# Geometry of quantum discrete Painlevé equations

Isomonodromic Deformations, Painlevé Equations, and Integrable Systems

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## Plan

1. Introduction/motivation for quantization

2. Geometry of the classical (cont/disc) Painlevé equations

3. Quantization through affine Weyl group

4. $\tau$ variables

# 1. Introduction/motivation : Why quantize?

• **Quantum IMD** = **conformal field theory**. This relation has been known for a long time.

• **The Schlesinger system**

$n \times n$ **Lax form**: $Y = Y(z,t) \in \mathbb{C}^n$.

$$\frac{\partial}{\partial z} Y = \mathcal{A} Y, \qquad \mathcal{A} = \sum_{a=1}^{N} \frac{A_a(t)}{z - t_a},$$

$$\frac{\partial}{\partial t_a} Y = \mathcal{B}_a Y, \qquad \mathcal{B}_a = -\frac{A_a(t)}{z - t_a}.$$

**Compatibility**: $[\frac{\partial}{\partial z} - \mathcal{A}, \frac{\partial}{\partial t_a} - \mathcal{B}_a] = 0$

$\rightarrow$ **Schlesinger system** [Schlesinger (1912)]

- **Schlesinger system** is a Hamiltonian system

$$\frac{\partial}{\partial t_a} A_b = \{H_a, A_b\}, \quad H_a = \sum_{b(\neq a)=1}^{N} \frac{\mathrm{tr}(A_a A_b)}{t_a - t_b},$$
$$\{(A_a)_{ij}, (A_b)_{kl}\} = \delta_{ab}\Big((A_a)_{il}\delta_{kj} - (A_a)_{kj}\delta_{il}\Big).$$

- Quantization : $\{*, *\} \to \frac{1}{\hbar}[*, *]$,

  $H_a \to$ Gaudin Hamiltonian

  Schlesinger system $\to$ **KZ equation**

$$\hbar \frac{\partial}{\partial t_a} \Psi(t) = \sum_{b(\neq a)=1}^{N} \frac{\Omega_{ab}}{t_a - t_b} \Psi(t).$$

[Knizhnik (89)][Reshetkhin (92)] [Harnad (96)] . . . [Nekrasov-Tsymbaliuk, 2103.1261] [Saebyeok-Lee-Nekrasov, 2103.17186].

- **Garnier system**

Scalar Lax form for $\psi = \psi(z,t)$:

$$\psi_{zz} + u(z,t)\psi = 0,$$

$$\psi_t = A(z,t)\psi_z - \tfrac{1}{2}A_z(z,t)\psi.$$

Their **compatibility** is given by

$$u_t = \{u, H\}, \quad H = \int uA\,dz,$$

$$\{u(z), u(w)\} = \Big(\frac{1}{2}\partial_z^3 + 2u(z)\partial_z + u_z(z)\Big)\delta(z-w),$$

- This Poisson structure is the classical Virasoro algebra.

• Fuchsian eq: $\mathbb{P}^1 \setminus \{(N+3)\text{pts}\}$

$$u(z) = \sum_{a=1}^{N+3} \left[\frac{c_a}{(z-t_a)^2} - \frac{H_a}{z-t_a}\right] + \sum_{i=1}^{N} \left[\frac{-3/4}{(z-q_i)^2} + \frac{p_i}{z-q_i}\right]$$

The compatibility $\to$ $N$-**Garnier system** [Garnier (1912)] ($P_{\mathrm{VI}}$ for $N=1$)

$$\frac{\partial q_i}{\partial t_a} = \frac{\partial H_a}{\partial p_i}, \quad \frac{\partial p_i}{\partial t_a} = -\frac{\partial H_a}{\partial q_i}.$$

• Hamiltonians $H_a = H_a(q,p,t)$ are determined by the conditions (i) $z = q_i$ are **apparent singularity** and (ii) $x = \infty$ is non-singular.

• The quantization of the Lax pair for $N$-Garnier system is given by Virasoro CFT with $N+3$ primaries + $(N+1)$ level 2 degenerate fields.

▲ **Relation to Gauge theory** (AGT or BPS/CFT correspondence).

IMD $\leftrightarrow$ CFT $\leftrightarrow$ gauge theory

• **Example.** $N \times N$ Schlesinger system on $\mathbb{P}^1$ with $k$ regular singular points with the spectral type (=multiplicity of eigenvalues)

$$(1^N), \underbrace{(1, N-1), \ldots, (1, N-1)}_{k-2}, (1^N).$$

$\rightarrow$ **FST system** [Fuji-Suzuki (2010)]($k = 4$), [Tsuda (2010)] ($k \geq 4$)
($N = 2, k = 4 \rightarrow P_{\mathrm{VI}}$, and $N = 2, k \geq 5 \rightarrow$ Garnier system.)

• FST system corresponds to $4d$ gauge theory, $G = SU(N)^{\otimes k-3}$, $N_{\mathsf{f}} = 2N$, $N_{\mathsf{bf}} = k - 4$. [Gavrylenko, Iorgov, Lisovyy, 1806.08650].

## Motivation for the quantization of IMD

- Since IMD equations are Hamiltonian system, it is natural to consider their quantization.

- They are related to CFT.

- The recent developments in gauge/string theories offer further motivation to quantize the IMD.

## The aim of this talk

- To consider the quantization of discrete Painlevé equations.

# 2. Geometry of the classical (cont/disc) Painlevé equations

### ▲ The original six (or eight) Painlevé equations

$$\begin{array}{ccccccccc} P_{\mathrm{VI}} & \to & P_{\mathrm{V}} & \to & P_{\mathrm{III}_1} & \to & (P_{\mathrm{III}_2}) & \to & (P_{\mathrm{III}_3}) \\ & & & \searrow & & \searrow & & & \\ & & & & P_{\mathrm{IV}} & \to & P_{\mathrm{II}} & \to & P_{\mathrm{I}} \end{array}.$$

are **non-autonomous** Hamiltonian systems

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad t' = \epsilon.$$

- In the autonomous limt ($\epsilon \to 0$), $H(q,p,t)$ is conserved.

## ▲ Hamiltonian $H_J$ for $P_J$ $(\epsilon = 1)$

$$H_{\mathrm{VI}} = \frac{q(q-1)(q-t)}{t(t-1)}\{p^2 - (\frac{a}{q-t} + \frac{b}{q-1} + \frac{c}{q})p\} + \frac{d(q-t)}{t(t-1)},$$

$$H_{\mathrm{V}} = t^{-1}\{q(q-1)p(p+t) - (\alpha_1+\alpha_3)qp + \alpha_1 p + \alpha_2 tq\},$$

$$H_{\mathrm{III}_1} = t^{-1}\{p(p-1)q^2 + (\alpha_1+\alpha_2)qp + tp - \alpha_2 q\},$$

$$H_{\mathrm{III}_2} = t^{-1}\big(p^2q^2 + q + pt + \alpha_1 pq\big),$$

$$H_{\mathrm{III}_3} = t^{-1}\big(p^2q^2 + pq + q + \frac{t}{q}\big),$$

$$H_{\mathrm{IV}} = qp(p-q-t) - \alpha_1 p - \alpha_2 q,$$

$$H_{\mathrm{II}} = \frac{p^2}{2} - \big(q^2 + \frac{t}{2}\big)p - aq, \qquad H_{\mathrm{I}} = \frac{p^2}{2} - 2q^3 - tq.$$

[Ohyama-Kawamuko-Sakai-Okamoto (2006)]

## ▲ Correspondence to gauge theory

▲ The Painlevé equations

$$
\begin{array}{ccccccccc}
P_{\mathrm{VI}} & \to & P_{\mathrm{V}} & \to & P_{\mathrm{III}_1} & \to & P_{\mathrm{III}_2} & \to & P_{\mathrm{III}_3} \\
 & & & \searrow & & \searrow & & & \\
 & & & & P_{\mathrm{IV}} & \to & P_{\mathrm{II}} & \to & P_{\mathrm{I}}
\end{array}
$$

correspond to the $4d$, $\mathcal{N}=2$, $SU(2)$ **gauge theory**

$$
\begin{array}{ccccccccc}
SW_4 & \to & SW_3 & \to & SW_2 & \to & SW_1 & \to & SW_0 \\
 & & & \searrow & & \searrow & & & \\
 & & & & AD_2 & \to & AD_1 & \to & AD_0
\end{array}
.
$$

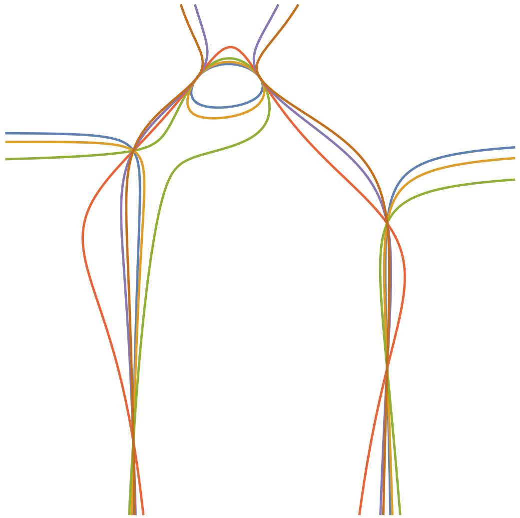
- $SW_{N_f}$: [Seiberg-Witten (1995)]. $AD_n$: [Argyres-Douglas (1995)].

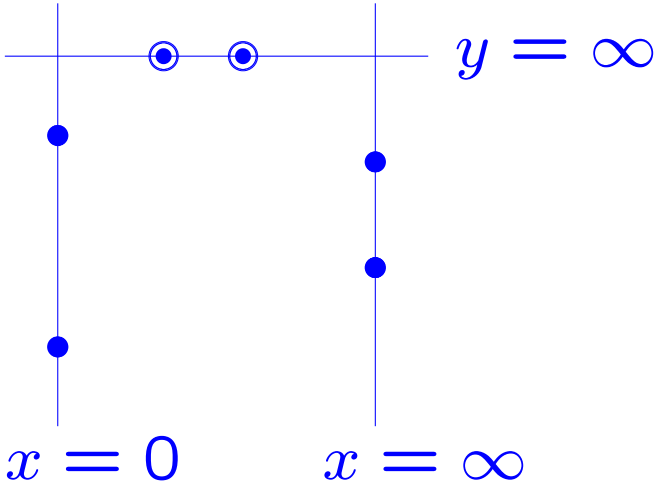
▲ An easy way to see the correspondence is to compare the geometry.

**Example.** $P_{\text{VI}} \leftrightarrow SW_4$ case: In variables $(x,y)=(q,pq)$, the equation for the level set $H_{\text{VI}}=u$ is written as

$$x(y-b_1)(y-b_2)-((1+t)y^2+b_5y+b_6)+\frac{t}{x}(y-b_3)(y-b_4)=u.$$

This is a family of elliptic curves known as the Seiberg-Witten curve for $SW_4$:





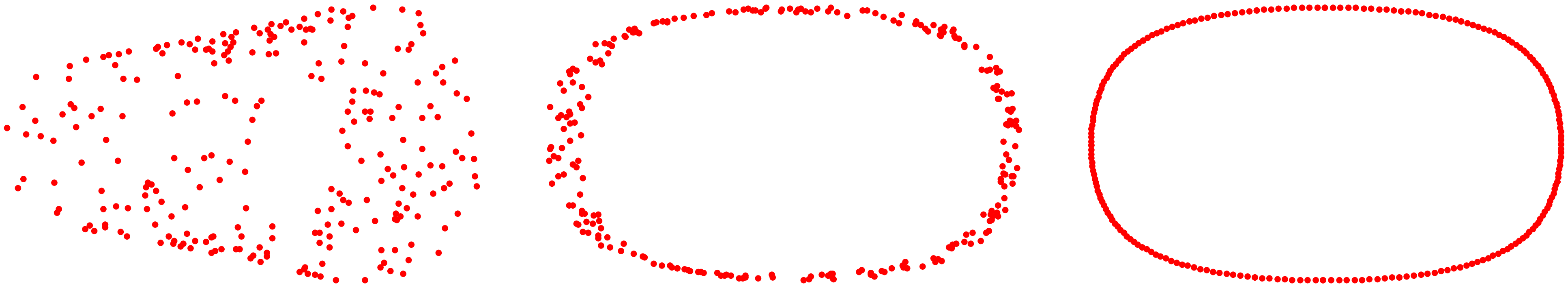

- For all the equations $P_{\text{J}}$, similar geometry is known [Okamoto (70's)][Sakai (2001)][Kajiwara et al, nlin/0403009]. They are 8-points blow up of $\mathbb{P}^1 \times \mathbb{P}^1$.

## ▲ The geometric structures in discrete cases.

• **Example.** Discrete Painlevé equation with $A_1^{(1)}$-symmetry

$$T : (a; x, y) \mapsto \left(pa; a\frac{x+y}{x+ay}y,\ \frac{x+ay}{x+y}\frac{1}{x}\right)$$

• For an initial data $(a, x, y) \in \mathbb{R}^3_{>0}$, the orbits in $(\log x, \log y)$ coordinates are

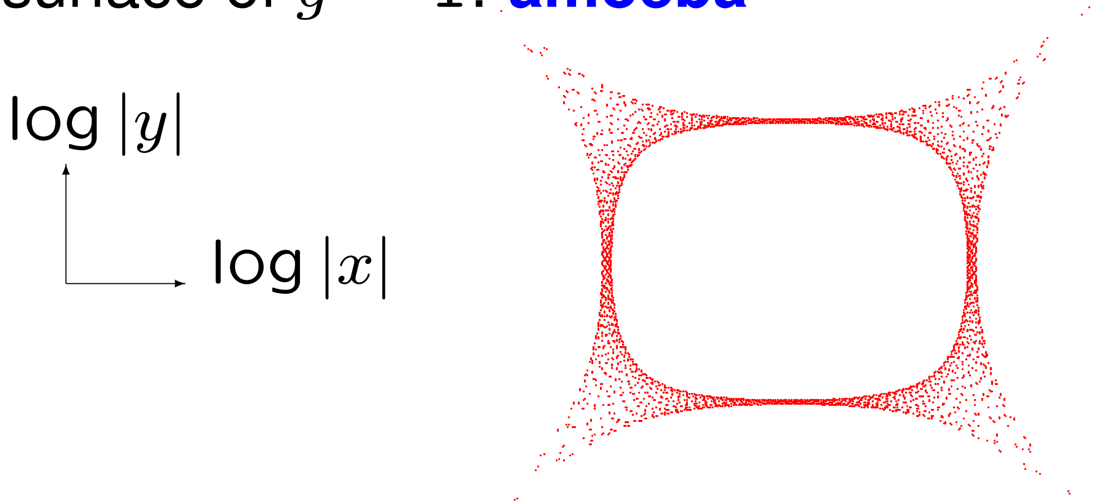


$p = 1.01$ $\qquad$ $p = 1.001$ $\qquad$ $p = 1$

• In the **autonomous** limit ($p \to 1$), the system admits an algebraic integral:

$$H(x, y) := x + \frac{a}{x} + y + \frac{1}{y} = u \quad \text{(constant)}.$$

• For complex initial values $x, y \in \mathbb{C}$, the level set $H(x, y) = u$ is a Riemann surface of $g = 1$: **amoeba**



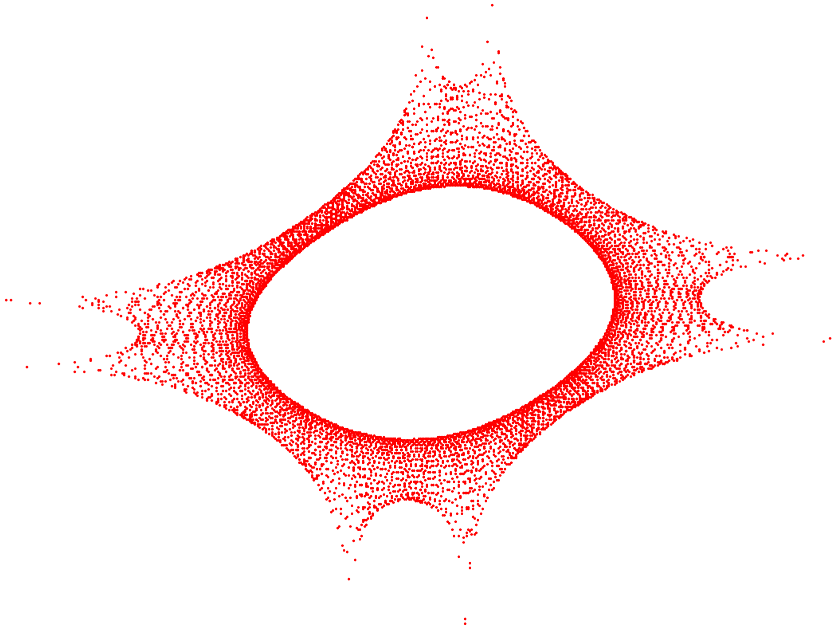

The previous real orbit is the inside boundary of this amoeba.

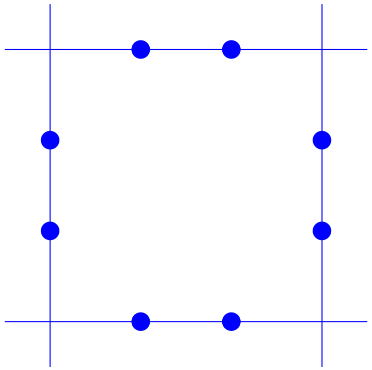
- **Example.** $D_5^{(1)}$ case: $q$-$P_{\mathrm{VI}}$ [Jimbo-Sakai (1996)]

$$T : \left( \begin{matrix} a_1, a_2, a_3, a_4 \\ a_5, a_6, a_7, a_8 \end{matrix} ; x, y \right) \mapsto \left( \begin{matrix} a_1, a_2, pa_3, pa_4 \\ a_5, a_6, pa_7, pa_8 \end{matrix} ; \overline{x}, \overline{y} \right), \ p = \frac{a_1 a_2 a_7 a_8}{a_3 a_4 a_5 a_6},$$

$$\overline{y} = \frac{a_5 a_6}{y} \frac{(x+a_3)(x+a_4)}{(x+a_1)(x+a_2)}, \quad \overline{x} = \frac{a_1 a_2}{x} \frac{(\overline{y}+pa_7)(\overline{y}+pa_8)}{(\overline{y}+a_5)(\overline{y}+a_6)}.$$

- The orbit for autonomous case: $p = 1$





- Conserved curve $H(x,y)=u$ for autonomous $q$-$P_{\mathrm{VI}}$ ($p=1$):

$$H=\frac{(x+a_1)(x+a_2)}{x}y+\{(a_5+a_6)x+\frac{a_1a_2(a_7+a_8)}{x}\}+\frac{(x+a_3)(x+a_4)}{x}\frac{a_5a_6}{y}$$

$$=\frac{(y+a_5)(y+a_6)}{y}x+\{(a_1+a_2)y+\frac{a_5a_6(a_3+a_4)}{y}\}+\frac{(y+a_7)(y+a_8)}{y}\frac{a_1a_2}{x}.$$

$\leftrightarrow$ 5d SU(2) Seiberg-Witten curve e.g. [Bao,Mitev,Pomoni,Taki,Yagi (1310.3841)]

- The parameters $a_1, a_2, \ldots, a_8$

$\leftrightarrow$ Positions of the "tentacles" of the amoeba.

## ▲ Sakai's classification [Sakai (2001)]

$$
\begin{tikzcd}
\text{ell} & E_8 \\
 & & & & & & & & A'_1 \arrow[dr] & \\
q & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_5 \arrow[r] & A_4 \arrow[r] & A_{2+1} \arrow[r] & A_{1+1} \arrow[r] \arrow[ur] & A_1 & A_0 \\
\text{add} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[rr] & & D_4 \arrow[r] & A_3 \arrow[r] \arrow[dr] & A_{1+1} \arrow[r] \arrow[dr] & A_1 \arrow[r] & A_0 \\
 & & & & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0
\end{tikzcd}
$$

• The (additive, $q$, elliptic)-difference cases correspond to (4d, 5d, 6d) gauge theories on ($\mathbb{R}^4$, $\mathbb{R}^4 \times S^1$, $\mathbb{R}^4 \times T^2$).

• Only the cases in red admit continuous (differential) deformation.

• The equations in the list are of genus 1 which correspond to rank 1 gauge theory.

## 3. Quantization through Affine Weyl group

• To quantize the (cont/disc) Painlevé equations, we will use the **affine Weyl group approach**.

Construct a birational representation of an affine Weyl group, and study a translation $T$ as a discrete flow.

• Standard methods to find suitable birational representation are

(i) Lie theory: classical [Noumi-Y. (2000)]. quantum [G.Kuroki 1206.3419].

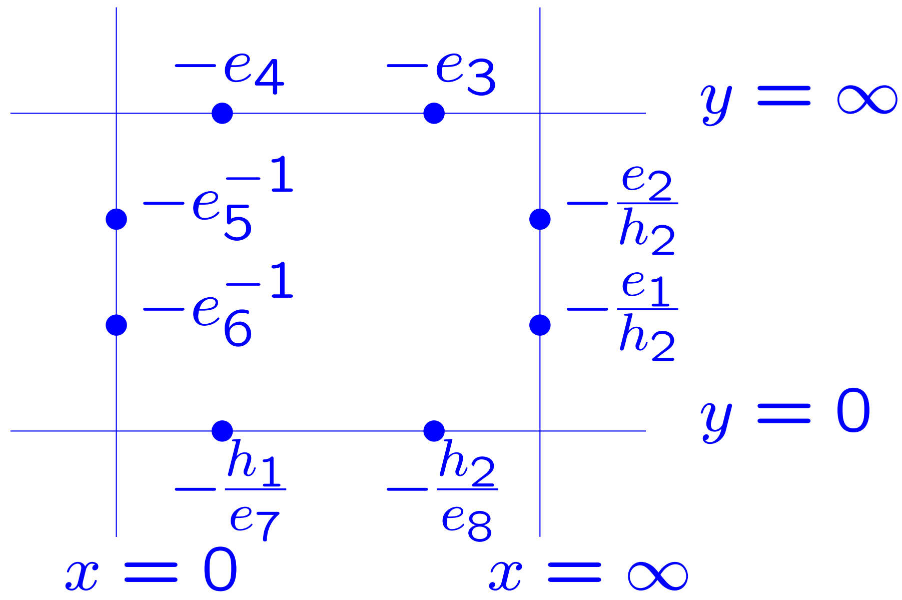
(ii) Rational surface: [Coble (1929)] [Sakai (2001)].

(iii) Cluster algebra: [Berstein-Gavrylenko-Marshakov, 1711.02063] [Masuda-Okubo-Tsuda, (2021)],...

• The quantization of the method (ii) is the main subject of this talk.

• **Example.**

Let $X$ be a blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 8 points. Picard group $\mathrm{Pic}(X)$ is generated by $H_1$, $H_2$, $E_1$, ..., $E_8$. ($\rightarrow$ associated parameters: $h_1, h_2, e_1, \ldots, e_8$)




• The **affine Weyl group** $W(D_5^{(1)})$.

$$
\begin{array}{ccccccc}
 & & s_0 & & s_4 & & \\
 & & | & & | & & \\
s_1 & - & s_2 & - & s_3 & - & s_5
\end{array}
\quad \Bigg| \quad
\begin{aligned}
s_i^2 &= 1, \\
s_i s_j &= s_j s_i, \quad (s_i \quad s_j), \\
s_i s_j s_i &= s_j s_i s_j, (s_i - s_j).
\end{aligned}
$$

$W(D_5^{(1)})$ acts on $X$ (birationally on $\mathbb{P}^1 \times \mathbb{P}^1$).

• The explicit actions $s_i$ on $K = \mathbb{C}(h_1, h_2, e_1, \ldots, e_8, x, y)$:

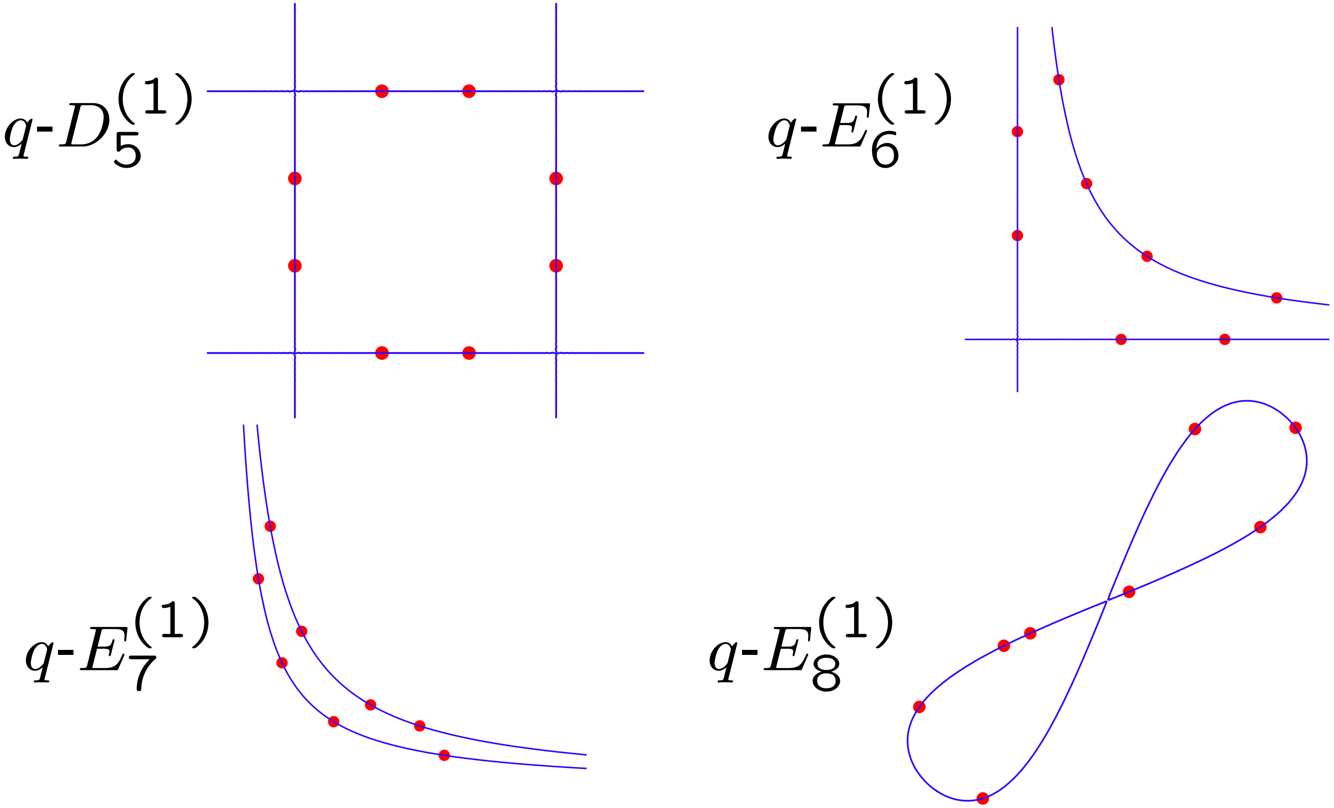
$$\boxed{\begin{aligned}
&s_0 = \{e_7 \leftrightarrow e_8\}, \quad s_1 = \{e_3 \leftrightarrow e_4\},\\
&s_2 = \{e_3 \to \frac{h_1}{e_7},\ e_7 \to \frac{h_1}{e_3},\ h_2 \to \frac{h_1 h_2}{e_3 e_7},\ y \to \frac{1 + \frac{e_7}{h_1}x}{1 + \frac{x}{e_3}} y\},\\
&s_3 = \{e_1 \to \frac{h_2}{e_5},\ e_5 \to \frac{h_2}{e_1},\ h_1 \to \frac{h_1 h_2}{e_1 e_5},\ x \to x\frac{1 + \frac{h_2}{e_1}y}{1 + e_5 y}\},\\
&s_4 = \{e_1 \leftrightarrow e_2\}, \quad s_5 = \{e_5 \leftrightarrow e_6\}.
\end{aligned}}$$

• Actions on $\{h_i, e_i\}$ are the standard **'linear'** reflections on $\mathrm{Pic}(X)$ (written in multiplicative variables: $h_i \sim e^{H_i}, e_i \sim e^{E_i}$).

$\to$ The actions on x,y are their **natural birational lift** to $\mathbb{P}^1 \times \mathbb{P}^1$.

• The Weyl group relations hold true also when x, y are **non-commutative**: $yx = qxy$ [Hasegawa(2007)]

## ▲ A standard realization for $E_n^{(1)}$:

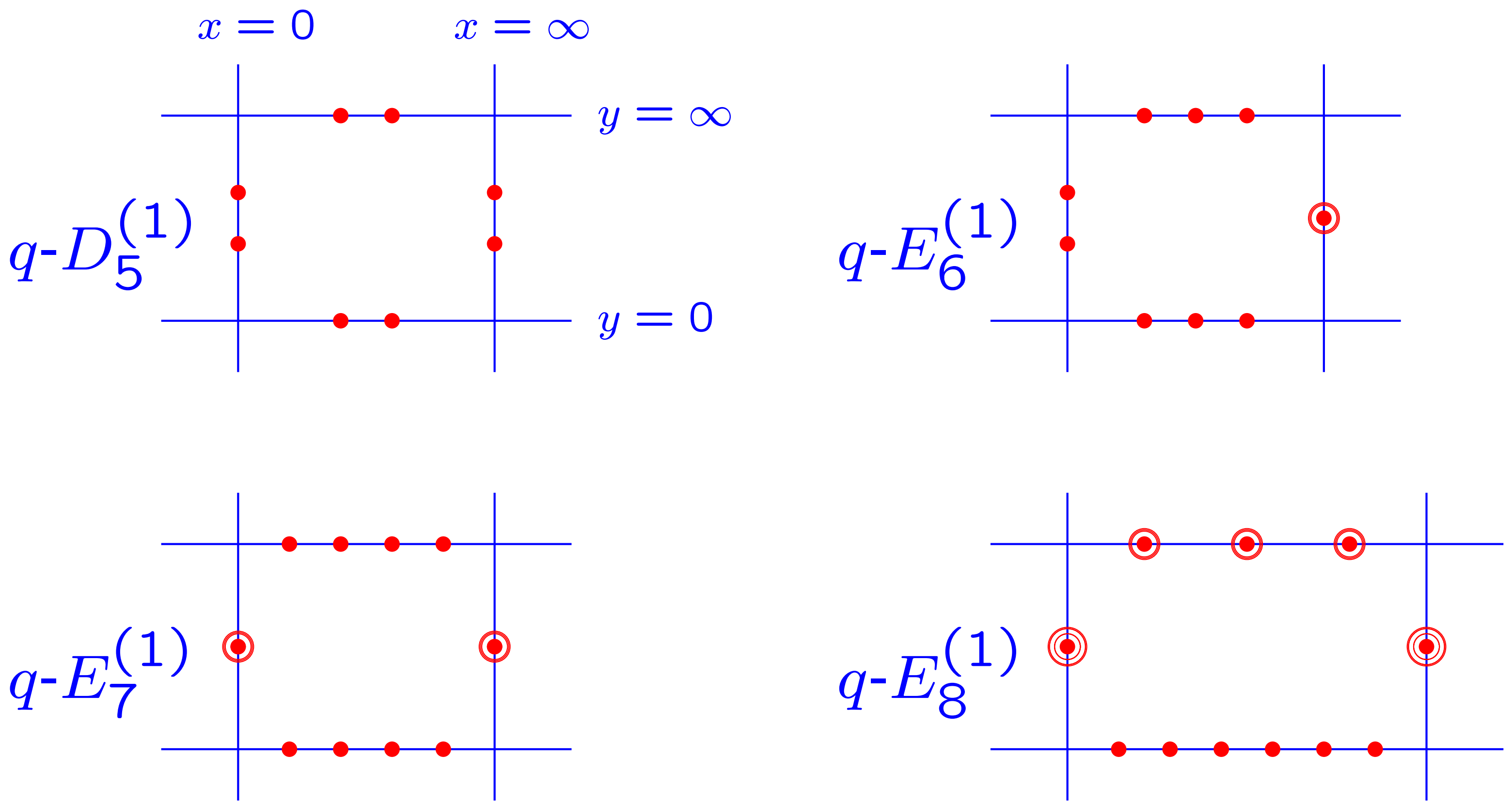



- For $D_5^{(1)}$, we have $\omega = \frac{dx \wedge dy}{xy} \rightarrow$ Poisson bracket $\{\log x, \log y\} = 1$.

But for $E_n^{(1)} \rightarrow$ quantization is not so easy.

**e.g.** $\{x, y\} = xy(xy - 1)$, (for $E_6^{(1)}$).

▲ We will take **another realization**.




• These curves for $q$-$E_n^{(1)}$ are of high degree but still $g=1$ due to the **multiple singularities**.

• We will consider the case $q$-$E_8^{(1)}$. [Moriyama-Y. (arXiv:2104.06661)]

• **Thm.** Let $k=\mathbb{C}(h_1,h_2,e_1,\ldots,e_{11})$. On a skew field $K=k(x,y)$ with $yx=qxy$, we have the following representation of $W(E_8^{(1)})$.

$$s_0=\{e_{10}\to\frac{h_2}{e_{11}},\ e_{11}\to\frac{h_2}{e_{10}},\ h_1\to\frac{h_1h_2}{e_{10}e_{11}},\ x\to x\frac{1+y\frac{h_2}{e_{10}}}{1+ye_{11}}\},$$

$$s_1=\{e_8\leftrightarrow e_9\},\quad s_2=\{e_7\leftrightarrow e_8\},$$

$$s_3=\{e_1\to\frac{h_1}{e_7},\ e_7\to\frac{h_1}{e_1},\ h_2\to\frac{h_1h_2}{e_1e_7},\ y\to\frac{1+x\frac{e_7}{h_1}}{1+\frac{x}{e_1}}y\},$$

$$s_4=\{e_1\leftrightarrow e_2\},\quad s_5=\{e_2\leftrightarrow e_3\},\quad s_6=\{e_3\leftrightarrow e_4\},$$

$$s_7=\{e_4\leftrightarrow e_5\},\quad s_8=\{e_5\leftrightarrow e_6\}.$$

$$\begin{array}{ccccccccccccccc} & & & & s_0 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ s_1 & - & s_2 & - & s_3 & - & s_4 & - & s_5 & - & s_6 & - & s_7 & - & s_8. \end{array}$$

• To apply the representation to Painlevé equation, we want to compute the action of **translations**.

For $E_8^{(1)}$ case, we have $(2\times)$120 directions. Each of them is given by **58** simple reflections $\to$ **too big!** - How can we understand them?

• In commutative case, we have the following factorization

$$\boxed{w(x) = \frac{A}{B}, \quad w(y) = \frac{C_1 C_2 \cdots C_6}{D_1 D_2 D_3}, \quad w \in W(E_8^{(1)}).}$$

Here $A, B, C_i, D_i$ are some **polynomials** in $x, y$. They are complicated for general $w$, but have a simple geometric characterization. [Kajiwara et.al (2003)]

• To understand these polynomials, a **lift of the rep. including tau-variables** is essential. Its quantization is our main problem.

# 4. $\tau$ variables

- In addition to $\{h_i, e_i, x, y\}$, we introduce variables ($\tau$-**variables**)

$$\sigma_1, \sigma_2, \tau_1, \ldots, \tau_{11}.$$

- We put the following $q$-commutation relations:

$$\boxed{\begin{gathered} yx = qxy \\ \sigma_i h_j = q^{H_i.H_j}\, h_j \sigma_i, \quad \tau_i e_j = q^{E_i.E_j}\, e_j \tau_i, \end{gathered}}$$

$H_1.H_2 = H_2.H_1 = 1$, $E_i.E_j = -\delta_{ij}$. Other cases are commutative.

- The variables $\sigma_i, \tau_i$ and the parameters $h_i, e_i$ are non-commutative.

- **Thm.** One can extend the representation of $W(E_8^{(1)})$ on variables $h_i, e_i, x, y$ including $\sigma_i, \tau_i$ as

$$s_0 = \{\tau_{10} \to (1+ye_{11})\frac{\sigma_2}{\tau_{11}},\ \tau_{11} \to \frac{\sigma_2}{\tau_{10}}(1+y\frac{h_2}{e_{10}}),\ \sigma_1 \to (1+ye_{11})\frac{\sigma_1\sigma_2}{\tau_{10}\tau_{11}}\},$$

$$s_1 = \{\tau_8 \leftrightarrow \tau_9\}, \quad s_2 = \{\tau_7 \leftrightarrow \tau_8\},$$

$$s_3 = \{\tau_1 \to (1+x\frac{e_7}{h_1})\frac{\sigma_1}{\tau_7},\ \tau_7 \to \frac{\sigma_1}{\tau_1}(1+\frac{x}{e_1}),\ \sigma_2 \to \frac{\sigma_1\sigma_2}{\tau_1\tau_7}(1+\frac{x}{e_1})\},$$

$$s_4 = \{\tau_1 \leftrightarrow \tau_2\}, \quad s_5 = \{\tau_2 \leftrightarrow \tau_3\}, \quad s_6 = \{\tau_3 \leftrightarrow \tau_4\},$$

$$s_7 = \{\tau_4 \leftrightarrow \tau_5\}, \quad s_8 = \{\tau_5 \leftrightarrow \tau_6\}.$$

(The actions on $\{h_i, e_i, x, y\}$ are the same as before.)

- **Reduced actions** $r_i$

$$\begin{array}{ll} r_i(u) = s_i(u), & u = h_j, e_j \\ r_i(u) = u, & u = x, y, \\ r_i(u) = s_i(u)|_{x=y=0}, & u = \sigma_j, \tau_j \end{array}$$

The actions $r_i$ on $\{\sigma_j, \tau_j\}$ are just a copy of the 'linear' actions on $\{h_j, e_j\}$.

$$r_0 = \{\tau_{10} \to \frac{\sigma_2}{\tau_{11}},\ \tau_{11} \to \frac{\sigma_2}{\tau_{10}},\ \sigma_1 \to \frac{\sigma_1\sigma_2}{\tau_{10}\tau_{11}}\},$$

$$r_1 = \{\tau_8 \leftrightarrow \tau_9\}, \quad r_2 = \{\tau_7 \leftrightarrow \tau_8\},$$

$$r_3 = \{\tau_1 \to \frac{\sigma_1}{\tau_7},\ \tau_7 \to \frac{\sigma_1}{\tau_1},\ \sigma_2 \to \frac{\sigma_1\sigma_2}{\tau_1\tau_7}\},$$

$$r_4 = \{\tau_1 \leftrightarrow \tau_2\}, \quad r_5 = \{\tau_2 \leftrightarrow \tau_3\}, \quad r_6 = \{\tau_3 \leftrightarrow \tau_4\},$$

$$r_7 = \{\tau_4 \leftrightarrow \tau_5\}, \quad r_8 = \{\tau_5 \leftrightarrow \tau_6\}.$$

• The actions $s_i$ can be realized as the adjoint actions.

• **Thm.** On variables $e_i, h_i, \tau_i, \sigma_i, x, y$, we have

$$s_i = \mathrm{Ad}(G_i) \circ r_i,$$

$$G_0 = \frac{(\frac{h_2}{e_{10}} y; q)^+_\infty}{(e_{11} y; q)^+_\infty}, \quad G_3 = \frac{(\frac{1}{e_1} x; q)^+_\infty}{(\frac{e_7}{h_1} x; q)^+_\infty}, \quad G_i = 1 \quad (i \neq 0, 3), \qquad (1)$$

where $(z; q)^+_\infty = \prod_{i=0}^\infty (1 + q^i z)$ is the $q$-factorial.

• The braid relations for $s_i$ follow from the quantum dilogarithm identity for the $q$-factorial.

• The representation has a remarkable regularity.

• **Thm.** For any $w \in W(E_8^{(1)})$, we have

$$w(\tau_i) = F_{i,w}(x,y) \times (\text{monomial of } \{\sigma_j, \tau_j\}),$$

where $F_{i,w}(x,y)$ is a non-commutative **polynomial** in $x, y$ (cf. "Laurent phenomena", "singularity confinement").

• When $q = 1$, the polynomial $F_{i,w}$ can be determined by its bidegree $(d_1, d_2)$ and multiplicity $m_k$ at $p_k$.

• We will formulate the analog of such characterization for quantum case $(q \neq 1)$.

**Example.** For $w = s_0s_3s_4s_0s_2s_3s_2s_1s_0s_2s_4s_3$, we have

$$w(e_{11}) = \frac{h_1^2h_2^2}{e_1e_2e_7e_8e_{10}^2e_{11}},$$

$$w(\tau_{11}) = F(x,y)\frac{\sigma_1^2\sigma_2^2}{\tau_1\tau_2\tau_7\tau_8\tau_{10}^2\tau_{11}},$$

and

$$\begin{aligned} F(x,y) &= (1+\tfrac{x}{e_1q})(1+\tfrac{x}{e_2q}) + ({\color{blue}*} + {\color{blue}*}x + {\color{blue}*}x^2)\ {\color{red}y} \\ &\quad + {\color{blue}*}\ (1+\tfrac{e_7}{h_1}x)(1+\tfrac{e_8}{h_1}x)\ {\color{red}y^2} \\ &= (1+e_{11}y)(1+w(e_{11})y) + {\color{red}x}\ (1+\tfrac{h_2}{e_{10}}y)({\color{blue}*}+{\color{blue}*}y) \\ &\quad + {\color{blue}*}\ {\color{red}x^2}\ (1+\tfrac{h_2}{e_{10}}y)(1+\tfrac{{\color{red}q}h_2}{e_{10}}y). \end{aligned}$$

Note that $(d_1,d_2) = (2,2)$, $(m_i) = (1,1,\ldots,0,2,1)$.

• **Def.** For a data $\lambda = (d_i, m_i)$, we define a $q$-difference operator $F_\lambda(x,y)$ by the following two expressions:

$$\boxed{\begin{aligned} F_\lambda &= \sum_{i=0}^{d_1} x^i \prod_{t=i}^{m_{11}-1} (1+q^t e_{11} y) \prod_{t=d_1-m_{10}}^{i-1} (1+q^t \frac{h_2}{e_{10}} y)\, U_i(y), \\ &= \sum_{i=0}^{d_2} \prod_{k=1}^{6} \prod_{t=i-m_k}^{-1} (1+q^t \frac{1}{e_k} x) \prod_{k=7}^{9} \prod_{t=0}^{i-d_2+m_k-1} (1+q^t \frac{e_k}{h_1} x)\, V_i(x)\, y^i, \end{aligned}}$$

Here $U_i, V_i$ are polynomials with suitable degrees specified by the condition: $\deg_x F = d_1$ and $\deg_y F = d_2$.

• The 1st [or 2nd] expression for $F_\lambda$ shows the **non-logarithmic** singularities around $x = 0, \infty$ [or $y = 0, \infty$], as the $q$-difference operator: $y\psi(x) = \psi(qx)$ [or $x\psi(y) = \psi(q^{-1}y)$].

• **Thm.** For $\lambda = (d, m)$ s.t. $w(e_i) = h_1^{d_1} h_2^{d_2}/(e_1^{m_1} \cdots e_{11}^{m_{11}})$, the quantum polynomial $F_\lambda$ is unique (under the normalization $F_\lambda(0,0) = 1$). Moreover, we have

$$w(\tau_i) = F_{i,w}(x, y) \times (\text{monomial of } \{\sigma_j, \tau_j\}).$$

This shows the regularity of $F_{i,w}$ and its geometric characterization.

• From this, the birational action on $x, y$ can also be computed as

$$w(x) = w(\frac{\tau_{11}}{\tau_{10}}), \quad w(y) = w(\frac{\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6}{\tau_7\tau_8\tau_9}).$$

• **A key fact for the proof**: The non-logarithmic property of $F_{i,w}$ is preserved under the Weyl group actions.

This fact follows from a realization of the Weyl group actions as the adjoint actions. [Moriyama-Y, 2104.06661]

• **Bilinear equations.** Consider the $4+5+2$ "seed" equations

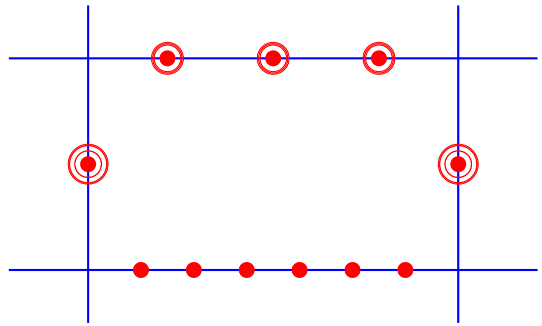
$$\boxed{\begin{aligned}
&\tau(e_{10})\tau(\tfrac{h_2}{e_{10}}) = \tfrac{h_2}{e_{10}}\tau(\tfrac{h_2}{e_i})\tau(e_i) + \tau(\tfrac{h_2}{e_j})\tau(e_j),\\
&\tau(\tfrac{h_2}{e_{11}})\tau(e_{11}) = e_{11}\tau(\tfrac{h_2}{e_i})\tau(e_i) + \tau(\tfrac{h_2}{e_j})\tau(e_j),\\
&\tau(e_i)\tau(\tfrac{h_1}{e_i}) = \tfrac{1}{e_i}\tau(\tfrac{h_1}{e_{11}})\tau(e_{11}) + \tau(\tfrac{h_1}{e_{10}})\tau(e_{10}),\\
&\tau(\tfrac{h_1}{e_j})\tau(e_j) = \tfrac{e_j}{h_1}\tau(\tfrac{h_1}{e_{11}})\tau(e_{11}) + \tau(\tfrac{h_1}{e_{10}})\tau(e_{10}),\\
&\tau(\tfrac{h_2}{e_1})\tau(e_1) = \ldots = \tau(\tfrac{h_2}{e_6})\tau(e_6),\\
&\tau(\tfrac{h_2}{e_7})\tau(e_7) = \ldots = \tau(\tfrac{h_2}{e_9})\tau(e_9).
\end{aligned}}$$

By taking copies of these relations by the action $w \in W(E_8^{(1)})$ such as $w(\tau(\lambda)) := \tau(w \cdot \lambda)$, we obtain **infinite system of bilinear equations** for the $\tau$-variables on $E_8$ lattice.

• **Thm.** The **overdetermined system** defined above is consistent and has a solution given by $\tau(\lambda) = F_\lambda(x,y)\tau^\lambda$.

## ▲ Quantum mirror curve.

• For generic parameters $(h_i, e_i)$, the curve $C$ of bi-degree $(6,3)$ with multiplicities $m_i = (1^6 2^3 3^2) = (1,\dots,1,2,2,2,3,3)$ is unique (multiple lines: $g(x,y) = x_0^3 x_1^3 y_0^2 y_1 = 0$).



• For special parameters:

$$p := \frac{h_1^6 h_2^3}{(e_1 \cdots e_6)(e_7 e_8 e_9)^2 (e_{10} e_{11})^3} = 1,$$

$\rightarrow$ the curve $C$ form **a pencil** $\lambda f(x,y) + \mu g(x,y) = 0$.
$\rightarrow$ The quantum discrete $E_8^{(1)}$ Painlevé equation reduces to an autonomous integrable system where the pencil gives **the algebraic integral**.

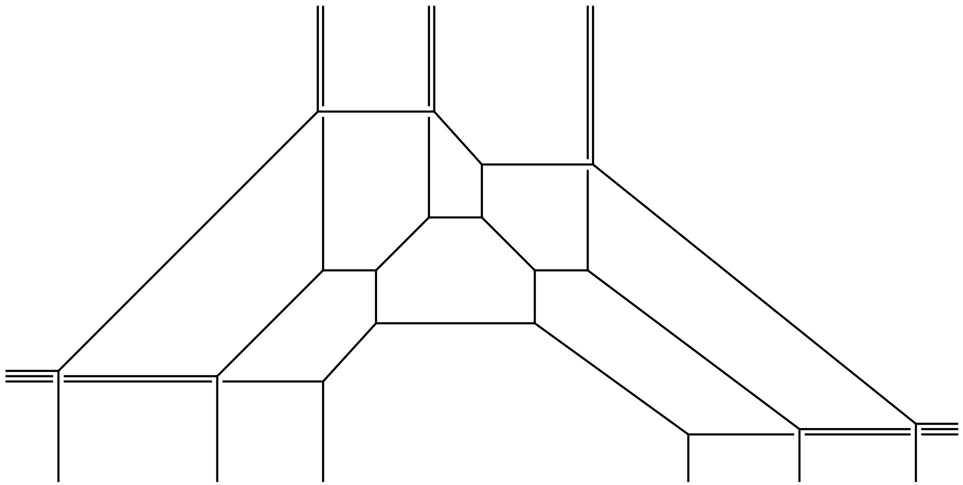
- From $W(E_8^{(1)})$ symmetry, one can determine the curve explicitly.

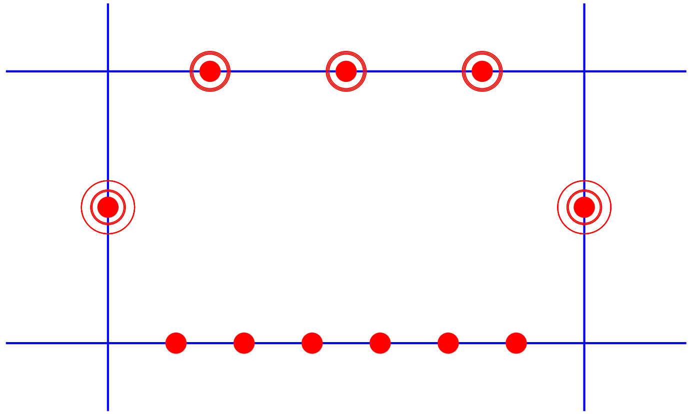
$$\boxed{\lambda(\sum_{i=0}^{3} C_i(x)y^i) + \mu x^3 y = 0}.$$

$$\boxed{\begin{aligned}
C_3(x) &= q^3 e_{11}^3 \prod_{i=7}^{9} (1+\frac{e_i}{h_1}x)(1+q\frac{e_i}{h_1}x), \\
C_2(x) &= q e_{11}^2 \prod_{i=7}^{9} (1+\frac{e_i}{h_1}x)\{[3]_q + qxA_{-1} + q\kappa A_1 x^2 + [3]_q \kappa x^3\}, \\
C_1(x) &= e_{11}\{[3]_q + [2]_q A_{-1}x + (\kappa A_1 + A_{-2})x^2 + \frac{\kappa}{q}(\kappa A_2 + A_{-1})x^4 \\
&\quad + \frac{[2]_q \kappa^2 A_1}{q^2} x^5 + \frac{[3]_q \kappa^2}{q^3} x^6\}, \qquad C_0(x) = \prod_{i=1}^{6} (1+\frac{1}{qe_i}x),
\end{aligned}}$$

$$[k]_q = \frac{1-q^k}{1-q}, \quad A_{\pm 1} = \sum_{i=1}^{9} a_i^{\pm 1}, \quad A_{\pm 2} = \sum_{1\le i<j\le 9} (a_i a_j)^{\pm 1},$$

$$a_i = e_i\ (1 \le i \le 6), \quad a_i = \tfrac{h_1}{e_i}\ (7 \le i \le 9) \quad \kappa = \frac{e_7 e_8 e_9 e_{10} e_{11}}{h_1^2 h_2}.$$





• The curve $C$ was first obtained by S.Moriyama [arXix:2007.05148] as a quantization of the classical $5d$ $E_8$ SW curve [Kim-Yagi (2015)].

• As a $q$-difference operator, the curve should be related to the trigonometric **Ruijsenaars van-Diejen operator** of type $E_8$ [Takemura (2018)] [Noumi-Ruijsenaars-Y (2020)][Chen-Haghighat-Kim-Sperling-Wang (2021)].

• There appear a good application of IMD to quantum spectral problems [Berstein-Gavrylenko-Grassi, (2105.00985)]. It will be useful also for the discrete cases.

## ▲ Summary

- Geometry of classical and quantum Painlevé equations are reviewed in relation with the gauge theory.
- We constructed a **quantum** birational rep. of affine Weyl group $W(E_8^{(1)})$.
- A **lift** of the rep. including the **tau variables** is also obtained.
- **Regularity** and the geometric characterization of the polynomial $F$ (quantum $\tau$ quotient) is proved.
- **Bilinear form** of the $qp$-$E_8^{(1)}$ ($q$-quantum $p$-difference) Painlevé equation is given.
- The quantum **mirror curve** of type $q$-$E_8^{(1)}$ is derived from its symmetry.

**Thank you!**