4.5. Hypothesis. $a^t = a^*$ for all $a \in M(\mathfrak{S})$.

We leave the verification of this hypothesis in all the cases, where a^t is known to us, to the reader (see [1, Sec. 11]).

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BAXTER'S EQUATIONS AND ALGEBRAIC GEOMETRY

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UDC 517.9+513.015.7

In the beginning of the 1970s Baxter (see [1-5]) integrated the quantum-mechanics model of a magnetic proposed by Heisenberg [6] and given the name of XYZ model.

This model describes a system of N interacting particles with a spin equal to 1/2. Its Hamiltonian H acts in the Hilbert state space $\mathfrak{H}_N$,

$$\mathfrak{H}_N = \prod_{n=1}^{N} \otimes \mathfrak{h}_n, \quad \mathfrak{h}_n = C^2,$$

and has the following form:

$$H = -\frac{1}{2} \sum_{n=1}^{N} (J_x \sigma_n^1 \sigma_{n+1}^1 + J_y \sigma_n^2 \sigma_{n+1}^2 + J_z \sigma_n^3 \sigma_{n+1}^3). \tag{1}$$

Here J_x, J_y, and J_z are real constants, and σ_n^j are spin operators,

$$\sigma_n^j = I \otimes \ldots \otimes \sigma^j \otimes \ldots \otimes I,$$

where σ^j are Pauli matrices, $j = 1, 2, 3$; $\sigma^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

G. M. Krzhizhanovskii Moscow Energetics Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 15, No. 2, pp. 22-35, April-June, 1981. Original article submitted October 17, 1980.

Baxter's essential aim was to investigate the so-called eight-vertex two-dimensional model of classical statistical physics. It turned out that this problem is intimately connected with the quantum XYZ model.

The development of methods for the inverse problem of the dispersion theory applied to the study of quantum systems undertaken by L. D. Faddeev and his disciples allowed them to notice at a certain stage a remarkable parallelism between Baxter's basic formulas and formulas arising within the framework of the method of the inverse problem on the quantum level. This, as well as a series of other considerations, finally led Faddeev, Takhtadzhyan, and Sklyanin to the formulation of the quantum method of the inverse problem. This method includes in a natural way all the essential achievements of classical statistical physics and of the theory of one-dimensional quantum systems, thus the ideas of Kramers–Wannier, Onsager, Baxter, Bethe's Ansatz, and many others.

The basic relation in the quantum method of the inverse problem is the set of Baxter's equations

$$R(\mathscr{L}\otimes\mathscr{L}^1)=(\mathscr{L}^1\otimes\mathscr{L})R, \tag{2}$$

where $\mathscr{L}$ and $\mathscr{L}^1$ are (2×2) matrices whose elements are operators in the two-dimensional space $\mathfrak{h}=C^2$. The tensor product is considered within the algebra of (2×2) matrices with operators as coefficients. The matrix R is a numerical (4×4) matrix.

The elements of the matrix $\mathscr{L}$ are labelled with two pairs of indices: $\mathscr{L}_{j\beta}^{i\alpha}$, i, j, α, $\beta = 1, 2$. The Latin indices refer to the blocks of $\mathscr{L}$, and the Greek ones are indices of elements of blocks.

To each matrix $\mathscr{L}$ there corresponds a monodromy matrix $\mathscr{T}$, which is a (2×2) matrix whose elements are operators in the space $\mathfrak{H}_N$ (here, and further on, we shall essentially adhere to the terminology and notations of [8]),

$$\mathscr{T}_{j,\beta_1,\ldots,\beta_N}^{i,\alpha_1,\ldots,\alpha_N}=\mathscr{L}_{j_1\beta_1}^{i\alpha_1}\cdot\mathscr{L}_{j_2\beta_2}^{j_1\alpha_2}\ldots\mathscr{L}_{j\ \ \beta_N}^{j_{N-1}\alpha_N}. \tag{3}$$

In all formulas it will be understood that the summation is carried out with respect to repeated indices.

If $\mathscr{T}$ and $\mathscr{T}^1$ are the monodromy matrices constructed according to the matrices $\mathscr{L}$ and $\mathscr{L}^1$ satisfying (2), they also satisfy the relation

$$R(\mathscr{T}\otimes\mathscr{T}^1)=(\mathscr{T}^1\otimes\mathscr{T})R. \tag{4}$$

The name of transfer matrix T is given to the operator tr $\mathscr{T}$ in $\mathfrak{H}_N$, where the trace is taken in the ring of (2×2) matrices with operators as elements.

To every solution of Eqs. (4) there correspond commuting transfer matrices $[T, T^1] = 0$. To verify it, it suffices to take the trace of the equality (4).

Baxter found the solutions of (2) for matrices of a special form corresponding to the interaction within the the eight-vertex model

$$\mathscr{L}=\begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}. \tag{5}$$

It turns out that elements of such matrices (up to a common factor) can be parametrized with the help of elliptic functions with three parameters λ, η, k,

$$a=\operatorname{sn}(\lambda+2\eta),\quad b=\operatorname{sn}\lambda,\quad c=\operatorname{sn}2\eta,\quad d=k\operatorname{sn}2\eta\cdot\operatorname{sn}\lambda\cdot\operatorname{sn}(\lambda+2\eta), \tag{6}$$

where $\operatorname{sn}\lambda = \operatorname{sn}(\lambda, k)$ is Jacobi's elliptic sine (see [9]) with a modulus k.

With fixed parameters η and k, the transfer matrices corresponding to $\mathscr{L}(\lambda)$ commute for different values of λ, i.e.,

$$[T(\lambda),\ T(\mu)]=0. \tag{7}$$

It follows from (7) that the coefficients of the decomposition of $T(\lambda)$ in λ commute among themselves, and hence so do any functions of them. Remarkably, it was found that among these operators we have the Hamiltonian of the XYZ model

$$H=-\operatorname{sn}2\eta\frac{d}{d\lambda}\ln T(\lambda)|_{\lambda=0}+\frac{1}{2}J_zNI_N. \tag{8}$$

The presence of an infinite collection of operators being integrals of the system and commuting with H contributes to the interest of the XYZ model.

In the present paper, Baxter's equations (2) are discussed without any restrictions whatsoever concerning the form of matrices $\mathcal{L}$. In the first section we construct for any even-dimensional matrix its parametrization by means of some algebrogeometric data analogous to Baxter's parameters. This parametrization of the tensor $R(\mathcal{L}\otimes\mathcal{L}^1)$ allows us to introduce in a natural way the concept of the rank of solutions of Baxter's equations. It is found that all the solutions of rank 1 up to a "calibration equivalence" and some simple symmetries coincide with Baxter's solutions. In the third section, all the remaining solutions are found. Their rank is 2.

It is interesting to note that for Baxter's solutions it is characteristic that matrices satisfying (2) are parametrized by elliptic functions with the same modulus, while the equations themselves, after parametrization, are transformed into variations on the addition theorems for elliptic functions. For solutions of rank 2, the elliptic functions no longer have the same modulus. Thus Eqs. (2) yield relations between elliptic functions with different moduluses.

1. "Vacuous" Vectors and Algebraic Curves

Let $\mathcal{L}$ be an arbitrary even-dimensional matrix. We shall regard it as a (2×2) matrix whose elements are $(n\times n)$ matrices. Accordingly, the matrix elements of $\mathcal{L}$ will be labeled with two pairs of indices, $\mathcal{L}_{j\beta}^{i\alpha}$, $1\le i,\ j\le n$, $1\le\alpha,\ \beta\le 2$. The operator $\mathcal{L}$ acts in the space C^{2n} generated by vectors of the form $X\otimes U$, where the vector U is two-dimensional, and X is n-dimensional, their coordinates being U_α and X_i, respectively. In what follows, unless otherwise stated, all the vectors are assumed to be normed so that their last coordinate is equal to 1, i.e., $X_n = U_2 = 1$.

By a "vacuous" vector of the operator $\mathcal{L}$ we shall understand a vector $X\otimes U$ which under the action of $\mathcal{L}$ is transformed into a vector that is also a tensor product, i.e.,

$$\mathcal{L}(X\otimes U) = h(Y\otimes V), \tag{9}$$

where h is a number. In terms of coordinates, (9) takes the form

$$\mathcal{L}_{j\beta}^{i\alpha}\, X_i\, U_\alpha = h\, Y_j V_\beta. \tag{9'}$$

We multiply (9) on the left by the covector $(\tilde{V}^\beta) = (1, -v)$ orthogonal to V; thus $\tilde{V}^\beta V_\beta = 0$.

Here $v = V_1$ and we put similarly $u = U_1$. Then

$$LX = 0, \tag{10}$$

where L is the operator with coordinates

$$L_j^i = \tilde{V}^\beta \mathcal{L}_{j\beta}^{i\alpha} U_\alpha. \tag{11}$$

For the existence of a vector X satisfying (10) it is necessary and sufficient that u and v should satisfy the algebraic relation

$$P(u, v) = \det L = 0. \tag{12}$$

This equation determines in C^2 an algebraic curve Γ.

In the general situation it can be assumed that for almost all u the roots v_i $(1\le i\le n)$ of (12) are different. Then to every point $z\in\Gamma$, i.e., to every pair $z = (u, v)$ satisfying (12), there corresponds a unique vector X satisfying (10) and $X_n = 1$.

All the other coordinates X_i are rational functions of u and v and, therefore, meromorphic functions of z on Γ.

As in the theory of "finite-zone" integration [10], one shows that the number of poles of X(z) is equal to $N = g + n - 1$, where g is the genus of the curve Γ.

Equation (12) is of the form

$$P(u, v) = \sum a_{ij}\, u^i v^j = 0, \quad 1\leqslant i,\ j\leqslant n. \tag{13}$$

In the general situation, the genus of the curve Γ determined by (13) is $g = (n-1)^2$ (see [12]).

The function h and the vector Y(z) are determined by (9'). Since $V_2 = 1$, we have

$$\mathcal{L}_{j2}^{i\alpha} X_i(z)\, U_\alpha(z) = h(z)\, Y_j(z). \tag{14}$$

The function h(z) is equal to the left side of (14) with $j = n$.

THEOREM 1. In the general situation the operator $\mathscr{L}$ is uniquely determined up to a multiplicative numerical constant by the coefficients a_{ij} of the polynomial (13) and by the meromorphic vectors X(z) and Y(z); the divisors of the poles of the latter have a degree $n^2 - n = g + n - 1$ and satisfy the equivalence condition

$$D_X + D_U \sim D_Y + D_V. \tag{15}$$

Proof. The equivalence condition of the two divisors means that there exists a function h(z) meromorphic on Γ whose pole divisor coincides with one divisor, while the divisor of the zeros coincides with the other. The function h is determined, up to a multiplicative constant, by its zeros and poles.

To every divisor D there corresponds an associated linear space M(D) of functions having at the points of D poles with degrees of multiplicity not exceeding the degrees of multiplicity in D of these points.

The Riemann–Roche theorem states that

$$\dim M(D) \geqslant N - g + 1, \tag{16}$$

and that if the degree N of the divisor D (i.e., the number of points of D counted with their multiplicities) is not smaller than g, then for the divisors in the general situation equality is achieved in (16) (see [11]).

The degree of the divisor $D_X + D_U$ is equal to $n^2 = g + 2n - 1$. A base in the space of functions associated with this divisor is formed by the functions $X_i(z)U_\alpha(z)$. Another base in the same space consists of the functions $h(z)Y_j(z)V_\beta(z)$.

Thus, the matrix $\mathscr{L}$ connects these two bases with each other. To find $\mathscr{L}$ when X, U, Y, and V are given, it suffices to use (9') with an arbitrary choice of points z_s $(1 \le s \le 2n)$ in a general situation.

COROLLARY. To the matrices $\mathscr{L}$ and $\tilde{\mathscr{L}}$ there correspond the same polynomials P(u, v) and "vacuous" vectors with equivalent divisors $D_X \sim D_{X'}$ and $D_Y \sim D_{Y'}$ if, and only if, they are connected by the relation

$$\tilde{\mathscr{L}}^{i\alpha}_{j\beta} = G^i_p \mathscr{L}^{p\alpha}_{q\beta} G'^q_j, \tag{17}$$

where G and G' are $(n \times n)$ matrices.

This follows from the fact that if the divisors D_X and $D_{X'}$ are equivalent, then the "vacuous" vectors X and X' are connected by the relation $G'X'(z) = f(z)X(z)$.

Now we consider the case when the polynomial P(u, v) corresponding to the matrix $\mathscr{L}$ has identically multiple roots v_i for all u, i.e., $P(u, v) = \tilde{P}^l(u, v)$, where $\tilde{P}(u, v)$ is a polynomial of degree n' in either variable, and $n = n'l$. This means that for any point (u, v) of the curve Γ given by the equation $\tilde{P}(u, v) = 0$ the rank of the matrix L is $n - l$. Thus l linearly independent solutions X^a $(a = 1, \ldots, l)$ of (10) form an l-dimensional bundle, i.e., a bundle of rank l over the curve Γ.

We normalize $X^a = (X^a_i)$ by the condition $X^a_i = \delta^a_i$, $1 \le i \le l$. In the theory of "finite-zone" integration the concept of the rank of a solution was introduced in [13, 14]. In analogy with [13], we establish the following analytic properties of the vectors X^a. All the coordinates X^a_i are meromorphic functions on Γ having $N = l(g + n' - 1)$ poles $\gamma_1, \ldots, \gamma_N$. The rank of the matrix of residues of X^a_i at its poles is equal to 1, i.e., there exist such vectors $\alpha_s = (\alpha_{sa})$ that $\operatorname{res}_{\gamma_s} X^a_i = \alpha_{sa} \operatorname{res}_{\gamma_s} X^l_i$. The set of parameters (γ_s, α_s) is called the set of "Tyurin's parameters" or the matrix divisor since, according to [15] it defines uniquely the fitted bundle of rank 1 (stable in the sense of Mumford).

THEOREM 1'. Let be given the polynomial $\tilde{P}(u, v)$ and thereby the curve Γ. For any meromorphic matrices $X^a_i(z)$ and $Y^a_j(z)$ $(1 \le a \le l)$ whose matrix divisors satisfy the equivalence relation

$$D_X + D^l_U \sim D_Y + D^l_V, \tag{15'}$$

there exists a unique $(2n \times 2n)$ matrix $\mathscr{L}$ such that

$$\mathscr{L}^{i\alpha}_{j\beta} X^a_i U_\alpha = Y^b_j V_\beta g^a_b,$$

where $g^a_b(z)$ is the matrix that brings about the equivalence (15'). This means that g^a_b has poles at the points of the pole divisor U(z) and at the points γ_s. Here, $\operatorname{res}_{\gamma_s} g^a_b = \alpha_{sa} \operatorname{res}_{\gamma_s} g^l_b$. Moreover, g^a_b has zeros at the poles of V(z), and at the points γ'_s it satisfies the relation $\beta_{sb} g^a_b(\gamma'_s) = 0$. The matrix divisors $D_X = (\gamma_s, \alpha_{sa})$, $D_Y = (\gamma'_s, \beta_{sb})$ have a degree $N = l(g + n' - 1)$, $1 \le s \le N$.

COROLLARY. The matrix divisors $D_X \sim D_{X'}$ and $D_Y \sim D_{Y'}$ are equivalent if, and only if, the corresponding matrices $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are connected by relation (17).

2. Baxter's Equations

Let $\mathscr{L}$ and $\mathscr{L}^1$ be (4×4) matrices. According to Theorem 1, these matrices are determined by the polynomials

$$P(u, v) = \Sigma a_{ij} u^i v^j = 0, \quad P_1(u, v) = \sum a^1_{ij} u^i v^j = 0, \quad 0 \leqslant i, j \leqslant 2, \tag{18}$$

and by the meromorphic functions $x(z)$, $y(z)$, $x^1(z^1)$, $y^1(z^1)$ having two poles each and satisfying condition (15). Curves Γ and Γ^1 determined by Eqs. (18) are of genus 1, i.e., are elliptic curves. Furthermore,

$$\mathscr{L}(X(u, v) \otimes U) = h(u, v)(Y(u, v) \otimes V), \tag{19}$$

$$\mathscr{L}^1(X^1(u^1, v^1) \otimes U^1) = h_1(u^1, v^1)(Y^1(u^1, v^1) \otimes V^1). \tag{20}$$

Here, capitals denote two-dimensional vectors whose second coordinate is equal to 1, while the first coordinate is denoted by the corresponding lower-case letter. Points of the curve Γ will be denoted either simply by z, or by (u, v); in the latter case it will be automatically assumed that the pair (u, v) satisfies the equation that defines Γ.

Consider the tensors Λ_1 and Λ_2 given by

$$\Lambda_1 = \Lambda_{1, pq\beta}^{ij\alpha} = \mathscr{L}^{1k\gamma}_{p\beta} \mathscr{L}^{l\alpha}_{q\gamma} R^{ij}_{kl}, \tag{21}$$

$$\Lambda_2 = \Lambda_{2, pq\beta}^{ij\alpha} = R^{kl}_{pq} \mathscr{L}^{i\gamma}_{k\beta} \mathscr{L}^{1, j\alpha}_{l\gamma}. \tag{22}$$

Each of these tensors can be regarded as a (2×2) matrix whose elements are (4×4) matrices. The indices α and β are external, while the pairs (i, j) and (p, q) are internal.

According to Theorem 1, to the tensors Λ_i there correspond curves $\hat{\Gamma}_i$ determined by the equations

$$Q_i(u, w) = 0, \quad i = 1, 2, \tag{23}$$

of degree 4 in each variable.

If the triplet u, v, w of numbers satisfies the conditions

$$P(u, v) = 0, \; P_1(v, w) = 0, \tag{24}$$

it follows from (19) and (20) that

$$\Lambda_1(\hat{X}(u, w) \otimes U) = h(u, v) h_1(v, w)(Y^1(v, w) \otimes Y(u, v) \otimes W), \tag{25}$$

where

$$\hat{X}(u, w) = R^{-1}(X^1(v, w) \otimes X(u, v)).$$

Consequently, (u, w) satisfies the equation $Q_1(u, w) = 0$, and the vectors $\tilde{X}(u, w)$ and $Y^1(v, w) \otimes Y(u, v)$ are "vacuous" vectors of Λ_1.

Similarly, we shall consider triplets u, $\tilde{v}$, w of numbers satisfying the equations

$$P_1(u, \tilde{v}) = 0, \quad P(\tilde{v}, w) = 0. \tag{26}$$

Then

$$\Lambda_2(X(\tilde{v}, w) \otimes X^1(u, \tilde{v}) \otimes U) = h_1(u, \tilde{v}) h(\tilde{v}, w) \; (\hat{Y}(u, w) \otimes W), \tag{27}$$

where

$$\hat{Y}(u \;\; w) = R(Y(\tilde{v}, w) \otimes Y^1(u, \tilde{v})). \tag{28}$$

Thus, the pairs u, w entering in triplet (26) satisfy the equation $Q_2(u, w) = 0$.

Baxter's equations are nothing else but the equality $\Lambda_1 = \Lambda_2$.

Consequently, the following proposition has been proved:

LEMMA 1. If $\mathscr{L}$, $\mathscr{L}^1$, and R satisfy Baxter's equations, the polynomials P and P_1 "commute" in the sense of compositions, i.e., Eqs. (24) and (26) define the same curve $\hat{\Gamma}$ with the equation

$$Q(u, w) = Q_1(u, w) = Q_2(u, w) = 0.$$

LEMMA 2. The polynomial $Q(u, w)$ is reducible, i.e., is decomposable into a product of two polynomials.

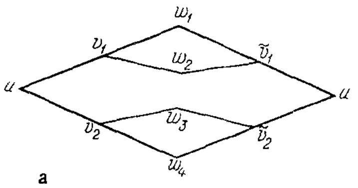
Proof. If Q is irreducible, then for almost all values of u the equation $Q(u, w) = 0$ has four distinct roots. Thus, the tensor Λ is of rank 1, and, according to the results of the preceding section, to every point of $\hat{\Gamma}$ there correspond "vacuous" vectors of the tensor $\Lambda = \Lambda_1 = \Lambda_2$,

$$\Lambda (\hat{X} (u, w) \otimes U) = f (u, w) (\hat{Y} (u, w) \otimes W). \tag{29}$$

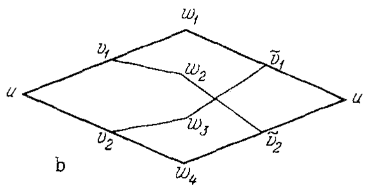
Comparing the "vacuous" vectors in (25) and (27), we find the equalities

$$\begin{aligned} R (X (\tilde{v}, w) \otimes X^1 (u, \tilde{v})) &= g (u, w) \ (X^1 (v, w) \otimes X (u, v)), \\ R (Y (\tilde{v}, w) \otimes Y^1 (u, \tilde{v})) &= g_1 (u, w) (Y^1 (v, w) \otimes Y (u, v)). \end{aligned} \tag{30}$$

Since the polynomials $P(u, v)$ and $P_1(u, v)$ "commute" in the sense of composition, the structure of the points (u, v, w) and $(u, \tilde{v}, w)$ can be of two types, which are visually represented by the graphs below:



a



b

Segments connect pairs satisfying corresponding equations.

We shall show that the structure of type a) is impossible. Indeed, in this case to the pair $x^1(u, \tilde{v}_1)$ and $x(u, v_1)$, which by (30) satisfies the equation $P_R(x(u, v_1), x^1(u, \tilde{v}_1)) = 0$ of the "vacuous" curve of the matrix R, there correspond two "vacuous" vectors $X(\tilde{v}_1, w_1)$ and $X(\tilde{v}_1, w_2)$, which contradicts their uniqueness.

In the case b), the set (w_1, w_2, w_3, w_4) is split up into two pairs, (w_1, w_4) and (w_2, w_3), that are singled out invariantly by the condition that the values of v and $\tilde{v}$ which correspond to w_i and w_j formed into one pair should be different.

Thus, the curve $\hat{\Gamma}$ splits up into two elliptic curves, $\hat{\Gamma}'$ and $\hat{\Gamma}''$, while the polynomial $Q(u, w)$ is decomposed into a product $Q(u, w) = Q'(u, w) \cdot Q''(u, w)$. The points of the curves $\hat{\Gamma}'$ and $\hat{\Gamma}''$ are the pairs $((u, w_1), (u, w_4))$ and $((u, w_2), (u, w_3))$, respectively. Hence the proof of the lemma is complete.

Consider the curve $\hat{\Gamma}'$. To any point of it, say (u, w_1), there correspond uniquely points of the curves Γ and Γ_1, (u, v_1) and $(u, \tilde{v}_1)$, respectively. Consequently, the curves Γ and Γ_1 are isomorphic to $\hat{\Gamma}'$. Any elliptic curve can be parametrized with points z of the complex torus with periods $(1, \tau)$ where $\operatorname{Im} \tau > 0$.

Let $u(z)$ and $w(z)$ be a parametrization of the points of $\hat{\Gamma}'$; then $(u(z), v(z))$ and $(u(z), \tilde{v}(z))$ are a parametrization of the solution of (18). Since $(v(z), w(z))$ satisfies the equation $P_1(v(z), w(z)) = 0$, we have $v(z) = u(z - \eta)$ and correspondingly $\tilde{v}(z) = u(z - \eta_1)$.

With the parameter z, Eqs. (30) take the form

$$R (X (z - \eta_1) \otimes X^1 (z)) = g (z)(X^1 (z - \eta) \otimes X (z)). \tag{31}$$

Since the divisors of the left and right sides of this equation have to be equivalent, we have $\eta_1 = \eta + \frac{m + n\tau}{2}$, where m and n are integers.

The equivalence of two divisors γ_i and γ_i' on an elliptic curve means that $\sum \gamma_i \equiv \sum \gamma_i'$ modulo the periods; with a shift by a vector η the divisor of the poles of a function having two poles is changed by 2η.

According to (30), functions $Y(z)$ and $Y^1(z)$ satisfy the same relation as $X(z)$ and $X^1(z)$. Thus, they coincide up to a shift, i.e.,

$$Y (z) = X (z + \eta_2), \tag{32}$$
$$Y^1 (z) = X^1 (z + \eta_2). \tag{33}$$

Finally, Eqs. (19) and (20) take the form

$$\mathscr{L}\,(X\,(z)\otimes U\,(z)) = h\,(z)\,(X\,(z+\eta_2)\otimes U\,(z-\eta)), \tag{34}$$

$$\mathscr{L}^1\,(X^1\,(z)\otimes U\,(z)) = h_1\,(z)(X^1\,(z+\eta_2)\otimes U\,(z-\eta_1)). \tag{35}$$

The equivalence relation (15) means that

$$\eta_2=\eta+\frac{m'+n'\tau}{2}.$$

Now we consider the inverse problem.

Let be given an elliptic curve Γ, i.e., a modulus k, arbitrary functions $x(z)$, $x^1(z)$, and $u(z)$ having each two poles on the curve, and also some points η, η_1, and η_2, differing by half-periods. Then relations (31), (34), and (35) define uniquely up to a multiplicative constant the matrices $\mathscr{L}$, $\mathscr{L}^1$, and R.

Remark. In the case of n = 2, i.e., of matrices of order 4, giving polynomial (13) is equivalent to giving the modulus of the elliptic curve Γ and two functions with two poles on it, because for any such functions $u(z)$ and $v(z)$ there exists such a unique polynomial P of the second degree in each of the variables that $P(u(z), v(z)) = 0$.

THEOREM 2. Matrices $\mathscr{L}$, $\mathscr{L}^1$, and R defined by (31), (34), and (35) satisfy Baxter's equations. These solutions exhaust all the solutions in a general situation for which the polynomial $Q(u, w)$ corresponding to the tensor Λ has not two identically degenerate double roots.

Under the assumptions that the polynomial $Q(u, w)$ has four different roots w_i for almost all u, we have found that its roots are parametrized with elliptic functions in the following way:

$$u(z)\;\begin{cases} v_1=u(z-\eta) & \begin{cases} w_1=u(z-\eta-\eta_1)\\ w_2=u(z-\eta+\eta_1)\end{cases}\\[2ex] v_2=u(z+\eta) & \begin{cases} w_3=u(z+\eta-\eta_1)\\ w_4=u(z+\eta+\eta_1)\end{cases}\end{cases}$$

Here we assume that $u(z)$ is an even function, for this can always be obtained by a shift of the origin of coordinates. Insofar as η and η_1 differ by half a period, $w_2 = w_3$.

Thus, as an addition to Lemma 2, it can be stated that in a general situation the polynomial $Q(u, w)$ has either one or two doubly degenerated roots.

We shall describe these two types of solutions of Baxter's equations as solutions of rank 1 and 2, respectively.

In the case of rank 1, arguing as under the assumption of simple roots of polynomial $Q(u, w)$, we arrive at relations (31), (34), and (35). This proves that the conditions of Theorem 2 are necessary.

To prove the thesis of the theorem, it remains to be proved that the vacuous vectors of the tensors Λ_1 and Λ_2 coincide. The coincidence of the vacuous vectors corresponding to simple roots w_1 and w_4 of the polynomial $Q(u, w)$ follows from (31). To a double root $w_2 = w_3$ there corresponds a two-dimensional subspace of "vacuous" vectors. Comparing (25) and (27), we find that a base in this space is formed by the vectors

$$X\,(-z)\otimes X^1\,(z),\qquad X\,(z-\eta)\otimes X^1\,(-z+\eta) \tag{36}$$

and by the vectors

$$R^{-1}\,(X^1\,(-z)\otimes X\,(z)),\quad R^{-1}\,(X^1\,(z-\eta)\otimes X\,(-z+\eta)). \tag{37}$$

Thus, for $\mathscr{L}$, $\mathscr{L}^1$, and R to satisfy Baxter's equations it is necessary and sufficient that, apart from (31), (34), and (35) holding, the vectors (36) and (37) should be linearly dependent, that is

$$R\,(X\,(-z)\otimes X^1\,(z)) = \alpha\,(z)\,(X^1\,(-z)\otimes X\,(z)) + \beta\,(z)(X^1\,(z-\eta)\otimes X\,(-z+\eta)), \tag{38}$$

where α and β are meromorphic functions.

The fact that (31) implies relation (38) follows from a more general assertion:

Assertion. If an arbitrary four-dimensional matrix R is defined by the relations

$$R\,(X\,(z)\otimes U\,(z)) = h\,(z)(Y\,(z)\otimes V\,(z)), \tag{39}$$

then we have

$$R\,(X(z)\otimes_i U(z')) = \alpha(z)(Y(z)\otimes V(z')) + \beta(z)(Y(z'-\mu)\otimes V(z+\mu)),$$

where the vector μ coincides, up to half a period, with a vector equivalent to $(D_X - D_V)$.

We shall leave this assertion without a proof, showing merely that the matrices $\mathscr{L}$, $\mathscr{L}^1$, and R obtained by "calibration transformations" can be reduced either to Baxter's already known solutions or to solutions obtained from them by multiplying them by a certain matrix.

Let G_X, G_{X^1}, and G_U be arbitrary two-dimensional matrices; then a "calibration transformation" transforms an arbitrary solution $\mathscr{L}$, $\mathscr{L}^1$, R of Baxter's equations into a solution of the same equations, viz.,

$$\tilde{\mathscr{L}} = (G_X\otimes G_U)\,\mathscr{L}\,(G_X^{-1}\otimes G_U^{-1}), \tag{40}$$

$$\tilde{\mathscr{L}}^1 = (G_{X^1}\otimes G_U)\,\mathscr{L}^1\,(G_{X^1}^{-1}\otimes G_U^{-1}), \tag{41}$$

$$\tilde{R} = (G_{X^1}\otimes G_X)\,R\,(G_X^{-1}\otimes G_{X^1}^{-1}). \tag{42}$$

Without loss of generality, we can assume that the divisor of the poles of u(z) is equivalent to the divisor of the poles of sn z. Let the divisors of the poles of x(z) and $x^1(z)$ be equivalent to those of sn (z + λ) and sn (z + μ). There exist matrices G_X, G_{X^1}, and G_U unique up to a proportionality and such that

$$G_U U(z) = f_1(z)\,\mathrm{Sn}\,z,\quad G_X X(z) = f_2\mathrm{Sn}\,(z+\lambda),\quad G_{X^1}X^1(z) = f_3(z)\,\mathrm{Sn}\,(z+\mu), \tag{43}$$

where the vector Sn z is given by $\mathrm{Sn}\,z = \begin{pmatrix}\sqrt{k}\,\mathrm{sn}\,z\\ 1\end{pmatrix}$.

With the help of these matrices we shall pass from the matrices $\mathscr{L}$, $\mathscr{L}^1$, and R defined by (31), (34), and (35) to the calibration-equivalent matrices $\tilde{\mathscr{L}}$, $\tilde{\mathscr{L}}^1$, and $\tilde{R}$. Relation (31) becomes

$$\tilde{R}\,(\mathrm{Sn}\,(z+\lambda-\eta_1)\otimes\mathrm{Sn}\,(z+\mu)) = \tilde{g}(z)(\mathrm{Sn}\,(z+\mu-\eta)\otimes\mathrm{Sn}\,(z+\lambda)). \tag{44}$$

If $\eta_1 = \eta$, then $\tilde{R}$ is a Baxter matrix, for this relation coincides with the formula (4.27) in [8] for Baxter matrices. Relations (34) and (35) are transformed similarly. After a "calibration transformation," (38) becomes the formula (4.28) of [8].

COROLLARY. If $\eta = \eta_1 = \eta_2$, then R, $\mathscr{L}$, and $\mathscr{L}^1$ are calibration-equivalent to Baxter's solutions.

With shifts by half-periods, the elliptic sine is so transformed (see [9]) that

$$\mathrm{Sn}\left(z+\frac{1}{2}\right) = G_1\,\mathrm{Sn}\,z,\qquad G_1 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \tag{45}$$

$$\mathrm{Sn}\left(z+\frac{\tau}{2}\right) = G_2\,\mathrm{Sn}\,z\cdot\sqrt{k}\,\mathrm{sn}\,z,\qquad G_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{46}$$

When the vector η_1 is shifted with respect to η by half-periods $\Pi_1 = 1/2$ and $\Pi_2 = \tau/2$, then, according to (45), (46), and (44), the Baxter matrices are transformed as follows:

$$T_i\colon\ \tilde{\mathscr{L}}, \tilde{\mathscr{L}}^1, \tilde{R} \to \tilde{\mathscr{L}}, (1\otimes G_i)\,\tilde{\mathscr{L}}^1, \tilde{R}\,(G_i\otimes 1). \tag{47}$$

Since matrices G_1 and G_2 commute up to a multiplicative numerical constant, the transformations T_i determine a projective representation of the group of half-periods in the matrix space.

Similarly, shifts by half-periods of the vector η_2 determine a projective representation of the group $Z_2 \times Z_2$ in which the generators act as follows:

$$T_i'\colon\ \tilde{\mathscr{L}}, \tilde{\mathscr{L}}^1, \tilde{R} \to \tilde{\mathscr{L}}\,(G_i\otimes 1), \tilde{\mathscr{L}}^1\,(G_i\otimes 1), \tilde{R}. \tag{48}$$

COROLLARY. All the solutions of rank 1 of Baxter's equations are calibration-equivalent either to Baxter's solutions or to solutions obtained from them by means of projective transformations of groups $Z_2 \times Z_2$, the action of the generators being given by (47) and (48).

3. Solutions of Rank 2

The essential purpose of the present section is to prove that all the solutions of rank 2 of Baxter's equations are calibration-equivalent to the solutions found by Felderhof [16].*

*The existence of solutions other than those of Baxter and also their parametrization [see (79) in the present paper] was communicated to the author by Zamolodchikov, who obtained them independently, but substantially later than Felderhof.

As shown in the preceding section, besides solutions of the Baxter type, Eqs. (2) can have solutions of rank2, i.e., solutions for which the polynomial Q(u, v) corresponding to the tensor $\Lambda = \Lambda_1 = \Lambda_2$ has two doubly generated roots for all u,

$$Q(u, w) = \hat{Q}^2(u, w). \tag{49}$$

Consider Eqs. (24) defining the polynomial Q(u, w). If u_1 and u_2 are roots of the polynomial $\hat{Q}(u, w)$ with a fixed w, while v_1 and v_2 are roots of $P_1(v, w)$, then it follows from (49) that

$$P(u_k, v_m) = 0, \quad k, m = 1, 2. \tag{50}$$

We represent the polynomial P(u, v) in the form

$$P(u, v) = \sum_{i=0}^{2} r_i(u) v^i = \sum_{i=0}^{2} q_i(v) u^i.$$

It follows from (50) that on the curve Γ corresponding to $\mathscr{L}$, there corresponds to every value of the function r(u) a unique value of the function q(v), where $r(u) = r_2(u)/r_0(u)$, $q(v) = q_2(v)/q_0(v)$. Thus on Γ these functions are connected by a rational transformation, and the curve itself is determined by the equation

$$p(r(u), q(v)) = 0, \quad p(r, q) = rq - \alpha r - \beta q + \gamma. \tag{51}$$

The curve Γ_1 that corresponds to the matrix $\mathscr{L}^1$ is determined by an analogous equation

$$p^1(r^1(u), q^1(v)) = 0, \quad p^1 = rq - \alpha_1 r - \beta_1 q + \gamma_1. \tag{52}$$

Since the polynomials P and P_1 commute in the sense of composition, the polynomials p and p' commute in the same sense, too. Moreover, it follows from it that

$$r(u) = q(u) = r'(u) = q^1(u). \tag{53}$$

If to the matrix $\mathscr{L}$ there corresponds the polynomial P(u, v), then to the calibration-equivalent matrix $\tilde{\mathscr{L}}$ there corresponds the polynomial

$$\tilde{P}(u, v) = P\left(\frac{au+b}{cu+d}, \frac{av+b}{cv+d}\right)(cu+d)^2(cv+d)^2,$$

where $G_U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Using rational changes of variables, we obtain finally the following result:

LEMMA 3. If $\mathscr{L}$ and $\mathscr{L}^1$ are solutions of rank2, then, up to a calibration equivalence, it can be assumed that the corresponding Γ and Γ_1 are determined by the equations

$$P(u, v) = u^2v^2 - \alpha u^2 - \beta v^2 + 1 = 0, \tag{54}$$
$$P_1(u, v) = u^2v^2 - \alpha_1 u^2 - \beta_1 v^2 + 1 = 0, \tag{55}$$
$$\alpha + \beta = \alpha_1 + \beta_1. \tag{56}$$

Relation (56) is necessary and sufficient for compositions (24) and (26) of polynomials (54) and (55) to coincide.

The choice of the pair of indices of the tensor Λ which are regarded as external is arbitrary. Hence, corresponding to the tensor $\Lambda^{ij\alpha}_{pq\beta}$, it is possible to construct a collection of polynomials $Q^{mn}(u, v)$, where m, n = 1, 2, 3 are the numbers of the corresponding upper and lower indices. For instance,

$$Q^{12}(u, w) = \det(W^q \Lambda^{ij\alpha}_{pq\beta} U_i) = 0 \tag{57}$$

(notations of Sec. 1). Condition (57) is necessary and sufficient for the existence of such vectors $X_{j\alpha}$ and $Y_{p\beta}$ that

$$\Lambda^{ij\alpha}_{pq\beta} X_{j\alpha} U_i = h Y_{p\beta} W_q. \tag{58}$$

In the new notations the old polynomial Q(u, w) becomes $Q^{33}(u, w)$.

Let the matrix R be defined by the relation

$$R(X_R(z) \otimes U_R(z)) = g(z)(Y_R(z) \otimes V_R(z)), \tag{59}$$

where $z \in \Gamma_R$ is a point of the elliptic curve. We introduce such polynomials P_R^{12} and P_R^{21} that

$$P_R^{12}(x_R(z), v_R(z)) = 0, \tag{60}$$

$$P_R^{21}(u_R(z),\ y_R(z)) = 0. \tag{61}$$

If $\mathscr{L}$ and $\mathscr{L}^1$ are determined, as previously, by (19) and (20), then the polynomials P^{11} and P_1^{11} are defined so that

$$P^{11}(x(z),\ y(z)) = 0, \quad P_1^{11}(x^1(z),\ y^1(z)) = 0. \tag{62}$$

In analogy with the preceding section, substituting i and q for the indices α and β, we find that the polynomial $Q^{12}(u, w)$ of the tensor Λ is determined by the compositions

$$P^{11}(u,\ v) = 0, \quad P_R^{12}(v,\ w) = 0, \tag{63}$$

$$P_R^{12}(u,\ \tilde{v}) = 0, \quad P^{11}(\tilde{v},\ w) = 0. \tag{64}$$

Considering the polynomial $Q^{21}(u, w)$, we find that the compositions of the polynomials P^{21} and P_1^{11} commute. Similarly to Lemma 3, we prove:

LEMMA 4. Up to a calibration equivalence of the matrices $\mathscr{L}$, $\mathscr{L}^1$, and R, it can be assumed that the corresponding polynomials are

$$P^{11}(u,\ v) = u^2v^2 - \gamma u^2 - \delta v^2 + 1 = 0, \tag{65}$$

$$P_1^{11}(u,\ v) = u^2v^2 - \gamma_1 u^2 - \delta_1 v^2 + 1 = 0, \tag{66}$$

$$P_R^{12}(u,\ v) = u^2v^2 - \gamma_R u^2 - \delta_R v^2 + 1 = 0, \tag{67}$$

$$P_R^{21}(u,\ v) = u^2v^2 - \gamma_R^1 u^2 - \delta_R^1 v^2 + 1 = 0, \tag{68}$$

where

$$\gamma + \delta = \gamma_R + \delta_R, \quad \gamma^1 + \delta^1 = \gamma_R^1 + \delta_R^1. \tag{69}$$

The solutions of Eq. (54) can be parametrized with elliptic functions of modulus $k = 1/\sqrt{\alpha\beta}$ as follows:

$$u(z) = \frac{\mathrm{sn}(z,\ k)}{\sqrt{\alpha}}, \quad v(z) = \frac{\mathrm{cn}(z,\ k)}{\sqrt{\beta}\ \mathrm{dn}(z,\ k)}. \tag{70}$$

This can be verified by simply substituting these expressions in (56) and taking into account the well-known identities

$$\mathrm{sn}^2 z + \mathrm{cn}^2 z = 1, \quad k^2 \mathrm{sn}^2(z,\ k) + \mathrm{dn}^2(z,\ k) = 1.$$

All the necessary relations between the elliptic functions used here and further on can be found in [9].

Since (54) and (65) determine the same curve, we have $\gamma\delta = \alpha\beta$.

Using an analogous parametrization of Eq. (65), which can differ from that of (69) by some vector η and by a reflection, we find that the matrix $\mathscr{L}$ is determined by the following equality:

$$\mathscr{L}\left(\begin{pmatrix} \frac{\mathrm{sn}(\eta - z)}{\sqrt{\gamma}} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{\mathrm{sn}\, z}{\sqrt{\alpha}} \\ 1 \end{pmatrix}\right) = h \begin{pmatrix} \frac{\mathrm{cn}'(z - \eta)}{\sqrt{\delta}\ \mathrm{dn}(z - \eta)} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{\mathrm{cn}\, z}{\sqrt{\beta}\ \mathrm{dn}\, z} \\ 1 \end{pmatrix}. \tag{71}$$

This matrix depends, up to a multiplicative constant, on the parameters α, β, γ, δ, η, where $k = 1/\sqrt{\alpha\beta} = 1/\sqrt{\gamma\delta}$, i.e.,

$$\mathscr{L} = \mathscr{L}(\alpha,\ \beta,\ \gamma,\ \delta,\ \eta), \quad \alpha\beta = \gamma\delta.$$

From (71) it is possible to obtain an explicit expression for the elements of the matrix

$$\mathscr{L} = \begin{pmatrix} e^2 a & 0 & 0 & ed \\ 0 & e^2 b & ec & 0 \\ 0 & ec & b & 0 \\ ed & 0 & 0 & a' \end{pmatrix}, \tag{72}$$

where

$$\begin{aligned} a &= \alpha\ \mathrm{sn}\,\eta\ \mathrm{dn}\,\eta, & a' &= \beta\ \mathrm{sn}\,\eta\ \mathrm{dn}\,\eta, & d &= \mathrm{sn}\,\eta\ \mathrm{cn}\,\eta, \\ b &= \sqrt{\alpha\beta}\ \mathrm{cn}\,\eta, & c &= \sqrt{\alpha\beta}\ \mathrm{dn}\,\eta, & e &= \sqrt{\gamma/\alpha}. \end{aligned} \tag{73}$$

It can be verified that the elements of $\mathscr{L}$ satisfy the relation

$$aa' + b^2 - c^2 - d^2 = 0. \tag{74}$$

Consequently, for $\mathscr{L}, \mathscr{L}^1$, and R to be solutions of rank 2 of Baxter's equations it is necessary that they should be of the form

$$\mathscr{L} = \mathscr{L}(\alpha, \beta, \gamma, \delta, \eta), \quad \mathscr{L}^1 = \mathscr{L}(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1),$$
$$R = R(\gamma_R^1, \delta_R^1, \gamma_R, \delta_R, \eta_R),$$

where $R_{pq}^{ij}(\alpha, \beta, \gamma, \delta, \eta) = \mathscr{L}_{qp}^{ij}(\alpha, \beta, \gamma, \delta, \eta)$, and that the parameters of these matrices should be connected by relations (56) and (69).

We return to the vacuous vectors of Λ corresponding to the curve $\hat{\Gamma}$ given by the equation $Q^{33}(u, w) = 0$. Comparing the bases in two-dimensional spaces of these vectors for every point of $\hat{\Gamma}$ which yield the equalities (25) and (26), we find that the following relations ought to be satisfied:

$$R(X(\pm\tilde{v}, w) \otimes X^1(u, \pm\tilde{v})) = \alpha_\pm (X^1(v, w) \otimes X(u, v)) + \beta_\pm (X^1(-v, w) \otimes X(u, -v)), \tag{75}$$
$$R(Y(\pm\tilde{v}, w) \otimes Y^1(u, \pm\tilde{v})) = \tilde{\alpha}_\pm (Y^1(v, w) \otimes Y(u, v)) + \tilde{\beta}_\pm (Y^1(-v, w) \otimes Y(u, -v)). \tag{76}$$

The mapping $u, v \to -1/v, 1/u$ determines an automorphism of the fourth-order curves given by Eqs. (54) and (55), which is seen to be a shift by one-quarter of the period. It follows from (71) that

$$y\left(\frac{1}{v}, -\frac{1}{u}\right) = -\frac{1}{x(u, v)}.$$

The functions $x^1(u, \tilde{v})$ and $y^1(u, \tilde{v})$ are similarly connected with each other. From (75) and (76) we find that then the matrix R should satisfy

$$M^{-1}RM = hR^{-1}, \tag{77}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and h is a constant.

For R of the form (72) to satisfy (77) it is necessary and sufficient that we should have $e^2 = 1$ or $\gamma_R^1 = \gamma_R$, $\delta_R^1 = \delta_R$.

Considering instead of Q^{33} the polynomials Q^{12} and Q^{21}, we obtain analogous relations for the parameters of the matrices $\mathscr{L}$ and , viz. $\alpha = \gamma$, $\beta = \delta$, $\alpha_1 = \gamma_1$, $\beta_1 = \delta_1$.

THEOREM 3. For $\mathscr{L}, \mathscr{L}^1$, and R to be solutions of rank 2 of Baxter's equations, it is necessary that they should be reducible by calibration transformations to matrices of the form

$$\mathscr{L} = \mathscr{L}(\alpha, \beta, \eta), \ \mathscr{L}^1 = \mathscr{L}(\alpha_1, \beta_1, \eta_1), \quad R = R(\alpha_R, \beta_R, \eta_R), \tag{78}$$

where $\mathscr{L}(\alpha, \beta, \eta) = \mathscr{L}(\alpha, \beta, \alpha, \beta, \eta)$, $\alpha + \beta = \alpha_1 + \beta_1 = \alpha_R + \beta_R$. For any matrix $\mathscr{L}(\alpha, \beta, \eta)$ there exists a one-parameter family of matrices $\mathscr{L}^1$ and R satisfying Baxter's equations.

All the statements of this theorem, except for the last one, were proved above.

A direct proof of the latter is rather long and cumbersome. It can be avoided by combining the results obtained here with those of Felderhof.

Formulas (73) with $\alpha = \gamma$ yield a parametrization of all matrices (up to multiplicative constants) of the form (72) ($e = \pm 1$) whose elements satisfy relation (74). Felderhof discussed Baxter's equations precisely for matrices of this type. He found that for these matrices there exists a parametrization

$$a = \mathrm{dn}\,\theta + \rho\,\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta, \ a' = \mathrm{dn}\,\theta - \rho\,\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta, \quad b = \varepsilon\,\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta,$$
$$c = \mathrm{cn}\,\theta, \quad d = \mathrm{sn}\,\theta\,\mathrm{dn}\,\theta, \quad \varepsilon^2 = \rho^2 + k^2. \tag{79}$$

Moreover, if λ and the modulus k of the elliptic functions are constant, the matrices $\mathscr{L}(\theta)$, $\mathscr{L}^1 = \mathscr{L}(\theta')$, $R_{pq}^{ij} = \mathscr{L}_{qp}^{ij}(\theta - \theta')$ satisfy Baxter's equations.

COROLLARY. All the solutions of rank 2 of Baxter's equations are calibration-equivalent to Felderhof's solutions.

It is important to note that the parametrizations (79) and (73) are two different parametrizations of the same matrices. It would be most interesting to find out which algebrogeometric object corresponds to the parametrization (79) to the extent to which the concept of vacuous vectors corresponds to the parametrization (73). We stress once more that, as opposed to the case of Baxter's solutions, the modulus of the curves of vacuous vectors takes different values.

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TANGENTIAL SINGULARITIES

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UDC 517.27

By tangential singularities we mean singularities of the position of a surface in affine or projective space with respect to its tangentially different dimensions.

The goal of the present paper is the classification of tangential singularities of a smooth hypersurface and of the family of level surfaces of a smooth function.

A line can have, with a surface in general position in three-dimensional space, order of tangency 1, 2, 3, 4 (with a hypersurface in $\mathbf{R}^n$, up to $2n-2$; with a level surface, up to $2n-1$).

We classify points of a surface according to orders of tangency of their tangents and according to the disposition of sets of points with different orders in their neighborhoods.

In the case of surfaces in three-dimensional space our classification consists of seven classes. A smooth curve of parabolic points (p_1) divides a surface in general position into elliptic (e) and hyperbolic (h_1) domains. In the hyperbolic domain there is singled out a curve (h_2), on which asymptotic lines have inflection (curvature zero). This curve on a surface in general position is smoothly immersed; on it there are singled out isolated points of double inflection of asymptotic lines (h_3), points of transversal self-intersection (h_4) and points of tangency of lines of inflection with a parabolic line (p_2).

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 15, No. 2, pp. 36-49, April-June, 1981. Original article submitted November 12, 1979.