SYMPLECTIC STRUCTURES ON THE MODULI SPACES OF CURVES AND BUNDLES

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1. Introduction

Let X be a compact Riemann surface of genus $g > 1$ and $\pi_1 = \pi_1(X)$. By the uniformization theorem,

$$X \simeq \Gamma\backslash\mathbb{H}, \quad \text{where} \quad \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

is the Lobachevsky plane, and $\Gamma \simeq \pi_1$ is a Fuchsian group.

Let E be a stable vector bundle of degree 0 and rank n over X. By the Narasimhan-Seshadri theorem, there is an irrep $\rho : \pi_1 \to \mathrm{U}(n)$ such that

$$E \simeq E_\rho = \pi_1\backslash\mathbb{H} \times \mathbb{C}^n,$$

where π_1 acts on $\mathbb{H} \times \mathbb{C}^n$ by $(z, v) \mapsto (\gamma z, \rho(\gamma)v)$. Moreover, $E_{\rho_1} \simeq E_{\rho_2}$ iff $\rho_1 \simeq \rho_2$.

Teichmüller space is $$T_g = \mathrm{Hom}_0(\pi_1, G)/G,$$ where $G = \mathrm{PSL}(2, \mathbb{R})$ and "0" stands for Fuchsian representations; it is connected component of $\mathrm{Hom}(\pi_1, G)$ with Euler class $2g-2$. (W. Goldman Ph.D.) $$\dim_{\mathbb{R}} T_g = 6g - 6.$$ The modular group is $$\mathrm{Mod}_g = \mathrm{Aut}(\pi_1)/\mathrm{Inn}(\pi_1).$$ T_g is a symplectic manifold with the Goldman form ω_G.	The moduli space $\mathscr{N}$ of rank n and degree 0 stable vector bundles over X is $$\mathscr{N} = \mathrm{Hom}_0(\pi_1, G)/G,$$ where $G = \mathrm{U}(n)$ and "0" stands for irreducible unitary representations of π_1. $$\dim_{\mathbb{R}} \mathscr{N} = 2n^2(g - 1) + 2.$$ There is no modular group in this case! $\mathscr{N}$ is a symplectic manifold with the Goldman form ω_G.

In general, let G be a reductive Lie group. The character variety is

$$\mathscr{K} = \mathrm{Hom}_0(\pi_1, G)/G,$$

where "0" stands for stable points. $\mathscr{K}$ is smooth and

$$T_\rho\mathscr{K} = H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\rho}).$$

The Goldman symplectic form ω_G is defined by

$$\omega_G(\chi_1,\chi_2) = \langle \chi_1 \cup \chi_2 \rangle([X]), \quad \text{where} \quad [X] \in H_2(\pi_1, \mathbb{Z}).$$

Using R. Fox free differential calculus

$$\boxed{D(ab) = D(a)\varepsilon(b) + aD(b)}$$

we have for $\pi_1 = \langle a_1, b_1, \dots, a_g, b_g \rangle / R_g$, where $R_g = \prod_{k=1}^{g} [a_k, b_k]$:

$$\boxed{[X] = \sum_{k=1}^{g} \left\{ \left(\frac{\partial R_g}{\partial a_k}, a_k \right) + \left(\frac{\partial R_g}{\partial b_k}, b_k \right) \right\}}$$

and

$$\boxed{\omega_G(\chi_1,\chi_2) = -\sum_{k=1}^{g} \left\{ \left\langle \chi_1\left(\#\frac{\partial R}{\partial a_k}\right), \chi_2(a_k) \right\rangle + \left\langle \chi_1\left(\#\frac{\partial R}{\partial b_k}\right), \chi_2(b_k) \right\rangle \right\}}$$

2. Complex structure

Cauchy-Riemann operator is

$$\bar\partial - \mu\partial$$

where μ is Beltrami differential, a $(-1,1)$-form on X. Holomorphic functions satisfy Beltrami equation

$$\frac{\partial f}{\partial \bar z} - \mu \frac{\partial f}{\partial z} = 0.$$

Nontrivial deformations:

$$\mu \in \mathscr{H}^{0,1}(X, TX),$$

harmonic $(-1,1)$-forms with respect to the hyperbolic metric on X.
Holomorphic tangent space

$$T_X T_g = \mathscr{H}^{0,1}(X, TX),$$

holomorphic cotangent space

$$T_X^* T_g = \mathscr{H}^{1,0}(X, T^*X),$$

holomorphic quadratic differentials.
Bers coordinates

$$\frac{\partial f^{\varepsilon\mu}}{\partial \bar z} = \begin{Bmatrix} \varepsilon\mu(z) & z \in \mathbb{H} \\ \overline{\varepsilon\mu(\bar z)} & z \in \overline{\mathbb{H}} \end{Bmatrix} \frac{\partial f^{\varepsilon\mu}}{\partial z},$$

$$f^{\varepsilon\mu} \circ \gamma = \gamma^{\varepsilon\mu} \circ f^{\varepsilon\mu}$$

where $\gamma^{\varepsilon\mu} \in \Gamma^{\varepsilon\mu}$, a Fuchsian group.

Cauchy-Riemann operator is

$$\bar\partial - M$$

where M is $\operatorname{End} E$-valued $(0,1)$-form on X. Holomorphic functions satisfy

$$\frac{\partial F}{\partial \bar z} = F(z)M(z).$$

Nontrivial deformations:

$$M \in \mathscr{H}^{0,1}(X, \operatorname{End} E),$$

harmonic $(0,1)$-forms.
Holomorphic tangent space

$$T_X \mathscr{N} = \mathscr{H}^{0,1}(X, \operatorname{End} E),$$

holomorphic cotangent space

$$T_X^* \mathscr{N} = \mathscr{H}^{1,0}(X, \operatorname{End} E),$$

the space of Higgs fields. *Bers coordinates for bundles* (L.T. & P. Zograf, 1989)

$$\frac{\partial F^{\varepsilon}}{\partial \bar z} = \varepsilon F^{\varepsilon}(z) M(z), \quad z \in \mathbb{H},$$

$$F^{\varepsilon} \circ \gamma = \rho^{\varepsilon}(\gamma) F^{\varepsilon} \rho(\gamma)$$

where $\rho^{\varepsilon} : \pi_1 \to \mathrm{U}(n)$ is irreducible.

Important: families $\Gamma^{\varepsilon\mu}$ and ρ^{ε} are not holomorphic in ε.

3. Kähler form

Hodge inner product on tangents spaces determines Kähler metrics on moduli spaces.

Weil-Petersson metric on T_g with the symplectic form ω_{WP}.	Narasimhan-Atiyah-Bott metric on $\mathscr{N}$ with the symplectic form ω_{NAB}.

Simple theorem (using Eichler-Shimura periods)

On the Teichmüller space T_g, $\boxed{\omega_G = \omega_{WP}}$ (W. Goldman, 1984)	On the moduli space $\mathscr{N}$, $\boxed{\omega_G = -4\omega_{NAB}}$

4. Affine bundles

$$\mathscr{P}_g \to T_g,$$

holomorphic affine bundle over T^*T_g. Fibres $\mathscr{P}_g(X)$ are holomorphic projective connections

$$\frac{d^2}{dz^2} + \frac{1}{2}R$$

over $\{X\} \in T_g$.

Canonical section

$$s_F : T_g \to \mathscr{P}_g,$$

given by the Fuchsian uniformization.

The section s_F is not holomorphic:

$$\boxed{\bar{\partial} s_F = -\sqrt{-1}\omega_{WP}}$$

(L.T. & P. Zograf, 1985).

Fuchsian section gives a real-analytic isomorphism

$$\mathscr{P}_g \simeq T^*T_g.$$

The monodromy map

$$Mon : \mathscr{P}_g \to \mathrm{Hom}_0(\pi_1, G_{\mathbb{C}})/G_{\mathbb{C}},$$

where $G_{\mathbb{C}} = \mathrm{PSL}(2, \mathbb{C})$.

$$\mathscr{A} \to \mathscr{N},$$

holomorphic affine bundle over $T^*\mathscr{N}$. Fibres $\mathscr{A}(E)$ are $(1,0)$-type zero curvature connections

$$\nabla = d + A$$

in $\{E\} \in \mathscr{N}$.

Canonical section

$$s_{NS} : \mathscr{N} \to \mathscr{A},$$

given by the Narasimhan-Seshadri theorem.

The section s_{NS} is not holomorphic:

$$\boxed{\bar{\partial} s_{NS} = -2\sqrt{-1}\omega_{NAB}}$$

(L.T. & P. Zograf, 1986).

Narasimhan-Seshadri section gives a real-analytic isomorphism

$$\mathscr{A} \simeq T^*\mathscr{N}.$$

The Riemann-Hilbert correspondence

$$RH : \mathscr{A} \to \mathrm{Hom}_0(\pi_1, G_{\mathbb{C}})/G_{\mathbb{C}},$$

where $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$.

5. Holomorphic sections

Here is the main difference between Teichmüller spaces and moduli spaces of stable vector bundles.

Affiine bundle $\mathscr{P}_g \to T_g$ has a family of global holomorphic sections, parametrized by T_g and given by the Bers simultaneous uniformization theorem: the *quasi-Fuchsian* sections $s_{qF} : T_g \to \mathscr{P}_g$.

Local construction from pair $X, \bar{X}$ by keeping $\bar{X}$ fixed and varying X:

$$\frac{\partial f_{\varepsilon\mu}}{\partial \bar{z}} = \begin{Bmatrix} \varepsilon\mu(z) & z \in \mathbb{H} \\ 0 & z \in \overline{\mathbb{H}} \end{Bmatrix} \frac{\partial f_{\varepsilon\mu}}{\partial z},$$

$$f_{\varepsilon\mu} \circ \gamma = \gamma_{\varepsilon\mu} \circ f_{\varepsilon\mu}$$

where $\gamma_{\varepsilon\mu} \in \Gamma_{\varepsilon\mu}$, a quasi-Fuchsian group, it depends holomorphically on ε. In general, start with the pair $X, \bar{Y}$ by keeping $\bar{Y}$ fixed and varying X,

$$X_{\varepsilon\mu} = \Gamma_{\varepsilon\mu} \backslash \Omega_{\varepsilon\mu}, \quad Y = \Gamma_{\varepsilon\mu} \backslash \Omega^*_{\varepsilon\mu};$$

quasi-Fuchsian projection connection (parametrized by Y):

$$s_{qF} = \mathscr{S}(\pi^{-1}), \quad \pi : \Omega \to X,$$

$\mathscr{S}$ is the Schwarzian derivative.

Affine bundle $\mathscr{A} \to \mathscr{N}$ has no global holomorphic sections. Namely, such $s : \mathscr{N} \to \mathscr{A}$ gives

$$\bar{\partial}(s_{NS} - s) = -2\sqrt{-1}\omega_{NAB}$$

— a contradiction since $[\omega_{NAB}] \neq 0$.

Local holomorphic sections.

For $\{E\} \in \mathscr{N}$ and $\nabla = d + A \in \mathscr{A}(E)$, realize E as a local system E_σ, where σ is a holonomy of ∇.

For $M \in H^{0,1}_{\mathrm{dR}}(X, \operatorname{End} E_\sigma)$ the normalized solution $F(z)$ of

$$\frac{dF}{d\bar{z}}(z) = F(z)M(z)$$

satisfies

$$F \circ \gamma = \sigma_\mu(\gamma) F \sigma(\gamma)^{-1}$$

and for small enough μ determines a family $\sigma_\mu : \pi_1 \to \mathrm{GL}(n, \mathbb{C})$ of irreps, holomorphic in Bers coordinates.

In coordinate chart U at $E \simeq E_\sigma$ realize each bundle as a quotient bundle E_{σ_μ}.

Let $d + A_{\sigma_\mu}$ be a connection in E_{σ_μ}, associated with the connection $d + 0$ in $\mathbb{H} \times \mathbb{C}^n \to \mathbb{H}$.

The family $\{d + A_{\sigma_\mu}\}$ determines a holomorphic section $\mathcal{S}_\sigma$ of $\mathscr{A} \to \mathscr{N}$ over $U \subset \mathscr{N}$.

In analogy with the Teichmüller theory, we call connections $\{d + A_{\sigma_\mu}\}$ *quasi-unitary*.

Global holomorphic section $s : T_g \to \mathscr{P}_g$ allow to identify

$$\mathscr{P}_g \simeq T^*T_g \quad \text{by} \quad \mathscr{P}_g \ni R \mapsto R - s \in T^*T_g,$$

and to pull back holomorphic Liouville symplectic form ω_L on T^*T_g to $\mathscr{P}_g$.
Q. When the pullback of ω_L by two holomorphic sections s_1 and s_2 give the same symplectic form on $\mathscr{P}_g$?
A. When the sections s_1 and s_2 satisfy $\partial(s_1 - s_2) = 0$.

Likewise, for each coordinate chart U local holomorphic section s_σ pulls back the holomorphic Liouville symplectic form ω_L on $T^*\mathscr{N}$ to $U \subset \mathscr{A}$.
Q. When these local pullbacks of ω_L define a global $(2,0)$-form on $\mathscr{A}$?
A. When on $U_1 \cap U_2$ the sections s_{σ_1} and s_{σ_2} satisfy $\partial(s_{\sigma_1} - s_{\sigma_2}) = 0$.

6. The reciprocity

The quasi-Fuchsian reciprocity (C. McMullen 2000, L.T. & L.P. Teo, 2003) $\partial(s_F - s_{qF}) = 0,$ $\bar\partial(s_F - s_{qF}) = -2\sqrt{-1}\omega_{WP}.$	*The quasi-unitary reciprocity* for vector bundles (L.T., 2021) $\partial(s_{NS} - s_\sigma) = 0,$ $\bar\partial(s_{NS} - s_\sigma) = -2\sqrt{-1}\omega_{NAB}.$
The proof uses q.c. mappings and Poincaré series for automorphic forms of weight 4.	The proof uses Hodge theory (for forms of weight 2 the series is divergent).

7. Pullback of the Goldman form

Put $G = \mathrm{PSL}(2,\mathbb{C})$. The following statement, made by S. Kawai and proved in [3], is often called "Kawai theorem" (see [4] for extra remarks).

Theorem 1. *The pullback to $\mathscr{P}_g$ of the holomorphic Goldman form ω_G on the character variety $\mathrm{Hom}_0(\pi_1, G)/G$ by the monodromy map Mon is $\sqrt{-1}$ times the pullback of the holomorphic Liouville form ω_L on T^*T_g by the quasi-Fuchsian section.*

Put $G = \mathrm{GL}(n,\mathbb{C})$. The following result is proved in [4].

Theorem 2. *The pullback to $\mathscr{A}$ of the holomorphic Goldman form ω_G on the character variety $\mathrm{Hom}_0(\pi_1, G)/G$ by the Riemann-Hilbert correspondence is $-2\sqrt{-1}$ times the pullback of the holomorphic Liouville form ω_L on $T^*\mathscr{N}$ by the quasi-unitary sections.*

References

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