POTENTIALS WITH ZERO COEFFICIENT OF REFLECTION ON A BACKGROUND OF FINITE-ZONE POTENTIALS

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Very recently, the class of finite-zone potentials u(x) for the Sturm–Liouville operator $-(d^2/dx^2) + u(x)$ has been studied from various points of view (see [1, 2, 3]). In this paper, we give an algebraic geometrical construction of potentials of which both the finite-zone and the well-known rapidly decreasing potentials with zero coefficient of reflection are particular cases. In the general case, these potentials correspond to potentials without reflection on a background of finite-zone potentials. The construction of a scattering theory for asymptotically finite-zone potentials will be given in a succeeding paper.

It should be noted that, for nonreflective potentials, the idea of the present construction coincides with the idea of interpolation [4], to which the author was directed by A. B. Shabat and which stimulated further investigation.

1. Let E be a rational function with simple poles on a smooth algebraic curve X. A complex function u(x), $x \in (a, b)$, has regular analytic properties if there exist π: Y → X, a two-sheeted covering of X, and a function $\Psi(x, P)$, $P \in Y$, such that: 1°) except at the poles of $\widetilde{E} = \pi^*E$, the function is meromorphic, and its poles do not depend on x; 2°) in a neighborhood of the poles of $\widetilde{E}$, $\Psi(x, P)\, e^{i\sqrt{\widetilde{E}}(x-x_0)}$ is a regular function with value 1 at these poles;

3°) $$-\Psi''(x, P) + u(x)\Psi(x, P) = \widetilde{E}(P)\Psi(x, P). \tag{1}$$

Before formulating the first theorem, we introduce, for every effective divisor $D = \sum k_s P_s \geqslant 0$, i.e., $k_s \geq 0$, on Y the concept of an admissible divisor. Let T be the involution of Y which transposes the sheets, $D^+ = T^*D$, and $-2D_\infty$ be the divisor of poles of $\widetilde{E}$. We denote by $\mathfrak{L}_-(\mathfrak{D})$ the subspace of functions odd with respect to T^* in the linear space $\mathfrak{L}(\mathfrak{D})$ of the divisor $\mathfrak{D} = D + D^+ + D_\infty$. We recall that the linear space of a divisor is the space of rational functions for which the sum of the given divisor with their divisors of zeros and poles is an effective divisor. We admit a divisor $d \geq 0$ for D if $\deg d = \dim \mathfrak{L}_-(\mathfrak{D}) - 1$, while $\dim(\mathfrak{L}_-(\mathfrak{D}) \cap \mathfrak{L}(\mathfrak{D} - d)) = 1$.

<u>THEOREM 1.</u> A function $\Psi(x, P)$ which satisfies conditions 1° and 2° satisfies Eq. (1) with some potential u(x) if and only if there exists a divisor $d = \sum l_s \varkappa_s$, admissible for its divisor D of poles, such that

$$\frac{d^i}{dz^i}(\Psi(\Psi^+)^{-1})\big|_{\varkappa_s} = 1, \qquad i = 0, \ldots, l_s - 1. \tag{2}$$

Here $\Psi^+ = T^*\Psi$.

Under the premises of the theorem, for every x the Wronskian $F = \Psi'\Psi^+ - \Psi\Psi^{+\prime} \in \mathfrak{L}_-(\mathfrak{D})$. The assertion of the theorem is equivalent to the fact that F does not depend on x. Here Eq. (2) holds at the zeros of F. (We agree to choose, between the possibilities, a divisor d, admissible for D, on the upper sheet.) Conversely, by the definition of d, it follows from (2) that F is constant.

<u>Definition.</u> The divisors D, d will be called the scattering data for u(x).

<u>THEOREM 2.</u> For an arbitrary set of scattering data, the inverse problem is solvable if and only if E is a function with one simple pole on a rational curve.

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The proof of Theorem 2 follows from comparison of the dimension of the space formed by the functions $\Psi(x, P)$, which satisfy conditions 1° and 2° and have divisor of poles D, for fixed x, and the number of Eqs. (2), i.e., deg d.

On a hyperelliptic curve of genus g, for a divisor D of degree N, there exists an admissible divisor d if and only if $N \leq g$. Here deg d = N − g. To finite-zone potentials corresponds the condition deg D = g. Then deg d = 0, and u(x) is uniquely determined by the divisor D. To the case of nonreflective potentials corresponds a hyperelliptic curve of genus 0.

Remark. It is easy to obtain the "formula for traces" for u(x) given by the divisors D and $d = \sum \varkappa_s$ on the hyperelliptic curve $\Gamma_g \left(y^2 = \prod_{i=1}^{2g+1} (E - E_i) \right)$:

$$u(x) = \sum_{i=1}^{2g+1} E_i + 2 \sum_{s=1}^{N-g} \tilde{E}(\varkappa_s) - 2 \sum_{k=1}^{N} \gamma_k(x).$$

Here the $\gamma_k(x)$ are the values of $\tilde{E}$ at the zeros of $\Psi(x, P)$.

COROLLARY. Let $-\infty = E_0 \leqslant \ldots \leqslant E_{2g+1} < \infty$ be real, the $\varkappa_s$ lie in the intervals (E_{2n}, E_{2n+1}), and the points of D lie such that one is in each of the intervals obtained; then u(x) will be a smooth real function as $x \to \pm\infty$, exponentially approaching finite-zone potentials $u_\pm(x)$. The potential $u_+(x)$ is given by the effective divisor equivalent to D − d, and $u_-(x)$ by the divisor equivalent to $D - d^+$.

Thus, the divisor d determines a displacement on the set of finite-zone potentials. For it to be zero ($u_+(x) = u_-(x)$), it is necessary that deg d ≥ g + 1. (Our attention was directed by V. B. Matveev to the presence of a displacement in the case of soliton perturbations of single-zone potentials, the study of which from other points of view was undertaken in [5].)

2. In this paragraph, we give explicit formulas for a k-soliton potential on the background of an n-zone potential and also an analog of the superposition laws for nonreflective potentials [1].

Let the potential u(x) be given by the divisors $D = P_1 + \ldots + P_{n+k}$ and $d = \varkappa_1 + \ldots + \varkappa_k$ on the hyperelliptic curve Γ_n. We denote by $u_i(x)$ the n-zone potentials given by the divisors $P_1 + \ldots + P_{n-1} + P_{n+i}$, $0 \leqslant i \leqslant k$; the $\Psi_i(x, P)$ are their corresponding Bloch functions.

THEOREM 3. Let $K(x) = \int_{x_0}^{x} u(x)\,dx$, $K_i(x) = \int_{x_0}^{x} u_i(x)\,dx$; then $K(x) = \sum_{i=0}^{k} a_i(x) K_i(x)$, where the $a_i(x)$ are the solutions of the system

$$\sum_i a_i(x) (\Psi_i(x, \varkappa_s) - \Psi_i^+(x, \varkappa_s)) = 0, \qquad \sum_i a_i(x) = 1. \tag{3}$$

The functions $a_i(x)$ are rational functions of the $\Psi_i(x, \varkappa_s) - \Psi_i^+(x, \varkappa_s)$. These, in turn, can be expressed rationally in terms of single-soliton potentials on a background of n-zone potentials. Only the awkwardness of the expressions obtained forces us to confine ourselves to the formulation in the theorem.

THEOREM 4. K(x) is a rational function of integrals of n-zone potentials and of single-soliton potentials on a background of n-zone potentials.

To obtain effective formulas in the case of k-soliton perturbations of single-zone potentials, it is necessary to use, in addition to Theorem 3, the fact that the Bloch function corresponding to a single-zone potential given by a point z_0 is

$$\Psi(x, z) = \frac{\sigma(z - z_0 - i(x - x_0))}{\sigma(z - z_0)} e^{i\zeta(z)(x - x_0)}.$$

Here, $\sigma(z) = \sigma(z \mid \omega, \omega')$ and $\zeta(z) = \zeta(z \mid \omega, \omega')$ are the Weierstrass σ- and ζ-functions.

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