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METHODS OF ALGEBRAIC GEOMETRY IN THE THEORY OF NON-LINEAR EQUATIONS

I. M. Krichever

Contents

Introduction

The mechanism of integrating the Korteweg-de Vries equation by the method of the inverse scattering problem, which was proposed in [1] (Gardner, Green, Kruskal, Miura), was interpreted from various points of view by Lax [2], Zakharov and Faddeev [3] and Gardner [4]. Beginning with the paper by Zakharov and Shabat [5], many other physically important equations were found that can be integrated by this method over the class of rapidly decreasing functions. Among them are the following, all familiar in mathematical physics: the non-linear Schrödinger equation $\alpha u_t = u_{xx} \pm |u|^2 u$ ([5], [6]), the "sine-Gordon" equation $u_{xt} = \sin u$ ([7], [11], [12]), the Kadomtsev–Petviashvili equation

$$\frac{3}{4}\beta^2 \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left[\alpha \frac{\partial v}{\partial t} + h\frac{\partial v}{\partial x} + \frac{1}{4}\left(\frac{\partial^3 v}{\partial x^3} + 6v\frac{\partial v}{\partial x}\right)\right] = 0 \text{ ([8])},$$

and many others [9]–[17].

A method for finding them was developed by Zakharov and Shabat [18], [19].

The use of scattering theory restricted the method to the class of functions that decrease rapidly in the spatial variable. The periodic problem required essentially new ideas, the first of which were derived by Novikov in [20]. (Some of the results of this paper were also obtained simultaneously by Lax [21].) In subsequent papers by Dubrovin [44], Dubrovin and Novikov [54], Its and Matveev [36], and Lax [47] a theory was constructed of the so-called finite-zone periodic and conditionally periodic solutions of the K-dV equation and their profound algebraic-geometrical nature was discovered.[1]

In a series of papers Marchenko and Ostrovskii ([23], [24], [25]) obtained results on the approximation of arbitrary periodic potentials by finite zone potentials with the same period.[2] Novikov and Dubrovin were the first to introduce the general concept of a finite-zone linear differential operator, for which the Bloch eigenfunction (or the Flock function) is defined on a Riemann surface of finite genus (an algebraic curve). A survey of these results and a full bibliography are contained in [26].

The author has proposed an algebraic-geometrical construction of a broad class of periodic and conditionally periodic solutions of the general Zakharov–Shabat equation $L_t - A_y = [A, L]$, which makes it possible to express them explicitly in terms of the Riemann θ-function. In particular, the non-stationary Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}-u(x, y, t)\psi=0.$$

is incidentally solved by explicit formulae for the so constructed solutions $u(x, y, t)$ of the physically important Kadomtsev–Petviashvili equation [27]. In addition, this construction gives a solution of the problem of classifying commutative rings of differential operators in one variable, in the first instance when the ring contains a pair of operators of relatively prime order (see [28], [29], [30]).

A fruitful discussion of these problems took place in Gel'fand's seminar at the Moscow State University, after talks the author gave in November – December 1975. Drinfeld [31] indicated an abstract-algebraic exposition of the author's construction, which gave rise to some useful generalizations and, in particular, made advances in the problem of classifying commutative rings of differential operators on the real line, without assuming that they are prime in pairs. A complete solution of this problem was then obtained by the author (see §2 of this survey). Gel'fand and Dikii have investigated the Hamiltonian structure of equations of Lax type $L_t = [L, A]$ in which L

[1] Some of the results of Novikov–Dubrovin–Matveev–Its were later obtained by McKean and van Moerbeke [22].

[2] An approximation in the class of conditionally periodic potentials clearly follows from [36], [44], but to establish an approximation with the same period is difficult.

and A are operators with scalar coefficients and the order of L is greater than two. The corresponding analogue to the Hamiltonian formalism of Gardner–Zakharov–Faddeev proved to be rather complicated. The results are given in [32] and [33]. So far the Hamiltonian formalism even for the stationary equations of the type of the Novikov equation $[L, A] = 0$ solved by the author, which give rise to commutative algebras, has not been worked out (see Appendix 1).

The concept of a finite-zone differential operator can be generalized to the case of several independent variables. Roughly speaking, a linear differential operator in n variables is said to be k-algebraic if it has a family of eigenfunctions, parametrized by points of a k-dimensional complex algebraic variety M^k with "good" analytical properties, similar to the properties of Bloch functions of a finite-zone one-dimensional Schrödinger operator. The broadest is the case $k = 1$ for an arbitrary number of variables.

Dubrovin, Novikov, and the author have solved the inverse problem of the reconstruction of a 1-algebraic (weakly algebraic, in the terminology of [34]) two-dimensional Schrödinger operator. They have shown that systems of compatible 1-algebraic operators with a common variety M^1 in the two-dimensional case form an analogue to a commutative algebra. The commutator of any pair of operators from such an "algebra" can be factored on the right by one and the same Schrödinger operator H:

$$[L_i, L_j] = D_{ij}H, \qquad [L_i, H] = D_iH,$$

where the D_{ij} and D_i are linear differential operators.

§4 contains an investigation of k-algebraic ($k > 1$) linear differential operators. As yet we have no solutions of the inverse problems for them. This leads to interesting new problems in algebraic geometry.

Finally, in the concluding section of the survey we give an account of the results of Moser, McKean and Airault, who have discovered a remarkable connection between the behaviour of singularities of rational and elliptic solutions of the K-dV equation and the motion of n particles on a straight line [51].

§1. The Akhiezer function and the Zakharov–Shabat equations

We consider the non-linear partial differential equations for the coefficients of the operators

$$(1.1) \qquad L_1 = \sum_{\alpha=0}^{n} u_\alpha(x, y, t)\frac{\partial^\alpha}{\partial x^\alpha}, \quad L_2 = \sum_{\beta=0}^{m} v_\beta(x, y, t)\frac{\partial^\beta}{\partial x^\beta},$$

which are equivalent to the operator equation

$$(1.2) \qquad \left[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}\right] = 0, \quad \text{where} \quad [A, B] = AB - BA.$$

Zakharov and Shabat [19] were the first to study these equations. They developed a method of obtaining certain exact solutions that are rapidly decreasing as $|x| \to \infty$.

In this survey we limit ourselves, for the sake of definiteness, to operators with scalar coefficients. All the results carry over easily to the general case of matrix coefficients (see [29], [30]).

In the subsequent construction of exact solutions of the Zakharov–Shabat equations a central role is played by the concept of an Akhiezer function.

LEMMA 1.1. *For each regular complex curve $\mathfrak{R}$ of genus g with a distinguished point P_0 and a non-special effective divisor of degree g (that is, for a set of g points $p_1, \ldots, p_g$ in general position) there exists a unique function $\psi(x, y, t, P)$, $P \in \mathfrak{R}$, having the following properties.*

1°. *Except at P_0 it is meromorphic, with poles at $p_1, \ldots, p_g$.*

2°. *Near P_0 it can be represented in the form*

$$(1.3)\qquad \psi(x, y, t, P) = \exp(kx + Q(k)y + R(k)t)\left(1 + \sum_{s=1}^{\infty} \xi_s(x, y, t)k^{-s}\right),$$

where $k^{-1} = k^{-1}(P)$ is a fixed local parameter, $k^{-1}(P_0) = 0$, and $Q(k) = q_m k^m + \ldots + q_0$ and $R(k) = r_n k^n + \ldots + r_0$ are polynomials.

Functions of this kind were first considered by Akhiezer [35] in the case of the hyperelliptic curve $w^2 = \prod_{i=1}^{2n+1}(E - E_i)$, with $P_0, p_1, \ldots, p_n$ as branch points.

Without proving the lemma, we pass on to the main theorem of this section, which was first established in [28].

THEOREM 1.1. *For each Akhiezer function there exist unique operators L_1 and L_2 of the form* (1.1) *such that*

$$L_1\psi = \frac{\partial}{\partial t}\psi, \qquad L_2\psi = \frac{\partial}{\partial y}\psi.$$

PROOF. For any formal series (1.3) there is a unique operator L_1 such that

$$L_1\psi(x, y, t, P) \equiv \frac{\partial}{\partial t}\psi(x, y, t, P) \,(\mathrm{mod}\, O(k^{-1})\, e^{kx+Q(k)y+R(k)t}).$$

Its coefficients can be found from a system of equations equivalent to this congruence:

$$(1.4)\qquad \sum_{\alpha=0}^{n} u_\alpha \sum_{l=0}^{\alpha} C_\alpha^l \frac{\partial^{\alpha-l}}{\partial x^{\alpha-l}} \xi_{s+l} = \sum_{i=0}^{n} r_i \xi_{s+i}, \qquad (\xi_j = 0,\ j < 0).$$

Then $u_n = r_n$, $u_{n-1} = r_{n-1}$, $u_{n-2} = r_{n-2} - nr_n \frac{\partial}{\partial x}\xi_1, \ldots$

The purpose of the compact curve $\mathfrak{R}$ is that an exact equation for the Akhiezer function can be derived from the congruence above. This is a characteristic feature in the solution of all the following inverse problems.

We consider the function $\left(L_1 - \frac{\partial}{\partial t}\right)\psi(x, y, t, P) = 0$. It satisfies all the requirements defining the Akhiezer function except one. The expansion of the regular factor for an exponent in P_0 begins with $O(k^{-1})$. From the uniqueness of $\psi(x, y, t, P)$ it follows that this function vanishes. The operator L_2 can be found similarly.

COROLLARY. *The operators so constructed satisfy the equation* $\left[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}\right] = 0$.

PROOF OF THE COROLLARY. The kernel of the operator $\left[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}\right]$ contains a one-parameter family of functions $\psi(x, y, t, P)$. Since the operator itself contains differentiation only with respect to x, its kernel; if it is not zero, is finite-dimensional. This contradiction proves the assertion of the corollary.

We consider an important example of the construction of solutions of the Kadomtsev–Petviashvili equation, according to this scheme.

Let $Q(k) = q_2k^2 + q_0$, $R(k) = r_3k^3 + r_1k + r_0$. By what has been proved, each regular complex curve $\mathfrak{R}$ of genus g with a distinguished point P_0 and a non-special effective divisor of degree g defines the operators

$$L_1 = q_2\left(\frac{\partial^2}{\partial x^2} + v_0(x, y, t)\right), \quad L_2 = r_3\left(\frac{\partial^3}{\partial x^3} + u_1(x, y, t)\frac{\partial}{\partial x} + u_0(x, y, t)\right),$$

which satisfying (1.2). Eliminating u_1 and u_2 from the equivalent system of equations we obtain for $v(x, y, t)$ the equation

$$\frac{3}{4}\beta^2\frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left\{\alpha\frac{\partial v}{\partial t} + h\frac{\partial v}{\partial x} + \frac{1}{4}\left(\frac{\partial^3 v}{\partial x^3} + 6v\frac{\partial v}{\partial x}\right)\right\} = 0,$$

where $\alpha = q_2^{-1}$, $\beta = d_3^{-1}$.

From (1.4) it follows that $v = q - 2\frac{\partial}{\partial x}\xi_1$. To find an explicit expression for $v(x, y, t)$, we express $\psi(x, y, t, P)$ in terms of the Riemann θ-function. In passing we also prove Lemma 1.1. Its [58] first obtained corresponding expressions for the case of a hyperelliptic curve with a branch point P_0.

On the regular algebraic curve $\mathfrak{R}$ of genus g we fix a basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$, with the intersection matrix

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}.$$

We now introduce a basis of holomorphic differentials ω_i on $\mathfrak{R}$, normalized by the conditions $\oint_{a_i}\omega_k = \delta_{ik}$. We denote by B the matrix of b-periods:

$B_{ik} = \oint\limits_{b_i} \omega_k$. This matrix is known to be symmetric and to have positive-definite imaginary part.

The integer linear combinations of vectors in C^g with coordinates δ_{ik} and B_{ik} form a lattice, which determines the complex torus $J(\mathfrak{R})$, the so-called Jacobian variety of the curve.

Let $\bar{P}$ be the distinguished point on $\mathfrak{R}$; then there is a well-defined mapping $\omega\colon \mathfrak{R} \to J(\mathfrak{R})$ the coordinates of $\omega(P)$ are $\int\limits_{\bar{P}}^{P} \omega_k$.

From B we construct the Riemann θ-function, the entire function of g complex variables

$$\theta(u_1, \ldots, u_g) = \sum_{m \in Z^g} \exp(\pi i (Bm, m) + 2\pi i (m, u)),$$

where $(m, u) = m_1 u_1 + \ldots + m_g u_g$.

It has the following easily verifiable properties:

$$(1.5) \quad \begin{cases} \theta(u_1, \ldots, u_j + 1, \ldots, u_g) = \theta(u_1, \ldots, u_j, \ldots, u_g), \\ \theta(u_1 + B_{1k}, \ldots, u_g + B_{gk}) = e^{-\pi i B_{kk} - 2\pi i u_k}\theta(u_1, \ldots, u_g). \end{cases}$$

In addition, for any non-special effective divisor $D = \sum\limits_{j=1}^{g} p_j$ of degree g there is a vector $W(D)$ such that the function $\theta(\omega(P) + W(D))$ defined on $\mathfrak{R}$, cut along the cycles a_i, b_j, has exactly g zeros, which coincide with the p_j (see [37]).

We denote by ω_2, ω_Q, and ω_R respectively, the normalized Abelian differentials of the second kind [38] that have a unique singularity at P_0 of the form $-\frac{\partial z}{z^2}$, $dQ\left(\frac{1}{z}\right)$, and $dR\left(\frac{1}{z}\right)$ in the local parameter $z(P)$. Let $2\pi i U_1$, $2\pi i U_2$, and $2\pi i U_3$ be the vectors of their b-periods.

From (1.5) it follows that the function

$$\exp\left\{\int\limits_{\bar{P}}^{P} (x\omega_2 + y\omega_Q + t\omega_R)\right\} \frac{\theta(\omega(P) + U_1 x + U_2 y + U_3 t + W(D))}{\theta(\omega(P) + W(D))}$$

does not change its value in a circuit around the cycles a_i and b_j and is, therefore, well defined. Normalizing it at P_0 we obtain $\psi(x, y, t, P)$ in a form first suggested by Its.

Expanding it near P_0 we arrive at the following formula for the solutions of the Kadomtsev–Petviashvili equation

$$(1.6) \qquad v(x, y, t) = q + 2\frac{\partial^2}{\partial x^2} \ln \theta(U_1 x + U_2 y + U_3 t + W),$$

where W is an arbitrary point of the Jacobian of the curve.

If $U_2 = 0$ or $U_3 = 0$, which means that there is on $\mathfrak{R}$ a function with a unique pole of the second or third order at P_0, then $v(x, y, t)$ satisfies either the K-dV equation or one of the variants of the equation of the non-linear string [9]

$$\pm\frac{\partial^2 v}{\partial y^2}\mp\frac{\partial^2 v}{\partial x^2}+\frac{1}{4}\frac{\partial^4 v}{\partial x^4}+\frac{3}{2}\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right)=0,\quad \left(h=-\frac{3}{4}\beta^2=\pm 1\right).$$

In the first case $\mathfrak{R}$ is hyperelliptic, and (1.6) reduces to the Matveev–Its formula [36].

From the expression for $\psi(x, y, t, P)$ and the fact that the U_i determine rectilinear portions on the Jacobian curve, we derive the following important corollary.

COROLLARY. *All the so constructed solutions of the Zakharov–Shabat equations are conditionally periodic functions (the surface* $\mathfrak{R}$ *is regular).*

§2. Commutative rings of differential operators

2.1. General properties. We consider the system of non-linear equations in the coefficients of the operators $L_1 = \sum_{\alpha=0}^{n} u_\alpha(x)\frac{d^\alpha}{dx^\alpha}$, $L_2 = \sum_{\beta=0}^{m} v_\beta(x)\frac{d^\beta}{dx^\beta}$, that is equivalent to the condition that they commute. It is assumed a priori that these are equations in the class of germs of matrix functions of a real variable $u_\alpha^{ij}(x)$, $v_\beta^{ij}(x)$, $1 \leqslant i, j \leqslant l$. It turns out that all these solutions admit a meromorphic continuation to the whole complex plane. Almost all the solutions are conditionally periodic.

Novikov and Dubrovin in [54] have integrated the equation $[L_1, L_2] = 0$ for the case of scalar operators and $n = 2$. Dubrovin [39], [40] has discussed the case of commuting matrix operators, one of which is of the first order. Recently, Manakov [41] has found an interesting new example of their application. He has shown [42] that the equations of motion of an n-dimensional rigid body are equivalent to the condition that the operators

$$L_1=I^{-1}\frac{d}{dt}-I^{-1}\Omega,\qquad L_2=I\frac{d}{dt}+\Omega I,$$

commute, where I is the inertia tensor. The present author in [29] and [30] has completely integrated the equations for the commutativity of matrix operators of relatively prime order.

We recall that within the framework of this survey we limit ourselves to the case of scalar operators since this permits the most complete and clear presentation of the ideas involved in applying the methods of algebraic geometry in the theory of non-linear equations. The matrix version gives rise to an insignificant technical modification of all the constructions.

Let us agree that all the relevant operators have constant leading coefficients. In addition, let $u_{n-1}(x) \equiv 0$. This can always be achieved by means of a gradient transformation.

The following proposition is the basis for the applicability of methods of algebraic geometry to solve equations of the Novikov type $[L_1, L_2] = 0$.

THEOREM 2.1. *There is a polynomial in two variables $Q(w, E)$ such that $Q(L_2, L_1) = 0$.*

Apparently, Shabat was the first to obtain a theorem of this kind for the case $n = 2$.

PROOF. The operator L_2 defines a linear operator $L_2(E)$ on the space $\mathcal{L}(E)$ of solutions of the equation $L_1 y = Ey$. Its matrix elements in the canonical basis

$$c_j(x, E); \quad \frac{d^r}{dx^r} c_j(x_0, E) = \delta_{rj}, \quad 0 \leqslant r, \quad j \leqslant n-1,$$

are polynomials in E. Let $Q(w, E) = \det (w \cdot 1 - L_2^{st}(E))$ be its characteristic polynomial. The kernel of $Q(L_2, L_1)$ contains $\mathcal{L}(E)$ for all E, hence, it is infinite-dimensional. Therefore, the operator itself is zero.

2.2. The case of one-dimensional fiberings. Operators of relative prime orders. First we consider the case when for almost all E the eigenvalues of $L_2(E)$ are distinct. Then to each point $P = (w, E)$ of the algebraic curve $\mathfrak{R}^2$ given by the equation $Q(w, E) = 0$ there corresponds a one-dimensional eigenspace of $L_2(E)$. This gives a one-dimensional fibering[1] over $\mathfrak{R}$. In each fibre over $\mathfrak{R} \setminus \infty$ we select a vector with first coordinate 1 in the basis $c_j(x, E)$. The remaining coordinates are all meromorphic functions on $\mathfrak{R}$ $\lambda_j(P)$. Since the $c_j(x, E)$ are entire functions in E, the joint eigenfunction $\psi(x, P) = \sum_{j=1}^{n-1} \lambda_j(P) c_j(x, E)$ of L_1 and L_2 is meromorphic in the affine part of $\mathfrak{R}$. Its poles do not depend on x.

To find the form of $\psi(x, P)$ at infinity, we construct for each operator germ the formal Bloch function.

LEMMA 2.1. *There is a unique solution, which we denote by $\psi(x, k; x_0)$, of the equation*

$$L_1 \psi(x, k) = k^n \psi(x, k) \tag{2.1}$$

in the space of formal series of the form

$$\psi(x, k) = \Big(\sum_{s=N}^{\infty} \xi_s(x) k^{-s} \Big) e^{k(x-x_0)} \tag{2.2}$$

(where N is an integer) with the "normalization conditions" $\xi_s = 0$, $s < 0$; $\xi_0(x) \equiv 1$, $\xi_s(x_0) = 0$. Any other solution of this kind is of the form

$$\psi(x, k) = \psi(x, k; x_0) A(k), \qquad A(k) = \sum_{s=N}^{\infty} A_s k^{-s}.$$

PROOF. Equating the coefficients of k^{-1}, $s \geqslant -n$ on both sides of (2.1) we obtain

[1] In his paper [31] Drinfeld took as the starting point an axiomatization of the properties of this fibering in contemporary abstract algebraic language.

$$\sum_{\alpha=0}^{n} u_\alpha \sum_{l=0}^{\alpha} C_\alpha^l \frac{d^{\alpha-l}}{dx^{\alpha-l}} \xi_{s+l} = \xi_{s+n}.$$

We find $\xi'_{s+n-1}(x)$ from the s-th equation, since it can easily be brought to the form $0 = n\xi'_{s+n-1}$ + (terms containing ξ_j, $j < s + n - 1$).

The operator L_2 leaves the solution space of (2.1) invariant. Hence, by the lemma just proved,

$$(2.3) \qquad (\psi^{-1}(x, k; x_0) L_2 \psi(x, k; x_0)) |_{x=x_0} = k^m + \sum_{s=-m+1}^{\infty} A_s k^{-s}.$$

In consequence, the coefficients of the left-hand side, which are polynomials in the $u_\alpha(x_0)$ and their derivatives, and in which the $v_\beta(x_0)$ occur linearly, give first integrals of the original equations. Inverting the first m integrals we obtain the following corollary.

COROLLARY 1. *The coefficients of L_2 are polynomials in the $u_\alpha(x)$, their derivatives, and the constants A_s, $-m \leqslant s \leqslant 0$.*

NOTE. To prove the corollary it is sufficient that L_1 and L_2 commute to within an operator of order $n - 2$.

From (2.3) it follows that $\psi(x, k; x_0)$ is an eigenfunction for all operators commuting with L_1.

COROLLARY 2. *The ring of operators that commute with a given one is commutative.*

(This was apparently first pointed out in [55].)

The functions $\psi(x, k_j; x_0)$, $k_j^n = E$ form a basis of $\mathcal{L}(E)$ consisting of eigenvectors for $L_2(E)$. Then $Q(w, E) = \prod_{j=0}^{n} (w - A(k_j))$. Hence, if n is relatively prime to m, then for large (and hence for almost all) values of E the eigenvalues of $L_2(E)$ are distinct. Therefore, the affine part of $\mathfrak{R}$ can be completed at infinity by the single point P_0 in the neighbourhood of which $(E(P))^{-1/n}$ is a local parameter. The expansion in this local parameter $\psi(x, P)$ has the form (2.2).

Thus, with each pair of commuting operators, and so also with the commutative ring generated by them, we can associate the complex curve $\mathfrak{R}$, the so-called spectrum of the operators, with the distinguished point P_0 and the joint eigenfunction $\psi(x, P)$, which is meromorphic away from P_0, with the divisor of the poles $p_1, \ldots, p_g$, where g is the genus of the curve $\mathfrak{R}$, which has the form (2.2) in the neighbourhood of P_0. Hence, $\psi(x, P)$ is a function of Akhiezer type.

By Lemma 1.1, the spectral data $\mathfrak{R}$, P_0, $D = \sum_{j=1}^{g} p_j$ uniquely define the Akhiezer function $\psi(x, P)$. For any function $E(P)$ having a pole only at P_0 (we denote the ring of such functions by $A(\mathfrak{R}, P_0)$) Theorem 1.1

associates with $\psi(x, P)\exp(E(P)t)$ the operator L such that $L\psi(x, P) = E(P)\psi(x, P)$. Its coefficients do not depend on t. And so $\mathfrak{R}$, P_0 and D determine a homomorphism λ form $A(\mathfrak{R}, P_0)$ into the ring of differential operators.

THEOREM 2.2. *For any commutative ring A of differential operators containing a pair of operators of relatively prime order there is a complex curve $\mathfrak{R}$ of genus g with a distinguished point P_0 and an effective divisor D of degree g such that λ: $A(\mathfrak{R}, P_0) \to A$ is an isomorphism.*

2.3. Multi-dimensional fiberings. General commutative rings. We now relax the condition that the operators are of relatively prime orders. The operator $L_2(E)$ can have multiple eigenvalues. This means that then $A(k) = \widetilde{A}(k^l)$, where l is the greatest common divisor of n and m.

To each point of $\mathfrak{R}$ given by the equation

$$Q(w, E) = \prod_{j=1}^{n_1} (w - \widetilde{A}(k_j)) = 0, \quad n_1 l = n,$$

there corresponds an l-dimensional subspace of eigenvectors of $L_2(E)$. This defines an l-dimensional fibering over $\mathfrak{R}$. In each fiber over $\mathfrak{R} \setminus \infty$ we select vectors such that $\frac{d^r}{dx^r}\varphi_i(x_0, P) = \delta_{ri}$, $0 \leqslant r, i \leqslant l-1$. As before, all the $\varphi_i(x, P)$ are meromorphic in the affine part of $\mathfrak{R}$ and the divisor of their poles D_i is of degree g.

To find the form of $\varphi_i(x, P)$ near the "point at infinity" P_0, we construct the matrix and function $\psi(x, P)$ whose columns are $\varphi_i(x, P)$, $\varphi_i^{(1)}(x, P)$, and $\varphi_i^{l-1}(x, P)$. The matrix function $\Psi'(x, P)\Psi^{-1}(x, P)$ does not depend on the choice of the base $\varphi_i(x, P)$, therefore, to find it in the neighbourhood of P_0 we can use the functions $\psi(x, \widetilde{k}_j; x_0)$, $\widetilde{k}_j^l = k$. In the local parameter $k^{-1}(P)$ it has the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & & \dots & 0 & 1 \\ k+\widetilde{u}_0, & \widetilde{u}_1 & & \dots & \widetilde{u}_{l-2} & 0 \end{pmatrix} + O(k^{-1}), \tag{2.4}$$

where the $\widetilde{u}_\alpha(x)$, $0 \leqslant \alpha \leqslant l-2$, are polynomials in the coefficients of L_1 and their derivatives.

We introduce the operator $\widetilde{L} = \Sigma\, \widetilde{u}_\alpha(x)\frac{d^\alpha}{dx^\alpha}$; $\widetilde{u}_l \equiv 1$, $\widetilde{u}_{l-1} = 0$. Let $\widetilde{c}_j(x, k)$ be the canonical basis of the space of solutions of $\widetilde{L}y = ky$. From (2.4) we deduce the following result.

LEMMA 2.2. *Near P_0 the function $\varphi_j(x, P)$ can be represented in the form*

$$\varphi_j(x, P) = \widetilde{c}_j(x, k)\left(1 + \sum_{s=1}^{\infty} \xi_j(x) k^{-s}\right). \tag{2.5}$$

We now consider the inverse problem of recovering commuting operators from $\mathfrak{R}$, P_0, D_1, ..., D_l, $\tilde{u}_0(x)$, ..., $\tilde{u}_{l-2}(x)$.

LEMMA 2.3. *Given a non-singular complex curve $\mathfrak{R}$ of genus g, a distinguished point P_0, and non-special effective divisors D_j of degree g, there exist unique functions $\varphi_j(x, P)$, meromorphic away from P_0, with the divisor of poles D_j, and having near P_0 the form* (2.5).

PROOF. Let ω be a normalized Abelian differential with a single pole at P_0 of the form $-\frac{dz}{z^2}$, $z(P) = k^{-1}(P)$. The function $c_j(x, \int_{\bar{p}}^{p} \omega)$ is defined on $\mathfrak{R}$ cut along the cycles a_i and has an essential singularity at P_0. We denote by $G_i(x, t)$, $t \in a_i$, the ratio of its values on the two edges of the cut.

We now pose Riemann's problem of finding a function $f_j(x, P)$ that is meromorphic on $\mathfrak{R}$ cut along the a_i, with the divisor of poles D_j, and satisfying the boundary condition on a_i

$$f_j^+(x, t) = G_i^{-1}(x, t)\, f_i^-(x, t); \qquad f_j(x, P_0) = 1.$$

The existence, uniqueness, and explicit construction of $f_j(x, P)$ in terms of the Cauchy kernel on $\mathfrak{R}$ and the Riemann θ-functions is contained in [37]. The required function is

$$\varphi_j(x, P) = c_j\left(x, \int_{\bar{p}}^{p} \omega\right) f_j(x, P).$$

THEOREM 2.3. *For each function $E(P) \in A(\mathfrak{R}, P_0)$ there is a unique operator L of order nl where n is the multiplicity of the pole $E(P)$, such that $L\varphi_j(x, P) = E(P)\varphi_j(x, P)$.*

PROOF. We construct from $\varphi_j(x, P)$ a matrix Akhiezer function $\Psi(x, P)$. By (2.4), in the neighbourhood of P_0, $\left(\frac{d^{\alpha l}}{dx^{\alpha l}} \Psi(x, P)\right) \Psi^{-1}(x, P)$ has the form $k^{\alpha} 1 + O(k^{\alpha-1})$ where 1 is the unit matrix. As in §1, we find the coefficients of the matrix operator $\bar{L} = \sum_{\alpha=0}^{n} w_{\alpha}(x) \frac{d^{\alpha l}}{dx^{\alpha l}}$ from the congruence

$$(\bar{L}\Psi)\, \Psi^{-1} \equiv E(P) \cdot \bar{L} \pmod{O(k^{-1})}.$$

From the uniqueness of the matrix function $\Psi(x, P)$ it follows that $\bar{L}(\Psi(x, P) = E(P)\Psi(x, P)$. Recalling that the columns of $\Psi(x, P)$ consist of the derivatives $\varphi_j(x, P)$, we find that the action of $\bar{L}$ on the column vectors is the same as that of L on $\varphi_j(x, P)$.

Thus, we have arrived at the following theorem.

THEOREM 2.4. *For any commutative ring A of differential operators there exist: a curve $\mathfrak{R}$ of genus g with a distinguished point P_0, a set of* *divisors $D_1, \ldots, D_l$ of degree g, a set of arbitrary functions $\widetilde{u}_0(x), \ldots, \widetilde{u}_{l-2}(x)$ such that the homomorphism λ: $A(\mathfrak{R}, P_0) \to A$ defined by them in accordance with Theorem 2.3 is an isomorphism.*

The curve $\mathfrak{R}$ is called the spectrum of A of multiplicity l. The problem of selecting commuting operators with coefficients that are polynomials in x is interesting. An example of such operators is constructed in [43]. Their joint spectrum is the elliptic curve $w^2 = E^3 - \alpha$. The multiplicity of the spectrum is 3.

§3. The two-dimensional Schrödinger operator and the algebras associated with it

Here we give an account of the basic ideas in the paper by Dubrovin, Novikov, and the author [34], in which the inverse problem of recovering from "algebraic" spectral data an operator depending essentially on some spatial variables was first posed and solved. (We recall that in the operators considered in §1 derivatives with respect to y occurred only to the first power.)

In this context a new problem arises naturally: to describe the subrings A of the ring $\mathcal{O}$ of differential operators in two variables whose quotient rings A (mod H) by the left principal ideal generated by the Schrödinger operator H in $\mathcal{O}$ are commutative, where $H = \frac{\partial^2}{\partial z \partial \bar{z}} + v(z, \bar{z}) \frac{\partial}{\partial \bar{z}} + u(z, \bar{z})$. We call such rings "commutative modulo H". This means that for arbitrary operators $L_1, L_2 \in A$ there are operators D_1, D_2, and D_3 such that

$$[L_1, L_2] = D_1H; \; [L_1, H] = D_2H; \; [L_2, H] = D_3H. \tag{3.1}$$

The latter equations are equivalent to a system of non-linear differential equations for the coefficients of L_1, L_2, H. As we shall see, their solutions can be expressed explicitly in terms of the Riemann θ-function.

As before, we assume that the leading terms of all the operators in question are homogeneous differential operators with constant coefficients.

THEOREM 3.1. *The operators L_1 and L_2 satisfying the compatibility equations* (3.1) *are connected by an algebraic relation, that is $Q(L_2, L_1)\varphi = 0$ on the solution space of $H\varphi = 0$.*

The theorem follows from the fact that on the solution space $\mathcal{L}(E)$ of the equation

$$L_1\varphi(z, \bar{z}, E) = E\varphi(z, \bar{z}, E); \quad H\varphi(z, \bar{z}, E) = 0 \tag{3.2}$$

L_2 defines a linear operator $L_2(E)$ whose matrix elements in a canonical basis are polynomials in E, (with $\dim \mathcal{L}(E) = 2n$). Then $Q(w, E) = \det(w \cdot 1 - L_2^{st}(E))$ is the characteristic polynomial of $L_2(E)$.

We assume that the eigenvalues of $L_2(E)$ are distinct for almost all E.

Then to each point of the algebraic curve $\mathfrak{R}$ given by the equation $Q(w, E) = 0$ there corresponds an eigenvector of $L_2(E)$ whose coordinates in the canonical base $c_j(z, \bar{z}, E)$ are all meromorphic functions $\lambda_j(P)$ on $\mathfrak{R}$.

The corresponding function $\psi(z, \bar{z}, P) = \sum \lambda_j(P)c_j(z, \bar{z}, P)$ is meromorphic on $\mathfrak{R}$ away from "infinity". Its divisor of poles is of degree g, the genus of $\mathfrak{R}$.

To find the behaviour of $\psi(z, \bar{z}, P)$ at "infinity", we construct, as before, the germ of the formal Bloch function. Without loss of generality, we may take the leading terms of L_1 and L_2 to be $\frac{\partial^n}{\partial z^n} + q_1 \frac{\partial^n}{\partial \bar{z}^n}$ and $\frac{\partial^m}{\partial z^m} + q_2 \frac{\partial^m}{\partial \bar{z}^m}$, respectively.

LEMMA 3.1. *There are unique formal solutions of* (3.2) *of the form*

$$
\left\{
\begin{aligned}
&\psi_1(z, \bar{z}, k_1) = e^{k_1 z}\Big(1 + \sum_{s=1}^{\infty} \xi_s(z, \bar{z})\, k_1^{-s}\Big), \quad k_1^n = E,\\
&\psi_2(z, \bar{z}, k_2) = e^{k_2 \bar{z}}\Big(\sum_{s=0}^{\infty} \chi_s(z, \bar{z})\, k_2^{-s}\Big), \quad q_1 k_2^n = E
\end{aligned}
\right. \tag{3.3}
$$

with the "normalization" conditions $\xi_s(0, 0) = \chi_s(0, 0) = 0$, $s \geqslant 1$, $\chi_0(0, 0) = 1$.

The series $\psi_1^{-1}L_2\psi_1 = k_1^m + O(k_1^{m-1})$ and $\psi_2^{-1}L_2\psi_2 = q_2 k_2^m + O(k_2^{m-1})$ are expansions of the eigenvalues of $L_2(E)$ in the neighbourhoods of the two "points at infinity" P_1 and P_2 of $\mathfrak{R}$, provided that n and m are relatively prime and $q_1^m \neq q_2^n$.

Local parameters in the neighbourhoods P_1 and P_2 are $(E(P))^{-\frac{1}{n}}$ and $(E(P)q_1^{-1})^{-\frac{1}{n}}$. In terms of these the expansion of $\psi(z, \bar{z}, P)$ has the form (3.3).

As in the case of the Akhiezer function the properties of $\psi(z, \bar{z}, P)$ make it possible to reconstruct it from the "algebraic" data.

LEMMA 3.2. *For any non-singular complex curve* $\mathfrak{R}$ *of genus* g, *with fixed local parameters* $w_1 = k_1^{-1}(P)$ *and* $w_2 = k_2^{-1}(P)$ *in the neighbourhoods of two distinguished points* P_1 *and* P_2 *and an effective non-special divisor* D *of degree* g, *there is a unique function* $\psi(z, \bar{z}, P)$ *that is meromorphic away from* P_1 *and* P_2 *with the divisor of poles* D *and whose expansion in the neighbourhood of* P_j *in the local parameter* $k_j^{-1}(P)$ *has the form* (3.3).

Two local parameters $w_j(P)$ and $w_j'(P)$ in the neighbourhood of P_j are said to be equivalent if $(w_j^{-1}w_j')(P_j) = 1$.

COROLLARY. *The function* $\psi(z, \bar{z}, P)$ *depends only on the equivalence class of* $w_j(P)$.

LEMMA 3.3. *There is a unique operator* $H = \frac{\partial^2}{\partial z \partial \bar{z}} + v(z, \bar{z})\frac{\partial}{\partial \bar{z}} + u(z, \bar{z})$ *such that* $H\psi = 0$.

Any operator L such that $L\psi = 0$ is divisible on the right by H, that is, $L = DH$.

PROOF. For any two series $\psi_1(z, \bar{z}, k)$ and $\psi_2(z, \bar{z}, k)$ of the form (3.3) there is an operator H such that $H\psi_1 \equiv 0(\mathrm{mod}\ O(k^{-1}))$ and

$H\psi_2 \equiv 0(\text{mod } O(1))$. In a standard way within the framework of our construction, it follows from the uniqueness of $\psi(z, \bar{z}, P)$ that $H\psi(z, \bar{z}, P)$ then vanishes identically.

LEMMA 3.4. *For any function $E(P) \in A(\mathfrak{R}, P_1, P_2)$ having poles only at P_1 and P_2 there is a unique operator L of the form*

$$\sum_{\alpha=0}^{n_1} u_\alpha(z, \bar{z}) \frac{\partial^\alpha}{\partial z^\alpha} + \sum_{\beta=1}^{n_2} v_\beta(z, \bar{z}) \frac{\partial^\beta}{\partial \bar{z}^\beta},$$

where the n_j are the orders of the poles of $E(P)$ at P_j, such that

$$L\psi(z, \bar{z}, P) = E(P)\psi(z, \bar{z}, P).$$

The coefficients of L and H can be expressed in the standard way by the Riemann θ-function. For example, for H we have [34]

$$v(z, \bar{z}) = -\frac{\partial}{\partial z}\log\left[\frac{\theta(U_1 z + U_2\bar{z} + V_1 + W)}{\theta(U_1 z + U_2\bar{z} + V_2 + W)}\right],$$

$$u(z, \bar{z}) = \frac{\partial^2}{\partial z\,\partial \bar{z}}\log\theta(U_1 z + U_2\bar{z} + W).$$

Let us summarize our results.

THEOREM 3.2. *For any ring A that is "commutative modulo H" and contains operators of relatively prime order with the leading terms $\frac{\partial^n}{\partial z^n} + q_1 \frac{\partial^n}{\partial \bar{z}^n}$ and $\frac{\partial^m}{\partial z^m} + q_2 \frac{\partial^m}{\partial \bar{z}^m}$, $q_1^m \neq q_2^n$, there exist a curve $\mathfrak{R}$ of genus g with two distinguished points P_1 and P_2, an equivalence class of local parameters in neighbourhoods of P_1 and P_2 and an effective non-special divisor of degree g such that the homomorphism λ: $A(\mathfrak{R}, P_1, P_2) \to A\,(\text{mod } H)$. defined by them is an isomorphism.*

Let us dwell on some open problems. We have constructed a class of Schrödinger operators with almost-periodic potentials for which the Bloch eigenfunctions can be found exactly at the zero energy level. It is not clear when the parameters of our construction can vary with the energy, in other words: if H is defined by $\mathfrak{R}$, the points P_1 and P_2, and the divisor D, is there a family of eigenfunctions of H with arbitrary energy ($H\psi = E\psi$, $E \neq 0$), parametricized by points of the algebraic curves $\mathfrak{R}(E)$? If there is, then how does one find $\mathfrak{R}(E)$ from the initial data? How can one find the space M^2 of fiberings over the complex plane C with fibres $\mathfrak{R}(E)$? Is there an algebraic variety $\hat{M}^2$, a compactification of M^2?

This last question is closely connected with the problem of selecting among the operators we have constructed the purely potential ones, that is, the operators of the form $H = \frac{\partial^2}{\partial z \partial \bar{z}} + u(z, \bar{z})$. Only in the class of these operators does the condition of reality of coefficients, which arises naturally within the framework of our constructions, turn into the condition of being Hermitian, which must hold for the physical Schrödinger operators.

LEMMA 3.5. *If on $\mathfrak{R}$ there is an anti-involution T_1 leaving D invariant and such that $T_1(P_1) = P_2$ and $T_1^* w_1 = w_2$, where w_1 and w_2 are local parameters near P_1 and P_2, then for the operator H constructed from these data the function $v(z, \bar{z})$ is purely imaginary, and the potential $u(z, \bar{z})$ is real. Hence, H becomes real after a gauge transformation.*

NOTE 1. The real solutions of the Zahkarov–Shabat equations can be distinguished similarly.

Apparently, H is a potential if and only if the original ring $\hat{A}$ is commutative, which implies the existence of the variety $\hat{M}^2$, that is, in our terminology, H is 2-algebraic.

NOTE 2. A necessary condition for H to be a real potential operator is the existence on $\mathfrak{R}$ of a second anti-involution such that $T_2^* w_1 = -\bar{w}_2$. As Novikov has pointed out, the presence of two anti-involutions T_1 and T_2 (under certain restrictions on the situation of D on the set of fixed points of T_1, which are indicated in [34], Lemma 3) is sufficient for the Bloch eigenfunction $\psi(z, \bar{z}, P)$, where $T_2 P = P$, to be bounded in z and $\bar{z}$. (We recall that $H\psi = 0$). The set of fixed points $T_2 P = P$ is called the "real Fermi-surface".

§4. The problem of multi-dimensional n-algebraic operators

Let A be a commutative ring of differential operators in n variables

$$L = \sum_{|\alpha| \leq l} u_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

where, as usual, $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

It is assumed that all the leading coefficients of the $u_\alpha(x)$, $|\alpha| = l$, are constant.

Suppose that the symbols of the leading terms of the $L_2 \in A$ $(i = 1, \ldots, n)$, the polynomials $P_i(k)$, $k = (k_1, \ldots, k_n)$, are algebraically independent. Then the quotient ring of the ring of polynomials $C[k_1, \ldots, k_n]$ by the ideal generated by $P_i(k) - E_i$ is finite-dimensional. We denote by $G_\alpha(x)$ $(\alpha = 1, \ldots, N)$ representatives of its generators.

LEMMA 4.1. *A basis $\mathscr{L}(E)$ of the solution space of the equations $L_i y = E_i y$ is formed by functions satisfying the "normalization" conditions*

$$G_\alpha\left(\frac{\partial}{\partial x}\right) c_\beta(x_0, E) = \delta_{\alpha\beta}, \quad E = (E_1, \ldots, E_n).$$

Each operator $L_0 \in A$ determines a linear operator $L_0(E)$ in $\mathscr{L}(E)$.

LEMMA 4.2. *The matrix elements* L_0^{st} *of* $L_0(E)$ *in the base* $c_\alpha(x, E)$ *are polynomials in the variables* E_i *and* E_i^{-1}.

THEOREM 4.1. *The operators* L_i, $0 \leqslant i \leqslant n$, *are connected by the algebraic relations* $Q(L_0, \ldots, L_n) = 0$, *where* $Q(w, E_1, \ldots, E_n) = = \det(w \cdot 1 - L_0^{st}(E))$.

If the symbol $P_0(k)$ of the leading terms of L_0 assumes almost always distinct values at the roots of the equations $P_i(k) = E_i$, then for almost all E the eigenvalues of $L_0(E)$ are distinct. As before, by associating with each point of the affine variety M^n given in $C \times C^n$ by the equation $Q(w, E) = 0$ the eigenvectors of $L_0(E)$, we obtain the following lemma.

LEMMA 4.3. *There is a meromorphic function* $\psi(x, m)$, $m \in M^n$, *that is an eigenfunction for each of the operators* L_i,

$$L_i\psi(x, m) = E_i(m)\psi(x, m), \quad L_0(\psi(x, m)) = w(m)\psi(x, m).$$

In contrast to the case $n = 1$, the compactification of M^n under which $\psi(x, m)$ has "good" properties in the neighbourhood D^∞ of the divisor of infinity., is not self-evident when $n > 1$. Here D^∞ denotes a divisor of the compact algebraic variety $\hat{M}^n$ for which $\hat{M}^n \setminus D^\infty$ is isomorphic to M^n.

THEOREM 4.2. *There exists the system of equations*

$$L_i\psi(x, k) = P_i(k)\psi(x, k) \ (i = 1, \ldots, n),$$

has a unique solution of the form

$$\psi(x, k) = e^{(k, x)}\left(\sum_{s=0}^{\infty} \xi_s(x, k)\right), \tag{4.1}$$

where $(k, x) = k_1x_1 + \ldots + k_nx_n$; *the* $\xi_s(x, k)$ *are homogeneous rational functions in* k *of degree* $-s$, *and* $\xi_0(0, k) = 1$; $\xi_s(0, k) = 0$, $s \geqslant 1$.

PROOF. The functions $\xi_s(x, k)$ can be found successively from the system of equations

$$\sum_{|\alpha| \leqslant l_i} u_{\alpha, i}(x) \sum_{|r| \leqslant |\alpha|} \frac{\partial^r P_i(k)}{\partial k^r} \frac{\partial^{\alpha - r}}{\partial x^{\alpha - r}} \xi_{s+|r|} = P_i(k)\xi_s.$$

From the s-th equation we find $\dfrac{\partial}{\partial x_j}\xi_s(x, k)$. Since the L_i commute, we can integrate these partial derivatives to find $\xi_s(x, k)$. Then $\xi_s(x, k)$ has the form $F(x, k)G^{-s}(k)$, where $G(k) = \det\left\| \dfrac{\partial P_i}{\partial k_j} \right\|$.

NOTE. If $\deg L_i = l_i$, $0 \leqslant i \leqslant n$, then by a gradient transformation the L_i can be reduced to the form $P_i\left(\frac{\partial}{\partial x}\right) + \widetilde{L}_i$, $\deg \widetilde{L}_i \leqslant l_i - 2$, if and only if $\xi_0(x, k) = \xi_0(x)$ does not depend on k.

COROLLARY 1. *The coefficients of the series* $\psi^{-1}(x, k)L_0\psi(x, k) = A(k)$ *are polynomials in the system of first integrals of the equations equivalent to the conditions* $[L_i, L_j] = 0$.

COROLLARY 2. *The ring of operators commuting with* L_i, $1 \leqslant i \leqslant n$, *is commutative, and* $\psi(x, m)$ *is an eigenfunction for all the operators* $L \in A$.

COROLLARY 3. *The characteristic polynomial* $Q(w, E)$ *is* $Q(w, E) = \prod_{\alpha} (w - A(k_\alpha))$, *where the* k_α *are the roots of the system of equations* $P_i(k) = E_i$.

Let us introduce a grading in the ring $C[w, E_1, \ldots, E_n]$, by ascribing to these variables the degrees $l_0, \ldots, l_n$, respectively.

COROLLARY 4. *Let* Q^0 *be a polynomial of degree* Nl_0 *connecting the* $P_i(k)$, *that is,* $Q^0(P_0(k), \ldots, P_n(k)) = 0$. *Then* $Q(w, E) = Q^0(w, E) + \widetilde{Q}(w, E)$, $\deg \widetilde{Q}(w, E) \geqslant Nl_0 - 1$.

We now describe the required compactification of M^n. To do this we regard the "weighted" projective space $CP(w)$, $w = (l_0, \ldots, l_n)$, as quotient space of $C^{n+2} \setminus \{0\}$ under the following action of the multiplicative group of complex numbers. A point $(z_0, \ldots, z_{n+1})$ is equivalent to $(t^{l_0}z_0, \ldots, t^{l_n}z_n, tz_{n+1})$, $t \neq 0$.

Then $\hat{M}^n$ can be specified in $CP(w)$ by the equation

$$0 = z_{n+1}^{Nl_0} Q\left(\frac{z_0}{z_{n+1}^{l_0}}, \ldots, \frac{z_n}{z_{n+1}^{l_n}}\right) = Q^0(z_0, \ldots, z_n) + z_{n+1}Q_1(z_0, \ldots, z_{n+1}).$$

The open subvariety of $\hat{M}^n$: $z_{n+1} \neq 0$ is isomorphic to M^n.

The regular mapping φ: $CP^n \to CP(w)$ defined in homogeneous coordinates by

$$\varphi(v_1, \ldots, v_{n+1}) = (\ldots, P_i(v_1, \ldots, v_n), \ldots, v_{n+1}),$$

establishes a birational isomorphism between the hyperplane $v_{n+1} = 0$ and the divisor D^∞ defined by the equations $z_{n+1} = 0$, $Q^0(z_0, \ldots, z_n) = 0$. Hence, the functions $k_i(m) = \dfrac{v_i}{v_{n+1}}$ are defined in a small neighbourhood of D^∞ in $\hat{M}^n$.

THEOREM 4.3. *Near* D^∞ *the function* $\psi(x, m)$ *can be expanded in the form* (4.1), *where* $k_i(m) = \dfrac{v_i}{v_{n+1}}$.

Our assumption is that the variety $\hat{M}^n$ and the divisor of poles $\psi(x, m)$ uniquely determine the commutative ring A. The solution of the inverse problem is complicated by the fact that $\hat{M}^n$ has singularities at the images under φ of the points $v_{n+1}=0$, $G(v_1, \ldots, v_n)=0$; $G(k)=\det\left\|\frac{\partial P_i}{\partial k_j}\right\|$. The dimension of the variety of singularities $n-2$. For $n=1$ this means that $\mathfrak{R}$ is smooth. Therefore, we can use the theory of Abelian differentials to recover $\psi(x, P)$. For $n > 1$ the theory of meromorphic differentials is not effective enough, even on smooth manifolds.

The answer to the problem we have discussed in the previous section of constructing potential Schrödinger operators must yield a solution to the inverse problem for the variety $\hat{M}^n$ whose equation $Q(w, E)=0$ has the following form: if $Q^0(P^0(k), \ldots, P_n(k))$ is an algebraic relation between the homogeneous polynomials

$$P_0(k)=\sum_{i=1}^{n} k_i^2, \text{ then } Q(w, E)=Q^0(w, E)+\widetilde{Q}(w, E), \quad \deg\widetilde{Q}\leqslant \deg Q^0-2.$$

APPENDIX 1

THE HAMILTONIAN FORMALISM IN EQUATIONS OF LAX AND NOVIKOV TYPE

The K-dV equation and its higher analogues determine flows on function spaces, which according to Gardner [4] and Zakharov and Faddeev [3] are formally of the Hamiltonian form $u_t=\frac{\partial}{\partial x}\frac{\delta I_n}{\delta u}$ on any space where the $I_n=\int L_n(u, u', \ldots)\,dx$ are meaningful and commute. Here I_n is the system of K-dV integrals first found in [45], and $\frac{\delta I}{\delta u}=\sum_{k=0}^{n}(-1)^k\frac{d^k}{dx^k}\frac{\partial L}{\partial u^{(k)}}$. The skew-symmetric operator $\frac{\partial}{\partial x}$ defines the Poisson bracket (the Gardner–Zakharov– Faddeev bracket) on the space of functionals, by the formula

$$\{\widetilde{F}_1, \widetilde{F}_2\}=\int\frac{\delta F_1}{\delta u}\frac{\partial}{\partial x}\frac{\delta F_2}{\delta u}\,dx; \quad \widetilde{F}_i=\int F_i(u, u', \ldots)\,dx \qquad (i=0, 1).$$

Zakharov and Faddeev [3] have shown that on the space of rapidly decreasing functions all the "higher K-dV analogues" are completely integrable Hamiltonian systems for which the scattering data of the Sturm–Liouville operator $-\frac{d^2}{dx^2}+u(x, t)$ are variables of "action-angle" type.

Novikov has shown that a displacement in x determines on the phase-space of solutions of the stationary equations $\sum_{n=0}^{N} c_n\frac{\delta I_n}{\delta u}=h$ a completely integrable finite-dimensional Hamiltonian flow [20]. A general proposition on the connection of Hamiltonian formalisms of stationary and non-stationary equations of the form $u_c = \frac{\partial}{\partial x}\frac{\delta I}{\delta u}$ was obtained by Novikov and Bogoyavlenskii [46].

Suppose that the flows X_i $(u_t = \frac{\partial}{\partial x}\frac{\delta I_i}{\delta u}), i = 1, 2$ commute. Hence, the fixed points T_h of the flow $X_1 \left(\frac{\delta I_i}{\delta u} = h\right)$ form an invariant set of X_2. We denote the restriction of X_2 to T_h by $\varphi_n(X_1, X_2)$. The commutativity of the flows is equivalent to the fact that $\{\widetilde{I}_1, \widetilde{I}_2\} = 0$ or

$$\frac{\delta I_1}{\delta u}\frac{\partial}{\partial x}\frac{\delta I_2}{\delta u} = \frac{d}{dx}\, Q(u, u', \ldots).$$

In K-dV theory such a construction of integrals in the stationary problem was proposed by Gel'fand and Dikii [48] and Lax [47].

THEOREM [46]. *The flow $\varphi_n(X_1, X_2)$ on the phase space T_h is Hamiltonian with the Hamiltonian* $-Q - h\frac{\delta I_2}{\delta u}$.

In the case of higher K-dV equations there are remarkable canonically conjugate variables in the phase space T, which were obtained in [49].

The coefficients of the formal series $V(u, k) = \sum_{i=0}^{\infty} b_i k^i$ satisfying the equation $-V''' + 4V'(u - \frac{1}{k}) + 2Vu' = 0$ (which is equivalent to the recurrent system of equations $4b'_{n+1} = -b''_n + 4b_n u + 2b_n u'$) are uniquely determined by the initial data

$$2c(k) = V''V - \frac{(V')^2}{2} - 2V^2\left(u - \frac{1}{k}\right).$$

The higher analogues of the K-dV equation have the form $u_t = b'_{n+1}$.

Let $W = \sum_{i=0}^{\infty} w_i k^i = -\frac{1}{2}\frac{V'}{V}$; then the variables $b_i w_{n-1}$ are canonically conjugate in the phase space T of solutions of the stationary equation $b_{n+1} = 0$. A shift in x defines a Hamiltonian flow in T with the Hamiltonian H_{n+1}, where $H = \sum_{i=0}^{\infty} H_i k^i = W^2 V + \frac{c(k)}{V} - V(u - \frac{1}{k})$.

In the language of the Bloch eigenfunction $\psi(x, P)$ of the finite-zone Sturm–Liouville operator: if $\chi(x, P) = -i\,(\log\psi)' = \chi_R + i\chi_I$, then

$\chi_R = \frac{\sqrt{c(k)}}{\prod_i (k-\gamma_i(x))} = \frac{\sqrt{c(k)}}{V}$, $\chi_I = -\frac{1}{2}(\log\chi_R)' = W = \sum w_i k^i$. Hence, the b_k are the elementary symmetric polynomials $b_k = \sigma_k(\gamma_1(x), \ldots, \gamma_n(x))$. Flashka and MacLaflin [50] have constructed canonically conjugate variables to the b_k, but the spectral meaning remains unclear.

Solutions of the Zakharov–Shabat equations, independent of the variable y are described by non-linear equations for the coefficients of the operators $L_1 = \sum_{\alpha=0}^{n} u_\alpha(x, t) \frac{\partial^\alpha}{\partial x^\alpha}$ and $L_2 = \sum_{\beta=0}^{m} v_\beta(x, t) \frac{\partial^\beta}{\partial x^\beta}$, which are equivalent to the operator equation

$$\text{(A.1.1)} \qquad \left[L_1, L_2 - \frac{\partial}{\partial t}\right] = 0 \Leftrightarrow \frac{\partial L_1}{\partial t} = [L_2, L_1].$$

Since the coefficients of L_2, which commutes with L_1 by (A.1.1), can be expressed to within operators of order $n - 2$ by polynomials in the derivatives $u_\alpha(x, t)$ and constants h_s, $-m \leqslant s \leqslant 0$ (see Corollary 1 to Lemma 2.1), these equations are equivalent to systems of equations for the functions $u_\alpha(x, t)$, known as *equations of Lax type*. In addition to what was indicated in §2, algorithms for the construction of operators L_2, commuting in this manner with L_1, are contained in [19] and [32]. In the latter paper, Gelfand and Dikii have shown that equations of Lax type can be represented in the form

$$\text{(A.1.2)} \qquad \frac{\partial u}{\partial t} = l \sum_{p=1}^{N} c_p \frac{\delta A_p}{\delta u},$$

where $u = (u_0, \ldots, u_{n-2})$, $\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_0}, \ldots, \frac{\delta}{\delta u_{n-2}}\right)$, and l is a skew-symmetric operator whose matrix elements are

$$l_{rs} = \sum_{\gamma=0}^{n-1-r-s} \binom{\gamma+r}{r} u_{r+s+\gamma+1} \left(-i\frac{\partial}{\partial x}\right)^\gamma - \binom{\gamma+r}{s} \left(i\frac{d}{dx}\right)^\gamma u_{r+s+\gamma+1}.$$

The construction of the integrals A_p of equations of Lax type uses the expansion in fractional powers of the resolvent of L_1.

The operator l determines the Poisson bracket (Gelfand–Dikii bracket) on the space of functionals, by the formula

$$\{\widetilde{F}_1, \widetilde{F}_2\} = -\int \left[\sum_{r,s} \left(l_{rs} \frac{\delta F_2}{\delta u_s}\right) \frac{\delta F_1}{\delta u_r}\right] dx.$$

The proof of the Jacobi identity for this bracket is non-trivial. Gelfand and Dikii have told the author that a complete proof, not only for scalar but also for matrix operators, is in [33]. As in the case of the "higher K-dV analogues", all the flows $u_t = l \frac{\delta A_p}{\delta u}$ commute among each other.

The Lagrangian nature of the equations for stationary solutions of equations of Lax type does not follow directly from (A.1.2), since it is necessarily connected with the inversion of the operator l. The latter equations, which indicate that for L_1 there is a commuting operator L_2 are called equations of Novikov type, $[L_1, L_2] = 0$. (The construction of polynomial integrals for these, and also the complete integration of equations of Novikov type due to the present author, were quoted in §2 of this survey.)

The Lagrangian part of the Novikov equations $\sum_{p=1}^{N} c_p \frac{\delta A_p}{\delta u}$ was considered explicitly by Gelfand and Dikii only under the additional assumption that the Lagrangian $\sum_{p=1}^{N} c_p A_p$ is non-degenerate. (This assumption seems to be equivalent to our requirement that the orders of the operators L_1 and L_2 be relatively prime. For these equations there is an algorithm for the construction integrals in involution. A count of the number of independent integrals must yield the complete integrability of the corresponding Hamiltonian system.[1]

As the solutions of the equations $[L_1, L_2] = 0$ show, when the orders of the operators are not relatively prime, an interesting variant of the Hamiltonian formalism with constraints must hold for the corresponding system.

APPENDIX 2

ELLIPTIC AND RATIONAL SOLUTIONS OF THE K-dV EQUATIONS AND SYSTEMS OF MANY PARTICLES

In October 1976 I received a preprint of the paper by Airault, McKean, and Moser [51] in which a remarkable connection is discovered between the evolution of poles of rational and elliptic solutions of the K-dV equation and the motion of a discrete system of interacting particles on a line.[2]

It is easy to show that all elliptic solutions of the K-dV equation are of the form $u(x, t) = \sum_{j=1}^{n} 2\wp(x - x_j(t))$, where $\wp$ is the Weierstrass function. The K-dV equation for them is equivalent to the system

$$x_j = 6 \prod_{k \neq j} \wp(x_j - x_k), \tag{A.2.1}$$

$$\sum_{k \neq j} \wp'(x_j - x_k) = 0, \tag{A.2.2}$$

where $x_j \neq x_k$ $(j = 1, \ldots, n)$.

[1] See the concluding remarks.

[2] In January 1977 Olshanitskii and Kolodzhevo pointed out to the author that in the paper [52] of G.V. and D.V. Chudnovskii the evolution of the poles of elliptic solutions of the K-dV and Burgers–Hopf equations and certain others is interpreted in terms of the motion of a Hamiltonian system of particles on a line. Some of their results on the K-dV equation overlap with the results of [51] reported here.

In this way the question of describing elliptic solutions of the K-dV equation rests upon that of describing L^n, given in C^n by the equations (A.2.2). Apart from the case $n = 3$, practically nothing is known about it. We do not even know the dimension of L^n. Apart from the degenerate cases of "travelling" waves $f(x - ct)$, elliptic solutions of the K-dV equation with three poles reduce to the two-zone solutions $u(x, t)$ (first found by Novikov and Dubrovin [54]) for which

$$x_1 + x_2 + x_3 = 0, \qquad t = \int_0^{x_1 - x_2} \frac{dz}{\sqrt{12\,(g_2 - 3\wp^2(z))}}$$

$$x_2 - x_3 = \frac{1}{2}\wp^{-1}\left[-\wp(x_1 - x_2) + \sqrt{g_2 - 3\wp^2(x_1 - x_2)}\right].$$

As is well known, the function x^{-2} is a degenerate form of the Weierstrass function.

Thus, if we let both periods of $\wp(x)$ tend to infinity, we obtain rational solutions of the K-dV equation of the form $2 \sum_{j=1}^{3} (x - x_j(t))^{-2}$.

However, we can obtain more complete results. It is easy to prove that rational solutions of the K-dV equation must be of the form $u(x, t) = 2 \sum_{j=1}^{n} (x - x_j(t))^{-2}$. The equations (A.2.1) and (A.2.2) for the rational case reduce to the system $\dot{x}_j = \sum_{k \neq j} 6(x_j - x_k)^{-2}$, $\sum_{k \neq j} (x_j - x_k)^{-3} = 0$ $(j = 1, \ldots, n)$.

What is remarkable is the fact that the variety of rational solutions of the K-dV equation is invariant under the flows X_i determined by the "higher K-dV analogues". We denote by X_i the images of these flows on the variety L^n. Since $\dim L^n \leqslant n$, there is a flow $\widetilde{X}_k$ that vanishes on L^n. Consequently, all rational solutions of the K-dV equation are stationary for one of the higher K-dV equations, that is, they form a separatrix family of finite-zone potentials of the Sturm–Liouville operator.

The expansion at infinity of the function $X_i u(x, t)$ has the form

$$c_i \left[\prod_{d=0}^{i} \left(n - \frac{d(d+1)}{2}\right)\right] x^{-2i+1} + O(x^{-2i}), \quad c_i \neq 0.$$

When we introduce the functions $\pi_k = \sigma_k(x_1, \ldots, x_n)$, then $\widetilde{X}_i \pi_k = 0$ if $k < 2i - 1$. Since the π_k, $k \leqslant n$ form a coordinate system on L^n, the flow $\widetilde{X}_i$ must vanish on L^n for $2i - 1 > n$. Now $X_i u(x, t) = 0$ only when n takes one of the values $d(d + 1)/2$. Otherwise, L^n is empty.

If $n = d(d + 1)/2$, then the closure of L^n is isomorphic to C^d. This isomorphism is determined by the mapping under which to $t_1, \ldots, t_d$ there correspond the poles of the function $u(x, 1)$, where $u(x, t)$ is the solution of the Cauchy problem with the initial data $u(x, 0) = d\,(d + 1)x^{-2}$ for the flow $t_1 X_1 + \ldots + t_d X_d$.

In [53] Moser has established the complete integrability of a system of particles on a line with the pair potential $2x^{-2}$. The Hamiltonian of this system is $H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i<k} 2(x_j - x_k)^{-2}$. He found a representation of Lax type: $B_t = [A, B]$ for the equations of motion of this system, where the matrix elements of A and B are

$$A_{jj} = p_j, \quad A_{jk} = i\,(x_j - x_k)^{-1}, \quad j \neq k,$$
$$B_{jj} = -i \sum_{k \neq j} (x_j - x_k)^{-2}, \quad B_{jk} = i\,(x_j - x_k)^{-2}, \quad j \neq k.$$

From this representation, obviously, $F_k = \operatorname{tr} B^k$ and $F_2 = H$ are integrals in involution. Hence, the flows defined in the phase space by the F_k commute. The set of fixed points of the initial system, that is, $\operatorname{grad} F_2 = 0$ or $p_j = 0$, $\sum_{k \neq j} (x_j - x_k)^{-3}$ $(j = 1, \ldots, n)$, is L^n. A direct comparison of the formulae shows that the flows on L^n corresponding to the motion of the poles of the solutions of the K-dV equation and the restriction of the flow $\operatorname{grad} F_3$ to L^n are the same. Apparently, there is a hitherto unproven proposition that the flows $\widetilde{X}_i$ and $(\operatorname{grad} F_i|_{L^n})$ coincide on L^n.

CONCLUDING REMARKS

1. After the main text of this survey had been sent to the printers, the author learned that Veselovoi has answered a number of the questions mentioned in Appendix 1. He proved that the kernel of the operator l is formed by linear combinations $\sum_{p=-n+2}^{0} c_p \frac{\delta A_p}{\delta u}$. Hence, the stationary equations $l \frac{\delta \mathcal{L}}{\delta u} = 0$, $\mathcal{L} = \sum_{p=1}^{N} c_p A_p$ are Lagrangian, $\frac{\delta}{\delta u} (\mathcal{L} - \sum_{p=-n+2}^{0} c_p A_p) = 0$. He proved that when N and n are relatively prime, the Lagrangian is non-degenerate and the equation $l \frac{\delta \mathcal{L}}{\delta u} = 0$ is an $(n - 1)$-parameter family of completely integrable Hamiltonian systems.

2. In [56] Petviashvili, using numerical computations stated a proposition on the existence of solutions for the Kadomtsev–Petviashvili equations.

Their explicit form was found by Matveev,

$$u(x, y, t) = \mp \frac{1 \pm 4y^2 - 4(x - 12t)^2}{2(x \mp 12t)^2 + y^2 \mp \frac{1}{4}}.$$

All N soliton solutions of this equation were found in the paper [57], by Bordag, Its, Matveev, Manakov, and Zakharov.

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log \det A,$$

where $A_{nn} = (x - i\nu_n y - (\xi_n + 3\nu_n^2 t))$, $A_{nm} = \dfrac{2}{\nu_n - \nu_m}$, $n \neq m$. For the solution to have no singularities the constants must be determined by $N = 2k$, $\mathrm{Re}\, \nu_n > 0$, $\nu_{n+k} = -\bar{\nu}_n$, $\xi_{n+k} = \bar{\xi}_n$.

It is interesting that the Kadomtsev–Petviashvili equation turns out to have no interaction of solitons even of phase shift type.

3. Very recently the author has discovered an algorithm for the construction of a broad class of rational and elliptic solutions of the Zakharov–Shabat equations. The evolution of the poles of these solutions, as in the case of rational solutions of the K-dV equation, is closely connected with the motion of systems of particles on a line.

4. Recently, the author proved that a function $u(x, y, t)$ is a rational solution of the Kadomtsev–Petviashvili equation, decreasing as $x \to \infty$, if and only if $u(x, y, t) = -2 \sum_{j=1}^{N} (x - x_j(y, t))^{-2}$ (where N is arbitrary), and that the dynamics of the poles $x_j(y, t)$ in the variable y coincides with the motion of the Moser system of particles with the Hamiltonian H (see Appendix 2), while in the variable t it coincides with the flow given by the Hamiltonian F_3. Explicit forms can be found for u. Thus, the theory of discrete integrable systems is covered by the theory of algebraic-geometrical solutions of the Zakharov–Shabat equations as a theory of special solutions.

NOTE IN PROOF. Very recently the author learned of three remarkable long forgotten papers: J. L. Burchnall and T. W. Chaundy, Proc. London Math. Soc. **21** (1922), 420–440; Proc. Royal Soc. London Ser. A **118** (1928), 557–573; Ser. A **134** (1931), 471; H. E. Baker, Proc. Royal Soc. London Ser. A **118** (1928), 573–580.

In these papers the problem of the classification of commutative algebras of ordinary scalar differential operators containing a pair of operators of relatively prime orders is posed and solved. For algebras of general type the problem is reduced to Abelian integrals, although finite formulae for the coefficients of the operators are not obtained. This result was rediscovered by the author and forms part of the results of [30]. Some degenerate cases are considered in the 1931 paper. In the 1922 paper commutative algebras are found containing a Sturm-Liouville operator; an algorithm is indicated for the reduction of a potential to hyperelliptic integrals. Formulae in terms of the θ-function discovered in the 70's were not known in [26], [36], [30]. It is natural to compare these results with the theory of exact periodic solutions of the K-dV equation ([26]) and its subsequent development, which is reflected in this survey.

1. The K-dV equation and its higher analogues are of Lax form. An important consequence of the results of Gardner–Zakharov–Faddeev in K-dV theory consists in the fact that all these systems commute, and as a result of this, the K-dV equation and higher K-dV equations define a simultaneous deformation of all commutative algebras containing a Sturm–Liouville operator. This fact was the starting point of the modern theory of periodic solutions of the K-dV equation [20].

2. The works of the 20's and 30's we have mentioned are entirely of local character in x. The periodicity (quasiperiodicity) of the coefficients of the operators is not obtained. Hence, the connection between commutative algebras and the Flock theory of linear equations with periodic coefficients is not noted, where the eigenfunction of the operators is determined non-locally in terms of the translation operator through a period. The key observation of the modern theory of the K-dV equation consists in the fact that Hill operators with finitely many of lacunae can automatically be embedded in a commutative algebra. The converse is also true [20], [44], [21], [22], [36]. The omission of this connection probably accounts for the fact that the remarkable results of the 20's were unknown in operator spectral theory and had no influence on the solution of direct and inverse problems. For example, these papers are not quoted in articles by Ince (1939–40) and Hochstadt (1965), which study special examples of periodic operators with finitely many lacunae.

3. In these old papers there is no discussion of all the problems concerning the construction of polynomial integrals, of the commutativity equations, of the theory of completely integrable Hamiltonian systems, of the temporal dynamics by virtue of the K-dV equation [26], nor of the algebraic-geometrical method of constructing exact solutions of the Zakharov–Shabat equations [28], [30]. A classification of commutative rings of matrix operators or of commuting scalar operators whose orders are not relatively prime is not achieved; nor are rings of multi-dimensional operators discussed (see §§3 and 4 of this survey).

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