Eigenfunctions for the elliptic Ruijsenaars difference operators

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Summary

On the basis of a collaboration with Edwin Langmann (KTH) and Junichi Shiraishi (Tokyo), I report recent progresses in understanding the joint eigenfunctions for the commuting family of elliptic Ruijsenaars difference operators. After reviewing some basic known facts regarding the Macdonald-Ruijsenaars operators in the trigonometric case, I propose two classes of joint eigenfunctions for the elliptic Ruijsenaars operators: (A) symmetric eigenfunctions around the torus that deform Macdonald polynomials, and (B) asymptotically free eigenfunctions in a certain asymptotic domain.

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Contents

Introduction

Fixing two bases $p, q \in \mathbb{C}^*$ $(|p| < 1, |q| < 1)$ and a parameter $t \in \mathbb{C}^*$, we consider the *elliptic Ruijsenaars q-difference operators* of type A, in n variables $x = (x_1, \ldots, x_n)$:

$$\mathcal{D}_x^{(r)}(p) = \sum_{I\subseteq\{1,\ldots,n\};\,|I|=r} t^{\binom{r}{2}} \prod_{i\in I,\, j\notin I} \frac{\theta(tx_i/x_j;p)}{\theta(x_i/x_j;p)} \prod_{i\in I} T_{q,x_i} \qquad (r = 0, 1, \ldots, n),$$

where

$$\theta(z;p) = (z;p)_\infty(p/z;p)_\infty, \quad (z;p)_\infty = \prod_{i=0}^{\infty}(1 - p^i z) \quad (z \in \mathbb{C}),$$

and for each $i = 1, \ldots, n$, T_{q,x_i} stands for the q-shift operator in the variables x_i: $T_{q,x_i} f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$.

By the pioneering work of Ruijsenaars (1987), it is known that these q-difference operators $\mathcal{D}_x^{(r)}(p)$ $(r = 1, \ldots, n)$ commute with each other. It is a fundamental problem to describe the joint eigenfunctions for this commuting family of q-difference operators:

$$\mathcal{D}_x^{(r)}(p)\psi(x;p) = \varepsilon^{(r)}(p)\psi(x;p) \qquad (r = 1, \ldots, n).$$

In this talk, we propose two classes of joint eigenfunctions which are represented by convergent power series in p.

(A) Symmetric eigenfunctions around the torus that deform Macdonald polynomials.

(B) Asymptotically free eigenfunctions in the domain $|x_1| \gg \cdots \gg |x_n| \gg |px_1|$.

We first review the trigonometric case, and then proceed to the elliptic case.

Part I

Macdonald-Ruijsenaars operators and their eigenfunctions

1. Macdonald-Ruijsenaars operators: trigonometric limit

In the limit as $p \to 0$, the elliptic Ruijsenaars operators $\mathcal{D}_x^{(r)}(p)$ reduce to the following q-difference operators with rational coefficients:

$$D_x^{(r)} = \sum_{I\subseteq\{1,\ldots,n\};\,|I|=r} t^{\binom{r}{2}} \prod_{i\in I,\, j\notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} \quad (r = 0, 1, \ldots, n).$$

These operators are expressed as follows in terms of the difference product $\Delta(x)$:

$$D_x^{(r)} = \sum_{I\subseteq\{1,\ldots,n\};\,|I|=r} \frac{T_{t,x}^{\epsilon_I}\Delta(x)}{\Delta(x)} T_{q,x}^{\epsilon_I} \qquad (r = 0, 1, \ldots, n), \quad \epsilon_I = \sum_{i\in I} \epsilon_i.$$

$$\Delta(x) = \prod_{1\le i<j\le n} (x_i - x_j) = \prod_{1\le i<j\le n} x_i(1 - x_j/x_i) = x^\rho \prod_{\alpha\in\Delta_+} (1 - x^{-\alpha}).$$

Root system of $\mathfrak{gl}_n$

$P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$: weight lattice   scalar product $\langle\ ,\ \rangle$: $P \times P \to \mathbb{Z}$; $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$.

$\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(i = 1, \ldots, n-1)$: simple roots $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i \subset P$: root lattice

$\Delta_+ = \left\{ \epsilon_i - \epsilon_j \;\middle|\; 1 \le i < j \le n \right\} \subset Q$: set of positive roots $\rho = \sum_{i=1}^n (n-i)\epsilon_i$

Notation of monomials

Identifying the weights $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i \in P$ with the multi-indices $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, we set $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$: plane wave / formal exponential $e(\langle\lambda, u\rangle) = e^{2\pi\sqrt{-1}\langle\lambda,u\rangle}$.

2. Macdonald polynomials (symmetric Laurent polynomials)

In the theory of Macdonald polynomials, it is known that, if the parameter $t \in \mathbb{C}^*$ is generic, then the commuting family of q-difference operators $D_x^{(r)}$ $(r = 1, \ldots, n)$ are simultaneously diagonalized on the ring $\mathbb{C}[x^{\pm}]^{\mathfrak{S}_n}$ of symmetric Laurent polynomials.

P_+: cone of dominant integral weights: $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in P$ is *dominant* if

$$\langle \alpha_i, \lambda \rangle \geq 0 \ (i = 1, \ldots, n-1) \quad \Longleftrightarrow \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$\mu \leq \lambda$: dominance order on P

$$\lambda - \mu \in Q_+ \quad \Longleftrightarrow \quad |\mu| = |\lambda|, \quad \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \ \ (i = 1, \ldots, n-1),$$

where $Q_+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$.

Theorem A: Suppose that $t \in \mathbb{C}^*$ satisfies the condition $t^k \notin q^{\mathbb{Z}_{<0}}$ $(k = 1, \ldots, n-1)$. Then, for each dominant integral weight $\lambda \in P_+$, there exists a unique symmetric Laurent polynomial $P_\lambda(x) = P_\lambda(x|q,t) \in \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}$ such that

(0) $P_\lambda(x) = x^\lambda +$ (lower oder terms with respect to $\leq$),

(1) For each $r = 1, \ldots, n$, $D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x) \quad (d_\lambda^{(r)} \in \mathbb{C})$.

We assume below that t is generic in the sense mentioned above. The eigenvalue $d_\lambda^{(r)}$ is the rth elementary symmetric function $e_r(t^\rho q^\lambda)$ of $t^\rho q^\lambda = (t^{n-1} q^{\lambda_1}, \ldots, t q^{\lambda_{n-1}}, q^{\lambda_n})$. The *Macdonald polynomials* $P_\lambda(x)$ $(\lambda \in P_+)$ form a $\mathbb{C}$-basis of $\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}$:

$$\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in P_+} \mathbb{C} P_\lambda(x)$$

Macdonald polynomials have various remarkable properties; we review some of them.

(a) Orthogonality relation for $P_\lambda(x)$:

We define the weight function $w_{\mathrm{Mac}}(x)$ by

$$w_{\mathrm{Mac}}(x) = \prod_{1\le i<j\le n} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \frac{(x_j/x_i;q)_\infty}{(tx_j/x_i;q)_\infty}, \qquad (z;q)_\infty = \prod_{i=0}^{\infty}(1-q^i z) \quad (|q|<1).$$

Suppose that $|t|<1$. For $\lambda,\mu\in P_+$,

$$\int_{\mathbb{T}^n} P_\lambda(x^{-1})\, P_\mu(x)\, w_{\mathrm{Mac}}(x)\, d\omega_n(x) = \delta_{\lambda,\mu} N_\lambda,$$

where

$$\mathbb{T}^n = \left\{\, x=(x_1,\ldots,x_n)\in(\mathbb{C}^*)^n \;\middle|\; |x_1|=\cdots=|x_n|=1 \,\right\},$$

$$d\omega_n(x) = \frac{1}{(2\pi\sqrt{-1})^n}\frac{dx_1\cdots dx_n}{x_1\cdots x_n}, \qquad \int_{\mathbb{T}} d\omega_n(x) = 1,$$

$$N_\lambda = n! \prod_{1\le i<j\le n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty}$$

The Macdonald-Ruijsenaars operators $D_x^{(r)}$ $(r=1,\ldots,n)$ are (formally) *self-adjoint* with respect to the weight function $w_{\mathrm{Mac}}(x)$.

(b) Combinatorial formula: When the weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n = (\lambda_1, \ldots, \lambda_n)$ is a partition, i.e. $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, the Macdonald polynomial $P_\lambda(x) = P_\lambda(x|q,t)$ is expressed as a sum over all non-decreasing sequences of partitions connection ϕ and λ by n steps:

$$P_\lambda(x) = \sum_{\phi=\mu^{(0)}\subseteq\mu^{(1)}\subseteq\cdots\subseteq\mu^{(n)}=\lambda} \prod_{k=1}^{n} \psi_{\mu^{(k)}/\mu^{(k-1)}}\ x_k^{|\mu^{(k)}/\mu^{(k-1)}|}$$

where $\psi_{\lambda/\mu} = \psi_{\lambda/\mu}(q,t)$ ($\mu \subseteq \lambda$, i.e. $\mu_i \leq \lambda_i$) stands for the *Pieri coefficients* defined by

$$\psi_{\lambda/\mu} = \prod_{1\leq i<j\leq n} \frac{(q^{\mu_i-\lambda_j+1}t^{j-i-1};q)_{\lambda_i-\mu_i}}{(q^{\mu_i-\lambda_j}t^{j-i};q)_{\lambda_i-\mu_i}} \prod_{1\leq i\leq j<n} \frac{(q^{\mu_i-\mu_j}t^{j-i+1};q)_{\lambda_i-\mu_i}}{(q^{\mu_i-\mu_j+1}t^{j-i};q)_{\lambda_i-\mu_i}}$$

$$(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a) \qquad (k=0,1,2,\ldots),$$

and $\psi_{\lambda/\mu} = 0$ unless λ/μ is a *horizontal strip* (i.e. $\lambda_{i+1} \leq \mu_i \leq \lambda_i$).
$\Longleftrightarrow$ *Tableau representation* (sum over all semi-standard tableaux of shape λ)

(c) Symmetry: Normalize $P_\lambda(x)$ setting $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(t^\rho)$ so that $\widetilde{P}_\lambda(t^\rho) = 1$. Then the values $\widetilde{P}_\lambda(t^\rho q^\mu)$ ($\mu \in P_+$) become symmetric with respecto λ and μ:

$$\widetilde{P}_\lambda(x) = \frac{P_\lambda(x)}{P_\lambda(t^\rho)} \quad\Longrightarrow\quad \widetilde{P}_\lambda(t^\rho q^\mu) = \widetilde{P}_\mu(t^\rho q^\lambda) \qquad (\lambda,\mu \in P_+).$$

3. Asymptotically free eigenfunctions (Macdonald function)

For a generic complex weight $\lambda \in P_{\mathbb{C}} = P \otimes_{\mathbb{Z}} \mathbb{C}$, there exists a unique *asymptotically free eigenfunction* $\psi_\lambda(x)$ with leading term x^λ in the domain $|x_1| \gg |x_2| \gg \cdots \gg |x_n|$.

(0) $\psi_\lambda(x) = x^\lambda + (\text{lower terms w.r.t. } \leq\ ) \in x^\lambda \mathbb{C}\{x_2/x_1, \ldots, x_n/x_{n-1}\}$,

(1) $D_x^{(r)} \psi_\lambda(x) = e_r(t^\rho q^\lambda) \psi_\lambda(x) \quad (r = 1, \ldots, n)$

(Cherednik 2009, van Meer-Stokman 2010, Noumi-Shiraishi 2012).

We introduce the variables $s = (s_1, \ldots, s_n)$ dual to $x = (x_1, \ldots, x_n)$, and relate them to the complex weights $\lambda \in P_{\mathbb{C}}$ through $s = t^\rho q^\lambda$, i. e. $s_i = t^{n-i} q^{\lambda_i}$ $(i = 1, \ldots, n)$. Also, we define the *modified Macdonald-Ruijsenaars operators* by

$$E_{x,s}^{(r)} = \sum_{I \subseteq \{1,\ldots,n\}\ |I|=r} B_I(x) s^{\epsilon_I} T_{q,x}^{\epsilon_I}, \quad B_I(x) = \prod_{\alpha \in \Delta_+} \frac{1 - t^{-\langle \epsilon_I, \alpha \rangle} x^{-\alpha}}{1 - x^{-\alpha}},$$

so that $E_{x,s}^{(r)} = x^{-\lambda} D_x^{(r)} x^\lambda$ under the identification $s = t^\rho q^\lambda$.

Theorem B: Suppose that $s \in (\mathbb{C}^*)^n$ satisfies the condition $s_j/s_i \notin q^{\mathbb{Z}}$ $(1 \leq i < j \leq n)$. Then, there exists a unique convergent power series $f(x; s)$ in $(x_2/x_1, \ldots, x_n/x_{n-1})$, with coefficients depending rationally on s, such that

(0) $f(x; s) = 1 + \sum_{\beta \in Q_+;\ \beta > 0} f_\beta(s) x^{-\beta} \in \mathbb{C}\{x_2/x_1, \ldots, x_n/x_{n-1}\}$,

(1) $E_{x,s}^{(r)} f(x; s) = e_r(s)\, f(x; s) \quad (r = 1, \ldots, n)$.

$f(x; s) = f(x; s|q, t)$: *Macdonald function* in the asymptotic domain $|x_1| \gg \cdots \gg |x_n|$;
$\psi_\lambda(x)$ is expressed as $\psi_\lambda(x) = x^\lambda f(x; t^\rho q^\lambda)$ in terms of the Macdonald function.

The Macdonald function $f(x;s)=f(x;s|q,t)$ $(|x_1|\gg|x_2|\gg\cdots\gg|x_n|)$ have various remarkable properties.

(a) Combinatorial formula: Let M_n be the set of all strictly upper triangular matrices $\theta=(\theta_{ij})_{1\le i<j\le n}$ with entries in $\mathbb{Z}_{\ge 0}$.

$$f(x;s)=\sum_{\theta\in M_n}c(\theta;s)\prod_{1\le i<j\le n}(x_j/x_i)^{\theta_{ij}},$$

$$c(\theta;s)=\prod_{1\le i<j\le k\le n}\frac{(q^{\theta_{i,>k}-\theta_{j,>k}}qs_j/s_i;q)_{\theta_{ik}}}{(q^{\theta_{i,>k}-\theta_{j,>k}}ts_j/s_i;q)_{\theta_{ik}}}\prod_{1\le i\le j<k\le n}\frac{(q^{\theta_{i,>k}-\theta_{j,\ge k}}qs_j/ts_i;q)_{\theta_{ik}}}{(q^{\theta_{i,>k}-\theta_{j,\ge k}}s_j/s_i;q)_{\theta_{ik}}},$$

where $\theta_{i,>k}=\sum_{k<l\le n}\theta_{i,l}$. By the specialization $s=t^{\rho}q^{\lambda}$, $\lambda\in P_+$, this formula reproduces the tableau representation of the Macdonald polynomials $P_\lambda(x)$ in [2].

(b) Symmetries: Normalize $f(x;s|q,t)$ by setting

$$\varphi(x;s|q,t)=f(x;s|q,t)\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(qs_j/ts_i;q)_\infty}$$

Then we have

$$\varphi(s;x|q,t)=\varphi(x;s|q,t),\quad \varphi(x;s|q,q/t)=\varphi(x;s|q,t)\prod_{1\le i<j\le n}\frac{(qx_j/tx_i;q)_\infty}{(tx_j/x_i;q)_\infty}\frac{(qs_j/ts_i;q)_\infty}{(ts_j/s_i;q)_\infty}.$$

Part II

Elliptic Ruijsenaars operators and their eigenfunctions

1. Elliptic Ruijsenaars difference operators

Fixing two bases $p, q \in \mathbb{C}^*$ ($|p| < 1, |q| < 1$) and a parameter $t \in \mathbb{C}^*$, we consider the *elliptic Ruijsenaars q-difference operators* of type A, in n variables $x = (x_1, \ldots, x_n)$:

$$\mathcal{D}_x^{(r)}(p) = \sum_{I \subseteq \{1,\ldots,n\};\, |I|=r} t^{\binom{r}{2}} \prod_{i \in I,\, j \notin I} \frac{\theta(tx_i/x_j; p)}{\theta(x_i/x_j; p)} \prod_{i \in I} T_{q,x_i} \qquad (r = 0, 1, \ldots, n),$$

where

$$\theta(z; p) = (z; p)_\infty (p/z; p)_\infty, \quad (z; p)_\infty = \prod_{i=0}^{\infty} (1 - p^i z) \quad (z \in \mathbb{C}),$$

and for each $i = 1, \ldots, n$, T_{q,x_i} stands for the q-shift operator in the variables x_i: $T_{q,x_i} f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$.

By the pioneering work of Ruijsenaars (1987), it is known that these q-difference operators $\mathcal{D}_x^{(r)}(p)$ ($r = 1, \ldots, n$) commute with each other. It is a fundamental problem to describe the joint eigenfunctions for this commuting family of q-difference operators:

$$\mathcal{D}_x^{(r)}(p)\psi(x; p) = \varepsilon^{(r)}(p)\psi(x; p) \qquad (r = 1, \ldots, n).$$

We propose two classes of joint eigenfunctions which are represented by convergent power series in p.

(A) Symmetric eigenfunctions around the torus that deform Macdonald polynomials.

(B) Asymptotically free eigenfunctions in the domain $|x_1| \gg \cdots \gg |x_n| \gg |px_1|$.

2. Elliptic deformation of Macdonald polynomials

We consider to diagonalize the commuting family of elliptic Ruijsenaars operators $\mathcal{D}_x^{(r)}$: $\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}[[p]] \to \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}[[p]]$ $(r = 1, \ldots, n)$ on the ring of formal power series in p with coefficients in $\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}$, supposing that $t \in \mathbb{C}^*$ is generic.

Joint eigenfunctions $\Longrightarrow$ elliptic deformations of Macdonald polynomials.

Theorem C: Let $\lambda \in P_+$ be a dominant integral weight.

(1) The initial value problem of the joint eigenfunction equations

$$\mathcal{D}_x^{(r)}(p)\mathcal{P}_\lambda(x;p) = \varepsilon_\lambda^{(r)}(p)\mathcal{P}_\lambda(x;p) \quad (r = 1, \ldots, n); \quad \mathcal{P}_\lambda(x;0) = P_\lambda(x)$$

has a formal solution $\mathcal{P}_\lambda(x;p) \in \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}[[p]]$, with uniquely determined formal eigenvalues $\varepsilon_\lambda^{(r)}(p) \in \mathbb{C}[[p]]$ $(r = 1, \ldots, n)$; $\mathcal{P}_\lambda(x;p)$ is determined uniquely up to multiplication by a formal power series $\gamma(p) \in \mathbb{C}[[p]]$, with $\gamma(0) = 1$.

(2) Suppose that $\mathcal{P}_\lambda(x;p)$ is *normalized*. Then, there exist a real constant $\tau \in (0,1)$ such that, for $|p| < \tau$, $\mathcal{P}_\lambda(x;p)$ is absolutely convergent in the domain

$$|p|/\tau < |x_j/x_i| < \tau/|p| \qquad (1 \le i < j \le n),$$

containing the torus $\mathbb{T}^n = \left\{ x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n \;\middle|\; |x_i| = 1 \ (i = 1, \ldots, n) \right\}$.

The formal joint eigenfunction $\mathcal{P}_\lambda(x;p) = \mathcal{P}_\lambda(x;p|q,t)$ has p-expansion of the form

$$\mathcal{P}_\lambda(x;p) = \sum_{k=0}^{\infty} p^k \mathcal{P}_{\lambda,k}(x); \quad \mathcal{P}_{\lambda,k}(x) = \sum_{\mu \in P_+;\, \mu \le \lambda + k\phi} \mathcal{P}_{\lambda,k;\mu} m_\mu(x) \in \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}, \quad \phi = \epsilon_1 - \epsilon_n.$$

We say that $\mathcal{P}_\lambda(x;p)$ is *normalized* if $\mathcal{P}_{\lambda,0;\lambda} = 1$, $\mathcal{P}_{\lambda,k;\lambda} = 0$ $(k > 0)$.

3. Asymptotically free eigenfunctions (Ruijsenaars function)

If a complex weight $\lambda \in P_{\mathbb{C}} = P \otimes_{\mathbb{Z}} \mathbb{C}$ is generic, there exists an asymptotically free joint eigenfunction $\psi_\lambda(x;p)$ with leading term x^λ in the domain $|x_1| \gg \cdots \gg |x_n| \gg |px_1|$.

(0) $\psi_\lambda(x;p) = x^\lambda + (\text{lower terms w.r.t. } \le_{\text{aff}}) \in x^\lambda \mathbb{C}\{px_1/x_n, x_2/x_1, \ldots, x_n/x_{n-1}\}$,

(1) $\mathcal{D}_x^{(r)}(p)\psi_\lambda(x;p) = \varepsilon_\lambda^{(r)}(p)\psi_\lambda(x;p) \quad (r = 1, \ldots, n)$.

Setting $s = t^\rho q^\lambda$, we introduce the *modified elliptic Ruijsenaars operators* $\mathcal{E}_{x,s}^{(r)}(p) = x^{-\lambda}\mathcal{D}_x^{(r)}(p)x^\lambda$, and consider the joint eigenvalue problem

$$\mathcal{E}_{x,s}^{(r)}(p)f(x;s;p) = \varepsilon^{(r)}(s;p)f(x;s;p) \quad (r = 1, \ldots, n), \;\; \psi_\lambda(x;p) = x^\lambda f(x;t^\rho q^\lambda;p).$$

Theorem D: Suppose that $s \in (\mathbb{C}^*)^n$ satisfies the condition $s_j/s_i \notin q^{\mathbb{Z}}$ $(1 \le i < j \le n)$.
(1) The linear operators $\mathcal{E}_{x,s}^{(r)}(p)$ acting on the ring $\mathbb{C}[[px_1/x_n, x_2/x_1, \ldots, x_n/x_{n-1}]]$ of formal power series have a joint eigenfunction $f(x;s;p)$ with leading term 1, with uniquely determined eigenvalues $\varepsilon^{(r)}(s;p) \in \mathbb{C}[[p]]$; $f(x;s;p)$ is determined uniquely up to multiplication by $\gamma(s;p) \in \mathbb{C}[[p]]$, $\gamma(s;0) = 1$.
(2) Suppose that $f(x;s;p)$ is *normalized*. Then, there exists a real constant $\sigma \in (0,1)$ such that, for $|p| < \sigma^n$, $f(x;s;p)$ is absolutely convergent in the domain

$$|px_1/x_n| < \sigma, \;\; |x_2/x_1| < \sigma, \ldots, \;\; |x_n/x_{n-1}| < \sigma.$$

We say that $f(x;s;p)$ is *normalized* if it has constant term 1 with respect to x.
$f(x;s;p)$: (stationary) *Ruijsenaars function* in the domain $|x_1| \gg \cdots \gg |x_n| \gg |px_1|$.

$$f(x;s;p) = \sum_{k=0}^{\infty} p^k f_k(x;s), \quad f_k(x;s) \in (x_1/x_n)^k \mathbb{C}\{x_2/x_1, \ldots, x_n/x_{n-1}\}.$$

4. Remarks on the construction of eigenfunctions

● Two classes of joint eigenfunctions

We have constructed two classes of joint eigenfunctions for the commuting family of elliptic Ruijsenaars operators.

(A) Symmetric eigenfunctions around the torus $\mathbb{T}^n$
$\Longrightarrow$ Elliptic deformation of Macdonald polynomials

$$\mathcal{P}_\lambda(x;p) = \mathcal{P}_\lambda(x;p|q,t) \in \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}[[p]] \qquad (\lambda \in P_+)$$

(B) Asymptotically free eigenfunctions and the Ruijsenaars function in the domain $|x_1| \gg \cdots \gg |x_n| \gg |px_1|$

$$\psi_\lambda(x;p) = \psi_\lambda(x;p|q,t) \in x^\lambda \mathbb{C}\{px_1/x_n, x_2/x_1, \ldots, x_n/x_{n-1}\} \quad (\lambda \in P \otimes_{\mathbb{Z}} \mathbb{C})$$

$$f(x;s;p) = f(x;s;p|q,t) \in \mathbb{C}\{px_1/x_n, x_2/x_1, \ldots, x_n/x_{n-1}\}$$

$$\psi_\lambda(x;p) = x^\lambda f(x;t^\rho q^\lambda;p)$$

● Construction of joint eigenfunctions

(a) Existence of formal solutions $\mathcal{P}_\lambda(x;p)$, $f(x;s;p)$
(perturbation of eigenvalue problems)

(b) Convergence of $f(x;s;p)$ (method of majorants for nonlinear recurrences)

(c) Integral transform from $f(x;s;p)$ to $\mathcal{P}_\lambda(x;p)$

● Integral transform

Ruijsenaars elliptic gamma function:

$$\Gamma(z;p,q) = \frac{(pq/z;p,q)_\infty}{(z;p,q)_\infty}, \quad (z;p,q)_\infty = \prod_{i,j=0}^{\infty}(1-p^i q^j z)$$

For the variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we define the kernel function $K(x, y; p)$ and a non-symmetric weight function $w(y; p)$ by

$$K(x,y;p) = \prod_{i,j=1}^{n} \frac{\Gamma(x_i/y_j;p,q)}{\Gamma(tx_i/y_j;p,q)}, \quad w(y;p) = \prod_{1\le i<j\le n} \theta(y_j/y_i;p)\frac{\Gamma(ty_j/y_i;p,q)}{\Gamma(qy_j/ty_i;p,q)}.$$

We denote by $d\omega_n(y)$ the normalized invariant n-form on the torus $\mathbb{T}^n$:

$$d\omega_n(y) = \frac{1}{(2\pi\sqrt{-1})^n}\frac{dy_1\cdots dy_n}{y_1\cdots y_n}, \quad \int_{\mathbb{T}^n} d\omega_n(y) = 1$$

For $r > 0$, $\sigma \in (0, 1)$, we define an n-cycle $C_{r,\sigma}$ in $(\mathbb{C}^*)^n$ by

$$C_{r,\sigma}: \quad |y_1| = r, \quad |y_2| = r\sigma, \quad \ldots, \quad |y_n| = r\sigma^{n-1}.$$

Theorem E: Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $\ell(\lambda) \le n$, and set

$$\lambda^\vee = (\lambda_n, \ldots, \lambda_1), \quad s^\vee = (t^\rho q^\lambda)^\vee = (q^{\lambda_n}, tq^{\lambda_{n-1}}, \ldots, t^{n-1}q^{\lambda_1}).$$

Then for a sufficiently small $\sigma \in (0, 1)$, we have

$$\int_{C_{r,\sigma}} K(x,y;p)w(y;p)\, y^{\lambda^\vee} f(y;s^\vee;p)d\omega_n(y) = b_\lambda(p)\,\mathcal{P}_\lambda(x;p).$$

● Orthogonality relation for $\mathcal{P}_\lambda(x;p)$

We define the symmetric weight function $w^{\mathrm{sym}}(x;p)$ by

$$w^{\mathrm{sym}}(x;p)=\prod_{1\le i<j\le n}\frac{\Gamma(tx_i/x_j;p,q)}{\Gamma(x_i/x_j;p,q)}\frac{\Gamma(tx_j/x_i;p,q)}{\Gamma(x_j/x_i;p,q)}.$$

Suppose that $|t|<1$. For $\lambda,\mu\in P_+$,

$$\int_{\mathbb{T}^n}\mathcal{P}_\lambda(x^{-1};p)\,\mathcal{P}_\mu(x;p)\,w^{\mathrm{sym}}(x;p)\,d\omega_n(x)=\delta_{\lambda,\mu}\mathcal{N}_\lambda(p).$$

Note that $w^{\mathrm{sym}}(x;p)=w^{\mathrm{sym}}(x;p,q|t)$ is symmetric with respect to exchanging p,q. We conjecture that our elliptic deformations $\mathcal{P}_\lambda(x;p|q,t)$ $(\lambda\in P_+)$ of Macdonald polynomials are symmetric with respect to (p,q), i.e. $\mathcal{P}_\lambda(x;p|q,t)=\mathcal{P}_\lambda(x;q|p,t)$.

● Symmetries of $f(x;s;p)$

$f(x;s;p)=f(x;s;p|q,t)$: the normalized Ruijsenaars function

(a) Symmetry with respect to the simultaneous rotation of indices for x and s variables:

$$f(x_1,\ldots,x_n;s_1,\ldots,s_n;p|q,t)=f(x_2,\ldots,x_n,px_1;s_2,\ldots,s_n,s_1;p|q,t).$$

(b) Transformation under the reflection $t\leftrightarrow q/t$ of the parameter t:

$$f(x;s;p|q,t)=\gamma(s;p|q,t)\prod_{1\le i<j\le n}\frac{\Gamma(tx_j/x_i;p,q)}{\Gamma(qx_j/tx_i;p,q)}\,f(x;s;p|q,q/t),$$

where $\gamma(s;p|q,t)\gamma(s;p|q,q/t)=1$, $\gamma(s;0|q,t)=1$.

5. Relation to the non-stationary Ruijsenaars function

From now on, we call the normalized eigenfunction $f(x;s;p|q,t)$ in the asymptotic domain $|x_1| \gg \cdots \gg |x_n| \gg |px_1|$ the *stationary Ruijsenaars function*, in order to distinguish from the non-stationary Ruijsenaars function below.

In one of the recent papers, Shiraishi [4] introduced his *non-stationary Ruijsenaars function* $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ in terms of an explicit formal power series of Nekrasov type, and formulated various conjectures related to this *function.* Shiraishi's function is defined as an infinite sum

$$f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \sum_{\lambda^{(1)},\ldots,\lambda^{(n)}\in\mathcal{P}} \prod_{i,j=1}^{n} \frac{N^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(ts_j/s_i;q,\kappa)}{N^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/s_i;q,\kappa)} \prod_{j=1}^{n}\prod_{i\geq 1}(px_{j+i}/tx_{j+i-1})^{\lambda^{(j)}_i}$$

over all n-tuples of partitions $(\lambda^{(1)},\ldots,\lambda^{(n)}) \in \mathcal{P}^n$, where the expression $N^{(k|n)}_{\lambda,\mu}(u;q,\kappa)$ $(k\in\mathbb{Z}/n\mathbb{Z})$,

$$\prod_{\substack{j\geq i\geq 1\\ j-i\equiv k \pmod n}} (uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q)_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\substack{j\geq i\geq 1\\ j-i\equiv -k-1 \pmod n}} (uq^{\lambda_i-\mu_j}\kappa^{i-j-1};q)_{\mu_j-\mu_{j+1}},$$

attached to a pair (λ,μ) of partitions is a kind of K-theoretic Nekrasov factors. This infinite sum can also be regarded as a generalization of the combinatorial formula (tableau formula) for the Macdonald polynomials.

When compared with the stationary Ruijsenaars function $f(x;s;p|q,t)$, Shiraishi's non-stationary Ruijsenaars function $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ depends on one extra parameter κ, which is considered as the scaling parameter in shifting the elliptic nome $p=e(\tau)$ of the elliptic curve.

With respect to the new parameter κ, Shiraishi's function

$$f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \sum_{\lambda^{(1)},\ldots,\lambda^{(n)}\in\mathcal{P}} \prod_{i,j=1}^{n} \frac{N^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(ts_j/s_i;q,\kappa)}{N^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/s_i;q,\kappa)} \prod_{j=1}^{n}\prod_{i\geq 1}(px_{j+i}/tx_{j+i-1})^{\lambda^{(j)}_i}$$

has an essential singularity at $\kappa = 1$. Let $\alpha(p|s,\kappa|q,t)$ be the constant term of $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ with respect to the x variables, and normalize $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ by dividing it by $\alpha(p|s,\kappa|q,t)$. Shiraishi's main conjecture claims that the essential singularity at $\kappa = 1$ would be regularized by this normalization, and that the value at $\kappa = 1$ would recover our stationary Ruijsenaars function $f(x;s;p|q,t)$; to be more specific,

$$f(x;s;p|q,t) = \left.\frac{f^{\widehat{\mathfrak{gl}}_n}(p^{\rho/n}x,p^{1/n}|s,\kappa|q,q/t)}{\alpha(p^{1/n}|s,\kappa|q,q/t)}\right|_{\kappa=1}.$$

The factors $p^{\rho/n}$, $p^{1/n}$ arise from the difference of realizations (principal or homogenous) of the affine Lie algebra $\widehat{\mathfrak{gl}}_n$.

Note that the coefficients of the stationary Ruijsenaars functions are determined by the recurrence relations arising from the elliptic Ruijsenaars operators. This means that both sides of Shiraishi's main conjecture are computable term by term. By computer experiments in terms of p-expansions, we confirmed that this conjecture holds up to some degrees within the capability of the computer.

In a recent paper [5] (Langmann-Noumi-Shiraishi), we proposed a new simpler explicit formula for the non-stationary Ruijsenaars function, in which $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ is expressed through an infinite variables version of the Macdonald function.

6. Future problems

There are many intriguing problems piled up around the subject of stationary and non-stationary Ruijsenaars functions.

(1) Prove Shiraishi's main conjecture.

(2) Find the non-stationary Ruijsenaars equation to be satisfied by $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$.

(3) Find a good explicit formula for the stationary Ruijsenaars function.

(4) What can one say about symmetries of those Ruijsenaars functions?

(5) What can one say about analytic properties of those Ruijsenaars functions?

$\vdots$

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