

  where $a(z) \not\equiv 0$, $b(z)$ and $c(z)$ are rational.

- If the hyper-order of $w(z)$ is less than one and "$w(z)$ has a lot of zeros" then the equation takes the form

$$w(z+1) - w(z-1) = \frac{(\lambda + \mu z)w'(z) + (\nu\lambda + \mu(\nu z - 1))w(z)}{w(z)^2},$$

  where λ, μ and ν are constants.

Singularity confinement

Grammaticos, Ramani and Papageorgiou (1991);

Ramani, Grammaticos and Hietarinta (1991)

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}$$

$$\begin{aligned}
y_{n-1} &= k, \\
y_n &= \theta + \epsilon, \qquad\qquad \theta = \pm 1 \\
y_{n+1} &= -\frac{a_n + \theta b_n}{2\theta}\epsilon^{-1} + O(1), \\
y_{n+2} &= -\theta + \frac{2\theta b_{n+1} - \theta b_n - a_n}{a_n + \theta b_n}\epsilon + O(\epsilon^2), \\
y_{n+3} &= \frac{a_n + \theta b_n}{2\theta}\left\{\frac{(a_{n+2} - a_n) - \theta(b_{n+2} - 2b_{n+1} + b_n)}{\theta(2b_{n+1} - b_n) - a_n}\right\}\epsilon^{-1} + O(1).
\end{aligned}$$

Confinement:

$$y_{n+1} + y_{n-1} = \frac{\alpha + \beta(-1)^n + (\gamma n + \delta) y_n}{1 - y_n^2}$$

Example of Hietarinta and Viallet

$$y_{n+1} + y_{n-1} = y_n + \frac{a}{y_n^2}$$

$$\begin{aligned}
y_{n-1} &= k + o(1), \\
y_n &= \epsilon, \\
y_{n+1} &= \epsilon^{-2} - k + \epsilon + O(\epsilon^2), \\
y_{n+2} &= \epsilon^{-2} - k + \epsilon^4 + O(\epsilon^5), \\
y_{n+3} &= -\epsilon + 2\epsilon^4 + O(\epsilon^5), \\
y_{n+4} &= k + o(1).
\end{aligned}$$

Exact calculations of degree growth

There are two equivalent definitions of the degree of a rational function.

Let $R(z) = \dfrac{P(z)}{Q(z)}$, where P and Q are polynomials with no common factors. Then

1. $\deg(R) = \max\{\deg(P(z)), \deg(Q(z))\}$.
2. Let a be any number in the extended complex plane $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Then the $\deg(R)$ is the number of pre-images of a in $\mathbb{CP}^1$ counting multiplicities.

For example, the degree of the rational function

$$\frac{2x^5 - 4x^4 + 2x^3 + x + 1}{x(x-1)^2} = \frac{x+1}{x(x-1)^2} + 2x^2$$

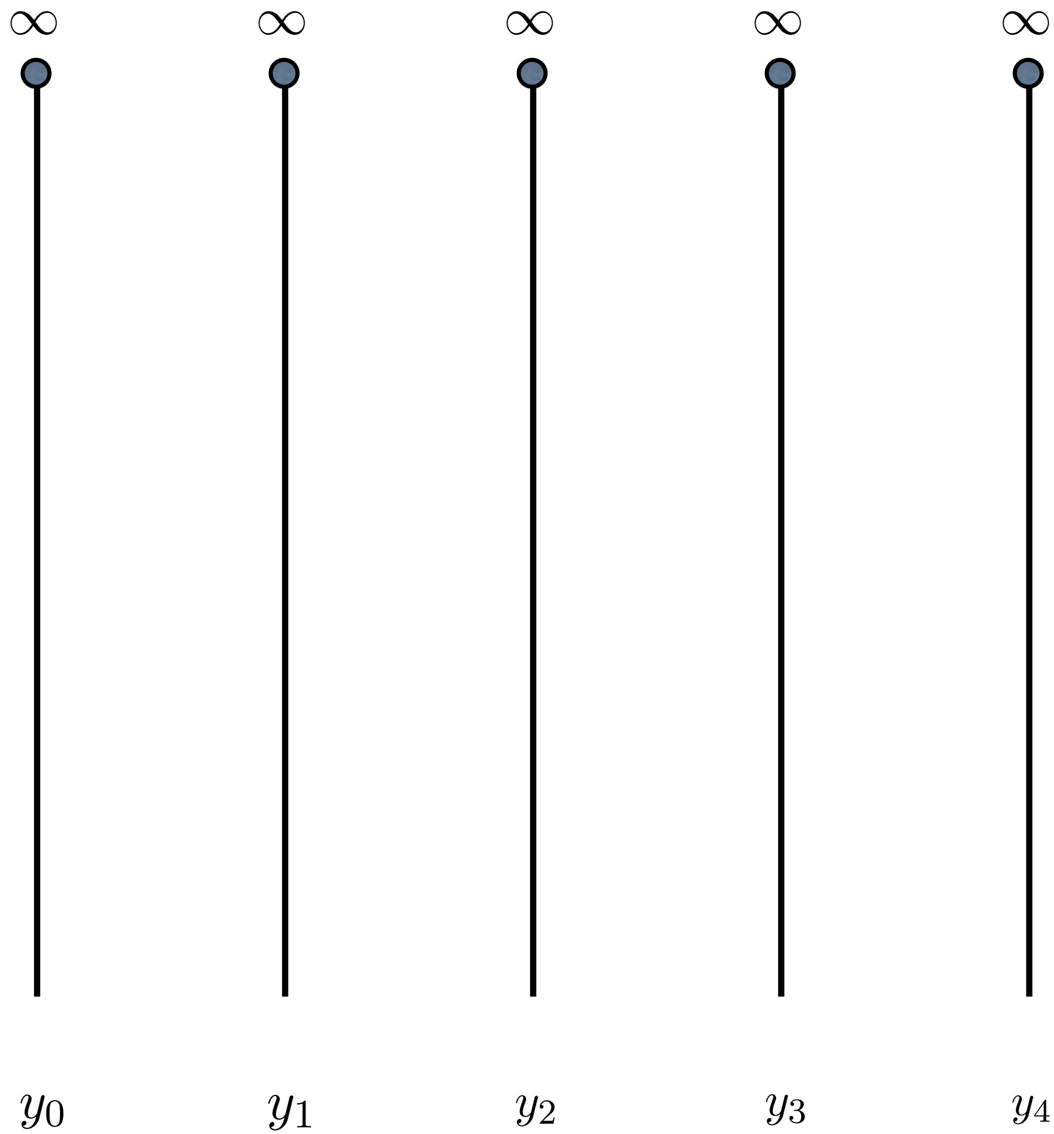
is 5.

Singularity confinement revisited

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}$$

$$\begin{aligned}
y_{n-1} &= k + o(1),\\
y_n &= \theta + \epsilon, \qquad \theta = \pm 1, \quad \epsilon = (z - z_0)^p f(z), \quad f \text{ analytic at } z_0,\ f(z_0) \neq 0\\
y_{n+1} &= -\frac{a_n + \theta b_n}{2\theta}\epsilon^{-1} + O(1),\\
y_{n+2} &= -\theta + \frac{2\theta b_{n+1} - \theta b_n - a_n}{a_n + \theta b_n}\epsilon + O(\epsilon^2),\\
y_{n+3} &= \frac{a_n + \theta b_n}{2\theta}\left\{\frac{(a_{n+2} - a_n) - \theta(b_{n+2} - 2b_{n+1} + b_n)}{\theta(2b_{n+1} - b_n) - a_n}\right\}\epsilon^{-1} + O(1).
\end{aligned}$$

Also, if $y_{n-1} \sim \alpha z$ and $y_n \sim \beta z$ as $z \to \infty$, then $y_{n+1} \sim -\alpha z$.
Take $y_0 = Az + B$ and $y_1 = Cz + D$, $AC \neq 0$.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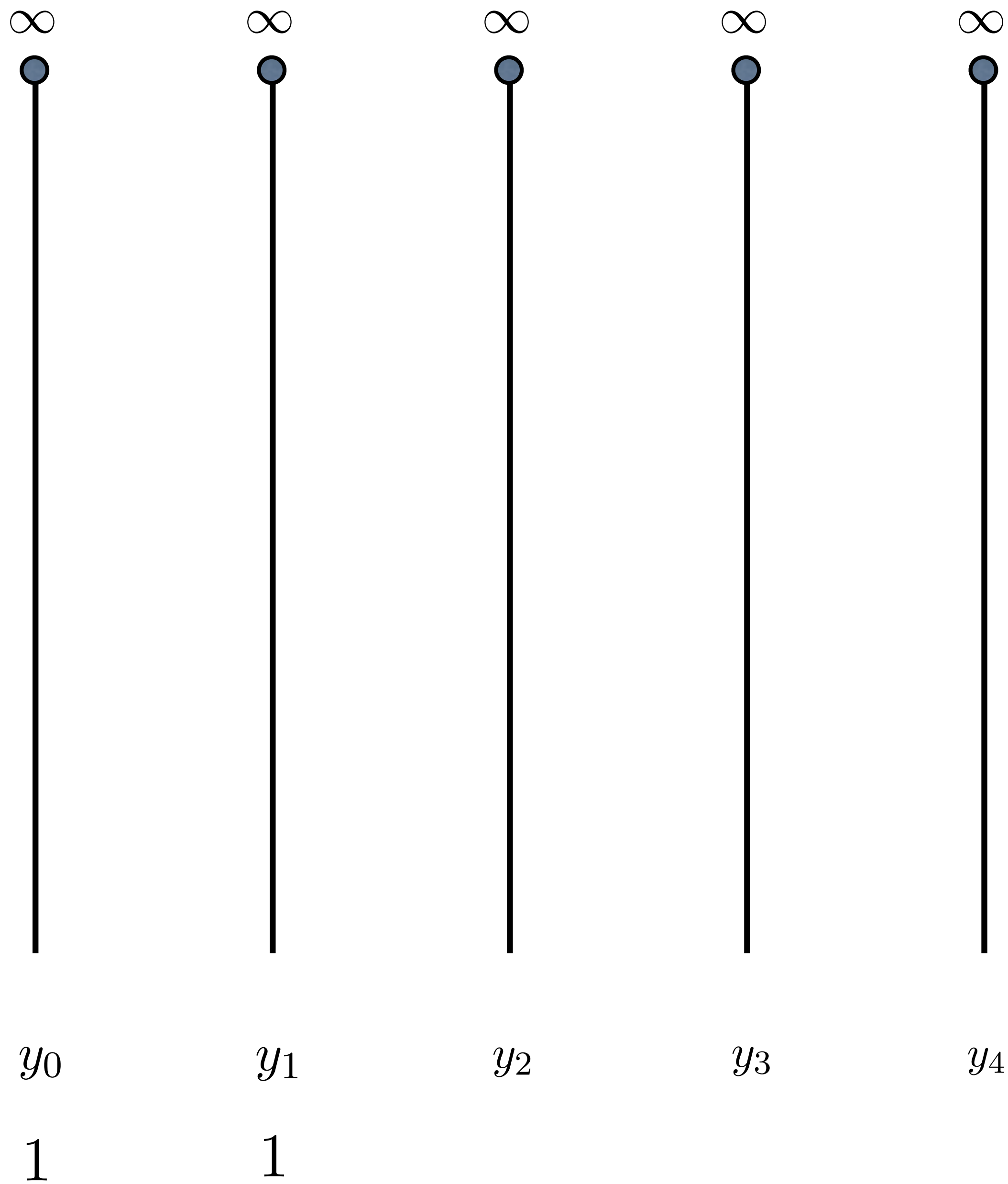

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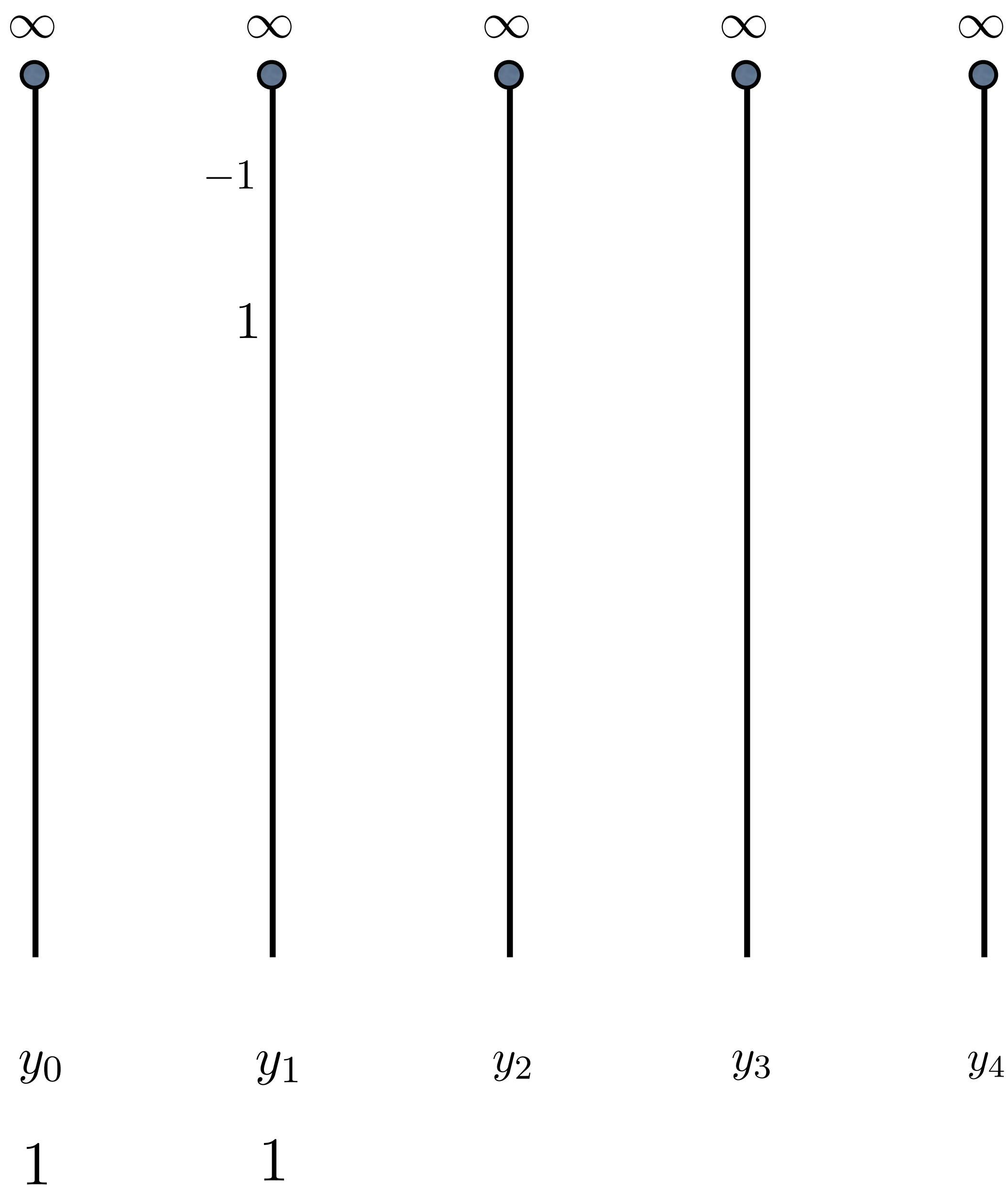
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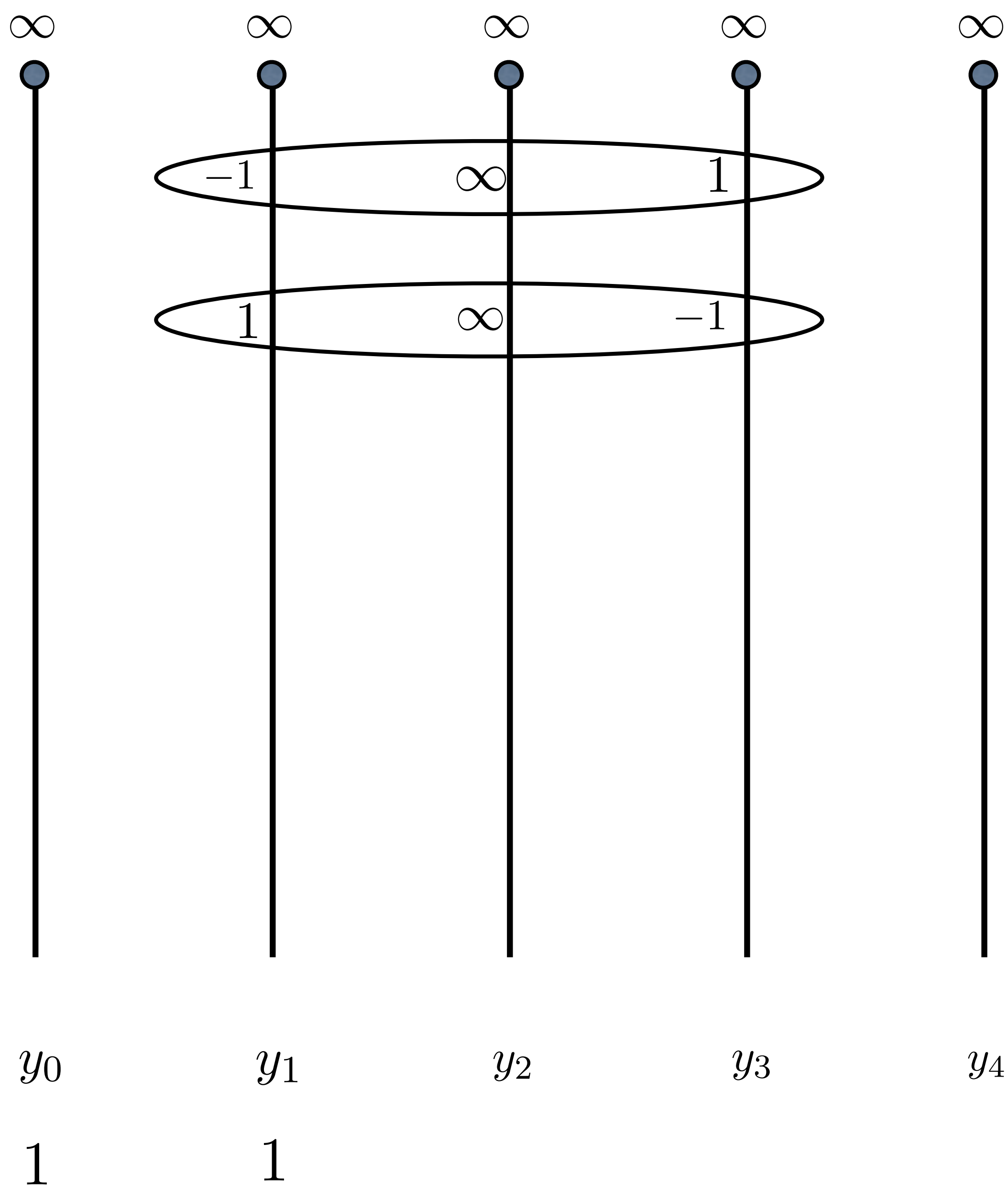
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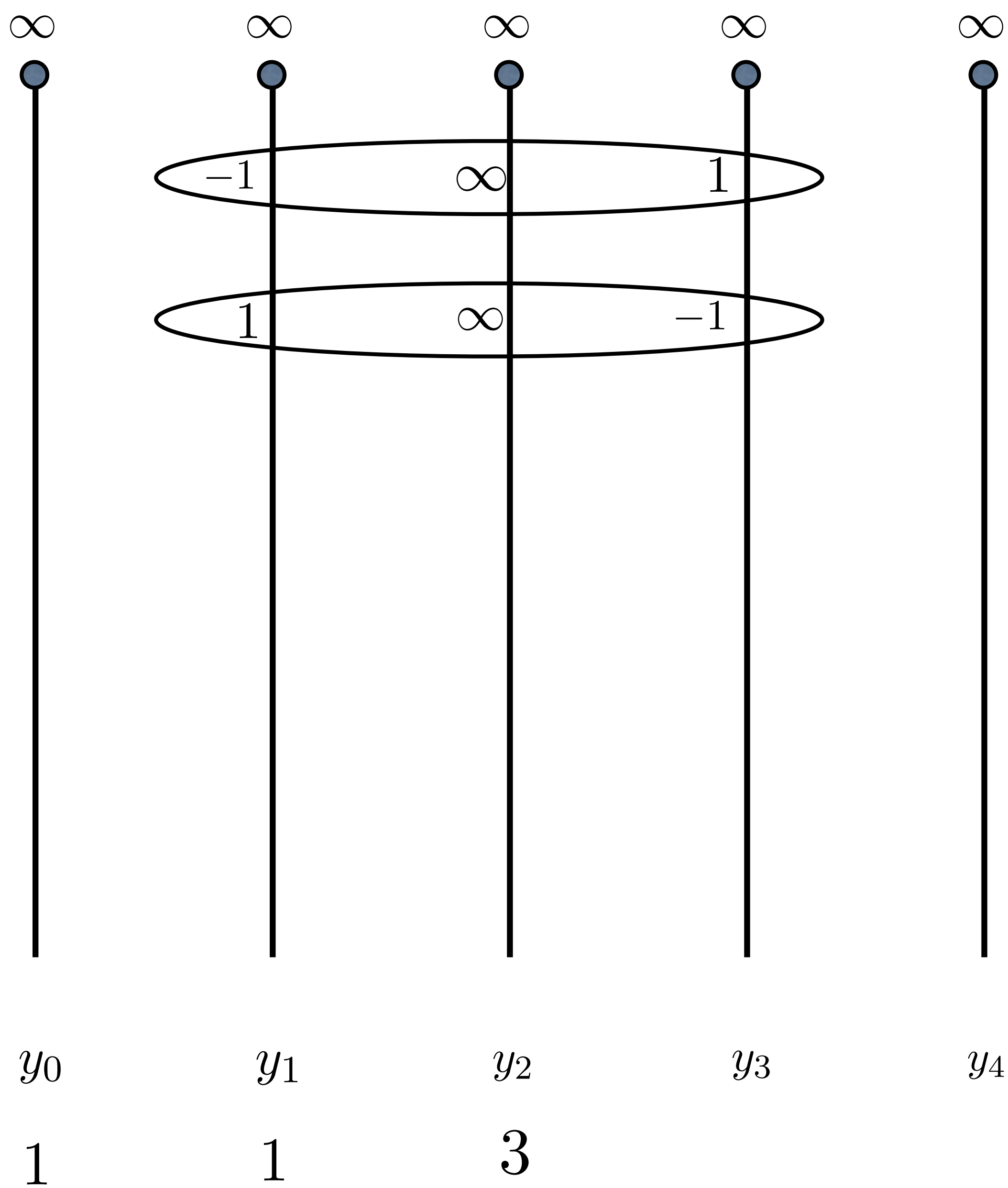

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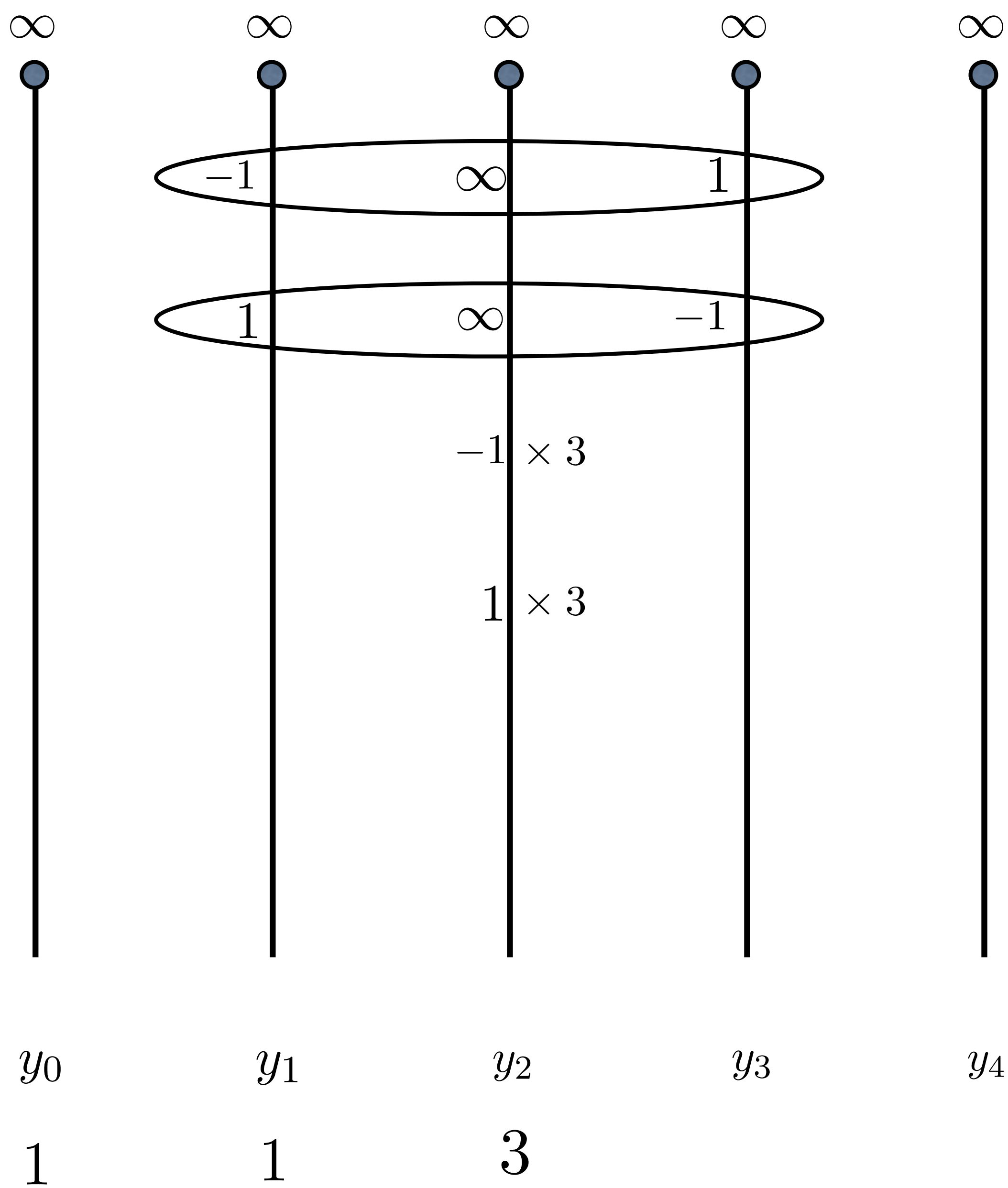

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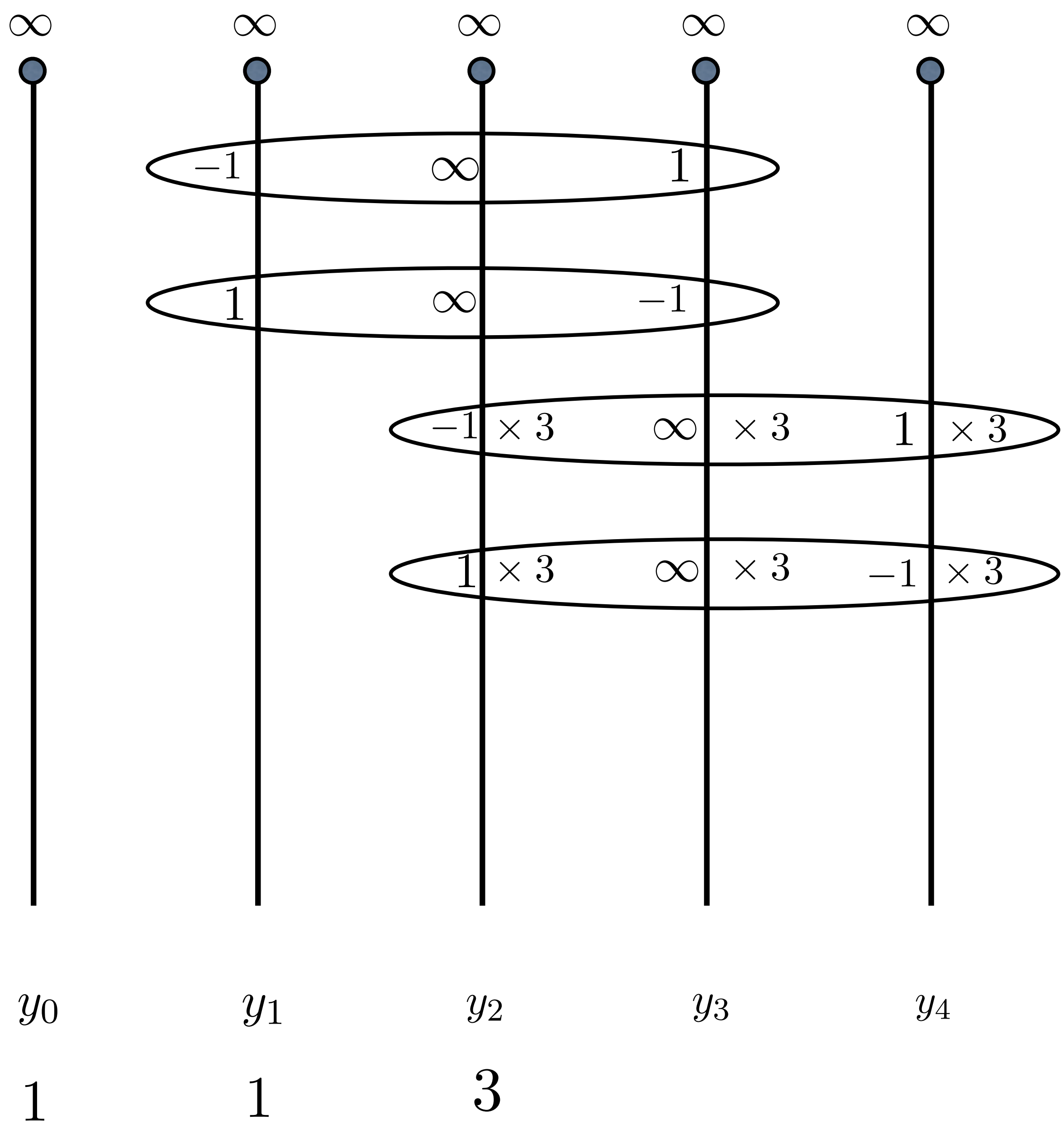

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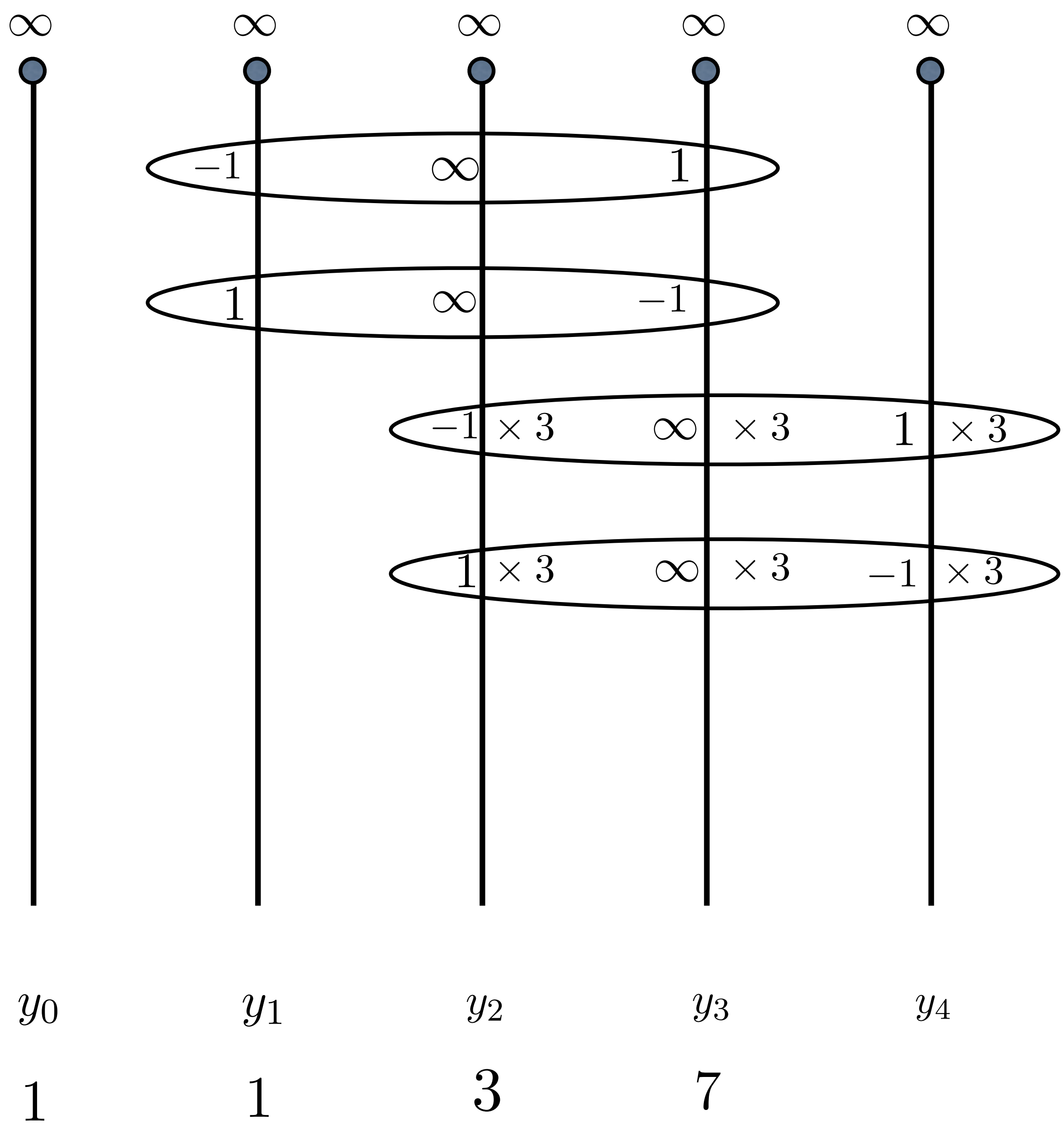

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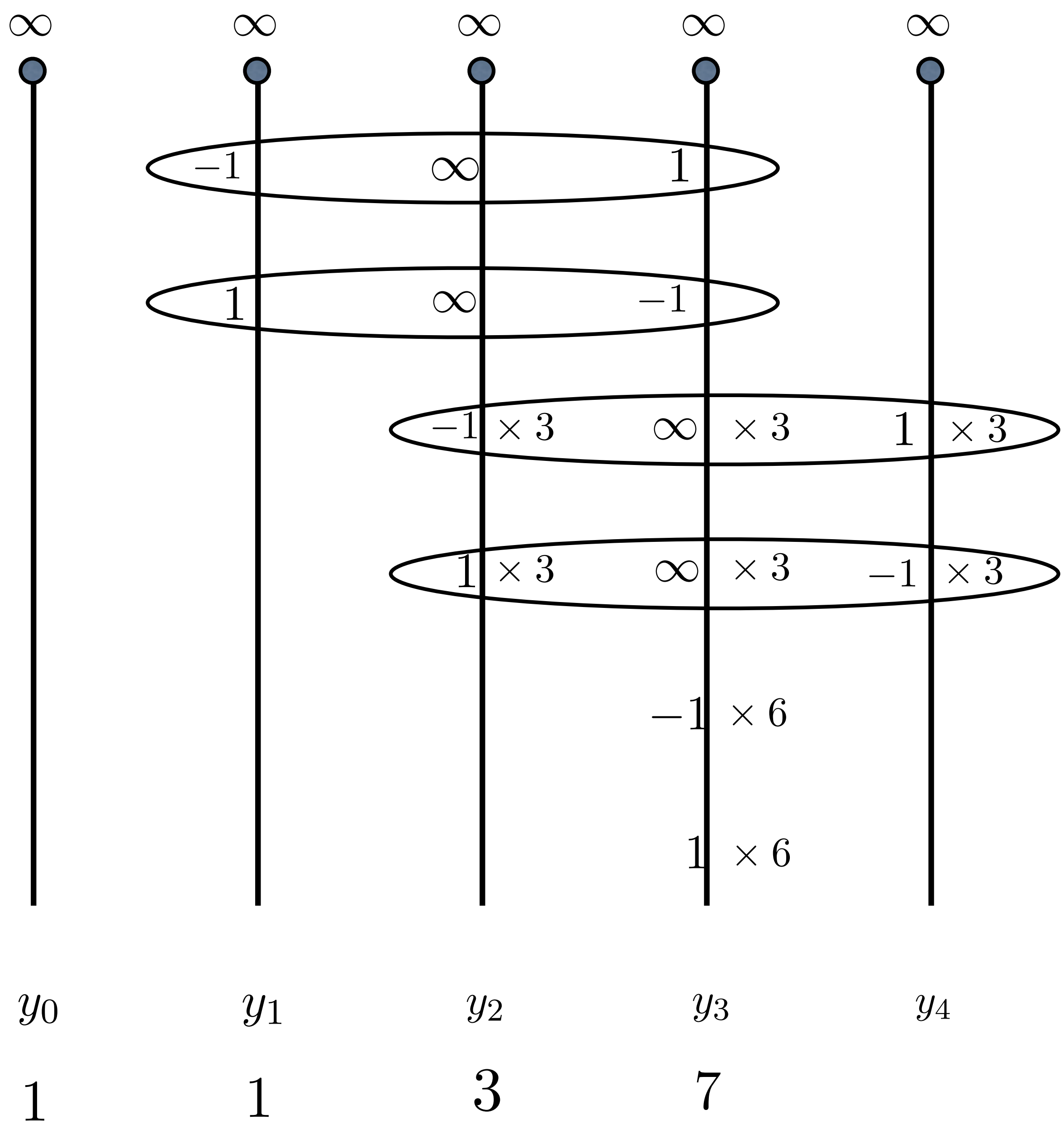
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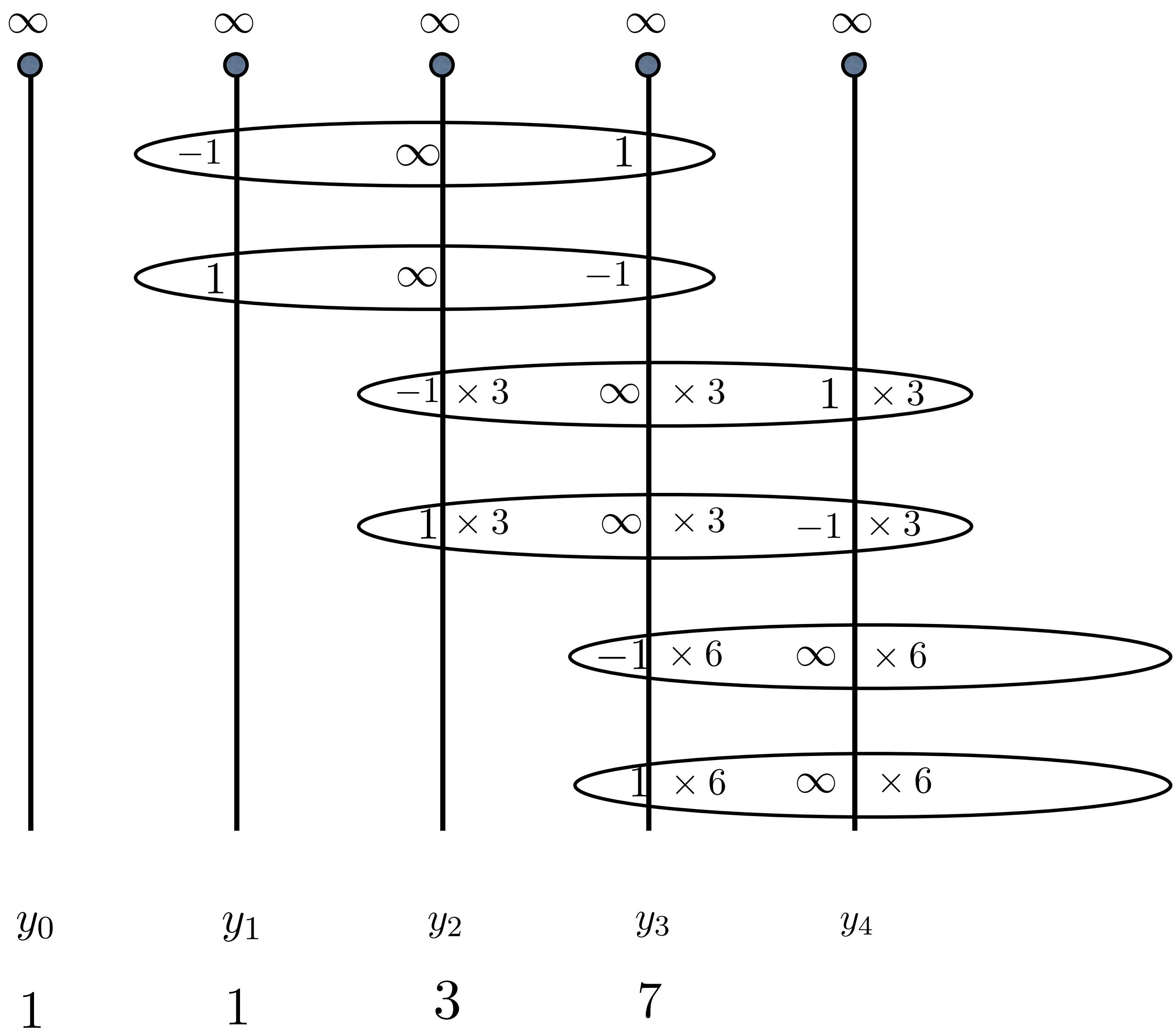

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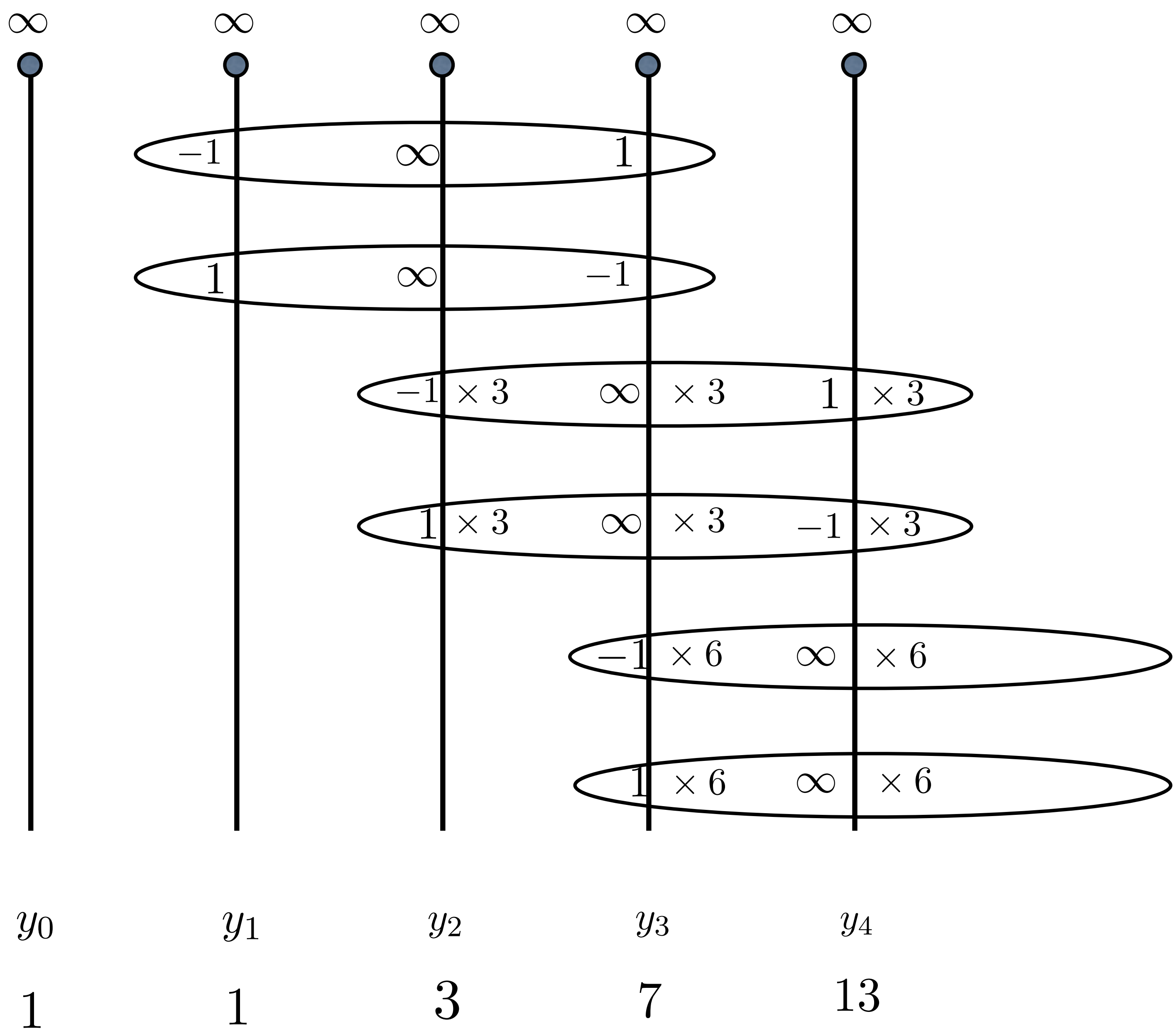
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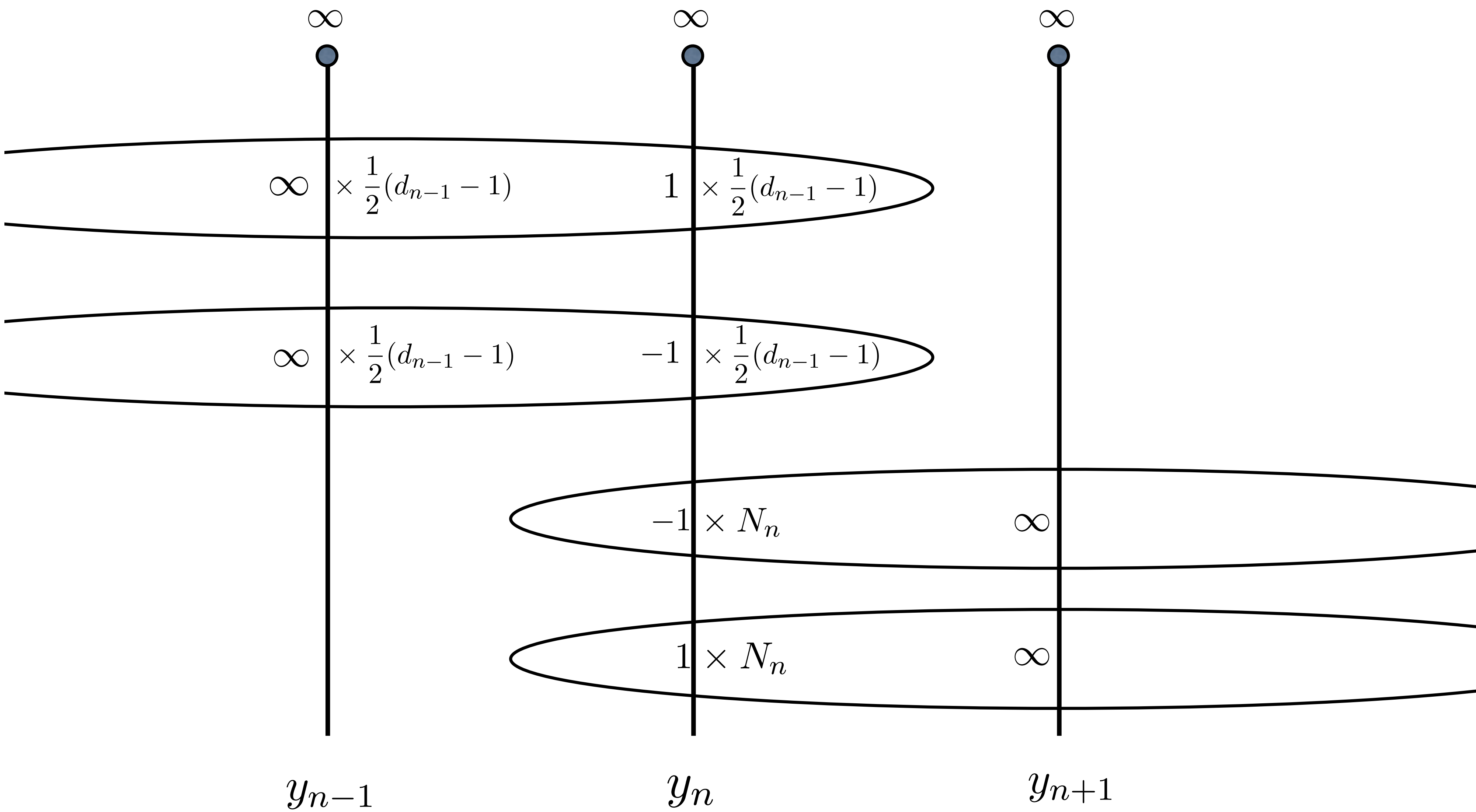


$$d_n = N_n + \frac{1}{2}\left(d_{n-1} - 1\right) \qquad d_{n+1} = 2N_n + 1$$

Exact formula for degrees

We have

$$d_{n+1} = 2N_n + 1 \quad \text{and} \quad d_n = N_n + \frac{1}{2}\left(d_{n-1} - 1\right).$$

Eliminating N_n gives

$$d_{n+1} - 2d_n + d_{n-1} = 2.$$

We also have the initial conditions $d_0 = d_1 = 1$. Hence

$$d_n = n(n-1) + 1.$$

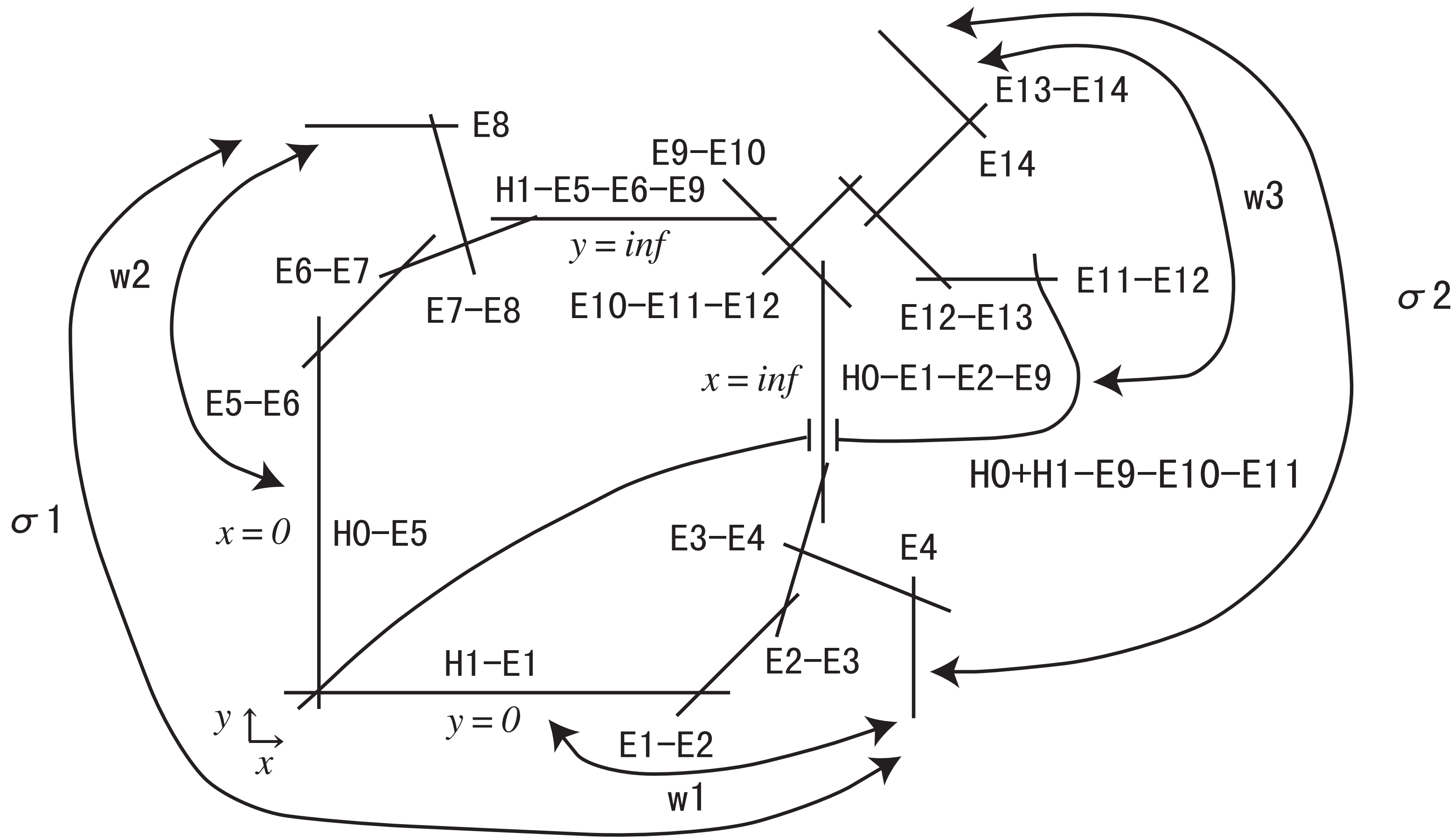
Example of Hietarinta and Viallet revisited

$$y_{n+1} + y_{n-1} = y_n + \frac{a}{y_n^2}$$

$$\begin{aligned} y_{n-1} &= k + o(1), \\ y_n &= \epsilon, \\ y_{n+1} &= \epsilon^{-2} - k + \epsilon + O(\epsilon^2), \\ y_{n+2} &= \epsilon^{-2} - k + \epsilon^4 + O(\epsilon^5), \\ y_{n+3} &= -\epsilon + 2\epsilon^4 + O(\epsilon^5), \\ y_{n+4} &= k + o(1). \end{aligned}$$

We will choose $y_0 \sim \alpha z + \beta$ and $y_1 \sim \gamma z + \delta$ as $z \to \infty$, where $\alpha\gamma(\alpha - \gamma) \neq 0$. Then y_n has a simple pole at $z = \infty$ for all n.

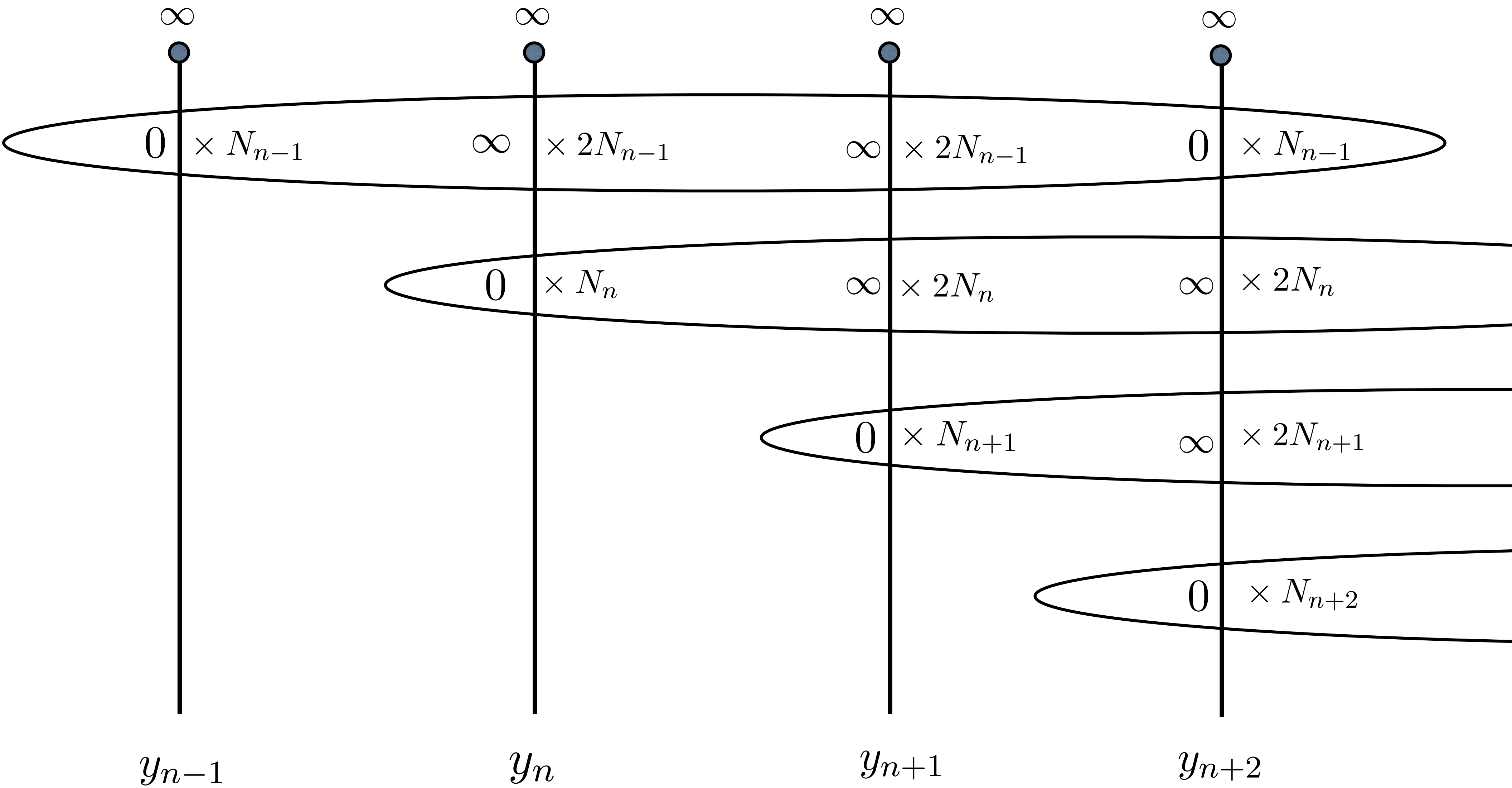
Takenawa's sequence of blow-ups for the Hietarinta-Viallet equation



He provided a rigorous proof that the algebraic entropy is

$$\frac{3+\sqrt{5}}{2}.$$

This value had been calculated using more heuristic methods by Hietarinta and Viallet.



$$d_{n+2} = N_{n+2} + N_{n-1}$$

$$d_{n+1} = 2(N_n + N_{n-1}) + 1$$

Substituting

$$N_n + N_{n-1} = (d_{n+1} - 1)/2 \quad \text{and} \quad N_{n+2} + N_{n-1} = d_{n+2}$$

in

$$(N_n + N_{n-1}) - (N_n + N_{n-3}) + (N_{n-2} + N_{n-3}) - (N_{n-1} + N_{n-2}) = 0$$

gives

$$d_{n+1} - 3d_n + d_{n-1} = 1.$$

Together with the initial conditions $d_0 = d_1 = 1$, this gives

$$d_n = \frac{\sqrt{5}-1}{\sqrt{5}} \left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+1}{\sqrt{5}} \left(\frac{3-\sqrt{5}}{2}\right)^n - 1.$$

It follows that the algebraic entropy is

$$\frac{3+\sqrt{5}}{2}.$$

Singularity confinement, discrete integrability and delay-differential equations

- Some form of singularity confinement underlies most "detectors" of integrability for discrete systems (rational surfaces approach, algebraic entropy, Nevanlinna theory, Diophantine integrability)
- Singularity (non-)confinement-type calculations underlie the more precise results from Nevanlinna theory for delay differential equations.
- Alex Stokes studied the way that singularity patterns vary with the multiplicity with which a solution hits singular values.
- In particular he observed that if a solution u of

$$au(z) + bu'(z) = u(z)\left[u(z+1) - u(z-1)\right]$$

is regular at $z = z_0 - 1$ and vanishes at $z = z_0$ with multiplicity m, then u has simple poles at $z_0 + 1, \ldots z_0 + 2m$, vanishes again at $z_0 + 2m + 1$ with multiplicity m and is neither 0 nor ∞ at $z = z_0 + 2m + 2$.

Delay equations with meromorphic solutions with hyper-order < 1
(with Risto Korhonen and Jun Wang — 2022)

Theorem Let w be a transcendental meromorphic solution of

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z,w)}{Q(z,w)}, \qquad (*)$$

where a is a rational function of z, $P(z,w)$ and $Q(z,w)$ are co-prime polynomials in w with rational coefficients in z. If $\rho_2(w) < 1$, then $R(z,w) = P(z,w)/Q(z,w)$ satisfies one of the following conditions

1. $\deg_w(R) \leq 1$;
2. $\deg_w(P) = \deg_w(Q) + 1 \leq 3$ and $w = 0$ is at most a simple root of Q;
3. $Q(z,w) = w\tilde{Q}(z,w)$, where $\tilde{Q}(z,0) \not\equiv 0$ and either $\deg_w(P) = \deg_w(\tilde{Q}) + 2 = 4$ or $\deg_w(P) \leq 2$ and $\deg_w(Q) \leq 1$.

Delay equations with meromorphic solutions with hyper-order < 1:
$\deg_w R(z,w) = 1$

- Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{\alpha(z)w(z) + \beta(z)}{w(z) - b(z)},$$

where $a \not\equiv 0$, $b \not\equiv 0$, α and β are rational functions of z and $\alpha \not\equiv -\beta/b$. If the hyper-order of $w(z)$ is less than one then "w has very few zeros, ignoring multiplicities":

$$\bar{N}\left(r, \frac{1}{w}\right) = S(r,w)$$

and

$$a(z+1)\left[a'(z+2) + b(z+3) - \alpha(z+2)\right] + a(z+2)[a'(z+1) + b(z) - \alpha(z+1)] = 0.$$

- In particular, if the equation is autonomous or if $N(r,1/w) = S(r,w)$, then $\rho_2(w) \geq 1$.

Delay equations with meromorphic solutions with hyper-order < 1:
$\deg_w P(z,w) = \deg_w Q(z,w) + 1 \leq 3$

Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{\alpha_3(z)w(z)^3 + \alpha_2(z)w(z)^2 + \alpha_1(z)w(z) + \alpha_0(z)}{w(z)^2 + b_1(z)w(z) + b_0(z)},$$

where $a \not\equiv 0$, $b \not\equiv 0$, α and β are rational functions of z and $w^2 + b_1 w + b_0 = 0$ has distinct non-zero roots. If the hyper-order of $w(z)$ is less than one then w satisfies the Riccati equation

$$w' = \frac{\alpha_3(z)}{a(z)}w^2 + \frac{\alpha_0(z)}{a(z)b_0(z)}w.$$

Delay equations with meromorphic solutions with hyper-order < 1:
$\deg_w P(z,w) = \deg_w \tilde{Q}(z,w) + 2 \leq 3$

- Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{\alpha_2(z)w(z)^2 + \alpha_1(z)w(z) + \alpha_0(z)}{w(z)[w(z) - b(z)]},$$

where $a \not\equiv 0$, $b \not\equiv 0$, α_0 and α_1 are rational functions of z and $\alpha_2 b^2 + \alpha_1 b + \alpha_0 \not\equiv 0$. If the hyper-order of $w(z)$ is less than one, then w has "a lot of simple zeros":

$$N\left(r, \frac{1}{w}\right) = T(r,w) + S(r,w)$$

and

$$a(z+1)\left[a'(z+2) + \alpha_2(z+2) - b(z+3)\right] = a(z+2)[a'(z+1) + \alpha_2(z+1) + b(z+1)].$$

- In particular, if the equation is autonomous, then $\rho_2(w) \geq 1$.

Delay equations with meromorphic solutions with hyper-order < 1:
$\deg_w P(z,w) = \deg_w \tilde{Q}(z,w) + 2 = 4$

- Here we just give an example.
- The function

$$w(z) = \tan(\pi z/4)$$

satisfies the equation

$$w(z+1) - w(z-1) + \frac{4}{\pi}\frac{w'(z)}{w(z)} = \frac{1 + 4w(z)^2 - w(z)^4}{w(z)[w(z)^2 - 1]}.$$

Lemma

Let w be a non-rational meromorphic solution of

$$P[z,w]=0,$$

where $P[z,w]$ is a differential-difference polynomial in w with rational coefficients and let $a_1(z),\ldots,a_k(z)$ be rational functions, satisfying $P[z,a_j]\not\equiv 0$ for $j=1,\ldots,k$. If there exist $s>0$ and $\tau\in(0,1)$ such that

$$\sum_{j=1}^{k} n\left(r,\frac{1}{w-a_j}\right)\leq k\tau n(r+s,w)+O(\log r),$$

then w has infinite order (in fact, the hyper-order of w is at least one).

Fast-growing solutions and singularity confinement

- Let $w(z)$ be a non-rational meromorphic solution of

$$w(z+1) - w(z-1) = \frac{a(z)w'(z) + b(z)w(z)}{w(z)^2} + c(z),$$

where $a(z) \not\equiv 0$, $b(z)$ and $c(z)$ are rational.

- Suppose that w has a simple zero at $z = \hat{z}$,

$$w(z-1) = K + O(z-\hat{z}), \qquad K \in \mathbb{C},$$
$$w(z) = \alpha(z-\hat{z}) + O((z-\hat{z})^2), \qquad \alpha \in \mathbb{C} \setminus \{0\}$$
$$w(z+1) = \frac{a(z)}{\alpha(z-\hat{z})^2} + \frac{b(z)}{\alpha(z-\hat{z})} + c(z) + K + O(z-\hat{z}),$$
$$w(z+2) = c(z+1) + O(z-\hat{z}),$$
$$w(z+3) = \frac{a(z)}{\alpha(z-\hat{z})^2} + \frac{b(z)}{\alpha(z-\hat{z})} + O(1),$$

where there can be at most finitely many $\hat{z}$ such that $c(\hat{z}+1) = 0$.

- Assume now that $c(z) \equiv 0$:

$$w(z+1) - w(z-1) = \frac{a(z)w'(z) + b(z)w(z)}{w(z)^2} + c(z),$$

- Suppose again that $w(z)$ has a pole at $z = \hat{z} + 1$, and that $w(\hat{z} - 1)$ is finite.

$$\begin{aligned}
w(z-1) &= K + O(z - \hat{z}), \qquad K \in \mathbb{C}, \\
w(z) &= \alpha(z - \hat{z}) + O((z - \hat{z})^2), \qquad \alpha \in \mathbb{C} \setminus \{0\} \\
w(z+1) &= \frac{a(z)}{\alpha(z - \hat{z})^2} + \frac{b(z)}{\alpha(z - \hat{z})} + O(1), \\
w(z+2) &= \alpha \left(1 - \frac{2a(z+1)}{a(z)}\right)(z - \hat{z}) + O((z - \hat{z})^2), \\
w(z+3) &= \frac{a(z)(a(z+2) - 2a(z+1) + a(z))}{(a(z) - 2a(z+1))\alpha(z - \hat{z})^2} + \frac{\gamma(z)}{\alpha(z - \hat{z})} + O(1),
\end{aligned}$$

where

$$\gamma(z) = \frac{a(z)b(z+2) - (2a(z+1) - a(z))b(z)}{a(z) - 2a(z+1)} - \frac{2a(z+2)[a(z)a'(z+1) - a(z+1)a'(z)]}{(a(z) - 2a(z+1))^2}.$$

Delay alternating QRT

- Differential-difference mKdV (Ablowitz and Ladik):

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = \frac{1}{2}(1+w_n^2)\Big\{(w_{n+2}+w_n)(1+w_{n+1}^2) \\ -(w_n+w_{n-2})(1+w_{n-1}^2) - 2(w_{n+1}-w_{n-1})\Big\}.$$

- Travelling wave reduction: $w_n(t) = u(z)$, where $z = n - ct$:

$$cu'(z) + \frac{1}{2}(1+u(z)^2)\Big\{(u(z+2)+u(z))(1+u(z+1)^2) \\ -(u(z)+u(z-2))(1+u(z-1)^2) - 2(u(z+1)-u(z-1))\Big\} = 0.$$

- When $c = 0$, this equation integrates to give

$$u(z+1) + u(z-1) = \frac{2u(z)+p(z)}{u(z)^2+1},$$

where p is an arbitrary period two function.

Summary

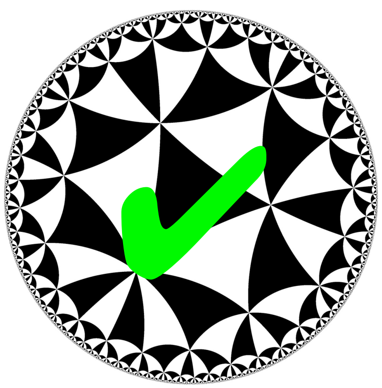
- Several delay Painlevé equations have been found.
- Such equations arise as symmetry reductions of integrable differential-difference equations.
- Others appear to arise as reductions of BTs for the Painlevé equations.
- The existence of finite order meromorphic solutions seems to be a reasonable characterisation of such equations.
- Some Lax pairs are known.

Complex Analysis video seminars

(CAvid)

The seminars are broadcast by Zoom
Tuesdays at 13:00 UTC
(6 am LA, 9 am New York, 2 pm London, 3 pm Paris, 6:30 pm New Delhi, 9 pm Beijing, 10 pm Tokyo)

Please e-mail r.halburd@ucl.ac.uk to register for the mailing list, which will be used to send Zoom links and passwords



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Stay safe
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