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HOLOMORPHIC BUNDLES OVER ALGEBRAIC CURVES AND NON-LINEAR EQUATIONS

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§1. Introduction

In the theory non-linear equations of Korteweg-de Vries type admitting for example, a Lax representation in the form

$$(1)\quad \frac{\partial L}{\partial t}=[A,\ L]\ \text{where}\ L=\sum_{i=0}^{n} u_i(x,\ t)\frac{\partial^i}{\partial x^i},\quad A=\sum_{i=0}^{m} v_i(x,\ t)\frac{\partial^i}{\partial x^i},$$

the most interesting multi-soliton and finite zone classes of exact solutions are singled out by the following condition: there exists an operator B commuting with L at $t = 0$:

$$(2)\qquad [L,\ B]=\Big[\sum_{i=0}^{n} u_i(x,\ 0)\frac{\partial^i}{\partial x^i},\quad \sum_{i=0}^{N} w_i(x)\frac{\partial^i}{\partial x^i}\Big]=0$$

(this restriction then holds automatically for any value of t). In the "rank 1 situation" (see below) if, for example, the orders of the operators L and B are mutually prime (and, in the matrix case, the eigenvalues of the matrices of the

[1] This survey is based on a lecture by the authors at the Soviet–American symposium on soliton theory (Kiev, September 1979).

leading coefficients of L and B are distinct), the "typical" solutions of (1) satisfying (2) are periodic or quasi-periodic functions of x and t. They can be expressed in terms of θ-functions of Riemann surfaces, and the periodic operator L has the remarkable spectral property that the Bloch spectrum is "finite zone".

Rapidly decreasing multi-soliton solutions (corresponding to non-reflexive potentials) and also rational solutions of (1) are obtained (see [1], [2], and [3]) from the periodic solutions by various limit passages. We recall the Burchnall–Chaundy Lemma (see [4]): Suppose that two commuting ordinary differential operators (2) are connected by an algebraic relation

$$R(L, B) = 0 \tag{3}$$

where $R(\lambda, \mu)$ is a polynomial with constant coefficients. A common eigenfunction of L and B

$$L\psi = \lambda\psi. \quad B\psi = \mu\psi \quad \psi = \psi(x, \lambda, \mu)$$

is then such that λ and μ lie on the Riemann surface (3), which we denote by the symbol Γ:

$$R(\lambda, \mu) = 0\,. \tag{4}$$

The pair (λ, μ) is thus a point $P \in \Gamma$.

DEFINITION. The multiplicity of the eigenfunction $\psi(x, \lambda, \mu) = \psi(x, P)$ on the Riemann surface (that is, the dimension of the eigenspace of ψ when $P \in \Gamma$ is fixed) is called the *rank* l of the commuting pair L, B.

In this way there arises an l-dimensional holomorphic vector bundle with base Γ.

All the results on the commutativity relations (2) and on exact solutions of equations of KdV type (1) obtained up to 1978, concern the rank $l = 1$.

It is important to emphasize that in the theory of "one-dimensional" systems of type (1) the condition (2) is imposed on the operator L itself in the Lax pair.

In [5] and [6] for certain physically important "two-dimensional" systems of KdV type an analogue was observed of the algebraic representation (1) in which L has the form

$$\begin{cases} L = \dfrac{\partial}{\partial y} - M, \\ \dfrac{\partial L}{\partial t} = [A, L] \leftrightarrow \left[\dfrac{\partial}{\partial y} - M, \dfrac{\partial}{\partial t} - A\right] = 0. \end{cases} \tag{5}$$

Here M and A are ordinary linear differential operators in x with coefficients depending on x, y, and t.

In the search for exact solutions of "two-dimensional" systems of the form (5) the authors introduced the following Ansatz, which reduces to a set of conditions involving an auxiliary pair of operators L_1 and L_2:

$$(6)\qquad \begin{cases} [L, L_i]=0 \quad (i=1, 2), & \left[\frac{\partial}{\partial t}-A, L\right]=0, \\ [L_1, L_2]=0, & L=\frac{\partial}{\partial y}-M. \end{cases}$$

Here L_1 and L_2 are ordinary linear differential operators (in x alone).

In contrast to one-dimensional systems (1), the orders of the operators L_1 and L_2 are arbitrary!

This class of solutions for commuting pairs L_1 and L_2 of rank 1 was discovered in [7], and for commuting pairs L_1 and L_2 of arbitrary rank in [8] and [9]. Solutions of rank $l > 1$ depend already on arbitrary functions of a single variable.

The most important example is the standard two-dimensional KdV (or KP) equation, where

$$M=\frac{\partial^2}{\partial x^2}-U(x, y, t), \quad A=\frac{\partial^3}{\partial x^3}-\frac{3}{2}U\frac{\partial}{\partial x}+W(x, y, t),$$

$$(7)\qquad \begin{cases} W_x=\frac{3}{4}U_y-\frac{3}{4}U_{xx}, \\ W_y=U_t-\frac{3}{4}U_{xy}-\frac{1}{4}U_{xxx}+\frac{3}{2}UU_x, \end{cases}$$

or

$$\frac{3}{4}U_{yy}=\frac{\partial}{\partial x}\left(U_t+\frac{1}{4}(6UU_x-U_{xxx})\right).$$

Solutions of rank $l = 1$ (that is, when the pair L_1, L_2 has rank 1) are, according to [10] of the form

$$(8)\qquad U(x, y, t)=\text{const}+2\frac{\partial^2}{\partial x^2}\log\theta(Ux+Vy+Zt+W),$$

where $\theta(v_1, \ldots, v_g)$ is the Riemann θ-function corresponding to the Riemann surface Γ (4).

For the case $l > 1$ even the study of the commutation condition $[L_1, L_2] = 0$ itself is very difficult. In [11] the problem of classifying such pairs L_1 and L_2 for arbitrary $l > 1$ was solved by reducing the computation of the coefficients to a certain Riemann problem.

In [8], [9], and [12] we developed a method by which in certain cases the Riemann problem can be avoided and explicit formulae for the coefficients of L_1 and L_2 of rank $l > 1$ can be obtained.

§2. A multi-point vector-valued analogue of the Baker–Akhiezer function

We consider a collection of $(l \times l)$ matrix-valued functions

$$\Psi_s(x, k) \ (s=1, \ldots, m), \ x=(x_1, \ldots, x_n)$$

such that $\psi_s(0, k) = 1$ and the matrices

$$(9)\qquad A_j^s(x, k)=\left(\frac{\partial}{\partial x_j}\Psi_s(x, k)\right)\Psi_s^{-1}(x, k)$$

are polynomials in k.

The $A_i^s(x, k)$ must satisfy the relations

$$\frac{\partial A_j^s}{\partial x_i} - \frac{\partial A_i^s}{\partial x_j} = [A_i^s, A_j^s]. \tag{10}$$

The specification of the functions A_i^s as polynomials in k satisfying (10) determines $\Psi_s(x,k)$ uniquely.

Now let Γ be an arbitrary non-singular Riemann surface of genus g, and $P_1, \ldots, P_m$ a set of points of Γ with local parameters $z_s = k_s^{-1}(P)$ in neighbourhoods of them. We select on Γ an unordered set of points

$$(\gamma) = (\gamma_1, \ldots, \gamma_{gl})$$

and a set (α) of complex $(l-1)$-vectors

$$\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,l-1}) \qquad (i = 1, \ldots, gl).$$

NOTE. The combined collection (γ, α) is called the "Turin parameters", since according to [13] they determine uniquely an l-dimensional holomorphic vector bundle that is stable in the sense of Mumford, of degree gl over Γ together with the equipment, that is, the set of holomorphic sections $\eta_1, \ldots, \eta_l$. The points $\gamma_1, \ldots, \gamma_{gl}$ are in fact the points of linear dependence of the sections η_i, and the $\alpha_{i,j}$ are the coefficients of the linear dependence

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{i,j}\eta_j(\gamma_i). \tag{11}$$

We consider the following problem: to find a vector-valued function $\psi(x, P)$ on Γ that is meromorphic except at $P_1, \ldots, P_m$ and such that:

1. the poles of $\psi(x, P) = (\psi_1, \ldots, \psi_l)$ are at the points γ_i and the residues $\psi_j(x, P)$ satisfy these relations

$$\operatorname{res}_{\gamma_i} \psi_j(x, P) = \alpha_{i,j} \operatorname{res}_{\gamma_i} \psi_l(x, P), \tag{12}$$

where $\alpha_{i,j}$ and γ_i do not depend on x;

2. $\psi(x, P)$ can be expanded in a neighbourhood of P_s as

$$\psi(x, P) = \left(\sum_{i=0}^{\infty} \xi_i(x) k_s^{-i}\right) \Psi_s(x, k_s). \tag{13}$$

When $l = 1$ the "bare functions" Ψ_s are exponentials and ψ is the m-point scalar analogue to the classical Baker–Akhiezer function.

Following the scheme of [11], which is based on the technique of [14] and [15], we obtain the general result.

THEOREM. *The dimension of the linear space of functions satisfying the requirements listed for fixed* x *is* l. *To determine* ψ *uniquely it is sufficient, for example, to specify its value at one point. The determination of* ψ *reduces to a system of linear singular integral equations on small contours (the boundaries of neighbourhoods of the points* $P_1, \ldots, P_m$*). The integral equations are solved separately for each* x*; the condition* (12) *on the residues and the specification of* $\psi(x, P_0)$ *uniquely distinguishes the required vector-valued function* $\psi(x, P)$

in the solution space of the singular equations.

We call a matrix $\Psi(x, P)$ whose rows are linearly independent solutions of the problem (12)–(13) a complete Baker–Akhiezer matrix-valued function. According to the theorem, $\Psi(x, P)$ is uniquely determined to within multiplication by a non-degenerate matrix-valued function $G(x)$:

$$\widetilde{\Psi}(x,\ P) \Rightarrow G(x)\Psi(x,\ P).$$

Apart from the Turin parameters (γ, α) the arbitrariness of the construction reduces to the choice of the matrix Ψ_s.

EXAMPLE 1.(see [8], [9]). THE KP EQUATION AND COMMUTING OPERATORS. We consider the single-point Baker–Akhiezer vector-valued function $\psi(x, y, t, P)$ with an essentially singular point P_0 on the Riemann surface Γ of genus g. It is determined by the Turin parameters (γ, α) and the matrix $\Psi_0(x, y, t, k)$. When $l = 1$ it is the classical Clebsch–Gordon–Baker function [16].

a) Let $l = 2$. We choose the matrix functions $A_i(x, y, t, k)$ $(i = 1, 2, 3)$, which define Ψ_0 by (1), in the form

$$A_1 = \begin{pmatrix} 0 & 1 \\ k-u & 0 \end{pmatrix}; \quad A_2 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -\frac{u_x}{4} & k+\frac{u}{2} \\ k^2-\frac{ku}{2}-\frac{u^2}{2}-\frac{u_{xx}}{4} & \frac{u_x}{4} \end{pmatrix},$$

where $u = u(x, y, t)$.

From the consistency equations (10) if follows that $u = u(x, t)$ does not depend on y and satisfies the KdV equation:

$$4u_t = u_{xxx} - 6uu_x.$$

b) Let $l = 3$. We choose A_i in the form

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k-w & -\frac{3}{2}u & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix},$$

$$A_2 = \begin{pmatrix} u & 0 & 1 \\ k-w+u_x & -\frac{u}{2} & 0 \\ -w_x+u_{xx} & k-w+\frac{u_x}{2} & -\frac{u}{2} \end{pmatrix}.$$

From (10) it follows that $u = u(x, y)$ does not depend on t and is a solution of the Boussinesque equation

$$3u_{yy} + u_{xxxx} - 6(uu_x)_x = 0.$$

c) When $l > 3$, the matrices $A_i(x, y, t, k)$ are chosen in the form

$$A_1=\begin{pmatrix}0 & 0 & \dots & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \dots & 0 & 0\\ u_0 & & \dots & u_{l-2} & 0\end{pmatrix}+\hat{\varkappa};\quad \hat{\varkappa}=\begin{pmatrix}0 & 1 & 0 & & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & & & \dots & 1\\ k & 0 & & & \dots & 0\end{pmatrix};$$

$$A_2=\hat{\varkappa}^2+a_2,\quad A_3=\hat{\varkappa}^3+a_3,$$

where a_2 and a_3 are $(l\times l)$-matrices not depending on k, whose elements are differential polynomials on $u_0,\dots,u_{1-2}$.

IMPORTANT PROPOSITION. *In all the preceding cases the Baker–Akhiezer vector-valued function ψ, which in a neighbourhood of P_0 has the form*

$$(14)\qquad \psi(x,\ y,\ t,\ P)=\Big(\sum_{s=0}^{\infty}\xi_s(x,\ y,\ t)\,k^{-s}\Big)\,\Psi_0(k,\ y,\ t,\ k),$$

$$\xi_0=(1,\ 0,\ \dots,\ 0);\quad \xi_s=(\xi_s^{(1)},\ \dots,\ \xi_s^{(l)}),$$

is annihilated by the pair of scalar operators (7):

$$(15)\qquad \Big(\frac{\partial}{\partial y}-M\Big)\psi=\Big(\frac{\partial}{\partial t}-A\Big)\psi=0,$$

where

$$M=\frac{\partial^2}{\partial x^2}+U,\quad A=\frac{\partial^3}{\partial x^3}+\frac{3}{2}U\frac{\partial}{\partial x}+W.$$

The coefficients of U and W do not depend on P and are determined by

$$l=2:\quad U=u(x,\ t)-2\xi_{1x}^{(2)},$$
$$l\geqslant 3:\quad U=-2\xi_{1x}^{(l)}.$$

CONCLUSION. $U(x, y, t)$ is a solution of the KP equation

$$(16)\qquad \frac{3}{4}U_{yy}=\frac{\partial}{\partial x}\Big\{U_t-\frac{1}{4}(6UU_x-U_{xxx})\Big\}.$$

Thus, we obtain a class of KP solutions depending on the data

$$\{\Gamma,\ P_0,\ \gamma,\ \alpha,\ u_0,\ \dots,\ u_{l-2}\}.$$

when $l=2$, $u_0(x, t)$ is a solution of the usual KdV equation.

A vector-valued function $\psi(x,0,0,P)=\psi(x,P)$ depending on a single variable x occurred in [11]. Its components consist of l common eigenfunctions of a pair of commuting scalar ordinary differential operators (in x):

$$(17)\qquad \begin{cases} L_1\psi_q(x,\ P)=\lambda(P)\,\psi_q(x,\ P),\\ L_2\psi_q(x,\ P)=\mu(P)\,\psi_q(x,\ P),\end{cases}$$

where λ and μ are arbitrary algebraic functions on the surface Γ, having a single pole at P_0 of order m and n. The orders of L_1 and L_2 are ml and nl, respectively. Thus, a commuting ring of operators of rank l can be classified by the surface Γ, the point P_0 with a local parameter, the set of Turin parameters $\gamma_1,\dots,\gamma_{gl}$, $(\alpha_1,\dots,\alpha_{gl})$, and the arbitrary functions $u_0(x),\dots,u_{l-2}(x)$. An operator L in this ring is given by an arbitrary algebraic function $\lambda(P)$ with a single pole at P_0.

We discuss later the problem of how to calculate the coefficients of these operators effectively (see [9], [12]).

All the relations (6) follow immediately from (15) and (17). And so the solutions we have found of the KP equation of rank l correspond to the Ansatz (6).

§3. The two-dimensional Schrödinger operator and two-point functions of Baker–Akhiezer type with separate variables

The problem of a natural generalization of equations of Lax type (1) to the case of operators L that depend essentially on several spatial variables is non-trivial. We note that for equations of KP type the corresponding operator contains $\frac{\partial}{\partial y}$ only to the first degree. It is known that for a potential in general position $u(x), x = (x_1, \ldots, x_n), n > 1$, there is no operator that "almost commutes" with $L = \Delta u$, that is, an operator A such that $[L, A]$ is multiplication by a function. This means that there are no non-trivial dynamical systems of the form $\dot{L} = [A, L]$, preserving the whole spectrum of the operator L. When $n > 1$, the eigenvalues of L are infinitely degenerate. Apparently, to recover L it is sufficient to be given the "inverse problem data" only for a single energy value; for example, for $E = 0$.

Deformations preserving the spectral characteristics for the single energy value $E = 0$ are described by an equation of the form

$$\frac{\partial L}{\partial t} = [A, L] + BL, \tag{18}$$

where B is a differential operator. Such equations were first considered in [17].

The inverse problem for the two-dimensional Schrödinger operator in a magnetic field with zero flux, that is, with periodic (or quasi-periodic) coefficients, was solved in [18] by using the data for a single energy level, in a class of operators analogous in a certain sense to finite-zone operators.

We recall the basic arguments leading to a statement of the inverse problem for the recovery of the operator

$$H = \left(i\frac{\partial}{\partial x} - A_1\right)^2 + \left(i\frac{\partial}{\partial y} - A_2\right)^2 + u(x, y).$$

Suppose that the potential $u(x, y)$ and the vector potentials $A_1(x, y)$ and $A_2(x, y)$ are periodic in x and y with periods T_1 and T_2. For the equation $H\psi = E\psi$ it is natural to select the Bloch eigenfunctions as those of the operator of displacement by the period

$$\psi(x + T_1, y) = e^{ip_1T_1}\psi(x, y),$$

$$\psi(x, y + T_2) = e^{ip_2T_2}\psi(x, y).$$

The numbers p_1 and p_2 are called quasimomenta. In three-dimensional space the simultaneous eigenvalues of the monodromy operators $\hat{T}_1$ and $\hat{T}_2$ and of the operator H form a two-dimensional submanifold. Its points are sets λ_1, λ_2, E for which there exist solutions of the equation $H\psi = E\psi$ such that $\psi(x + T_1, y) = \lambda_1 \psi(x, y)$, $\psi(x, y + T_2) = \lambda_2 \psi(x, y)$. We say that H has good analytic properties if this manifold M^2 for complex values of λ_1, λ_2, and E, is a two-dimensional analytic submanifold. Then the intersection of M^2 with the surface $E = E_0$ is an analytic curve $\mathfrak{R}(E_0)$, the so-called "complex Fermi surface".

H is said to be a finite-zone operator if the genus $\mathfrak{R}(E_0)$ is finite. In this case we can clarify the asymptotic behaviour of the Bloch functions for large values of the quasimomenta in the non-physical domain of complex p_1 and p_2. In this domain they must be subject to $p_1^2 + p_2^2 = O(1)$. Hence, the curve $\mathfrak{R}(E_0)$ is compactified by two points at infinity P_1 and P_2, in a neighbourhood of which the Bloch functions have the following asymptotic expansions:

$$\psi = e^{k_1(x+iy)}\left(\sum_{l=0}^{\infty} \xi_l(x,y)\, k_1^{-l}\right) \sim e^{k_1 z},$$

$$\psi = e^{k_2(x-iy)}\left(\sum_{l=0}^{\infty} \zeta_l(x,y)\, k_2^{-l}\right) \sim e^{k_2 \bar{z}},$$

where k_1^{-1} and k_2^{-1} are local coordinates in neighbourhoods of P_1 and P_2. Except at the points P_1 and P_2, the function $\psi(x, y, P)$, $P \in \mathfrak{R}$, is meromorphic and has g poles $\gamma_1, \ldots, \gamma_g$. The problem of recovering H from the curve $\mathfrak{R}$ with two distinguished points P_1 and P_2 and from the set $\gamma_1, \ldots, \gamma_g$ was solved in [18]. We draw attention to the important fact that the asymptotic behaviour of ψ near the points P_1 and P_2 depends on the distinct variables z and $\bar{z}$. Functions of Baker–Akhiezer type with this property are called "two-point functions with separate variables". The following formulae hold (for rank $l = 1$):

$$A_{\bar{z}} = A_1 + iA_2 = -\frac{\partial}{\partial z}\log\frac{\theta(U_1 z + U_2\bar{z} + V_1 + W)}{\theta(U_1 z + U_2\bar{z} + V_2 + W)};$$

$$A_z = A_1 - iA_2 = 0;\quad z = x + iy;\quad \bar{z} = x - iy;$$

$$u(x, y) = \frac{\partial^2}{\partial z \partial \bar{z}}\log\theta(U_1 z + U_2\bar{z} + W).$$

The constant vectors U_i and V_i depend only on P_1 and P_2, but W is determined by the divisor $\gamma_1, \ldots, \gamma_g$. Generally speaking, the operator H is not Hermitian. The choice of the parameters $\mathfrak{R}, P_1, P_2, \gamma_1, \ldots, \gamma_g$, for which H is Hermitian was obtained in [19].

The condition on H to be a finite-zone operator is not stable under a variation of the energy level. This means that if the genus of the complex Fermi-surface-curve $\mathfrak{R}(E)$ of the Bloch functions satisfying the equation $H\psi = E\psi$ is finite for one value $E = E_0$, then it becomes infinite even for neighbouring values. In the theory of the KdV equation a natural generalization of the language of theta-functions enables us to solve the inverse problem for operators whose Bloch eigenfunction is defined on a hyperelliptic curve of infinite genus [20]. Because of the instability of the finite-zone property, to develop a complete theory of the two-dimensional Schrödinger operator it is necessary to generalize the above construction to the case of infinite genus. The first task is to elucidate the asymptotic behaviour and the disposition of the poles of the Bloch functions for quasimomenta at a fixed energy value. We note that the corresponding asymptotic behaviour must be considered in the non-physical domain of complex values of the quasimomenta.

The following algebraic condition for the two-dimensional Schrödinger operator, which distinguishes finite-zone solutions of equations of Lax type, is an analogue of (2). Suppose that there are linear operators L_1 and L_2 such that the commutators have the form

$$[H,\ L_i] = B_i H; \quad [L_1,\ L_2] = B_3 H, \tag{19}$$

where B_1, B_2, and B_3 are differential operators.

The simultaneous eigenvalues of the operators

$$H\psi = 0, \quad L_i\psi = \lambda_i\psi \tag{20}$$

are connected by the algebraic relation

$$R(\lambda_1,\ \lambda_2) = 0, \tag{21}$$

where $R(\lambda, \mu)$ is a polynomial in two variables.

As in the theory of finite-zone solutions of equations of Lax type and their two-dimensional generalizations (6), we introduce the concept of the rank of the algebra of operators (19), which is defined as the multiplicity of the eigenvalues, that is, the number of linearly independent solutions of the equations (20). For an algebra of rank l, the simultaneous eigenfunctions form an l-dimensional holomorphic bundle over the curve Γ given by (21). The operators H constructed above correspond to algebras of rank 1.

It would be interesting to investigate the interrelation of the concepts of rank and the "generality of position" for an operator H with periodic coefficients. For finite-zone operators this interrelation is as follows. For fixed values of the orders of L_1 and L_2 the number of parameters determining the algebraic relation (3) for algebras of rank 1 is greater than the number of parameters determining these relations for algebras of rank $l > 1$. However, in addition to the parameters specifying Γ, an algebra of rank l depends on $2(l-1)$ arbitrary functions, hence, algebras of rank $l > 1$ are, generally speaking, not degenerations of algebras of rank $l = 1$.

We now give constructions of finite-zone operators H of rank l. Let $\Psi_1(z, k)$ and $\Psi_2(\bar{z}, k)$ be matrix-valued functions defined by the equations

$$(22)\qquad \begin{cases} \dfrac{\partial}{\partial z}\Psi_1(z,k)=A^1(z,k)\Psi_1(z,k),\\ \dfrac{\partial}{\partial \bar z}\Psi_2(\bar z,k)=A^2(\bar z,k)\Psi_2(\bar z,k),\end{cases}$$

where

$$(23)\qquad A^1=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & & \dots & 0 & 1\\ k_1+u_0 & u_1 & & \dots & u_{l-2} & 0\end{pmatrix};\quad A^2=\begin{pmatrix} 0 & 0 & & \dots & & k_2+v_0\\ 1 & 0 & & \dots & & v_1\\ 0 & 1 & 0 & \dots & & v_2\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \dots & 1 & 0 & v_{l-2}\\ 0 & 0 & \dots & 0 & 1 & 0\end{pmatrix},$$

$\Psi_i(0,k)=1$ and $u_i(z)$ and $v_i(\bar z)$ are arbitrary functions.

We consider the two-point Baker–Akhiezer vector-valued function $\psi(z,\bar z,P)$ on the Riemann surface Γ of genus g, corresponding to Turin parameters (γ,α) and having the following form in a neighbourhood of the two distinguished points P_1 and P_2:

$$(24)\qquad \psi(z,\bar z,P)=\Big(\sum_{s=0}^{\infty}\xi_s(z,\bar z)k_1^{-s}\Big)\Psi_1(z,k_1),$$

$$(25)\qquad \psi(z,\bar z,P)=\Big(\sum_{s=0}^{\infty}\zeta_s(z,\bar z)k_2^{-s}\Big)\Psi_2(\bar z,k_2).$$

We normalize it by the following condition: $\xi_0=(1,0,0,\dots,0)$; $\xi=(\xi^{(1)},\dots,\xi^{(l)};\ \zeta=(\zeta^{(1)},\dots,\zeta^{(l)})$.

Here $k_\varepsilon^{-1}=k_\varepsilon^{-1}(P)$, $\varepsilon=1$, or 2, are local parameters in neighbourhoods of P_1 and P_2.

PROPOSITION. *The Baker–Akhiezer vector-valued function satisfies the condition* $H\psi=0$, *where*

$$H=\frac{\partial^2}{\partial z\partial\bar z}+v(z,\bar z)\frac{\partial}{\partial\bar z}+u(z,\bar z)$$

is the two-dimensional Schrödinger operator with scalar coefficients

$$(26)\qquad \begin{cases} v(z,\bar z)=-\dfrac{\partial}{\partial z}\log\zeta_0^{(1)}(z,\bar z),\\ u(z,\bar z)=-\dfrac{\partial}{\partial\bar z}\xi_1^{(l)}(z,\bar z).\end{cases}$$

Only Hermitian operators H which for a choice of gauge correspond to the case of a real "magnetic field" $B=\partial v/\partial\bar z$ and "electric potential" $U=2u-\partial v/\partial\bar z$ are physically meaningful.

As was mentioned above, the conditions on the parameters of our construction of operators H of rank 1 corresponding to the operators being Hermitian, were obtained in [19]. Following the ideas there we now give similar conditions for $l=2$.

We consider curves Γ with an anti-holomorphic involution $\sigma\colon\Gamma\to\Gamma$

interchanging the distinguished points, $\sigma(P_1) = P_2$, and the local parameters $k_\varepsilon^{-1}(\sigma(k_1) = -\bar{k}_2)$. We define an Abelian differential ω of the third kind with simple poles at P_1 and P_2 and with residues ± 1, respectively. Such a differential exists and is determined to within the addition of an arbitrary holomorphic differential.

We choose such a differential ω, $\omega(P) = -\bar{\omega}(\sigma(P))$, that is odd with respect to σ. The dimension of the space of such differentials is equal to the dimension of the odd holomorphic differentials $\omega_1(P)$. Since multiplication by i carries even differentials into odd ones, this real dimension is equal to g. We denote by $\gamma_1, \ldots, \gamma_{2g}$ the zeros of $\omega(P)$. Since ω is odd, the set of points (γ) is invariant under σ, $\sigma(\gamma_i) = \gamma\sigma(i)$, where $\sigma(i)$ is a corresponding permutation of the indices.

EXAMPLE. Let Γ be the hyperelliptic curve in $\mathbf{C}^2$ given by

$$y^2 = \lambda \prod_{i=1}^{2g} (\lambda - \lambda_i),$$

where the set of complex numbers λ_i is invariant under the involution $\lambda \to \bar{\lambda}^{-1}$, and $\prod_i \lambda_i = 1$.

An anti-holomorphic involution on Γ interchanging the points $P_1 = 0, P_2 = \infty$, has the form

$$P = (y, \lambda) \to \sigma(P) = \left(-\frac{\bar{y}}{\bar{\lambda}^{g+1}}, \frac{1}{\bar{\lambda}} \right).$$

The Abelian differentials with poles at P_1 and P_2 have the form

$$\omega = \frac{d\lambda}{\lambda} + \sum_{i=0}^{g-1} c_i \frac{\lambda^i \, d\lambda}{y},$$

Where the c_i are constants. The condition on ω to be odd means that $c_i = -\bar{c}_{g-1-i}$.

Thus, $\gamma_1, \ldots, \gamma_{2g}$ are the zeros of the function

$$\frac{1}{\lambda} + \frac{1}{y} \left(\sum_{i=0}^{g-1} c_i \lambda^i \right)$$

on Γ.

With each point γ_i we associate a number α_i (we recall that $l = 2$) for which $\bar{\alpha}_i = -\alpha_{\sigma(i)}^{-1}$.

In addition to the choice of Turin parameters (γ, α), the vector-valued function $\psi(z, \bar{z}, P)$ was defined by two functions $u_0(z)$ and $v_0(\bar{z})$. Let $\overline{u_0(z)} = -v_0(z)$.

PROPOSITION. *These conditions on the parameters of the problem distinguish Hermitian operators H.*

SKETCH OF PROOF. We consider the scalar function

$$\varphi(z, \bar{z}, P) = \psi(z, \bar{z}, P)\psi^+(z, \bar{z}, \sigma(P))$$

(the dagger denotes Hermitian conjugation). From (23) for $l = 2$ and $\overline{u_0(z)} = -v_0(\bar{z})$ it easily follows that

$$\Psi_1(z, k)\Psi_2^+(\bar{z}, -\bar{k}) = 1.$$

Hence, $\varphi(z, \bar{z}, P)$ is a meromorphic function on the whole curve Γ. From the fact that $\bar{\alpha}_i = \alpha^{-1}_{\sigma(i)}$ it follows that the poles of φ at the points γ_i are simple. By definition of $\gamma_1, \ldots, \gamma_{2g}$, the differential $\varphi(z, \bar{z}, P)\,\omega(P)$ has a total of two poles at P_1 and P_2. Since the sum of its residues is zero, $\varphi(z, \bar{z}, P_1) = \varphi(z, \bar{z}, P_2)$. Calculating the values of φ at P_1 and P_2 we obtain

$$\varphi(z, \bar{z}, P_1) = \bar{\zeta}_0^{(1)}, \quad \varphi(z, \bar{z}, P_2) = \zeta_0^{(1)}.$$

Hence, by (26), $B(z, \bar{z})$ is real. It is easy to see that V is also real.

§4. Deformations of holomorphic bundles

As we have said above, in the general case the problem of calculating the vector analogue of the Baker–Akhiezer function Ψ reduces to a system of singular integral equations equivalent to the Riemann problem. However, we do not need the function Ψ. In the construction of the coefficients of linear operators and solutions of corresponding non-linear equations, the Riemann problem can sometimes be avoided. This possibility is based on the study of conditions on the Turin parameters (γ, α) generalizing rectilinear windings of Jacobi tori for rank 1.

As before, let Γ be a non-singular algebraic curve of genus g with distinguished points $P_1, \ldots, P_m$ and fixed local parameters $k_s^{-1}(P)$ in neighbourhoods of them. We consider the logarithmic derivative of the Baker–Akhiezer function $\Psi(x, P)$, which was defined in the preceding section from the "bare functions" $\Psi_s(x, k)$ and the Turin parameters (γ^0, α^0), the matrix functions $\chi_i(x, P)$ being such that

$$\left(\frac{\partial}{\partial x_i} - \chi_i(x, P)\right) \Psi(x, P) = 0. \tag{27}$$

The functions $\chi_i(x, P)$ are meromorphic on Γ, having poles at $P_1, \ldots, P_m$. In addition, the $\chi_i(x, P)$ have gl simple poles $\gamma_1(x), \ldots, \gamma_{gl}(x)$. The rank of the matrix of residues of the χ_i at the points γ_s is 1. Thus, at the point γ_s we define the $(l-1)$-vectors $\alpha_{sj}(x)$ $(j = 1, \ldots, l)$ so that the following relations hold for the matrix elements χ_i^{ab}:

$$\operatorname{res}_{\gamma_s} \chi_i^{ab} = \alpha_{sb} \operatorname{res}_{\gamma_s} \chi_i^{al}. \tag{28}$$

The parameters $\gamma(x)$ and $\alpha(x)$ satisfy the "deformation" equations

$$\frac{\partial}{\partial x_i} \gamma = -\operatorname{Sp} \chi_{i,0}(x), \tag{29}$$

(30) $$\frac{\partial}{\partial x_i}\alpha_j = -\sum_{a=1}^{l}\alpha_a\chi_{i,1}^{aj} + \alpha_j\left(\sum_{a=1}^{l}\alpha_a\chi_{i,1}^{al}\right),$$

where $\chi_{i,0}$ and $\chi_{i,1}$ are the coefficients of the expansion of $\chi_i(x, P)$ in a Laurent series in the neighbourhood of the pole $\gamma = \gamma_s(x)$ (the index s is here omitted for the sake of brevity):

(31) $$\chi_i(x, P) = \chi_{i,0}(x)(k-\gamma)^{-1} + \chi_{i,1}(x) + O(k-\gamma).$$

We denote by $u_{is}(x, k)$ matrices depending polynomially on k that are equal to the singularities of χ_i at P_s. This means that

(32) $$\chi_i(x, P) - u_{is}(x, k_s(P))$$

is a regular function near P_s.

PROPOSITION. *For any functions $u_{is}(x, k)$ depending polynomially on k and any $\gamma(x)$ and $\alpha(x)$ there exists a matrix-valued function $\chi_i(x, P)$ satisfying* (28) *and* (32). *It is uniquely determined by its value at any P_0, $\chi_i(x, P_0) = u_{i0}(x)$.*

The arbitrariness in the definition of $\chi_i(x, P)$ is connected with the fact that the matrix analogue of the Baker–Akhiezer function is determined by its singularities at the points $P_1, \ldots, P_m$ and by the Turin parameters only to within multiplication by a non-degenerate matrix.

The proof reduces to a simple calculation using the Riemann–Roch theorem of the dimension of the space of functions having simple poles at the points γ_s and poles of multiplicity n_i at the points P_i. This dimension is equal to the number of inhomogeneous linear equations equivalent to (28) and (32) and to the condition

$$\chi_i(x, P_0) = u_{i0}(x).$$

Let $\chi_i(x, P)$ be the matrix-valued function defined by the parameters

$$\{\gamma(x),\ \alpha(x),\ u_{si}(x, k),\ u_{i0}(x)\}.$$

PROPOSITION. *The conditions* (29) *and* (30) *are necessary and sufficient for the solution of* (27), *normalized by the condition* $\Psi(0, P) = 1$, *to be a Baker–Akhiezer function.*

For brevity we omit the index i, that is, we assume that $\Psi(x, P)$ depends only on the single parameter x.

PROOF. First of all, we prove that (29) and (30) are equivalent to $\Psi(x, P)$ being holomorphic at the points $\gamma_j(x)$.

Suppose that $\Psi(x, P)$ is holomorphic at $\gamma = \gamma_j(x)$. Then for any column Ψ^j of Ψ we have

(33) $$\sum_{a=1}^{l}\alpha_a\psi_a = 0,\quad \alpha_l = 1,\quad \Psi^j = (\psi_1, \ldots, \psi_l)^t,$$

as follows by equating to zero the coefficient of $(k-\gamma)^{-1}$ in (27). In addition,

$$\frac{\partial}{\partial x}\psi_a=\sum_b \chi_1^{ab}\psi_b+\sum_b \chi_0^{ab}\frac{\partial \psi_b}{\partial k}. \tag{34}$$

Differentiating (33) we obtain

$$\sum_a \alpha_{ax}\psi_a+\sum_a \alpha_a\psi_{ax}+\sum_a \alpha_a\gamma_x\frac{\partial\psi_a}{\partial k}=0$$

or, bearing (33) and (34) in mind,

$$\sum_a\left[\alpha_{ax}\psi_a+\alpha_a\left(\sum_b \chi_1^{ab}\psi_b+\chi_0^{ab}\frac{\partial\psi_b}{\partial k}\right)+\gamma_x\alpha_a\frac{\partial\psi_a}{\partial k}\right]= \tag{35}$$

$$=\sum_a\left(\alpha_{ax}+\sum_b \alpha_b\chi_1^{ba}\right)\psi_a=0.$$

The condition (29) is a simple consequence of the fact that the logarithmic derivative of det Ψ is equal to the trace of $\chi(x, P)$. Since the coefficients of ψ_a in (33) and (35) must be proportional, (30) holds.

Now we prove the sufficiency of (29) and (30). We consider the matrix

$$\tilde{\chi}=(\partial_x g)\,g^{-1}+g\chi g^{-1},$$

which is gauge equivalent to χ, where

$$g=\begin{pmatrix} \frac{\alpha_1}{k-\gamma} & \frac{\alpha_2}{k-\gamma} & \cdots & \frac{\alpha_{l-1}}{k-\gamma} & \frac{1}{k-\gamma} \\ 0 & 0 & \ldots 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0\ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

$$g^{-1}=\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & 1 \\ 0 & 0 & \ldots 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0\ldots & 0 & 0 \\ k-\gamma & -\alpha_{l-1} & \ldots & -\alpha_2 & -\alpha_1 \end{pmatrix}.$$

A direct verification shows that if (29) and (30) hold, then $\tilde{\chi}$ has no singularities at $k=\gamma$. Hence, the solution of

$$\frac{d}{dx}\tilde{\Psi}=\tilde{\chi}\tilde{\Psi}$$

has no singularities. But then neither has $\Psi=g^{-1}\tilde{\Psi}$, which satisfies (27).

To complete the proof we find the form of Ψ in a neighbourhood of the singular point P_s. To do this we raise the following Riemann problem:

To find a matrix-valued function $\Psi_s(x, k)$ that is holomorphic in k everywhere except in a neighbourhood of $k=\infty$ and can be represented near $k=\infty$ in the form

$$\Psi_s(x,\ k)=R(x,\ k)\Psi(x,\ k_s^{-1}(P)), \tag{36}$$

where the matrix-valued function

$$R(x, k) = \sum_{i=0}^{\infty} \xi_i(x) k^{-i}$$

is regular near $k = \infty$.

This problem has a unique solution such that $\Psi_s(x, 0) = 1$.

LEMMA. *The logarithmic derivative of Ψ_s is the polynomial*

$$\left(\frac{d}{dx} \Psi_s\right) \Psi_s^{-1} = \sum_{i=1}^{ns} w_{si}(x) k^i.$$

The lemma is proved by noting that $\left(\frac{d}{dx} \Psi_s\right) \Psi_s^{-1}$ has no singularities other than $k = \infty$, and by (36) and the definition of Ψ has a pole of order n_s at $k = \infty$.

Multiplying (36) by R^{-1} on the left we find that Ψ can be represented in the form (13) near P_s, that is, it is a matrix analogue of the Baker–Akhiezer function.

§5. Finite-zone solutions of the KP equation of rank 2 and genus 1

In this section we give explicit formulae for equations for the Turin parameters corresponding to finite-zone solutions of the KP equations of rank 2 and genus 1, that is, KP solutions connected with commuting operators L_4 and L_6 of orders 4 and 6. In general position, such operators are linked by the relations

$$L_6^2 = 4L_4^3 + g_1 L_4 + g_2 \tag{37}$$

and are determined by the constants g and g_2, the Turin parameters (γ, α) on the elliptic curve Γ defined by (37), and by a single arbitrary function $u_0(x)$ ([11]).

In this case the Turin parameters are a pair of points γ_1, γ_2 on the elliptic curve, with a complex number $\alpha_{11} = \alpha_1$, $\alpha_{21} = \alpha_2$ given at each of them.

According to §1, Example 1, the KP solution corresponding to the commutative algebra generated by L_4 and L_6 is determined by the set (γ, α) and an arbitrary solution $u_0(x, t)$ of the KdV equation.

The logarithmic derivative of the matrix analogue of the Baker–Akhiezer function $\Psi(x, y, t, P)$ corresponding to this solution has the following form near $\lambda = 0$:

$$\left(\frac{\partial}{\partial x} \Psi\right) \Psi^{-1} = \chi_1(x, y, t, \lambda) = \begin{pmatrix} 0 & 1 \\ k-u & 0 \end{pmatrix} + O(\lambda), \tag{38}$$

where $\lambda = k^{-1}$ is a parameter on the elliptic curve.

The form of the singularity of $\chi_1(x, y, t, \lambda)$ in the neighbourhood of $\lambda = 0$ and the specification of the parameters $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ determines χ_1 uniquely. Let us find its explicit form.

Any elliptic function can be represented in terms of the Weierstrass ζ-function [21]. We are looking for χ_1 in the form

$$\chi_1 = A\zeta(\lambda - \gamma_1) + B\zeta(\lambda - \gamma_2) + C\zeta(\lambda) + D,$$

where A, B, C, and D are matrices that do not depend on λ. The Weierstrass zeta-function is given by the series

$$\zeta(\lambda) = \lambda^{-1} + \sum_{m,\, n\neq 0} [(\lambda - \omega_{mn})^{-1} + \omega_{mn}^{-1} + \lambda\omega_{mn}^{-2}]; \qquad \omega_{mn} = m\omega + n\omega_1$$

or by the relation $\zeta'(\lambda) = -\wp(\lambda)$. The Weierstrass $\wp(\lambda)$-function has a unique pole of the second order at $\lambda = 0$. In contrast to $\wp(\lambda)$ $\zeta(\lambda)$ is not doubly-periodic.

A necessary and sufficient condition for χ_1 to be an elliptic function is

(39) $$A + B + C = 0.$$

From (38) it follows that $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. By the definition of χ_1, its residues at γ_1 and γ_2 are of rank 1, that is,

$$A = \begin{pmatrix} \alpha_1 a & a \\ \alpha_1 b & b \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_2 c & c \\ \alpha_2 d & d \end{pmatrix}.$$

Thus, $A = (\alpha_2 - \alpha_1)^{-1}\begin{pmatrix} 0 & 0 \\ \alpha_1 & 1 \end{pmatrix}$, $B = (\alpha_1 - \alpha_2)^{-1}\begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix}$. The free term in (38) is $\begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}$. Hence,

(40) $$D - A\zeta(\gamma_1) - B\zeta(\gamma_2) = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}.$$

Putting everything together, we obtain

(41) $$\chi_1 = \frac{1}{\alpha_2 - \alpha_1}\begin{pmatrix} 0 & 0 \\ \alpha_1 & 1 \end{pmatrix}\zeta(\lambda - \gamma_1) + \frac{1}{\alpha_1 - \alpha_2}\begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix}\zeta(\lambda - \gamma_2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\zeta(\lambda) + D,$$

where D is defined by (40). From (29),

(42) $$\begin{cases} \gamma_{1x} = -\operatorname{Sp} A = (\alpha_1 - \alpha_2)^{-1}, \\ \gamma_{2x} = -\operatorname{Sp} B = (\alpha_2 - \alpha_1)^{-1}. \end{cases}$$

The matrix $\chi_{1,1}$ which defines the dynamics of α_1 in x, is by virtue of (30)

$$\frac{1}{\alpha_1 - \alpha_2}\begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix}\zeta(\gamma_1 - \gamma_2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\zeta(\gamma_1) + D.$$

Consequently,

(43) $$\alpha_{1x} = \alpha_1^2 + u - \Phi(\gamma_1, \gamma_2).$$

Similarly,

(44) $$\alpha_{2x} = \alpha_2^2 + u + \Phi(\gamma_1, \gamma_2).$$

Here

$$\Phi(\gamma_1, \gamma_2) = \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2).$$

The expansions of the logarithmic derivatives $\Psi_y\Psi^{-1}$ and $\Psi_t\Psi^{-1}$ near $\lambda = 0$ are

(45) $$\chi_2 = \Psi_y\Psi^{-1} = \begin{pmatrix} k & 0 \\ v & k \end{pmatrix} + O(\lambda), \quad \lambda = k^{-1},$$

(46) $$\chi_3 = \Psi_t\Psi^{-1} = \begin{pmatrix} \omega_1 & k + \frac{u}{2} \\ k^2 - \frac{uk}{2} + \omega_2 & -\omega_1 \end{pmatrix} + O(\lambda).$$

As in the case of χ_1, the expansions (45) and (46) determine χ_2 and χ_3 uniquely, and an explicit representation for them as a sum of ζ-functions can be obtained; here the equations for the Turin parameters acquire the following form:

(47) $\gamma_{iy} = 1; \quad \alpha_{iy} = -v(x, y, t);$

(48) $\gamma_{it} = (-1)^i \left(\alpha_1\alpha_2 + \frac{u}{2}\right)(\alpha_1 - \alpha_2)^{-1};$

(49) $\alpha_{it} = -2\alpha_i\omega_1 + \alpha_i^2 \frac{u}{2} - \omega_2 - (-1)^i\left(\frac{u}{2} + \alpha_i^2\right)\Phi - \wp(\gamma_i).$

We introduce the notation $\gamma_1 = y + c(x, t)$; $\gamma_2 = y - c(x, t) + c_0$; $c_0 = \text{const}$; $\alpha_1 - \alpha_2 = z(x, t)$; $\alpha_1 + \alpha_2 = w(x, y, t)$; $\Phi = \Phi(y, c, c_0)$.

From the consistency condition of the flows in x, y, t, given by (42)–(44) and (47)–(49) we obtain

(50) $$\begin{cases} v = (\alpha_2 - \alpha_1)^{-1}(\wp(\gamma_2) - \wp(\gamma_1)), \\ \omega_1 = -\frac{u_x}{4} + \frac{1}{2}(\wp(\gamma_1) - \wp(\gamma_2))(\alpha_1 - \alpha_2)^{-1}, \\ \omega_2 = \omega_{1x} - \frac{u^2}{2} + \wp(\gamma_1) + \wp(\gamma_2). \end{cases}$$

In the new variables the equations themselves become

(51) $$\begin{cases} c_x = z^{-1}; \; z_x = zw - 2\Phi(y, c, c_0); \; c_y = z_y = 0; \\ c_t = 2z^{-1}(z^2 - \varphi); \\ u(x, y, t) = -\alpha_1^2 - \alpha_2^2 + \varphi(x, t) = -\frac{z^2 + w^2}{2} + \varphi(x, t); \\ w_x = -\frac{z^2 + w^2}{2} + 2\varphi(x, t). \end{cases}$$

Substituting in the equation for w_x the expression $w = (\log z)_x + 2\Phi z^{-1}$, we obtain

(52) $$\varphi(x, t) = \frac{1 + 3c_{xx}^2}{4c_x^2} + Qc_x^2 - \frac{1}{2}\frac{c_{xxx}}{c_x},$$

(53) $$u(x, y, t) = \frac{c_{xx}^2 - 1}{c_x^2} + 2\Phi c_{xx} + c_x^2(\Phi_c - \Phi^2) - \frac{1}{2}\frac{c_{xxx}}{c_x},$$

$$c_t = \frac{3}{8c_x}(1 - c_{xx}^2) - \frac{1}{2} Q c_x^3 + \frac{1}{4} c_{xxx}; \quad Q = \Phi_c + \Phi^2. \tag{54}$$

PROPOSITION. *Every solution $c(x, t)$ of* (54) *determines in accordance with* (53) *a solution of the* KP *equation that is periodic in y. If $c_x = z^{-1} \neq 0, z \neq 0$, then $u(x, y, t)$ is non-singular and bounded in x.*

A comparison of the constructions of solutions of the KP èquation by means of the vector analogue of the Baker–Akhiezer function and the equations for the Turin parameters shows that (54) is "latently isomorphic" to the KdV equation, although the isomorphism is somewhat hidden.

Now (54) is an integrable system, admitting a representation of zero curvature in which the operators depend algebraically on an auxiliary "spectral parameter" on an elliptic curve, in contrast to all previously known cases where λ enters rationally. This representation has the form

$$\chi_{1t} - \chi_{3x} + [\chi_1, \chi_3] = 0, \quad \chi_i = \chi_i(x, y, t, \lambda). \tag{55}$$

The representation (55) enables us to obtain the integrals of (54) in the usual way from the expansion of χ_1 in the spectral parameter λ. An analysis of general systems of the form (55) is given in the next section of the paper.

Let us consider the stationary solutions of (54) of the form $u(x + at, y)$, corresponding to solutions of the Boussinesque equation. A simple substitution (see [3], p. 309) enables us to obtain from them a more general solution of the KP equation of the type of a conoidal wave $u(x + a_1 t, y + b_1 t)$.

The substitution $z = h^{-2}(c)$ reduces (54) ($c_t = ac_x$) to the Hamiltonian form

$$\begin{cases} \dfrac{d^2h}{dc^2} = -\dfrac{\partial W(h, c)}{\partial h}, \\ W = -\dfrac{1}{2} Q(c, c_0) h^2 + ah^{-2} - \dfrac{1}{8} h^{-6}, \end{cases} \tag{56}$$

where $Q = \Phi_c + \Phi^2$ is an elliptic function. This system is completely integrable. From (55) it follows that it admits the commutation representation

$$\chi_{3x} = [\chi_1, \chi_3]. \tag{57}$$

Consequently, the quantity $R(\mu, \lambda) = \det(\mu 1 - \chi_3(x, \lambda))$ does not depend on x and is an integral of the equations

$$R(\mu, \lambda) = \det(\mu 1 - \chi_3(c, \lambda)) = \mu^2 - \wp'(\lambda) - I(c, c_0).$$

The corresponding integral $I(c, c_0)$ is

$$I(c, c_0) = -\frac{u}{2}\left(\frac{\alpha_1 - 2\alpha_2}{\alpha_1 - \alpha_2}\wp(\gamma_1) + \frac{2\alpha_1 - \alpha_2}{\alpha_1 - \alpha_2}\wp(\gamma_2) - \frac{u^2}{4}\right) + \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1 - \alpha_2}\wp'(\gamma_1) - \frac{\alpha_1}{\alpha_1 - \alpha_2}\wp'(\gamma_2)\right). \tag{58}$$

The equations (54) are parametricized by the constant c_0. The set of their stationary solutions for all c_0 is isomorphic to the space of Turin parameters.

The manifold of the level surface $I(c, c_0) = I$ is isomorphic to the three-dimensional Jacobi manifold $(J(\Gamma_2)$ of Γ_2, which is a two-sheeted covering of the initial elliptic curve and is given by the equation $R(\mu, \lambda) = 0$. The intersection of the level lines I = const and c_0 = const determines its odd part, the "Prym manifold" in the Jacobian Γ_2.

Thus, the variety of the moduli of holomorphic equipped bundles of rank 2 over an elliptic curve stratifies into two Abelian Prym varieties, corresponding to a covering of the elliptic curve.

RESULT. The conoidal waves of the KP equation of rank 2 and genus 1 can be expressed in terms of θ-functions of two complex variables; they do not coincide with the solutions of KP equations of genus 2 and rank 1, which can also be expressed by θ-functions of two variables.

These assertions follow directly from results in the Appendix.

To conclude this section we give an explicit formula for the operator L_4 that occurs in the commutative pair $[L_4, L_6] = 0$ of rank 2.

From the results of [11], §3, it follows that the commutative ring is uniquely determined by (42), (43), and (44), where $u(x)$ is an arbitrary function. There is, however, no need to solve these equations to obtain all commutative rings of rank 2 corresponding to an elliptic curve. If we choose $c(x)$ as an independent functional parameter, then the formulae (51) determine $\gamma_i(x)$, $\alpha_i(x)$, and $u(x)$. And so the specification of $c(x)$ uniquely determines by means of (41) the logarithmic derivative

(59) $$\Psi_x \Psi^{-1} = \chi_1(x, \lambda) = \begin{pmatrix} 0 & 1 \\ \chi_{21} & \chi_{22} \end{pmatrix},$$

where $\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_{1x} & \psi_{2x} \end{pmatrix}$; the ψ_i are eigenfunctions of the operator $L_4 \psi_i(x, \lambda) = \wp(\lambda)\psi_i(x, \lambda)$.

From (59), which indicates that $\psi_i'' = \chi_{21}\psi_i + \chi_{22}\psi_i'$, there follow the recurrence relations for the higher derivatives. For example,

$$\psi_i''' = \chi_{21}'\psi_i + \chi_{21}\psi_i' + \chi_{22}'\psi_i' + \chi_{22}(\chi_{21}\psi_i + \chi_{22}\psi_i).$$

To determine the coefficients of

(60) $$L_4 = \frac{d^4}{dx^4} + v_2(x)\frac{d^2}{dx^2} + v_1(x)\frac{d}{dx} + v_0(x)$$

we represent $L_4\psi_i$ by means of the succeeding formulae as $b_1(x, \lambda)\psi_i + b_2(x, \lambda)\psi_i'$. The functions $b_1(x, \lambda)$ and $b_2(x, \lambda)$ are meromorphic in λ and depend linearly on the coefficients of L_4. These latter can be found from a comparison of the Laurent expansions of b_1 and b_2 near $\lambda = 0$:

$$b_1(x, \lambda) = \lambda^{-2} + O(\lambda); \quad b_2(x, \lambda) = O(\lambda).$$

Having done this, we obtain

$$L_4 = L^2 + c_x [\wp(c+c_0) - \wp(c+c_1)] \frac{d}{dx} - \wp(c+c_0) - \wp(c+c_1); \tag{61}$$
$$L = \frac{d^2}{dx^2} + u(x).$$

§6. Equations of zero curvature for algebraic sheaves of operators

In the preceding section it was shown that the construction of finite-zone solutions of genus $g = 1$ and rank 2 of the KP equation leads to an integrable system admitting a representation of zero curvature, but in which the operators depend algebraically on a "spectral parameter", a point of an elliptic curve.

A general representation of similar type

$$u_t - v_x + [u, v] = 0 \tag{62}$$

indicates the compatibility of the equations

$$\left(\frac{\partial}{\partial x} - u(x, t, P)\right) \Psi(x, t, P) = 0, \tag{63}$$

$$\left(\frac{\partial}{\partial t} - v(x, t, P)\right) \Psi(x, t, P) = 0, \tag{64}$$

where P is a point of an algebraic curve Γ of genus g with distinguished points $P_1, \ldots, P_m$, and Ψ is a matrix analogue of the Baker–Akhiezer function.

Let $u(x, t, P)$ and $v(x, t, P)$ be matrix-valued functions determined, as in §4, by their singularities at the points P_s, and by their values at the fixed point P_0, with $u_0 = u(x, t, P_0)$ and $v_0 = v(x, t, P_0)$. The singularities of u and v at the points P_s, that is, the matrix functions

$$u_s = \sum_{i=1}^{n_s} u_{si}(x, t) k^i; \quad v_s = \sum_{i=1}^{m_s} v_{si}(x, t) k^i,$$

are polynomially dependent on k.

In the case of a curve of genus $g = 0$, v and u are rational functions of k. The equation (62), which must be satisfied for all k, is in this case clearly equivalent to the finitely many equations obtained by equating to zero the singular parts of $w = u_t - v_x + [u, v]$ at $P_1, \ldots, P_m$ and the value of w at P_0.

If the genus of Γ is $g \geqslant 1$, then, u and v, in addition to the singularities at P_s, have singularities connected with the Turin parameters (γ, α) and satisfying (28)–(30). Nevertheless, as before, the equations (62) are equivalent, as before, to equations connected only with the points $P_1, \ldots, P_m$.

PROPOSITION. *The system of equations* (63) *and* (64) *is compatible if and only if*

$$u_{0t} - v_{0x} + [u_0, v_0] = 0, \tag{65}$$

$$u_t - v_x + [u, v] = O(1) \,|_{P=P_s}. \tag{66}$$

These last equations mean that the function $w = u_t - v_x + [u, v]$ has no singularities at the points $P_1, \ldots, P_m$.

The number of matrix equations equations (65) and (66) is $M + N + 1$, where $M = \Sigma m_s$, $N = \Sigma n_s$. But the number of independent matrix valued functions defining u and v is $M + N + 2$. The indeterminacy of the system is due to its "gauge invariance". The transformation

$$u \to \partial_x g g^{-1} + g u g^{-1},$$
$$v \to \partial_t g g^{-1} + g v g^{-1}$$

where $g(x, t)$ is an arbitrary non-degenerate matrix, maps the solution set of (65)–(66) into itself.

SKETCH OF PROOF. We consider the matrix function $w = u_t - v_x\ [u, v]$. The equations (29), which define the dynamics of the poles $\gamma_s(x, t)$ of u and v, are equivalent to the fact that the function w, which a priori would have poles of the second order at the points γ_s, has actually simple poles at these points. A direct substitution of the Laurent expansions of u and v near $\gamma = y_s(x, t)$

$$u = \frac{u^0}{k-\gamma} + u^1 + u^2 (k-\gamma) + O((k-\gamma)^2),$$
$$v = \frac{v^0}{k-\gamma} + v^1 + v^2 (k-\gamma) + O((k-\gamma)^2)$$

in w shows that as a consequence of (30) there is a relation between the residues of the elements w^{ab} at the points γ_s:

$$\operatorname{res}_{\gamma_s} w^{ab} = \alpha_{sb} \operatorname{res}_{\gamma_s} w^{al}.$$

Hence, w is a function of the same type as u and v and so is uniquely determined by its singularities at the points P_s and the value $w(x, t, P_0)$. By hypothesis, these parameters are zero. Hence,

$$w = u_t - v_x + [u, v] = 0. \tag{67}$$

To complete the proof of the proposition it is sufficient to show that the pair of equations for γ and $\alpha = (\alpha_1, \ldots, \alpha_{l-1}, 1)$ is compatible. Since, by (67), $\operatorname{Sp} w = 0$, we have

$$\operatorname{Sp} u_t^0 - \operatorname{Sp} v_x^0 = 0 \leftrightarrow \gamma_{xt} = \gamma_{tx}.$$

To prove the compatibility of (30) for α_x and α_t we introduce the row vector $\beta = (\beta_1, \ldots, \beta_l)$ for which

$$\beta_x = -\beta u^1, \tag{68}$$

$$\beta_t = -\beta v^1. \tag{69}$$

The compatibility of this pair of equations is equivalent to that of the equations for α and $\alpha_i = \beta_i \beta_l^{-1}$. The compatibility of (68) and (69) means that

$$\beta(u_t^1 - v_x^1 + [u, v]) = 0. \tag{70}$$

By equating to zero the free term of the Laurent expansion of w at $\gamma = \gamma_s(x, t)$, we find that

$$u_t^1 - v_x^1 + [u^1, v^1] = [v^0, u^2] + [v^2, u^0]. \tag{71}$$

Thus, for (69) and (68) to be compatible it is sufficient that

$$\beta([u^0, v^2] + [u^2, v^0)] = 0. \tag{72}$$

This relation no longer contains derivatives in x and t. Let us use the following device. It is easy to construct a Baker–Akhiezer function $\widetilde{\Psi}(x, t, P)$ such that

$$\widetilde{u}(x_0, t_0, P) = u(x_0, t_0, P); \quad \widetilde{v}(x_0\ \ t_0, P) = v(x_0, t_0, P).$$

Here $\widetilde{u} = \widetilde{\Psi}_x \widetilde{\Psi}^{-1}$ and $\widetilde{v} = \widetilde{\Psi}_t \widetilde{\Psi}^{-1}$. Since for this function (69) and (68) are compatible,

$$\widetilde{\beta}([\widetilde{v}^0, \widetilde{u}^2] + |\widetilde{v}^2, \widetilde{u}^0]) = 0$$

for all x and t. For $x = x_0$, $t = t_0$ it coincides with (72).

§7. Appendix. Algebraic families of commuting flows

In [22] a λ-representation was found for the first time of the KdV equation and all its higher analogues, that is, a representation of the whole system in the form of equations of zero curvature of sheaves of operators

$$\left[\frac{\partial}{\partial t_i} \quad u_i(t, \lambda), \ \frac{\partial}{\partial t_j} - u_j(t, \lambda)\right] = 0.$$

depending polynomially on the spectral parameter λ, $t = (t_1, t_2, \ldots)$; $t_1 = x$, $t_2 = t$. In the more general situation of rational sheaves of operators or even of the algebraic sheaves defined above, an invariant separation of algebraic families analogous to (2) for equations of KdV type can be obtained as follows.

Let

$$L_i = \frac{\partial}{\partial t_i} \quad u_i(t, P)$$

be a set of algebraic sheaves of operators where the $u_i(t, P)$ are meromorphic matrix-valued functions of the type described earlier on an algebraic curve Γ of genus g. When $g = 0$, the u_i are rational functions on a Riemann sphere with constant poles (not depending on t).

DEFINITION 1. The set of operators L_i is called a *commutative family* if for any i and j the operators L_i and L_j commute:

$$\frac{\partial u_i}{\partial t_j} - \frac{\partial u_j}{\partial t_i} + [u_i, u_j] = 0, \tag{73}$$

DEFINITION. If there exists a matrix-valued function $w(t, P)$ algebraically dependent on P and such that

(74) $$\left[\frac{\partial}{\partial t_i} - u_i(t, P), \quad v \; t, P)\right] = 0,$$

then the commutative family is said to be *algebraic.*

The basic example of an algebraic family is the condition for the whole system to be stationary with respect to one of the variables

$$\frac{\partial u_j}{\partial t_i} = 0 \quad (j = 1, 2, 3 \ldots).$$

In this case $w = u_i$. However, a priori it is not necessary to assume that w is connected with the set $(u_1, \ldots, u_i, \ldots)$. The general case can be reduced to this.

The linear operators $L_i = \frac{\partial}{\partial t_i} - u_i\,(t, P)$ that occur in the algebraic family (if they have certain properties of being Hermitian) are "finite-zone or finitely lacunary" in the sense of spectral operator theory [22]. Therefore, these operators and the corresponding solutions of non-linear equations are called "finite-zone".

In relation to any of the equations (73) labelled by (i, j), the remaining equations labelled by (i, k) play the rôle of "higher KdV analogues". A priori they are all partial differential equations. However, the hypothesis of being algebraic ("of finite-zone" type) (74) reduces to the fact that these equations split into a collection of commuting systems of ordinary differential equations each in one variable, which can be expressed explicitly in the form of a finite-dimensional analogue of the Lax pair [74].

PROPOSITION. *If the operators L_i commute with w, then they commute among themselves, that is,* (73) *follows from* (74).

For a fixed number and order of poles of w the space of the corresponding matrices is finite-dimensional, and (74) determine commuting deformations of it. All the equations (74) have common integrals. Let $\Psi(t, P)$ be a solution of the equations

(75) $$\left(\frac{\partial}{\partial t_i} - u_i\right)\Psi(t, P) = 0; \quad \Psi(0, P) = 1.$$

From (74) it follows that

(76) $$w(t, P)\Psi(t, P) = \Psi(t, P)w(0, P).$$

Hence, the characteristic polynomial

(77) $$Q(\mu, P) = \det(\lambda 1 - w(t, P)) = 0$$

does not depend on t. Its coefficients are integrals of (74).

DEFINITION 3. An algebraic family is said to be *complete* if the flows defined by (74) cover the whole variety of levels of the integrals (77).

In general position, for almost all P the eigenvalues of $w(0, P)$ are distinct and the curve $\hat{\Gamma}$ defined by (77) is an l-sheeted covering of the initial curve Γ. To each point γ of $\hat{\Gamma}$ there corresponds a unique eigenvector $w(0, P)$ with first coordinate normalized to 1. The remaining coordinates $h_i(\gamma)$ are meromorphic functions on $\hat{\Gamma}$. The vector-valued function

$$\psi(t, \gamma) = \sum_{i=1}^{l} h_i(\gamma) \Psi_i(t, P),$$

where $\Psi_i(t, P)$ is the i-th column of the matrix $\Psi(t, P)$, has the following analytic properties.

1. Since $\Psi(t, P)$ is meromorphic except at $P_1, \ldots, P_m$, $\psi(t, \gamma)$ is meromorphic except at $P_i^j (j = 1, \ldots, l)$, the inverse image of P_i on $\hat{\Gamma}$. The poles of $\psi(t, \gamma)$ do not depend on t, and there are $g + l - 1$ of them, where g is the genus of $\hat{\Gamma}$.

2. From (74) and the fact that the characteristic polynomial Q does not depend on t it follows that the eigenvalues $u_i(t, P)$ for $P = P_s$ do not depend on t. Hence, near P_i^j the coordinates of $\psi(t, \gamma)$ have the form

$$\psi = \exp\left(\sum_{a} \lambda_a t_a k\right) \left(\sum_{s=0}^{\infty} \xi_s(t) k^{-s}\right),$$

where the λ_a are constants and $k^{-1} = k^{-1}(\gamma)$ are local parameters near P_i.

Thus, $\psi(t, \gamma)$ is a Baker–Akhiezer function of rank 1 and is uniquely determined by the divisor of the poles $\gamma_1, \ldots, \gamma_{g+l-1}$. In accordance with the general rules, $\psi(t, \gamma)$ can be expressed explicitly in terms of a θ-function. The matrix w for ψ is defined by

$$w(t, P)\psi(t, \gamma) = \mu(\gamma)\psi(t, \gamma),$$

where $\gamma = (P, \mu)$ is the inverse image of P on $\hat{\Gamma}$ given by (77).

If we identify the matrices w and AwA^{-1}, where A is a constant diagonal matrix, then the quotient variety of the integral levels of (77) is isomorphic to the Jacobian torus of the curve $J(\hat{\Gamma})$, while the equations (74) give rectilinear windings on these tori (see [18], Ch. III, §3).

In the theory of equations of KdV type the higher analogues formed complete algebraic families. Another example of a family of operators with two variables $t_1 = x + t'$, $t_2 = x - t'$, depending rationally on a parameter λ are of the operators of the form (78), which are used in [23] and [24] for the theory of chiral fields:

$$L_i = \frac{\partial}{\partial t_i} - \frac{A_i}{\lambda - a_i}. \tag{78}$$

Examples of algebraic families, containing arbitrary numbers of operators of the form (78) were considered by Garnier [25].[1] The starting point of [25] were the Schlesinger equations, which describe deformations of ordinary differential

[1] The authors are grateful to G. Flaschke and A. Newell, who brought this remarkable classic work to their attention.

equations that preserve the monodromy of singularities of these equations when $a_i \to t_i$ in (78).

Garnier considered equations of the type (74) and of the special form:

$$\left[\frac{\partial}{\partial t_i} - \frac{A_i}{\lambda - a_i}, \quad \sum_j \frac{A_j}{\lambda - a_j}\right] = 0, \tag{79}$$

where

$$w = \sum_{j=1}^{n} \frac{A_j}{\lambda - a_j} \qquad (j = 1, \ldots, n).$$

The family (79) is not complete. The number n of operators is substantially smaller than the genus of the curve $\hat{\Gamma}$, given by the equation

$$Q(\lambda, \mu) = \det\left(\sum_{j=1}^{n} \frac{A_j}{\lambda - a_j} - \mu \cdot 1\right) = 0.$$

Garnier used (79) to construct new integrable finite-dimensional systems. The system he discovered

$$\xi_i'' = \xi_i\left(\sum_{i=1}^{l} \xi_i \eta_i + a_i\right), \quad \eta_i'' = \eta_i\left(\sum_{i=1}^{l} \xi_i \eta_i + a_i\right)$$

coincides on distinct invariant hyperplanes $\xi_i = b_i \eta_i$ with the Neumann system of harmonic oscillators "forcibly" constrained to the sphere $\Sigma \xi_i^2 = 1$, [26], (which, of course, destroys the harmonic character), and also with an anharmonic system of oscillators [27].

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