Stationary measure for the open KPZ equation

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joint work with I. Corwin

Isomonodromic Deformations, Painlevé Equations, and Integrable systems
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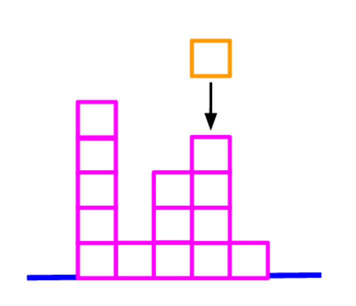
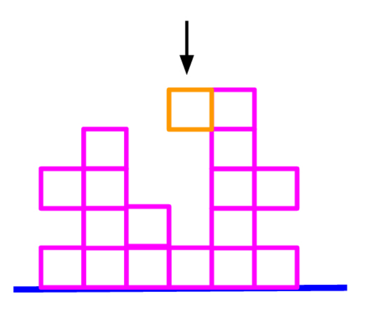
Random growth models (1+1)d

Random Deposition



Ballistic Deposition



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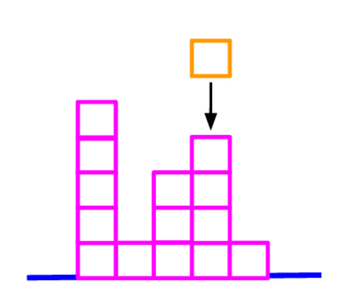

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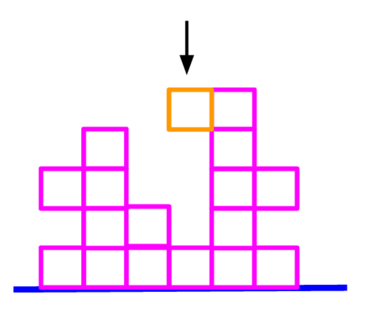
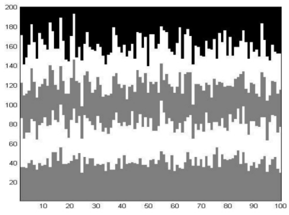
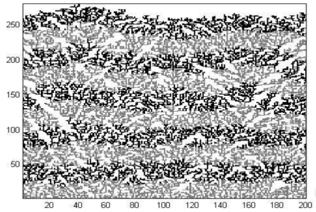






Open KPZ

KPZ: $\partial_T H(T,X) = \frac{1}{2}\partial_X^2 H(T,X) + \frac{1}{2}\big(\partial_X H(T,X)\big)^2 + \xi(T,X)$

Open KPZ

KPZ: $\partial_T H(T,X) = \frac{1}{2}\partial_X^2 H(T,X) + \frac{1}{2}\big(\partial_X H(T,X)\big)^2 + \xi(T,X)$

Open KPZ: for all $T > 0$, we impose with $u, v \in \mathbb{R}$

$$\partial_X H(T,X)\big|_{X=0} = u, \qquad \partial_X H(T,X)\big|_{X=1} = -v.$$

Stochastic heat equation (SHE)

SHE with inhomogeneous Robin boundary conditions:

$$\partial_T Z(T,X) = \tfrac{1}{2}\partial_X^2 Z(T,X) + \xi(T,X)Z(T,X),$$

$$T \geqslant 0 \text{ and } X \in [0,1];$$

$$\partial_X Z(T,X)\big|_{X=0} = (u - \tfrac{1}{2})Z(T,0), \quad \partial_X Z(T,X)\big|_{X=1} = -(v - \tfrac{1}{2})Z(T,0),$$

$$\text{for all } T > 0.$$

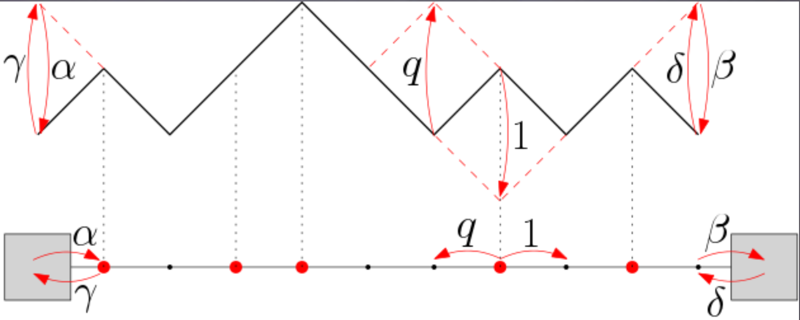
The Hopf-Cole solution to the open KPZ is defined as $H(T,X) := \log Z(T,X)$, which formally solves the open KPZ equation.

Definition: A stationary measure for the open KPZ equation is the law on a random function $H_0 : [0,1] \to \mathbb{R}$ with $H_0(0) = 0$ such that the law of $X \mapsto H(T,X) - H(T,0)$, as a process in X, is T-independent for all $T \geqslant 0$ if we start with $H(0,X) = H_0(X)$.

[Funaki–Quastel '15], [Hairer–Mattingly '18]

Open ASEP

Open ASEP (with system size $N = 10$) and its height function.



Height function

Height function is defined for $t \geqslant 0$ and $x \in \{0, \ldots, N\}$ as

$$h_N(t,x) := h_N(t,0) + \sum_{i=1}^{x} (2\tau_i(t) - 1) \text{ with } \tau_i(t) = 0 \text{ or } 1 \text{ and}$$

$$h_N(t,0) := -2\mathcal{N}_N(t),$$

where the net current $\mathcal{N}_N(t)$ equals the number of particles to enter into site 1 minus the number of particles to exit from site 1 up to time t.

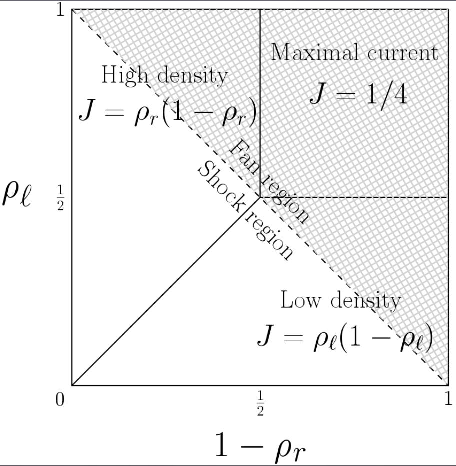
Open ASEP

Open ASEP is an irreducible continuous time Markov chain with finite state space. We will denote this by $\pi_N(\tau)$ its unique invariant measure. .
For a function f on the state space denote its expectation under the invariant measure by

$$\langle f \rangle_N := \sum_{\tau \in \{0,1\}^{\{1,\dots,N\}}} f(\tau) \cdot \pi_N(\tau).$$

$$\text{Stationary current } J_N := \frac{\langle \alpha(1-\tau_1) - \gamma\tau_1 \rangle_N}{1-q}$$

Phase diagram



[Derrida–Evans–Hakim– Pasquier '93], [Sandow '94],
[Uchiyama–Sasamoto–Wadati '03]

Law of Large Numbers

Theorem:

For τ distributed according to the invariant measure π_N the following limits hold for $x \in [0,1]$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{[Nx]} \tau_j = \begin{cases} \frac{1}{2}x, & \text{maximal current phase;} \\ \rho_l x, & \text{low density phase;} \\ \rho_r x, & \text{high density phase.} \end{cases}$$

Weakly asymmetric scaling

Let $u, v \in \mathbb{R}$

$$q = \exp\left(-\frac{2}{\sqrt{N}}\right);$$

$$\alpha = \frac{1}{1+q^u}, \qquad \beta = \frac{1}{1+q^v}, \qquad \gamma = \frac{q^{u+1}}{1+q^u}, \qquad \delta = \frac{q^{v+1}}{1+q^v}.$$

Define $H^N(X) := N^{-1/2} h_N(NX)$.

[Bertini–Giacomin '97], [Corwin–Shen '16], [Parekh '19]

Corwin-K. '21:

- WASEP-stationary measures: For $u, v \in \mathbb{R}$ we construct stationary measures $H_{u,v}$ for the open KPZ equation as the limits of $H_{u,v}^{(N)}(\cdot)$;
- Characterization: For $u, v \in \mathbb{R}$ with $u + v > 0$ we characterize $H_{u,v}$ through an explicit multi-point Laplace transform.

Theorem (Corwin-K. '21):

- WASEP-stationary measures: For $u, v \in \mathbb{R}$, the sequence of laws of $H^{(N)}_{u,v}(\cdot)$ are tight in the space of measures on $C[0,1]$. All subsequential limits $H_{u,v}$ are stationary measures for the open KPZ equation and are Hölder $1/2-$ almost surely.
- Coupling: Let $M \in \mathbb{Z}_{\geqslant 2}$, $u_1 \leqslant \cdots \leqslant u_M$ and $v_1 \geqslant \cdots \geqslant v_M$. For the corresponding collection $\{H_{u_i,v_i}\}_{i=1}^{M}$ of WASEP-stationary measures there exists a coupling such that for $0 \leqslant X \leqslant X' \leqslant 1$

$$H_{u_i,v_i}(X') - H_{u_i,v_i}(X) \leqslant H_{u_j,v_j}(X') - H_{u_j,v_j}(X).$$

- Characterization: For $u, v \in \mathbb{R}$ with $u + v > 0$, there is a unique WASEP-stationary measure $H_{u,v}$ for the open KPZ equation whose law is determined by its explicit multi-point Laplace transform.

Laplace transform

A simple case for $u, v > 0$ and $c \in (0, 2u)$:

$$\mathbb{E}\left[e^{-cH_{u,v}(1)}\right] = e^{c^2/4} \cdot \frac{\int\limits_0^\infty e^{-r^2} \cdot \frac{\left|\Gamma(\frac{c}{2}+u+ir)\Gamma(-\frac{c}{2}+v+ir)\right|^2}{\left|\Gamma(2ir)\right|^2} dr}{\int\limits_0^\infty e^{-r^2} \cdot \frac{\left|\Gamma(u+ir)\Gamma(v+ir)\right|^2}{\left|\Gamma(2ir)\right|^2} dr}.$$

$H_{u,v}(1)$ records the net height change across the interval $[0, 1]$.

Methods for Part III

Theorem (Bryc–Wesolowski '17):
In the fan region for $0 < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ the joint generating function of the stationary distribution of the ASEP

$$\left\langle \prod_{j=1}^{N} t_j^{\tau_j} \right\rangle_N = \frac{\mathbb{E}\left[\prod_{j=1}^{N}\left(1 + t_j + 2\sqrt{t_j}\, Y_{t_j}\right)\right]}{2^N \mathbb{E}\left[(1 + Y_1)^N\right]},$$

where $\{Y_t\}_{t \geqslant 0}$ is the Askey-Wilson process with parameters determined by the model.

Connection to Askey-Wilson polynomials
[Uchiyama–Sasamoto–Wadati '03]

The Askey-Wilson process:

It is a continuous Markov process on $[-1,1]$ for

$t \in [q^2; \min(q^{-2u}, 1/q^2)]$ with

$$\pi_t(y) = AW_y\left(q^u\sqrt{t}, -q\sqrt{t}, \frac{q^v}{\sqrt{t}}, -\frac{q}{\sqrt{t}}\right), \text{ and for } s < t,$$

$$p_{s,t}(x,y) = AW_y\left(q^u\sqrt{t}, -q\sqrt{t}, \frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right),$$

where $x = \cos\theta_x$ and

$$AW_x(a,b,c,d) = \frac{(q,ab,ac,ad,bc,bd,cd;q)_\infty}{2\pi(abcd;q)_\infty\sqrt{1-x^2}}\left|\frac{(e^{2i\theta})_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta})_\infty}\right|^2,$$

$q-$Pochhammer symbol: $(z;q)_\infty = (1-z)(1-qz)(1-q^2z)\dots$

Key asymptotic result

Proposition (Corwin–K '21):

Let $\mathcal{A}^+[\kappa, z] = -\frac{\pi^2}{6\kappa} - \left(z - \frac{1}{2}\right)\log\kappa - \log\left[\frac{\Gamma(z)}{\sqrt{2\pi}}\right]$ and

$$\mathcal{A}^-[\kappa, z] = \frac{\pi^2}{12\kappa} - \left(z - \frac{1}{2}\right)\log 2, \quad z \in \mathbb{C}, \kappa > 0.$$

Then for $q = e^{-\kappa}$ we have

$$\log(q^z; q)_\infty = \mathcal{A}^+[\kappa, z] + \mathrm{Error}^+[\kappa, z];$$

$$\log(-q^z; q)_\infty = \mathcal{A}^-[\kappa, z] + \mathrm{Error}^-[\kappa, z],$$

with good control over the error term in z and κ.

[Moak '84], [Olde Daalhuis '94], [Mcintosh '99], [Zhang '14]

Continuous dual Hahn Process

$$\text{Let } \mathcal{C}_{uv} := \begin{cases} 2 & \text{if } u \leqslant 0 \text{ or } u \geqslant 1, \\ 2u & \text{if } u \in (0,1). \end{cases}$$

For $u, v > 0$ and $s \in [0, \mathcal{C}_{uv})$ define a measure $\mathfrak{p}_s$ with density given by

$$\mathfrak{p}_s(r) := \frac{(v+u)(v+u+1)}{8\pi} \cdot \frac{\left|\Gamma\left(\frac{s}{2} + v + i\frac{\sqrt{r}}{2}\right) \cdot \Gamma\left(-\frac{s}{2} + u + i\frac{\sqrt{r}}{2}\right)\right|^2}{\sqrt{r} \cdot \left|\Gamma(i\sqrt{r})\right|^2} \mathbf{1}_{r>0}.$$

Continuous dual Hahn Process

For $a \in \mathbb{R}$ and $b = \bar{c} \in \mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Re}(b) = \mathrm{Re}(c) > 0$ let

$$\mathrm{CDH}(x; a, b, c) := \\ = \frac{1}{8\pi} \cdot \frac{\left|\Gamma\left(a + i\frac{\sqrt{x}}{2}\right) \cdot \Gamma\left(b + i\frac{\sqrt{x}}{2}\right) \cdot \Gamma\left(c + i\frac{\sqrt{x}}{2}\right)\right|^2}{\Gamma(a+b) \cdot \Gamma(a+c) \cdot \Gamma(b+c) \cdot \sqrt{x} \cdot \left|\Gamma\left(i\sqrt{x}\right)\right|^2} \mathbf{1}_{x>0}.$$

For $s, t \in [0, C_{u,v})$ with $s < t$ and $m, r \in (0, \infty)$ define

$$\mathfrak{p}_{s,t}(m, r) := \mathrm{CDH}\left(r; u - \frac{t}{2}, \frac{t-s}{2} + i\frac{\sqrt{m}}{2}, \frac{t-s}{2} - i\frac{\sqrt{m}}{2}\right).$$

Explicit characterization

$$\vec{X} = (X_0, \dots, X_{d+1}) \text{ where } 0 = X_0 < X_1 < \cdots < X_d \leqslant X_{d+1} = 1,$$
$$\vec{c} = (c_1, \dots, c_d) \text{ where } c_1, \dots, c_d > 0,$$
$$\vec{s} = (s_1 > \cdots > s_{d+1}) \text{ where } s_k = c_k + \cdots + c_d \text{ and } s_{d+1} = 0.$$

For any $d \in \mathbb{Z}_{\geqslant 1}$ provided that $s_1 < C_{u,v}$

$$\mathbb{E}\left[e^{-\sum_{k=1}^{d} c_k H_{u,v}(X_k)}\right] = \frac{\mathbb{E}\left[e^{\frac{1}{4}\sum_{k=1}^{d+1}(s_k^2 - \mathbb{T}_{s_k})(X_k - X_{k-1})}\right]}{\mathbb{E}\left[e^{-\frac{1}{4}\mathbb{T}_0}\right]},$$

where $\mathbb{T}_s$ is the continuous dual Hahn process.

Probabilistic description

$H_{u,v} : [0,1] \to \mathbb{R}$ is equal in law to $2^{-1/2}\mathbb{B} + \mathbb{X}$, where

$\mathbb{B} : [0,1] \to \mathbb{R}$ has the law of a standard Brownian motion;

$\mathbb{X} : [0,1] \to \mathbb{R}$ is independent of $\mathbb{B}$ and is absolutely continuous with respect to that of a Brownian motion with diffusion coefficient $1/2$ with an explicit Radon-Nikodym derivative.

[Bryc–Kuznetsov–Wang–Wesolowski '21],
[Barraquand–Le Doussal '21]