Isomonodromic defs, Painlevé eqs Integrable systems and Gauge Theory

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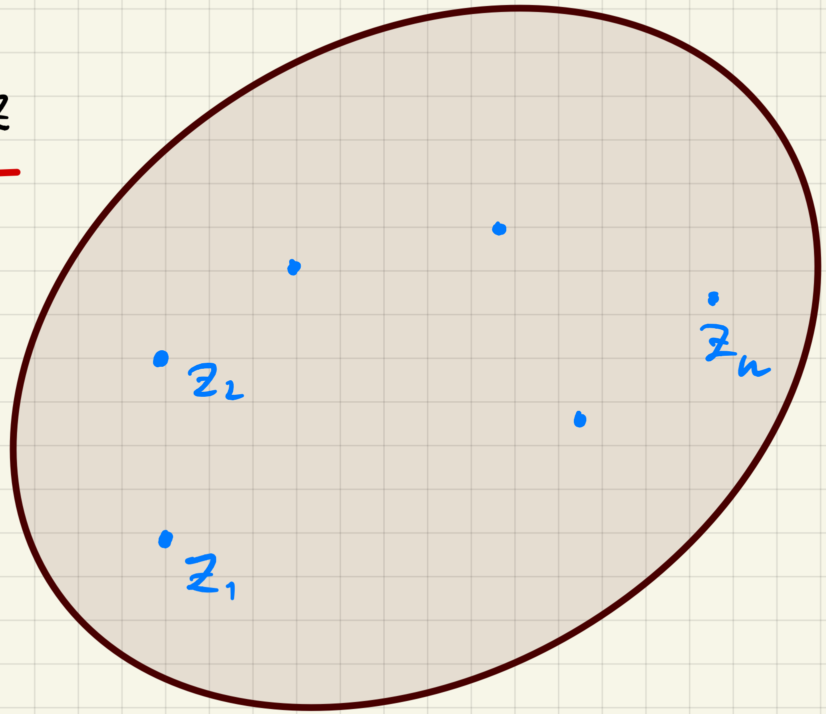
Isomonodromic deformations in genus zero

$$\nabla = \partial_z + A_z$$

$$A(z) = \sum_i \frac{A_i \, dz}{z - z_i}$$



Moving z_i without changing the gauge equivalence class of ∇

adjoint orbits

$$\mathcal{M} = \left(\mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_n \right) /\!/ G$$

$$= \left\{ (A_i) \;\middle|\; \sum_i A_i^{\in \mathcal{O}_i} = 0 \quad (A_i) \sim (h^{-1} A_i h) \right\}$$

Schlesinger flows on $\mathcal{P}$ generated by $H_i^{(2)} = \oint_{\text{around } z_i} \operatorname{tr} \mathbf{A}^2(z)$

obeying

$$\frac{\partial H_i^{(2)}}{\partial z_j} = \frac{\partial H_j^{(2)}}{\partial z_i} \qquad \left\{ H_i^{(2)}, H_j^{(2)} \right\} = 0$$

One can also study the "higher" flows, generated by

$$H_i^{(k,\ell)} = \oint_{\text{around } z_i} \operatorname{tr} \mathbf{A}^k(z) \, (z - z_i)^\ell$$

meaning is unclear

(Hitchin $_{g=0}$) Gaudin integrable system

$$\phi(z) = \sum_i \frac{\phi_i}{z - z_i}$$

fixed orbits

$\phi_i \in \mathcal{O}_i$

fixed

Given (z_i) $\quad P = (\mathcal{O}_1 \times \dots \times \mathcal{O}_n)/\!/G$

↪ algebraic integrable system

$$\boxed{R(\lambda, z) = \det(\phi(z) - \lambda) = 0}$$

Encodes Hamiltonians

For $G = SL(2)$ we can parametrize

$\mathcal{P}$ in several ways

$$\boxed{\omega = \sum_{a=1}^{n-3} d\ell_a \wedge d\theta_a}$$

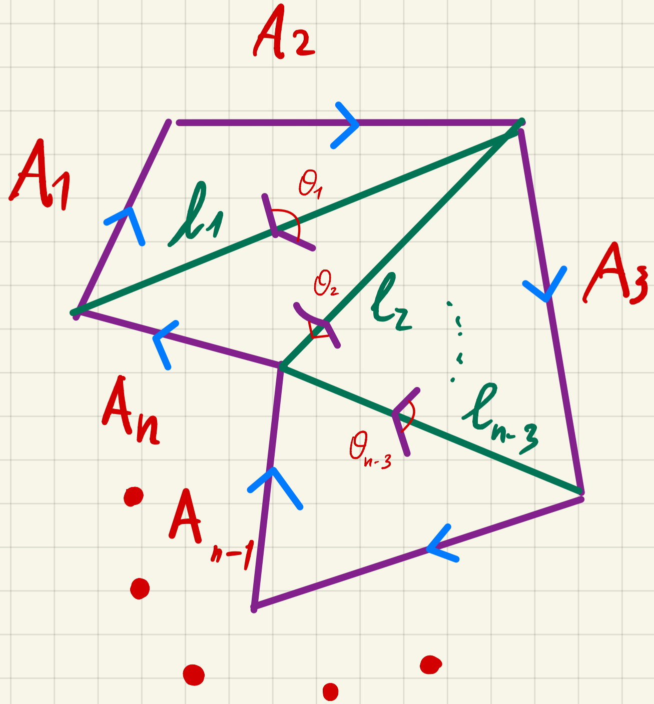
$$A_i \in \mathbb{C}^3$$

$$\parallel$$

$$\begin{pmatrix} u_i & v_i + \sqrt{-1}\, w_i \\ v_i - \sqrt{-1}\, w_i & -u_i \end{pmatrix}$$

$$u_i^2 + v_i^2 + w_i^2 = s_i^2$$

fixed $\in \mathbb{C}$



$\mathbb{C}$-Klyachko coordinates

rational relativistic integrable system

Separated variables (Sklyanin, Krichever)

"Tyurin parameters"

Choose a gauge $\quad A_n = \begin{pmatrix} S_n & 0 \\ 0 & -S_n \end{pmatrix} \quad z_n \to \infty$

("the other" $SL(2,\mathbb{C})$)

$$A(z) = \sum_i \frac{A_i}{z - z_i} = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}$$

Garnier - Painlevé Hamiltonians

$$b(z) = 6 \, \frac{\prod_{a=1}^{n-3} (z - w_a)}{\prod_{i=1}^{n-1} (z - z_i)} \qquad a(w_a) = p_a$$

$$\boxed{\omega = \sum_{a=1}^{n-3} dl_a \wedge d\theta_a = \sum_{a=1}^{n-3} dp_a \wedge dw_a}$$

These well-known stories are

two opposite quasiclassical limits

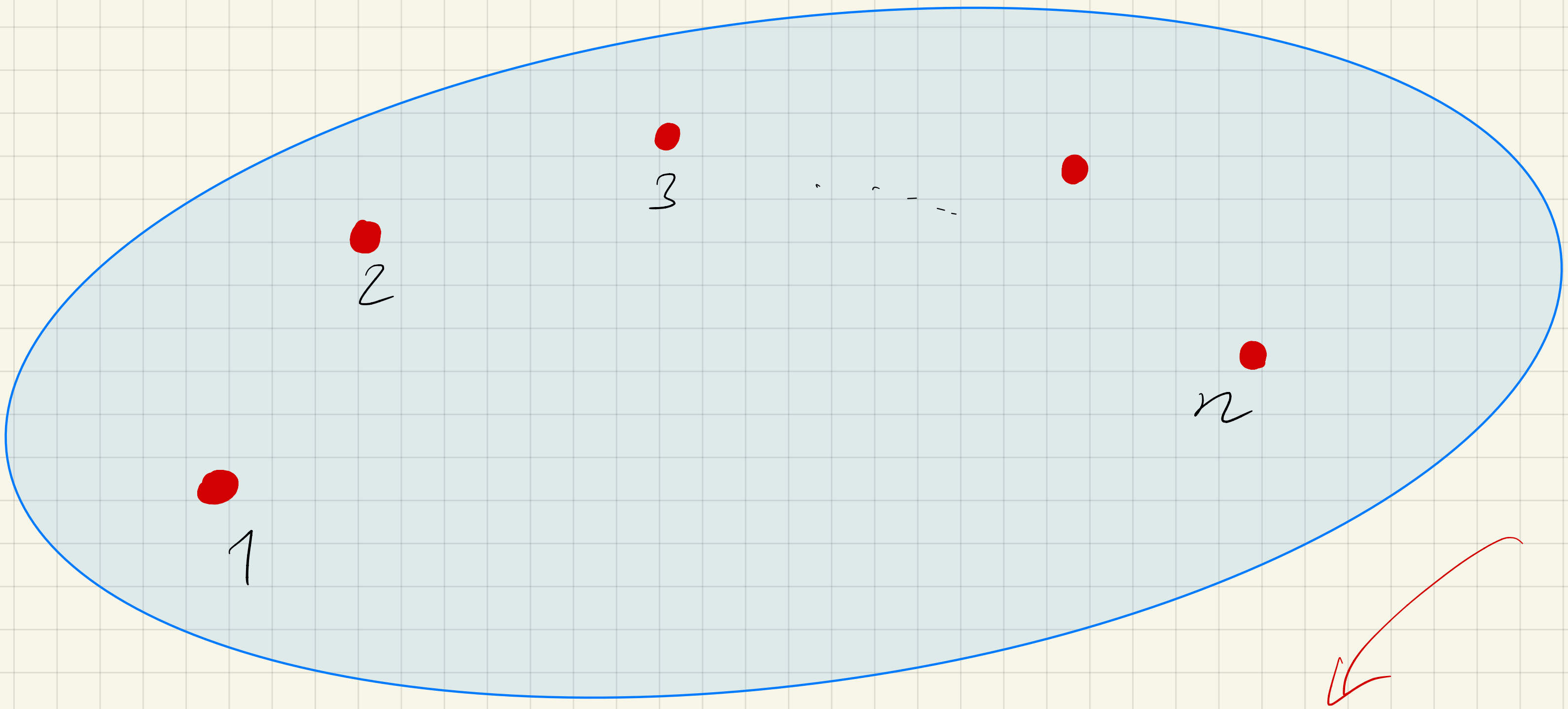
of KZ/BPZ system

$$(k+h^\vee)\frac{d}{dz_i}\Psi = \hat{H}_i\Psi$$

$$\hat{H}_i = \sum_{j\neq i}\frac{T_i^a\otimes T_j^a}{z_i-z_j}$$

In two dimensional CFT where

Ψ are conformal blocks



$k \in \mathbb{Z}_+$

integrable $\hat{g}_k$-reps

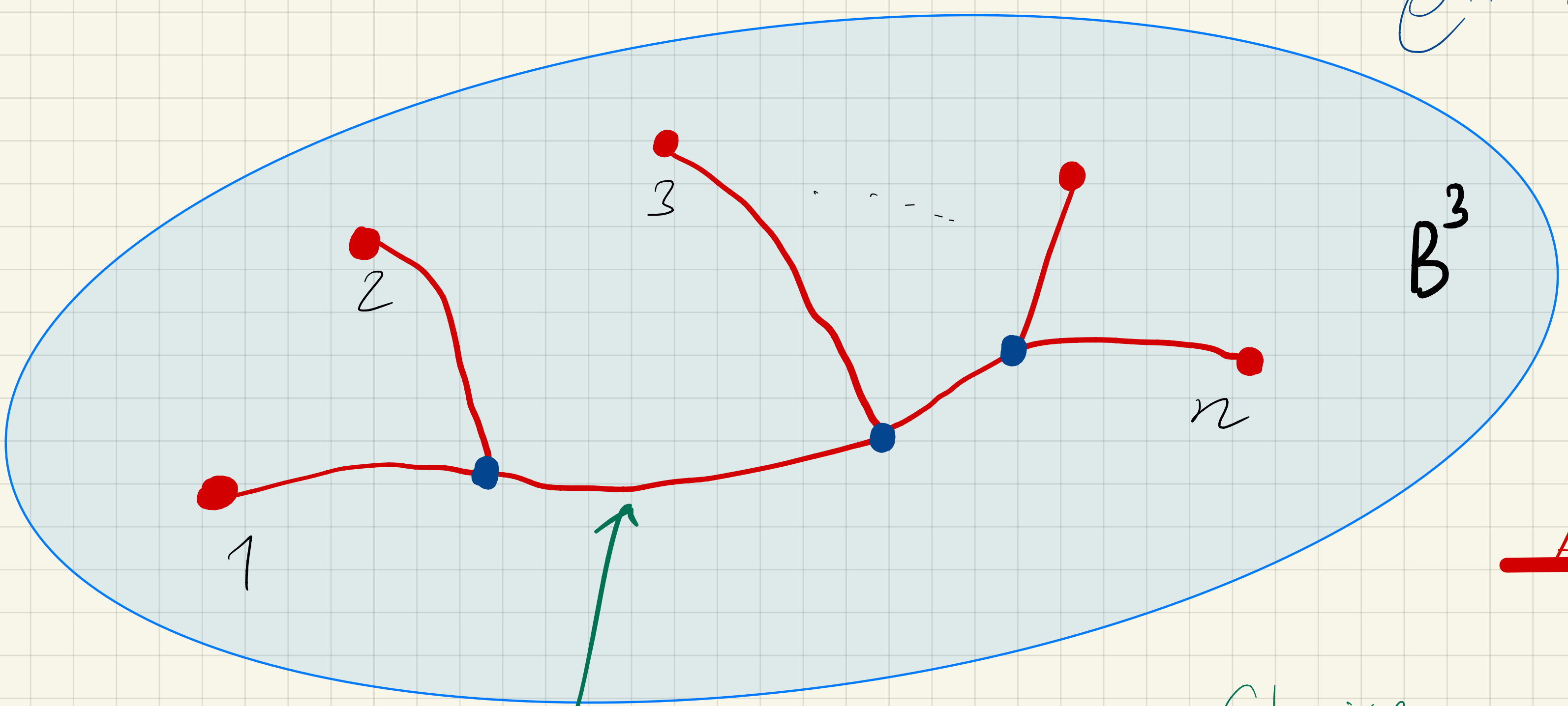
some g modules

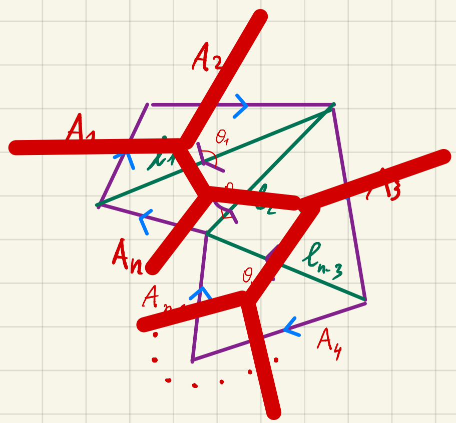
$$\Psi \in \left(V_1 \otimes V_2 \otimes \cdots \otimes V_n \right)^{g}$$

Related to 3d TFT (Chern-Simons)

$$S^2 = \partial B^3$$

$$e^{\frac{i}{4\pi}k\int \mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right)}$$





Wilson graph

Choice of diagonals

Integrable system arises in the non-unitary domain

$$k + h^{\vee} \to 0$$

Isomonodromy is ok $\quad k \to \infty$

KZ Reshetikhin '91
Harnad '93

Teschner

BPZ Lukyanov $\simeq$ '12
Litvinov
NN
Zamolodchikov

Most of interesting things happen when $k, s_i \in \mathbb{C}$

CS does not make sense

Wilson graph is not defined

Nevertheless, there is a theory which analytically continues conformal blocks of $\hat{\mathfrak{g}}$ to complex level, spins etc.

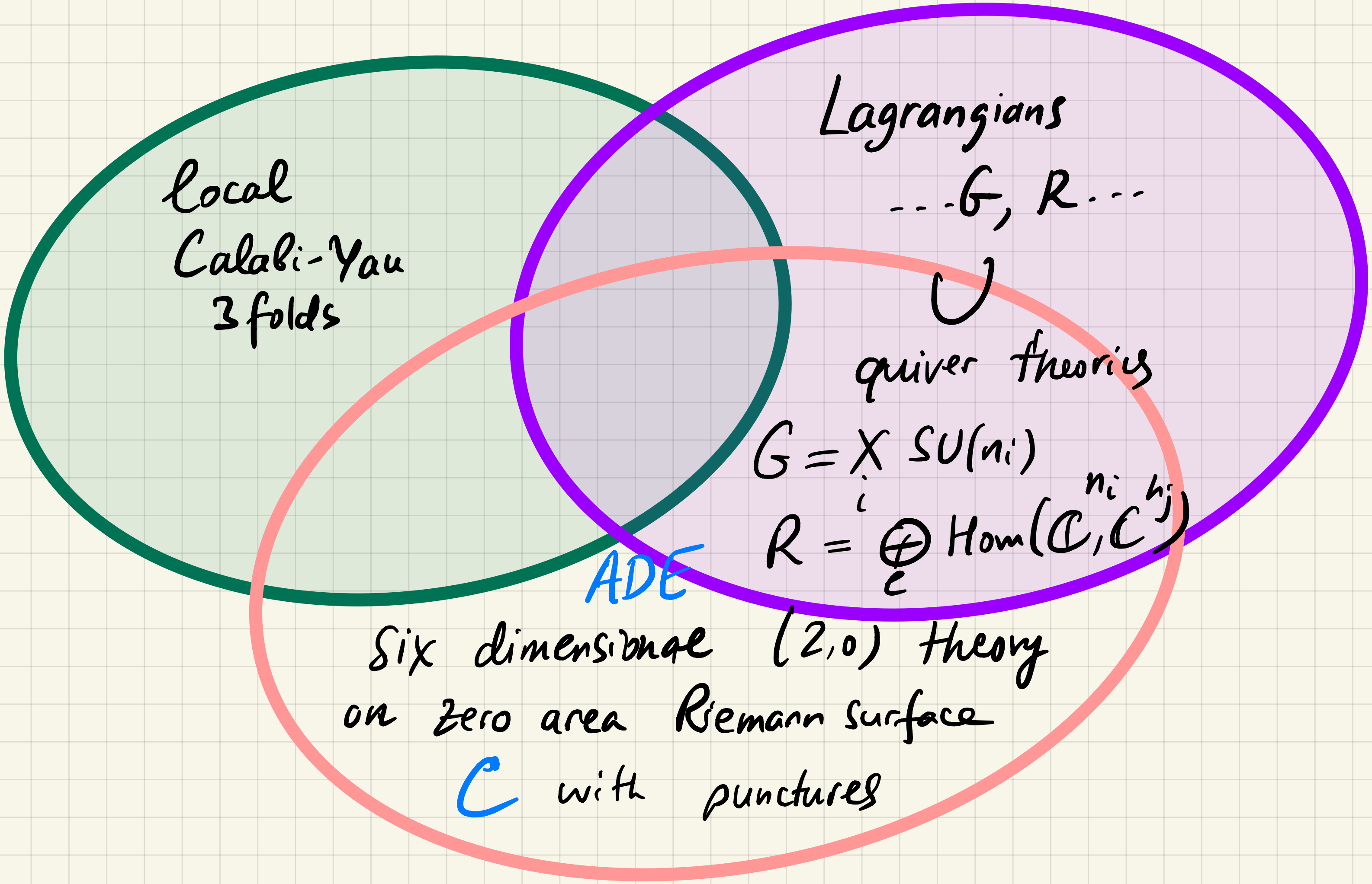
Allows to study Langlands duality

$$G \longleftrightarrow {}^L G$$

$$(k+h^\vee)(k^\vee+h) = 1$$

Four dimensional "super-Yang Mills"

(a class of $\mathcal{N}=2$ $d=4$ theories)



Computability

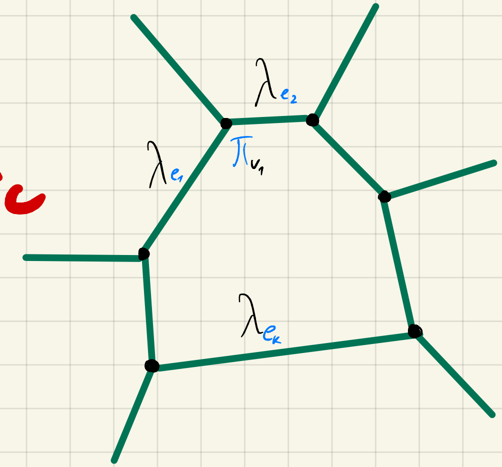
on CY_3 side

on quiver gauge theory side

for toric CY_3 X

DT/GW

count of holomorphic curves on X



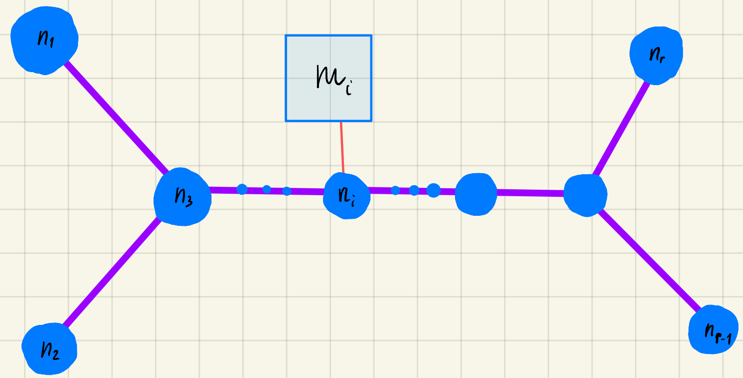
$$Z(t_e, \hbar, \varepsilon) = \sum e^{-t_e|\lambda_e|} \, e^{-\hbar|\pi_v|}$$

complicated rational function of ε

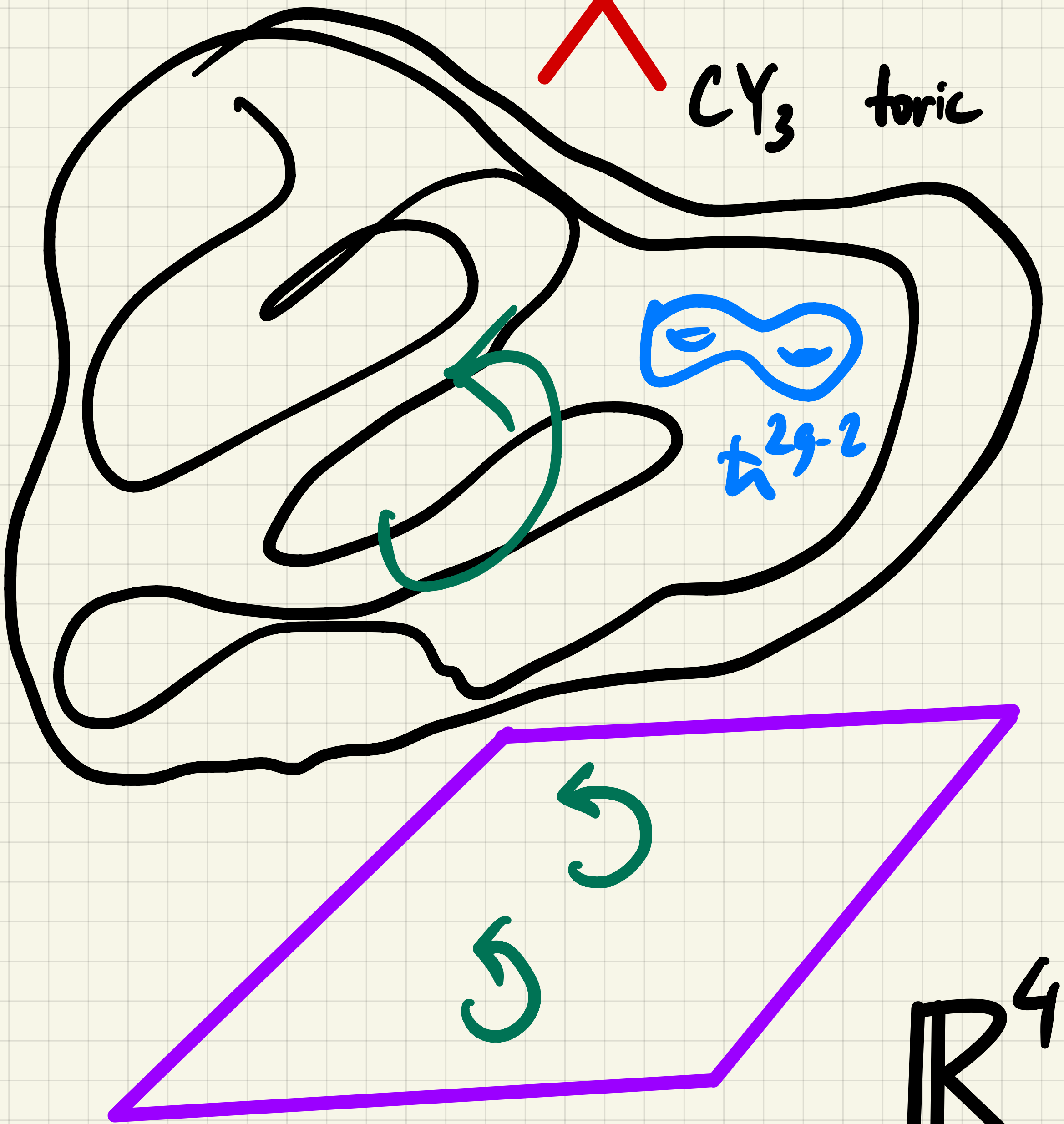
partitions λ_e	plane partitions π_v

$$Z(\vec{a}_i, \vec{m}_i, \tau_i, \varepsilon_1, \varepsilon_2) =$$

$$= \sum_{\vec{\lambda}_i = (\lambda_i^{(\alpha)})_{\alpha=1}^{n_i}} e^{2\pi i \tau_i (|\lambda_i^{(1)}| + \ldots + |\lambda_i^{(n_i)}|)} \times \prod_{\text{edges}} \text{rational function}(\vec{\lambda}_i, \vec{\lambda}_j) \times \prod_{\text{vertices}} \text{rational function}(\vec{\lambda}_i)$$



X CY_3 toric



ε parameters

$\to$ scales Ω_X

$\mathbb{C}^\times \hookrightarrow \mathbb{T}_X$ $\quad -\varepsilon_1-\varepsilon_2$

rotational symmetries

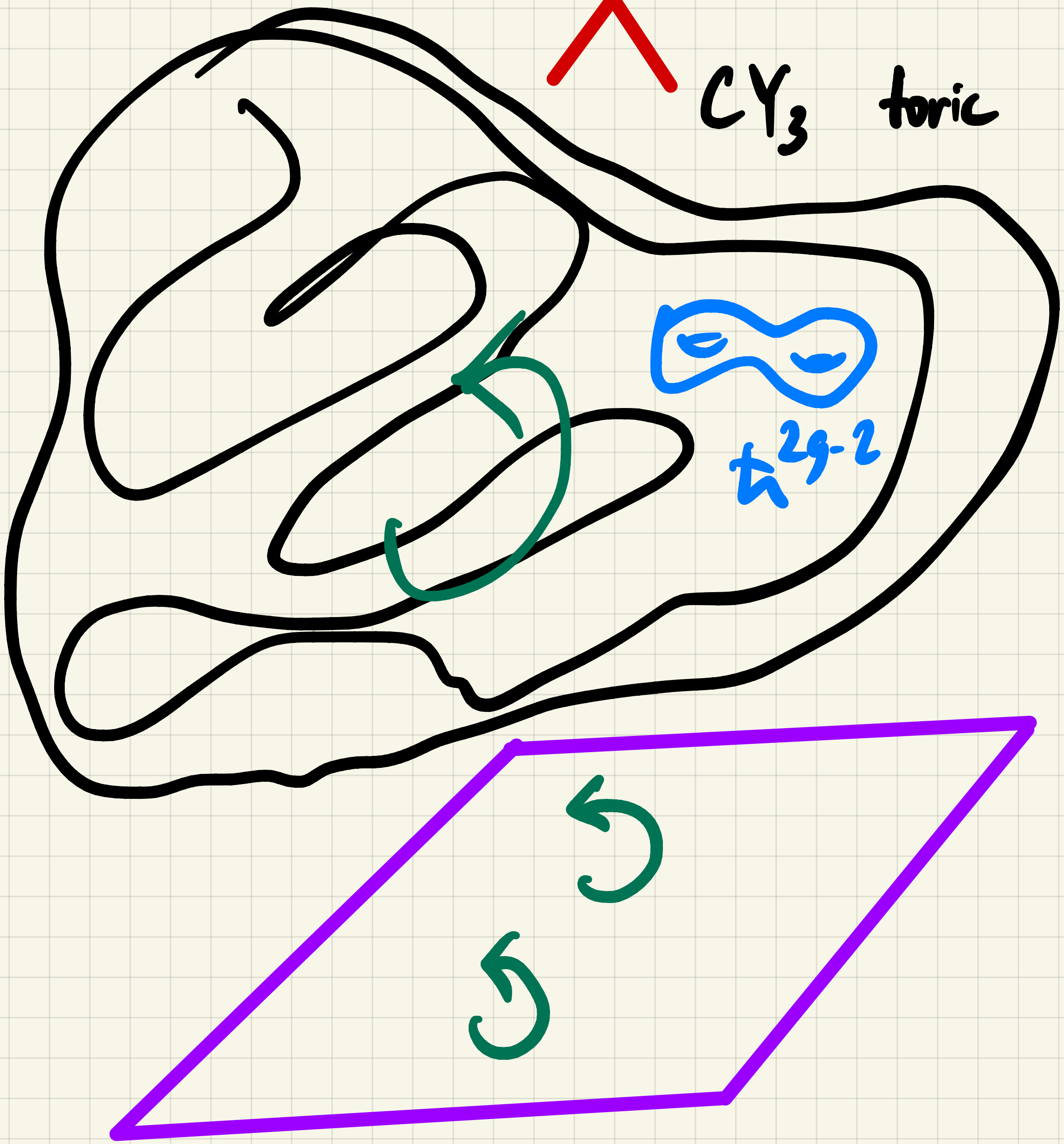
string coupling $\hbar^2 \sim \varepsilon_1\varepsilon_2$

$\mathbb{C}^\times \times \mathbb{C}^\times$

$\varepsilon_1 \quad \varepsilon_2$

$\mathbb{R}^4 \approx \mathbb{C}^2$

X CY_3 toric



ε parameters

$\to$ scales Ω_X

$\mathbb{C}^\times \hookrightarrow \mathbb{T}_X \hookleftarrow \mathbb{T}_X^{CY}$

rotational symmetries

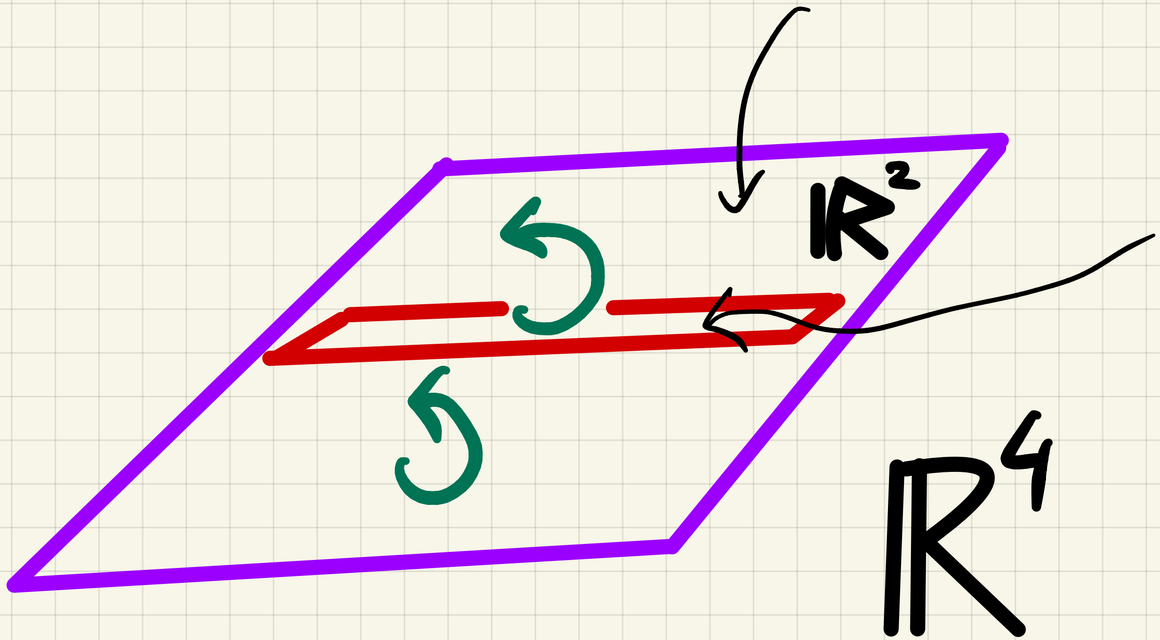
preserves Ω_X

string coupling

Back to Ψ

Surface defects

G gauge theory in the bulk



two dimensional sigma model on

$Y \curvearrowleft G$

Coupled to 4d gauge fields

Back to $\Psi(\vec{q}, \vec{a}, \vec{m}, \tau, \varepsilon_1, \varepsilon_2)$

Surface defect

Bulk gauge theory

parameters

$\vec{a}, \vec{m}, \tau, \varepsilon_1, \varepsilon_2$

gauge coupling G

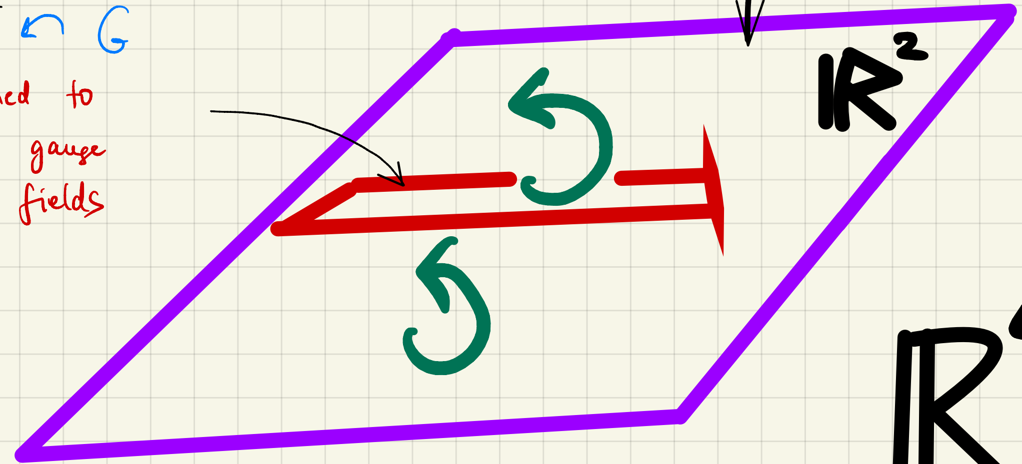
Choice of vacuum

masses R

two dimensional sigma model on

$Y \curvearrowleft G$

coupled to 4d gauge fields



$\mathbb{R}^4$

$\langle \mathcal{O}_k \rangle$

$\mathcal{L}$

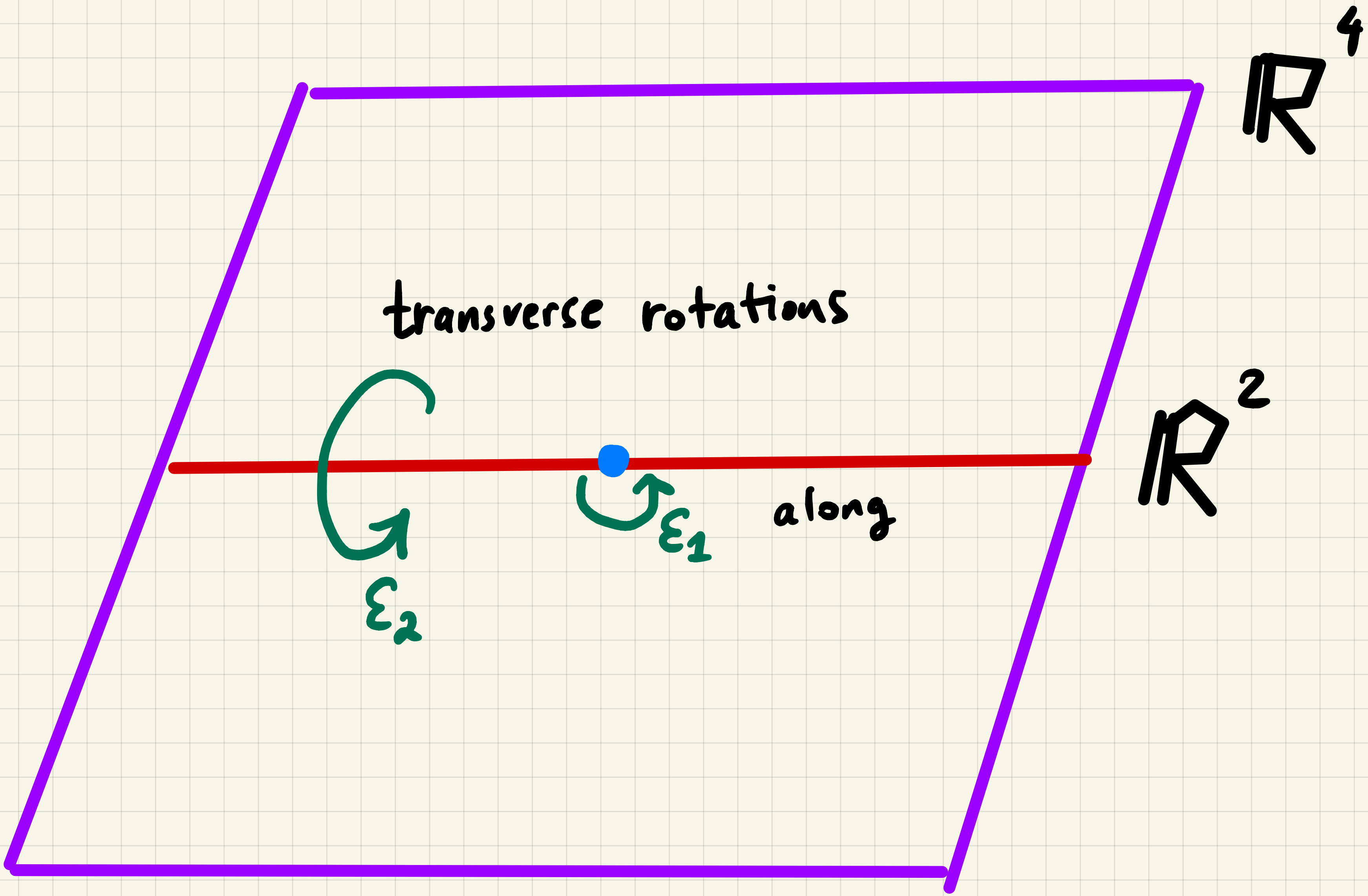
Lagrangian

local operators

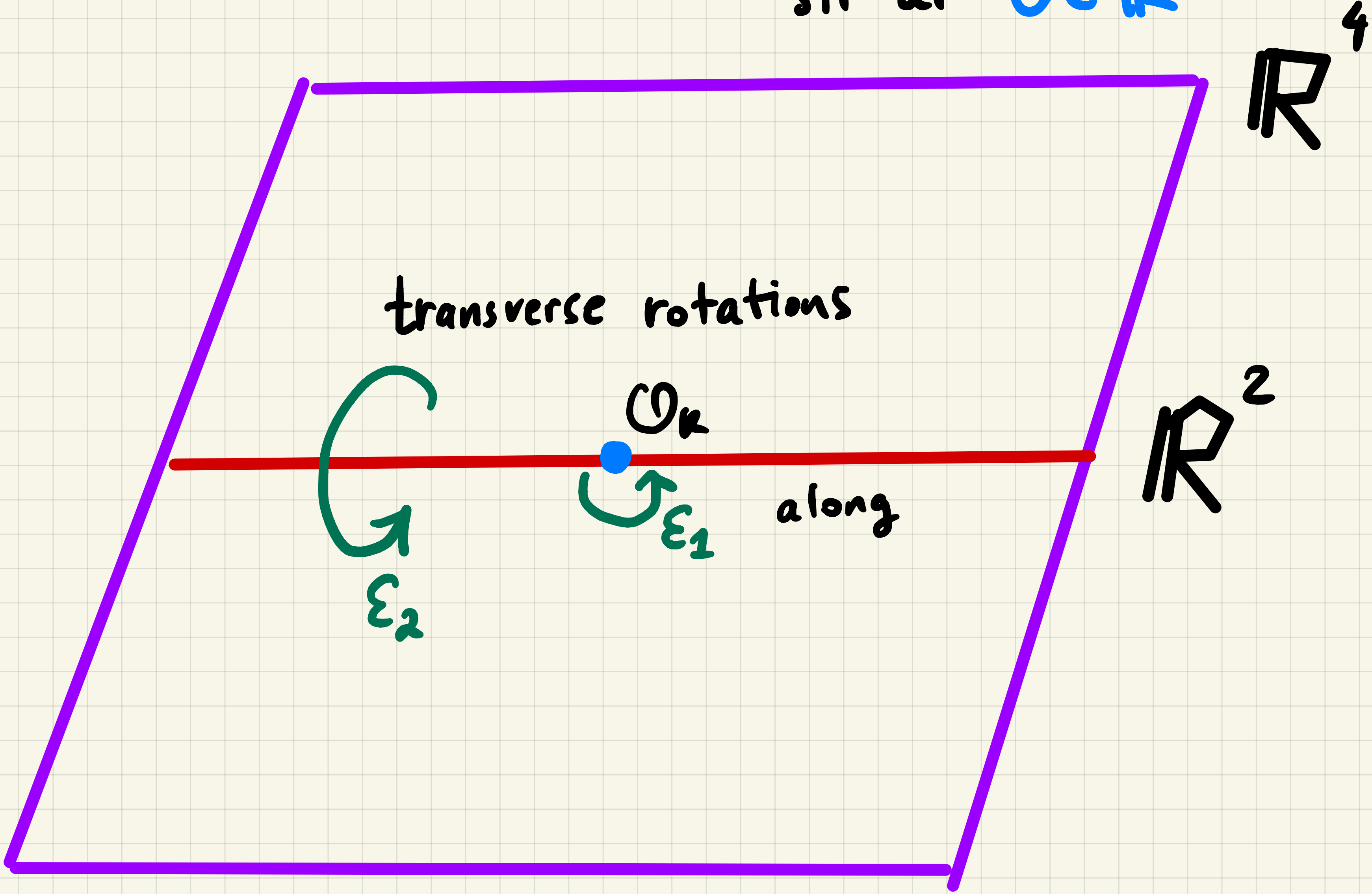
parameters of defect

$\vec{q}$ complexified θ-angles

Fully equivariant situation



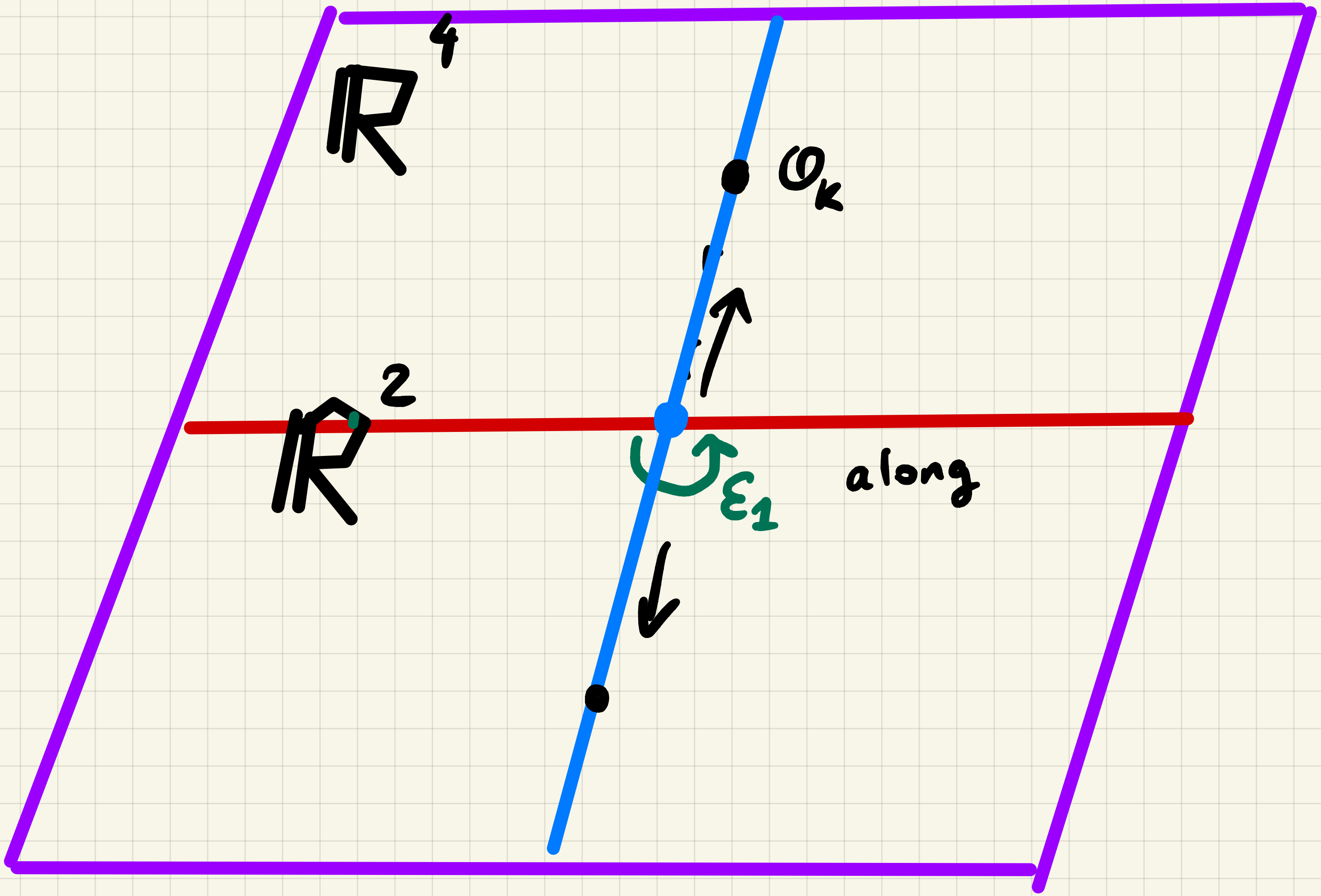
Fully equivariant situation all local operators sit at $0 \in \mathbb{R}^4$



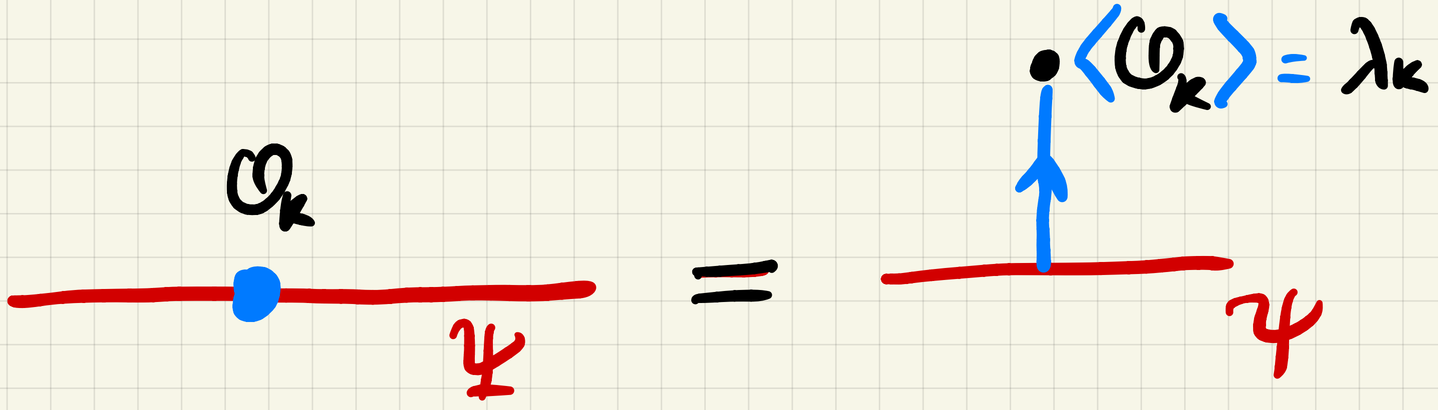
Partly equivariant situation

$\varepsilon_2 \to 0$ local operators can move in $\mathbb{R}^2$ $\Rightarrow$ commute

$\Rightarrow$ integrability



$$\mathcal{O}_k * \Psi = \lambda_k \Psi$$



Localization / compactification of appropriate moduli spaces

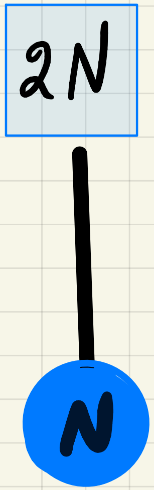
$$\Downarrow$$

Equation

$$\varepsilon_2 \varepsilon_1 \frac{d}{d\tau_i} \Psi = \hat{H}_i\left(\varepsilon_1 \frac{\partial}{\partial q}, q \,; \tau_i\right) \Psi$$

Best understood cases SU(N) theories

A_1

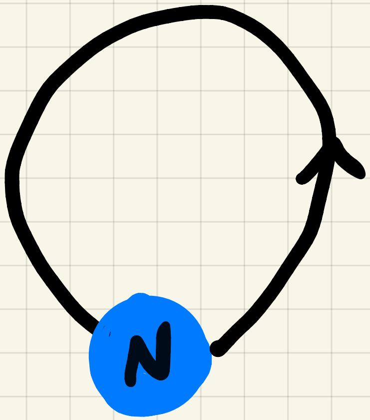


$m_1, \dots, m_{2N}$

$a_1, \dots, a_N$

τ

$\hat{A}_0$



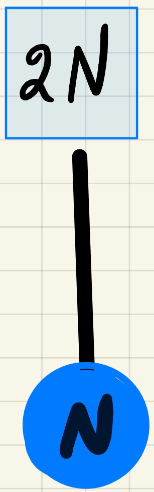
m

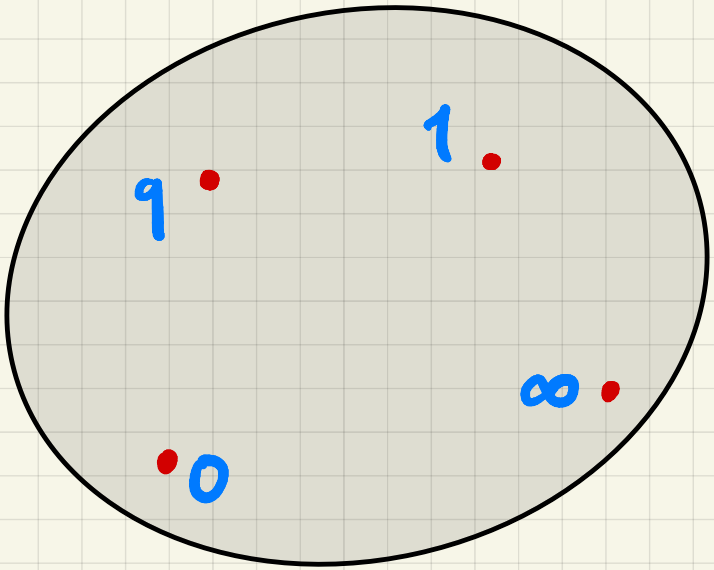
$a_1, \dots, a_N$

τ

Best understood cases $SU(N)$ theories

A_1





$$q = e^{2\pi i \tau}$$

$m_1, \dots, m_{2N}$

$a_1, \dots, a_N$

τ

$$\Psi \in \left(V^+_{(0)} \otimes V_{(q)} \otimes V_{(1)} \otimes V^-_{(\infty)} \right)^{\mathfrak{sl}_N}$$

$$\vec{\mu}_{(0)} = \left(\frac{m_1 - m^+}{\varepsilon_1}, \dots, \frac{m_N - m^+}{\varepsilon_1} \right)$$

lowest weight

$V^+_{(0)}$

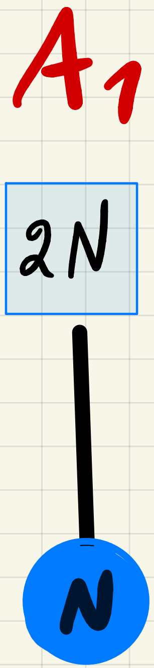
$$\vec{\mu}_{(\infty)} = \left(\frac{m_{N+1} - m^-}{\varepsilon_1}, \dots, \frac{m_{2N} - m^-}{\varepsilon_1} \right)$$

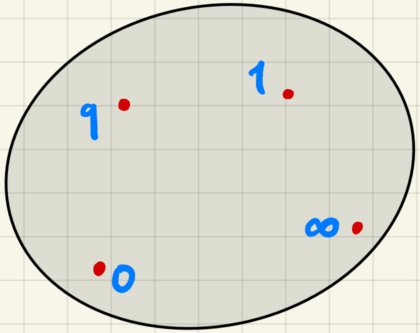
highest weight

$V^-_{(\infty)}$

Best understood cases

$SU(N)$ theories





$$q = e^{2\pi i \tau}$$

NN Tsymbaliuk

$$\Psi \in \left(V^+_{(0)} \otimes V_{(q)} \otimes V_{(1)} \otimes V^-_{(\infty)} \right)^{\mathfrak{sl}_N}$$

$$\vec{\mu}_{(0)} = \left(\frac{m_1 - m^+}{\varepsilon_1}, \dots, \frac{m_N - m^+}{\varepsilon_1} \right) \qquad \vec{\mu}_{(\infty)} = \left(\frac{m_{N+1} - m^-}{\varepsilon_1}, \dots, \frac{m_{2N} - m^-}{\varepsilon_1} \right)$$

lowest weight $V^+_{(0)}$ highest weight $V^-_{(\infty)}$

$m_1, \dots, m_{2N}$

$a_1, \dots, a_N$

τ

$$m^+ = \frac{m_1 + \dots + m_N}{N}$$

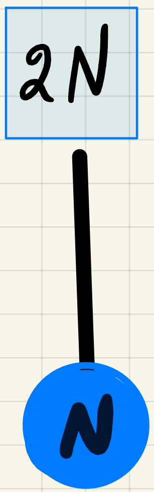
$$m^- = \frac{m_{N+1} + \dots + m_{2N}}{N}$$

$$V_{(q)} = \prod_{i=1}^N x_i^{\frac{m^+ + a_i}{\varepsilon_1}} \mathbb{C}[x_i, x_i^{-1}]^0$$

$$V_{(1)} = \prod_{i=1}^N \bar{x}_i^{-\frac{m^- - a_i}{\varepsilon_1}} \mathbb{C}[\bar{x}_i, \bar{x}_i^{-1}]^0$$

Best understood cases SU(N) theories

A_1



$m_1, \dots, m_{2N}$

$a_1, \dots, a_N$

τ

in the limit $\varepsilon_1 \to 0$

isomonodromic deformation of $\nabla = \partial + A$

$$A = \frac{A_{(0)}}{z} + \frac{A_{(q)}}{z-q} + \frac{A_{(1)}}{z-1}$$

$$\text{Eigenvalues}(A_{(0)}) = (m_1, \dots, m_N) - m^+ \cdot \mathbb{1}$$

$$\text{Eigenvalues}(A_{(q)}) = m^+ (1, \dots, 1, 1-N)$$

$$\text{Eigenvalues}(A_{(1)}) = m^- (1, \dots, 1, 1-N)$$

$$\text{Eigenvalues}(A_{(\infty)}) = (m_{N+1}, \dots, m_{2N}) - m^- \cdot \mathbb{1}$$

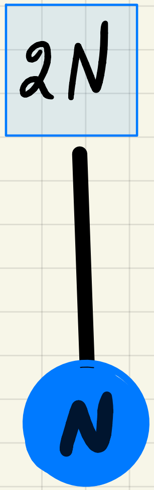
$$A_{(0)} + A_{(q)} + A_{(1)} + A_{(\infty)} = 0$$

Best understood cases $SU(N)$ theories

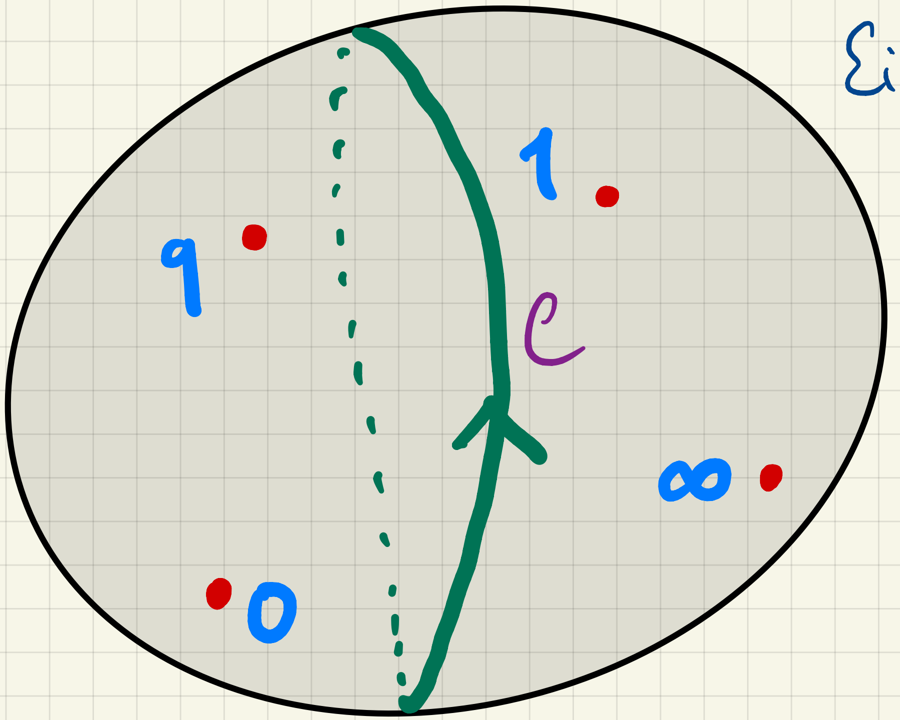
in the limit $\varepsilon_1 \to 0$

isomonodromic deformation of $\nabla = \partial + A$

A_1



$$A = \frac{A_{(0)}}{z} + \frac{A_{(q)}}{z-q} + \frac{A_{(1)}}{z-1}$$



$m_1, \dots, m_{2N}$

$a_1, \dots, a_N$

τ

$$\mathrm{Eigen}\left(P\exp \oint_C A \right) =$$

$$= \left(\exp 2\pi i \frac{a_k}{\varepsilon_2} \right)_{k=1}^{N}$$

The horizontal section

$$\tilde{\Psi}_z \sim e^{\frac{S}{\varepsilon_1}} \chi$$

$$\nabla \chi = 0$$

is a quasiclassical limit of the

5 - point conformal block

$$\tilde{\Psi}_z \in \left(V^+_{(0)} \otimes V_{(q)} \otimes V_{(1)} \otimes V^-_{(\infty)} \otimes \mathbb{C}^N_z \right)^{sl_N}$$

Obeying KZ eqs

$\nabla_z \tilde{\Psi}_z = 0 \to$ "horizontal"

$\nabla_q \tilde{\Psi}_z = 0 \to$ "isomonodromy"

5-point conformal block

$$\widetilde{\Psi}_z \in \left(V^+_{(0)} \otimes V_{(q)} \otimes V_{(1)} \otimes V^-_{(\infty)} \otimes \mathbb{C}^N_z\right)^{\mathfrak{sl}_N}$$

obeying KZ eqs

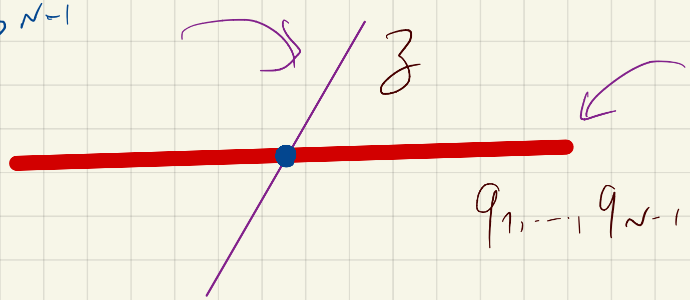
$\nabla_z \widetilde{\Psi}_z = 0 \to$ "horizontal"

$\nabla_q \widetilde{\Psi}_z = 0 \to$ "isomonodromy"

$\parallel$

expectation value of intersecting surface defects

$Y \simeq$ vector bundle $\downarrow \mathbb{CP}^{N-1}$

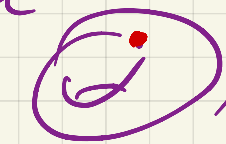


maximal orbit

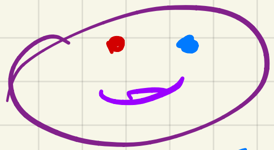
$Y =$ vector bundle over complete flag variety

NN, S. Jeong, N. Lee

$\hat{A}_0$ theory (aka $\mathcal{N}=2^*$ aka softly broken $\mathcal{N}=4$)

Ψ solves (1-point conformal block on )

$$\varepsilon_1 \varepsilon_2 \frac{d}{d\tau} \Psi = \left(\frac{1}{2} \sum_{i=1}^{N} \left(\varepsilon_1 \frac{\partial}{\partial q_i} \right)^2 + m(m+\varepsilon_1) \sum_{i<j} \wp(q_i - q_j ; \tau) \right) \Psi$$

At the moment the gauge theory construction of the 2-point block (horizontal section)  is UNKNOWN

isomonodromy problem is known, of course (Krichever)

Some uses of gauge theory

Separation of variables

Formulas for Ψ

Quantization conditions

New types of vertex operators

Hecke operators of (analytic) geometric Langlands and their $k \neq -N$ generalizations

Some uses of gauge theory

Separation of variables
Formulas for Ψ
Quantization conditions
New types of vertex operators Hecke

Blowup formulas in 4d $\Rightarrow$
relations between conformal blocks of
current algebras and W-algebras
$k \to (k_1, k_2) \to$ Kyiv formula (GIL)

Some uses of gauge theory

Interesting questions

- 5-categorification (what replaces CS?)
- $4d \to 5d$ (q- analogues)

Liouville $\longrightarrow$ $\Delta u = \sinh u$??

Bobenko's talk

$4d \to 6d$ (elliptic)

??

Some uses of gauge theory

Interesting questions

- 5-categorification (what replaces CS?)
- Knots, defects, relation to Smooth structures, volume conjecture?
- What replaces Isomonodromy for higher rank ?

(add higher times in Z)

$\sim t_k \mathrm{Tr}\,\phi^{k+2}$

Some uses of gauge theory

Interesting questions

- What replaces Isomonodromy for higher rank?
  (add higher times in Z)
  $\sim t_k \operatorname{Tr} \phi^{k+2}$

for general quiver theories?

$\text{Hitchin}(G, C) \longrightarrow$ moduli spaces of ADE instantons / monopoles on
$\cup$
opers $\rightsquigarrow$?? $\subset$ $\mathbb{R}^2 \times \mathbb{T}^2$ $\qquad$ $\mathbb{R}^2 \times S^1$

Some uses of gauge theory

Interesting questions

- for general quiver theories?

$\mathrm{Hitchin}(G,C) \longrightarrow$ moduli spaces of ADE instantons/monopoles on

$\cup$

opers $\rightsquigarrow$?? $\subset$ $\mathbb{R}^2 \times \mathbb{T}^2$ $\quad$ $\mathbb{R}^2 \times S^1$

- for CY_3?

dimers and integrable systems (Goncharov-Kenyon, Fock, Marshakov, Bershtein, Semenyakin, ...)

Some uses of gauge theory/string theory

for toric $X = CY_3$

$$\downarrow$$

$\left(\begin{array}{c}\text{moment map} \\ \text{for } \mathbb{T}_{CY}\end{array}\right) \quad \mathbb{R}^2 \times \mathbb{C}^1 \quad \left(\begin{array}{c}\text{canonical holomorphic} \\ \text{function}\end{array}\right)$

T-duality along $\mathbb{T}_{CY}$ $\quad X \longrightarrow T^2 \times$ hyperkähler S

$+$ NS5 wrapping a curve C

Phase space $= \{$ branes wrapping $C \subset S \}$

Thank you!