HOLOMORPHIC FIBERINGS AND NONLINEAR EQUATIONS. FINITE ZONE SOLUTIONS OF RANK 2

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I. The theory of finite zone solutions of the Korteweg–de Vries (K.d.V.) equation in one space variable (see the survey [1]) is well known, as well as its analogs such as the sine-Gordon equation, the Toda lattice equation, and so on. These solutions are naturally connected with the theory of holomorphic line bundles (with fibre C^1). Therefore, in the sequel we shall call them finite zone solutions of rank 1. The family of finite zone solutions of rank 1 for the K.d.V. equation in two space variables (the Kadomcev–Petviašvili (K.P.) equation) was obtained in [2].

In their recent papers [3] and [4] the authors discuss new perspectives on the inverse problem method, that are connected with the application of holomorphic vector bundles (with fibre C^1) over Riemann surfaces (algebraic curves). These papers are devoted to the following problems:

a) The problem, formulated in the twenties, of the effective classification and calculation of the coefficients of commuting linear ordinary differential operators whose orders are divisible by l [5]. The connection between this problem and l-dimensional bundles is very simple and follows naturally from the results in [2] and [6]. On the inefficient abstract-algebraic level some classification language was discussed in [7] and [8], while analytic constructions are given in [3]. In this paper explicit formulas are for the first time obtained for the coefficients of commuting operators of orders 4 and 6, which do not reduce to the rank 1 case (see Theorem 3).

b) The problem of constructing new large classes (depending on $l-1$ arbitrary functions of one variable) of exact solutions for the two-dimensional K.d.V. (K.P.) equation, and subsequently for other equations of mathematical physics in two space variables, admitting a commutative representation. A "latent", but apparently fundamental, connection between this problem and holomorphic fiberings was discovered by the authors in [4].

II. We recall the results obtained in [3] and [4]. A holomorphic fibering η of rank l, that is, a fibering with fibre C^l whose base is an algebraic curve Γ of genus g, where the determinant $\det\eta$ has the degree lg, is an algebraic-geometric object. The l-tuple $(\xi_1, \dots, \xi_l)$ of holomorphic sections, which in general depend on a collection of distinct points $\gamma_1, \dots, \gamma_{lg} \in \Gamma$ is called "equipment". We assume that this linear dependence has the form

$$\xi_l(\gamma_j) = \sum_{i=1}^{l-1} \alpha_{ij}\xi_i(\gamma_j), \quad j = 1, \dots, lg. \tag{1}$$

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The collection (γ_j, α_{ij}) is called the Tjurin parameters, defining a holomorphic vector bundle which is stable in the sense of Mumford [9].

These same parameters appear in the "clothes" of classical analysis. Following [4], we introduce the Baker–Ahiezer multiparameter vector $\psi = \{\psi_s(x_1, \dots, x_q; P; x_{10}, \dots, x_{q0})\}$, $1 \le s \le l$, where $P \in \Gamma$ and x_i and x_{i0} are numerical parameters. This vector-valued function is given by means of the following requirements:

1) all the coordinates ψ_s are meromorphic on Γ less P_0;

2) the poles of all the ψ_s do not depend on $(x_1, \dots, x_q)$, are located at the points $\gamma_1(x_0), \dots, \gamma_{lg}(x_0)$, and are of order one;

3) the residues ϕ_{sj} of the components $\psi_s(x, P, x_0)$ of the Baker–Ahiezer function at the poles γ_j are all proportional to the residue ϕ_{ij} with the coefficients $\alpha_{sj}(x_0)$ independent of $x = (x_1, \dots, x_q)$:

$$\phi_{sj}(x, x_0) = \alpha_{sj}(x_0)\phi_{lg}(x, x_0); \tag{2}$$

4) as $P \to P_0$ the vector-valued function $\psi = \{\psi_s\}$ is representable in the form

$$\psi = \left(\xi_0 + \sum_{s=1}^{\infty} \xi_s k^{-s}\right) \Psi_0(x, k; x_0), \tag{3}$$

where $\xi_0 = (1, 0, \dots, 0)$ and $\xi_s = \xi_s(x, x_0)$ are row vectors, and $z = k^{-1}(P)$ is a local parameter in the neighborhood of P_0. The $l \times l$ matrix Ψ_0 is given by means of the following requirements: the matrices

$$A_i(x, k) = \frac{\partial \Psi_0}{\partial x_i} \Psi_0^{-1}, \quad i = 1, \dots, q, \tag{4}$$

are polynomial in k; they satisfy the compatibility equations

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = [A_j, A_i]; \tag{5}$$

moreover, $\Psi_0(x_0, k; x_0) = 1$.

The above analytic properties uniquely define the vector-valued function $\psi(x, P, x_0)$, which by the same token is uniquely given by the quantities A_i, Γ, P_0, γ_j and α_{ij}.

For one variable, $q = 1$, such a function is constructed in [3], where it was established that for a specific choice of $A_1(x, k)$ the components of $\psi(x, P, x_0)$ are eigenfunctions of linear ordinary differential operators. Moreover, they correspond to the same eigenvalues, which by the same token turn out to be degenerate with multiplicity l; the orders of the operators are multiples of l.

The authors have shown ([4], §3) that a Baker–Ahiezer function ψ can be constructed which depends on $q = l(g + 1) - 1$ parameters $x_1, \dots, x_q$. It is likely that this number q is the maximum possible. The dimension of the moduli space of the equipped fiberings is equal to $l^2 g$, where g is the genus of Γ. For $l > 1$ we always have $q < l^2 g$. It follows from this that it is possible to construct q-parameter commutative groups of transformations of the moduli space whose orbits are not tori for $l > 1$. Consequently, the problem does not reduce to θ-functions. Thus, according to [3], for $l > 1$ the calculation of ψ requires the solution of a system of singular integral equations on a circle.

In [4] the authors have shown that in the important case $q = 3$, $x_1 = x$, $x_2 = y$, $x_3 = t$ the quantities A_1, A_2 and A_3 can be chosen so that the row vector ψ is annihilated by scalar operators whose form does not depend on l:

$$\left(\frac{\partial}{\partial t} - A\right)\psi = 0, \quad \left(\frac{\partial}{\partial y} - L\right)\psi = 0; \tag{6}$$
$$L = \frac{\partial^2}{\partial x^2} + u, \quad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + w.$$

Consequently, the following compatibility equation is satisfied:

$$\left[\frac{\partial}{\partial t} - A, \frac{\partial}{\partial y} - L\right] = 0. \tag{7}$$

By the same token the coefficients $u(x,y,t)$ and $w(x,y,t)$ satisfy the K.P. equation

$$0 = \frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + \frac{1}{4}\left(\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}\right)\right);$$
$$\frac{3}{4}\frac{\partial u}{\partial y} = \frac{3}{4}\frac{\partial^2 u}{\partial x^2} - \frac{\partial w}{\partial x}.$$

Definition. The solutions u and w constructed above are called finite zone solutions of genus g and rank l.

III. In view of the fact that it is impossible to simplify the calculation of ψ for nonsingular curves of genus $g \geq 1$, we develop methods of computing the solutions which do not require the preliminary calculation of the Baker–Ahiezer vector.

Lemma 1. *The Tjurin parameters* (γ_j, α_{ij})*, regarded as functions of* $x_0 = (x_{10}, \dots, x_{q0})$*, satisfy a compatible collection of differential equations with respect to the variables* x_{i0}*, whose right-hand sides can be algebraically defined in terms of* γ_j, α_{ij}*, the curve* Γ*, the point* P_0*, and the coefficients of the expansion in* $k^{-1} = z$ *of the matrices* $B_i(x,P)$ *at the point* P_0*, where* $B_i = \hat{\Psi}_{x_i}\hat{\Psi}^{-1}$*,* $\hat{\Psi}$ *being the matrix of the Wronskian for* ψ*.*

The computational algorithm for the right-hand sides can be obtained from [3], §3 and [4], §3. In the latter for $g = 1$ and $l = 2$ these right-hand sides are written in an unnecessarily complicated form and with some sign mistakes. The following proposition holds.

Lemma 2. *Let* $g = 1$*,* $l = 2$ *and suppose that the matrices* A_1, A_2 *and* A_3 *are chosen in the form given in* [4], §1, *Example* 1*. Then the quantities* $\gamma(x_0)$ *and* $\alpha(x_0)$ *satisfy the system of equations*

$$\begin{aligned} &\gamma_{ix} = (-1)^i(\alpha_2 - \alpha_1)^{-1}, \quad \alpha_{ix} = \alpha_i^2 + u + (-1)^i\Psi(\gamma_1, \gamma_2, P_0), \\ &\gamma_{iy} = 1, \quad \alpha_{iy} = -v(x,y,t), \\ &\gamma_{it} = (-1)^{i+1}(\alpha_1\alpha_2 + u/2)(\alpha_2 - \alpha_1)^{-1}, \end{aligned} \tag{8}$$

where we have performed the substitution $x_{10} = x_0 \to x$*,* $x_{20} = y_0 \to y$*,* $x_{30} = t_0 \to t$*,* $\alpha_{11} \to \alpha_1$*,* $\alpha_{21} \to \alpha_2$*. The quantity* $u(x,y,t)$ *satisfies the Kadomcev–Petviašvili equation by virtue of* (7) *and* $2v_x = u_y$*. The function* $\Phi(\gamma_1, \gamma_2, P_0)$ *has the form*

$$\Phi(\gamma_1, \gamma_2, P_0) = \zeta(\gamma_2 - \gamma_1) + \zeta(P_0 - \gamma_2) - \zeta(P_0 - \gamma_1), \tag{9}$$
$$\frac{d\zeta(z)}{dz} = -\wp(z), \quad \zeta(-z) = -\zeta(z), \quad (\wp'(z))^2 = 4\wp^3 + g_2\wp + g_3,$$

where $\wp(z)$ *is Weierstrass'* $\wp$*-function* [10].

We introduce the notation $\gamma_1 = y = c(x,t)$, $\gamma_2 = y - c(x,t) + c_0$, $c_0 = \text{const}$, $\alpha_1 - \alpha_2 = z(x,t)$, $\alpha_1 + \alpha_2 = w(x,y,t)$, $\Phi = \Phi(y,c,c_0)$.

From the addition theorem for elliptic functions [10] it follows that the quantity $Q = \partial\Phi/\partial c + \Phi^2$ does not depend on y. The equations (8) take the form

$$
\begin{gathered}
u(x,y,t) = -\alpha_1^2 - \alpha_2^2 + \phi(x,t) = -\frac{z^2 - w^2}{2} + \phi(x,t); \\
w_x = -\frac{z^2 + w^2}{2} + 2\phi(x,t).
\end{gathered}
\tag{10}
$$

Substituting the expression $w = (\log z)_x + 2\Phi z^{-1}$ in the equation for w_x, we obtain

$$
\begin{gathered}
\phi(x,t) = \frac{1 + 3c_{xx}^2}{4c_x^2} + Qc_x^2 - \frac{1}{2}\frac{c_{xxx}}{c_x}, \\
u(x,y,t) = -\frac{1}{4c_x^2} + \frac{1}{4}\frac{c_{xx}^2}{c_x^2} + 2\Phi c_{xx} + c_x^2(\Phi_c - \Phi^2) - \frac{1}{2}\frac{c_{xxx}}{c_x}. \\
c_t = \frac{3}{8c_x}(1 - c_{xx}^2) - \frac{1}{2}Qc_x^3 + \frac{1}{2}c_{xxx}.
\end{gathered}
\tag{11}
$$

It follows from [4] that the equation in t for $c(x,t)$ is "latently" isomorphic to the K.d.V. equation, but an explicit construction of this isomorphism has not been obtained.

Theorem 1. *The nonsingular solutions of the equation* (11), *bounded and smooth with respect to x and such that $c_x = z^{-1} \neq 0$ and $z \neq 0$, generate nonsingular solutions $u(x,y,t)$ of the K.P. equation which are periodic in y and bounded with respect to x. If the function $c(x,t)$ depends only on $x+at$, then the solution $u(x,y,t)$ of the K.P. equation depends on $(x+at,y)$. For $g = 1$ and $l = 2$ all the solutions of the K.P. equation depend nontrivially on x and y.*

We consider the question of nonsingular periodic solutions of the form $u(x+at,y)$ for $g = 1$ and $l = 2$. It follows from the foregoing arguments that to this end it is necessary to find a periodic solution of the equation (11), where $c = c(x+at)$, with $c_x \neq 0$ and $z = c_x^{-1} \neq 0$ for all x. We choose c as independent variable and make the susbtitution $z = h^{-2}(c)$. Then the equations (11) take the form

$$
\begin{gathered}
h'' = \frac{d^2h}{dc^2} = -\frac{\partial W(h,c)}{\partial h}, \\
W = -\frac{1}{2}Q(c,c_0)h^2 + ah^{-2} - \frac{1}{8}h^{-6},
\end{gathered}
\tag{12}
$$

where $Q(c,c_0) = \Phi_c + \Phi^2$ is an elliptic function. A qualitative analysis leads to the following conclusions.

Theorem 2. a) *The K.P. equation has a nonsingular periodic solution of genus $g = 1$ and rank $l = 2$, of the form $u(x+at,y)$ for $a \leq 0$, if and only if the equation $h'' = Qh$ has a solution without zeros.*

b) *For sufficiently large $a > 0$ the K.P. equation always has a nonsingular periodic solution of genus $g = 1$ and rank $l = 2$ which is of the cnoidal wave type and periodic in x,y,t. The calculation of these solutions reduces to finding the periodic nonvanishing positive solutions $h \neq 0$ of the equations* (12).

IV. For one variable, $q = 1$, in the case of genus $g = 1$ and rank $l = 2$ the solution of the equations (8) in x leads, using results obtained in [3], to the explicit calculation of nontrivial ordinary commuting operators L_4 and L_6 of orders 4 and 6, respectively.

Theorem 3. *The operator L_4 of rank 1 has the form (see the formulas (11))*

$$L_4 = L^2 - c_x[\wp(c+c_0) - \wp(c+c_1)]\frac{d}{dx} - \wp(c+c_0) - \wp(c+c_1),$$

where $L = d^2/dx^2 + u(x)$. The operator L_6 is connected with L_4 by the algebraic relation

$$L_6 = 4L_4^3 + g_2L_4 + g_3, \quad L_6 = 2L^3 + D.$$

where D is a third order operator.

The analysis for $g = 1$ and $l = 3$ is more complicated. The corresponding results will be published later.

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