NONLINEAR EQUATIONS AND ELLIPTIC CURVES

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The main ideas of global "finite-zone integration" are presented, and a detailed analysis is given of applications of the technique developed to some problems based on the theory of elliptic functions. In the work the Peierls model is integrated as an important application of the algebrogeometric spectral theory of difference operators.

INTRODUCTION

During the last 10-15 years one of the most powerful tools in the investigation of nonlinear phenomena has become the so-called method of the inverse problem which is applicable to a number of fundamental equations of mathematical physics. Development of this method (see [17, 43, 53] and the survey cited there) has led to the situation that the concept of solitons has become one of the fundamental concepts in modern mathematical and theoretical physics.

Starting from the work of Novikov [40], methods of constructing solutions of nonlinear equations which make extensive use of the apparatus of classical algebraic geometry of Riemann surfaces have developed rapidly and continue to develop within the framework of the method of the inverse problem. (Various stages in the development of algebrogeometric or "finite-zone" integration are described in the surveys [13, 15, 25, 35]). Methods of algebraic geometry make it possible to introduce in a natural way the concept of periodic and quasiperiodic analogues of soliton and multisoliton solutions and to obtain for them explicit expressions in terms of Riemann theta functions.

The purpose of the present work is to present the main ideas of global "finite-zone integration" and give a detailed analysis of applications of this technique to some problems based on the theory of elliptic functions.

The construction of Baker—Akhiezer functions [13, 25, 26, 35] — functions possessing specific analytic properties on Riemann surfaces — is a central link in the algebrogeometric constructions of solutions of nonlinear equations. It turns out that the concept of a vector analogue of a Baker—Akhiezer function introduced in [27, 37, 35, 36] does not trivialize even on the ordinary complex plane (a Riemann surface of genus zero). Moreover, it makes it possible [28] for equations admitting a representation of "zero curvature" of general position to construct all solutions without restriction to some fixed function class (rapidly decreasing, periodic, or other functions). For these equations an analogue can be proved of the D'Alembert representation of all solutions in the form of a nonlinear superposition of waves traveling along characteristics. The auxiliary linear Riemann problem plays the role of superposition.

The representation of "zero curvature" of pencils rationally depending on a spectral parameter which was set forth by Zakharov and Shabat [19] is perhaps the most general scheme for constructing nonlinear integrable equations which includes all known cases with the exception of some isolated examples. In the first chapter we present a general Ansatz distinguishing finite-zone solutions together with an exposition of the construction of the analogue mentioned above of the D'Alembert representation of solutions of such equations of general position

Together with the general algebrogeometric construction of "finite-zone" solution of nonlinear equations, the concept of finite-zone integration also includes important elements of the Floquet spectral theory of operators with periodic coefficients. In those cases where the auxiliary spectral problem for the nonlinear equation is self-adjoint the "finite-zone"

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potentials can be distinguished by the condition of a finite number of lacunae in the spectrum of the operator (which provided the name for these solutions). In this approach the Riemann surfaces arise as surfaces on which the Bloch functions — eigenfunctions of the linear operator and the monodromy operator (the operator of translation by a period) — become single-valued. An important observation of the modern theory is that the concepts of Bloch functions and functions of Baker–Akhiezer type coincide.

In the second chapter we present the algebrogeometric spectral theory of the Schrödinger difference operator and of the Sturm–Liouville equation with periodic potentials. This theory, which arose from demands of the theory of nonlinear equations, in recent years has found broad application in problems of solid-state physics connected with the theory of quasi-one-dimensional conductors [3-7, 10, 11, 30]. This theory is usually constructed on the basis of the Peierls model [41].

The Peierls model describes the self-consistent behavior of atoms of a lattice, which are characterized by the position on the line x_n and an internal degree of freedom v_n, and of the electrons.

The direct interelectron interaction is neglected in the model. The spectrum of the electrons is defined as the spectrum E_i^+ of the periodic Schrödinger operator

$$L\psi_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} = E\psi_n$$

with periodic boundary conditions

$$\psi_{n+N}(E_i^+) = \psi_n(E_i^+), \text{ when } c_n = e^{x_n - x_{n+1}}, \quad c_{n+N} = c_n.$$

If there are m electrons in the system, then at zero temperature the electrons occupy the m lowest levels of the spectrum (without consideration of spin degeneracy). The model takes into account the elastic deformation of the lattice.

The Peierls functional, which represents the total energy of the system arriving at one node, has the form

$$\mathcal{H} = \frac{1}{N}\left(\sum_{i=1}^{m} E_i^+ + \sum_{n=0}^{N-1} \Phi(c_n, v_n)\right),$$

where $\Phi(c, v)$ is the potential of elastic deformation. The problem consists in minimization of this nonlinear and nonlocal functional with respect to the variables v_n, c_n.

In the third chapter we present results of [7, 11, 29, 30] in which for some model potentials [for example, $\Phi = \varkappa\left(c^2 + \frac{v^2}{2}\right) - P\ln c$] it can be proved that the minimum is realized on configurations in which $c_n = f_1(n)$, $v_n = f_2(n)$, where $f_{1,2}$ are elliptic functions which can be given explicitly. The energy of the base state is found. Perturbations of integrable cases are considered.

CHAPTER 1

NONLINEAR EQUATIONS AND ALGEBRAIC CURVES

1. Representation of Zero Curvature

Beginning with the work of Lax [6], who showed that at the basis of the mechanism integrating the KdV equation

$$4u_t = 6uu_x + u_{xxx}, \tag{1.1}$$

discovered in [67], there lies the possibility of representing this equation in the form

$$\dot{L} = [A, L] = AL - LA, \tag{1.2}$$

where

$$L = \frac{\partial^2}{\partial x^2} + u(x, t); \quad A = \frac{\partial}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + \frac{3}{4}u_x, \tag{1.3}$$

all schemes of constructing integrable equations and their solutions are based on some analogue of the commutation representation (1.2).

The first and most natural generalization of Eq. (1.2) is to take for L and A there arbitrary differential operators

$$L=\sum_{i=0}^{n} u_i(x,t)\frac{\partial^i}{\partial x^i};\quad A=\sum_{l=0}^{m} v_l(x,t)\frac{\partial^l}{\partial x^l} \tag{1.4}$$

with matrix or scalar coefficients.

Suppose, further, to be specific that L and A are operators with scalar coefficients (the matrix case differs from the scalar case by minor technical complications). Then by a change of variable and the conjugation $\tilde{L} = fLf^{-1}$, $\tilde{A} = fAf^{-1}$, where f is a suitable function, it may be assumed that $v_m = u_n = 1$, $v_{m-1} = u_{n-1} = 0$. In this case Eqs. (1.2) form a system of $n + m - 2$ equations for the unknowns $u_i(x, t)$, $i = 0,\dots,n - 2$; $v_j(x, t)$, $j = 0,\dots,m - 2$. It turns out that from the first $m - 1$ equations obtained by equating to zero the coefficients of $\partial^k/\partial x^k$, $k = m + n - 3,\dots,n - 1$, in the right side of (1.2) the $v_j(x, t)$ can be found successively; they are differential polynomials in $u_i(x, t)$ and some arbitrary constants h_j (see, for example, [26]). By substituting the expressions obtained into the remaining $n - 1$ equations, we obtain a system of evolution equations for the coefficients of the operator L which are called equations of Lax type. There are a number of schemes (see, for example, [8, 18, 49, 53]) which realize in some manner a reduction of the general equation (1.2) to equations for the coefficients of the operator L.

System (1.2) represents a pencil of Lax equations parametrized by the constants h_j. For example, if

$$L=\partial^2+u,\quad A=\partial^3+v_1\partial+v_2,\quad \partial=\frac{\partial}{\partial x}, \tag{1.5}$$

then

$$v_1=\frac{3}{2}u+h_1,\quad v_2=\frac{3}{4}u_x+h_2, \tag{1.6}$$

and Eq. (1.2) is equivalent to the pencil of equations

$$4u_t=u_{xxx}+6uu_x+4h_1u_x \tag{1.7}$$

(for $h_1 = 0$ we obtain the standard KdV equation).

With each operator L there is connected an entire hierarchy of equations of Lax type which constitute reductions to equations for the coefficients of the operator L of Eqs. (1.2) with operators A of different orders. One of the most important facts in the theory of integrable equations is the commutativity of all equations contained in the common hierarchy.

For the KdV equations the corresponding equations are called "higher KdV equations." They have the form

$$u_\tau=\sum_{k=1}^{n} c_kQ_k(u,\dots,u^{2k+1})$$

and constitute a commutation condition of the Sturm–Liouville operator with the operators $\partial/\partial t - A$ [i.e., Eqs. (1.2)] where A has order $2n + 1$.

Another representation — a representation of Lax type for matrix functions depending on an additional spectral parameter — was first introduced and actively used for higher analogues of the KdV equation in [40].

For the general equation (1.2) such a λ-representation can be obtained in the following manner.

We consider the matrix problem of first order

$$\left[\frac{d}{dx}-\begin{pmatrix} 0 & 1 & 0\dots0 & 0\\ 0 & 0 & 1\dots0 & 0\\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot\\ 0 & 0 & 0\dots0 & 1\\ \lambda+u_0, & u_1\dots & u_{n-2} & 0\end{pmatrix}\right]Y=\left(\frac{d}{dx}+\tilde{L}(x,\lambda)\right)Y=0, \tag{1.8}$$

equivalent to the equation

$$Ly=\lambda y. \tag{1.9}$$

By acting with the operator A on the coordinates of the vector $Y_i = \partial^i y/\partial x^i$ and using Eq. (1.9) to express $\partial^n y$ in terms of lower order derivatives and the parameter λ, we find that on the space of solutions of (1.9) the operator $\partial/\partial t - A$ is equivalent to the operator

$$\left(\frac{d}{dt} + \tilde{A}(\lambda, x, t)\right) Y = 0, \tag{1.10}$$

where the matrix $\tilde{A}$ depends in polynomial fashion on the parameter λ. The matrix elements of $\tilde{A}$ are differential polynomials in $u_i(x, t)$ (polynomials in u_i and their derivatives).

From (1.2) it follows that

$$\left[\frac{\partial}{\partial x} + \tilde{L}, \frac{\partial}{\partial t} + \tilde{A}\right] = \frac{\partial}{\partial x}\tilde{A} - \frac{\partial}{\partial t}\tilde{L} + [\tilde{L}, \tilde{A}] = 0. \tag{1.11}$$

For the KdV equation the matrices of (1.11) $\tilde{L}$ and $\tilde{A}$ have the form

$$\tilde{L} = -\left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda - u & 0 \end{array}\right), \tag{1.12}$$

$$\tilde{A} = -\left(\begin{array}{cc} -\frac{u_x}{4}, & \lambda + \frac{u}{2} \\ \hline \lambda^2 - \frac{u}{2}\lambda - \frac{u^2}{2} - \frac{u_{xx}}{4}, & \frac{u_x}{4} \end{array}\right). \tag{1.13}$$

Thus, equations of Lax type are a special case of more general equations — equations admitting a representation of "zero curvature" which was introduced in [19], as already mentioned in the introduction, as a general scheme of producing one-dimensional integrable equations.

Let U(x, t, λ) and V(x, t, λ) be arbitrary rational matrix functions depending rationally on the parameter λ:

$$\begin{aligned} U(x, t, \lambda) &= u_0(x, t) + \sum_{k=1} \sum_{s=1}^{h_k} u_{ks}(x, t)(\lambda - \lambda_k)^{-s}, \\ V(x, t, \lambda) &= v_0(x, t) + \sum_{r=1}^{m} \sum_{s=1}^{d_r} v_{rs}(x, t)(\lambda - \mu_r)^{-s}. \end{aligned} \tag{1.14}$$

The compatibility condition for the linear problems

$$\left(\frac{\partial}{\partial x} + U(x, t, \lambda)\right) \Psi(x, t, \lambda) = 0, \tag{1.15}$$

$$\left(\frac{\partial}{\partial t} + V(x, t, \lambda)\right) \Psi(x, t, \lambda) = 0 \tag{1.16}$$

constitute an equation of "zero curvature"

$$U_t - V_x + [V, U] = 0, \tag{1.17}$$

which must be satisfied for all λ. It is equivalent to the system $\left(\sum_k h_k\right) + \left(\sum_r d_r\right) + 1$ of matrix equations for the unknown functions $u_{ks}(x, t)$, $v_{rs}(x, t)$, $u_0(x, t)$, $v_0(x, t)$. These equations arise by equating to zero all singular terms on the left side of (1.17) at the points $\lambda = \lambda_k$ and $\lambda = \mu_r$ and also the free term equal to $u_{0t} - v_{0x} + [v_0, u_0]$.

The number of equations is one matrix equation less than the number of unknown matrix functions. This indeterminacy is connected with the "gauge symmetry" of Eqs. (1.17). If g(x, t) is an arbitrary nondegenerate matrix function, then the transformation

$$\begin{aligned} U &\to \partial_x g \cdot g^{-1} + gUg^{-1}, \\ V &\to \partial_t g g^{-1} + gVg^{-1}, \end{aligned} \tag{1.18}$$

called a "gauge" transformation, takes solutions of (1.17) into solutions of the same equation.

A choice of conditions on the matrices U(x, t, λ) and V(x, t, λ) consistent with Eqs. (1.17) and destroying the gauge symmetry is called fixation of the gauge. The simplest gauge is given by the condition $u_0(x, t) = v_0(x, t) = 0$.

As in the case considered above of commutation equations of differential operators, Eqs. (1.17) are essentially generating equations for an entire family of integrable equations. Equations (1.17) can be reduced to a pencil of equations parametrized by arbitrary constants for the coefficients of $U(x, t, \lambda)$ alone. By changing the number and multiplicities of the poles of V, we hereby obtain hierarchies of commuting flows connected with $U(x, t, \lambda)$.

An important question for separating out of (1.17) some special equations is that of the reduction of these equations, i.e., the description of invariant submanifolds of (1.17). Restrictions of the equations of motion to these submanifolds written in suitable coordinates frequently lead to equations that are strongly different from the general form. Here the difference is manifest not only in the external form of the equations but also in the behavior of their solutions.

It should be noted that gauge transformations taking one invariant submanifold into another take the corresponding integrable systems into one another; each of these systems corresponds to different physical problems.

Leaving aside further analysis of questions of reduction and gauge equivalence of the systems, which can be found, for example, in the works [19, 18, 63], we henceforth consider Eqs. (1.17) globally, fixing the specific gauge in which $u_0 = v_0 = 0$.

We note further that questions of reduction and the description of invariant submanifolds of Eqs. (1.17) reduce to the description of various orbits of the coadjoint representation of the algebra of flows; the Hamiltonian theory of these equations is naturally introduced within this framework (see [57]).

Let $\Psi(x, t, \lambda)$ be a solution of Eqs. (1.15), (1.16) which are compatible if U and V are solutions of Eqs. (1.17). The matrix function $\Psi(x, t, \lambda)$ is uniquely determined if we fix the initial conditions, for example, $\Psi(0, 0, \lambda) = 1$. Here $\Psi(x, t, \lambda)$ is an analytic function of λ everywhere except for the poles λ_k, μ_r of the functions U and V at which it has essential singularities.

To clarify the form of the singularities Ψ at these points, we pose the following Riemann problem.

Find an analytic function $\Phi_\varkappa$ analytic for all $\lambda\neq\varkappa$ which in a neighborhood of the point $\lambda=\varkappa$ can be represented in the form

$$\Phi_\varkappa(x, t, \lambda)=R_\varkappa(x, t, \lambda)\Psi(x, t, \lambda), \tag{1.19}$$

where

$$R_\varkappa(x, t, \lambda)=\sum_{s=0}^{\infty}R_{\varkappa s}(x, t)(\lambda-\varkappa)^s \tag{1.20}$$

is a regular matrix function in a neighborhood of $\lambda=\varkappa$.

The condition that $\Phi_\varkappa$ be representable in the form (1.19) means that on a small circle Γ: $|\lambda-\varepsilon|=\varepsilon$ it is necessary to pose the standard Riemann problem of finding functions $\Phi_\varkappa$ and $R_\varkappa$ analytic outside and inside a contour and connected on Γ by relation (1.19). By definition, $\Phi_\varkappa$ inside the contour is equal to $R_\varkappa\Psi$.

From general theorems on the solvability of the Riemann problem [41] it follows that exists and is uniquely determined by the normalization condition

$$\Phi_\varkappa(x, t, \infty)=1. \tag{1.21}$$

Solution of the Riemann problem with any contour reduces in standard fashion to solution of a system of linear singular equations with Cauchy kernels [39].

If $\varkappa\neq\lambda_k$, $\varkappa\neq\mu_r$, then $\Phi_\varkappa$ is analytic everywhere and hence $\Phi_\varkappa=1$. Suppose $\varkappa$ coincides with one of the points λ_k or μ_r. We consider the logarithmic derivatives $\partial_x\Phi_\varkappa\Phi_\varkappa^{-1}$ and $\partial_t\Phi_\varkappa\Phi_\varkappa^{-1}$. They are regular functions of λ for $\lambda\neq\varkappa$. From (1.19) and Eqs. (1.15), (1.16) to which Ψ is subject it follows that these logarithmic derivatives have poles for $\lambda=\varkappa$ of multiplicity equal to the multiplicities of the poles of U and V at this point, respectively. Considering equality (1.21), we arrive finally at the following assertion.

LEMMA 1.1. The function $\Phi_\varkappa$ satisfies the equations

$$\partial_x \Phi_\varkappa(x, t, \lambda) = U_\varkappa(x, t, \lambda)\Phi_\varkappa(x, t, \lambda) = \left(\sum_{s=1}^{h} \tilde{u}_{\varkappa s}(x, t)(\lambda-\varkappa)^{-s}\right)\Phi_\varkappa, \tag{1.22}$$

$$\partial_t \Phi_\varkappa(x, t, \lambda) = V_\varkappa(x, t, \lambda)\Phi_\varkappa(x, t, \lambda) = \left(\sum_{s=1}^{d} \tilde{v}_{\varkappa s}(x, t)(\lambda-\varkappa)^{-s}\right)\Phi_\varkappa, \tag{1.23}$$

where h and d are the multiplicities of the poles of U and V at the point $\lambda=\varkappa$.

COROLLARY. If $\lambda_k \neq \mu_r$ for all k, r, then the functions

$$\Phi_{\lambda_k}(x, t, \lambda) = \Phi_{\lambda_k}(x, \lambda); \quad U_{\lambda_k}(x, t, \lambda) = U_{\lambda_k}(x, \lambda) \tag{1.24}$$

do not depend on t. Similarly,

$$\Phi_{\mu_r}(x, t, \lambda) = \Phi_{\mu_r}(t, \lambda); \quad V_{\mu_r}(x, t, \lambda) = V_{\mu_r}(t, \lambda). \tag{1.25}$$

It is evident from this assertion that in the case of noncoincident poles the Riemann problem (1.19) plays the role of separation of variables.

In the general case of coincident poles λ_k and μ_r this construction assigns to each solution of Eqs. (1.17) U, V a collection of functions $U_\varkappa$, $V_\varkappa$ defined from (1.22), (1.23):

$$U, V \to \{U_\varkappa, V_\varkappa, \varkappa=\lambda_k, \mu_r\}. \tag{1.26}$$

Here $U_\varkappa$, $V_\varkappa$ satisfy the same equations (1.17), but, in contrast to U and V, they have poles only at a single point.

Our next problem will be the construction of the transformation inverse to (1.26) and proof of the equivalence of Eqs. (1.17) with arbitrary rational functions to the collection of equations (1.17) with poles at single points.

Thus, suppose we have solutions $U_{\varkappa_r}$ and $V_{\varkappa_r}$ of Eqs. (1.17) with poles at the points $\lambda=\varkappa_r$, respectively. We denote by $\Phi_{\varkappa_r}(x, t, \lambda)$ solutions of Eqs. (1.22), (1.23) normalized by the condition $\Phi_{\varkappa_r}(0, 0, \lambda)\equiv 1$.

We denote by $\Psi(x, t, \lambda)$ a function analytic in λ everywhere except the points $\varkappa_r$, and representable in a neighborhood of these points in the form

$$\Psi(x, t, \lambda) = R_{\varkappa_r}(x, t, \lambda)\Phi_{\varkappa_r}(x, t, \lambda). \tag{1.27}$$

The construction of Ψ is equivalent to the solution of the Riemann problem on the collection of circles $|\lambda-\varkappa_r|=\varepsilon$ with centers at the points $\varkappa_r$.

LEMMA 1.2. There exists a unique solution Ψ of the problem posed which is normalized by the condition $\Psi(x, t, \infty) = 1$.

THEOREM 1.1. The function $\Psi(x, t, \lambda)$ satisfies Eqs. (1.15), (1.16) where U and V have the form (1.14) and

$$\begin{aligned} &\sum_{s=1}^{h_k} u_{ks}(x, t)(\lambda-\lambda_k)^{-s} \equiv R_{\lambda_k} U_{\lambda_k} R_{\lambda_k}^{-1} \pmod{O(1)}, \\ &\sum_{s=1}^{d_r} v_{rs}(x, t)(\lambda-\mu_r)^{-s} \equiv R_{\mu_k} V_{\mu_k} R_{\mu_k}^{-1} \pmod{O(1)}, \end{aligned} \tag{1.28}$$

where λ_k, μ_r are the points $\varkappa_r$ at which $U_{\varkappa_k}$ and $V_{\varkappa_r}$ have poles, respectively. The multiplicities h_k and d_r are equal to the multiplicities of the poles of $U_{\varkappa_r}$ and $V_{\varkappa_r}$, respectively. All solutions of Eqs. (1.17) are given by this construction.

The proof of the theorem reduces to considering the logarithmic derivatives of Ψ as in the derivation of the equations for $\Phi_\varkappa$.

As an example, we consider a case where arbitrary functions $u_{ks}(x)$, $1 \leqslant k \leqslant n$, $1 \leqslant s \leqslant h_k$; $v_{rs}(t)$, $1 \leqslant r \leqslant m$, $1 \leqslant s \leqslant d_r$ are given. Then the functions

$$U_{\lambda_k}(x, \lambda) = \sum_{s=1}^{h_k} u_{ks}(x)(\lambda-\lambda_k)^{-s},$$

$$V_{\mu_r}(t,\lambda)=\sum_{s=1}^{d_r}\tilde{v}_{rs}(t)(\lambda-\mu_r)^{-s}, \tag{1.29}$$

where $\lambda_k \neq \mu_r$ are arbitrary collections of points, uniquely determine the functions $\Phi_{\lambda_k}(x, \lambda)$ and $\Phi_{\mu_r}(t, \lambda)$ of (1.22), (1.23).

By Lemma 1.2 and Theorem 1.1 these data determine solutions of Eqs. (1.17) in which the poles of U and V do not coincide. Moreover, Theorem 1.1 asserts that this construction gives all solutions of such equations.

The simplest case of Eqs. (1.17) in which U and V have noncoincident poles are the equations of the principal chiral field

$$U_\eta=\frac{1}{2}[V,U],\quad V_\xi=\frac{1}{2}[U,V], \tag{1.30}$$

which are equivalent to the compatibility condition for the equations

$$\left(\partial_\xi+\frac{U}{\lambda-1}\right)\Psi=0;\quad \left(\partial_\eta-\frac{V}{\lambda+1}\right)\Psi=0, \tag{1.31}$$

where $\xi = x' - t'$, $\eta = x' + t'$ are conical variables.

Here $U(\xi, \eta)$ and $V(\xi, \eta)$ are points of the chiral field $G(\xi, \eta)$: $U = G_\xi G^{-1}$, $V = G_\eta G^{-1}$. Equation (1.30) gives

$$2G_{\xi\eta}=G_\xi G^{-1}G_\eta+G_\eta G^{-1}G_\xi. \tag{1.32}$$

These equations are Lagrangian with Lagrangian

$$L=\mathrm{Sp}\,(G_\xi G^{-1}G_\eta G^{-1}). \tag{1.33}$$

THEOREM 1.2. The construction presented gives all solutions of the equations of the principal chiral field. Here the initial conditions $u(\xi)$ and $v(\eta)$ in (1.29), which determine the solutions $U(\xi, \eta)$ and $V(\xi, \eta)$, coincide with the values of U and V on the characteristics: $u(\xi) = U(\xi, 0)$ and $v(\eta) = V(0, \eta)$.

As a second example, we consider the equation

$$u_{\xi\eta}=4\sin u. \tag{1.34}$$

It is equivalent to the compatibility condition for the problems

$$\partial_\xi\Psi=\left(\begin{array}{c|c}\frac{iu_\xi}{2} & 1\\ \hline \lambda^{-1} & -\frac{iu_\xi}{2}\end{array}\right)\Psi, \tag{1.35}$$

$$\partial_\eta\Psi=\left(\begin{array}{c|c}0 & \lambda e^{-iu}\\ \hline e^{iu} & 0\end{array}\right)\Psi. \tag{1.36}$$

As for the equations of the chiral field, the linear problems (1.35), (1.36) each have in the coefficients a noncoincident simple pole. However, system (1.35), (1.36) written in another gauge corresponds to distinguishing an invariant manifold in the general two-pole equation. This leads to minor alteration of the general construction which we present below for completeness.

Let $u(\xi, \eta)$ be an arbitrary solution of Eq. (1.34). Then there exists a unique solution of Eqs. (1.35), (1.36) such that $\Psi(0, 0, \lambda) = 1$. The function $\Psi(\xi, \eta, \lambda)$ for all ξ and λ is analytic in λ everywhere except the points $\lambda = 0$, $\lambda = \infty$. As before, in order to find the form of the essential singularities of Ψ at these points, we pose the following Riemann problem.

Find a function $\Phi_\infty(\xi, \eta, \lambda)$ analytic away from $\lambda = \infty$ which in a neighborhood of $\lambda = \infty$ can be represented in the form

$$\Phi_\infty(\xi,\eta,\lambda)=R_\infty(\xi,\eta,\lambda)\,\Psi(\xi,\eta,\lambda), \tag{1.37}$$

where R_∞ is analytic in this neighborhood.

The function Φ_∞ exists and is unique if the following normalization condition is imposed on it: $\Phi_\infty(\xi, \eta, 0)$ is a lower triangular matrix with ones on the diagonal, i.e., Φ_∞ for $\lambda = 0$

has the form

$$\Phi_\infty(\xi, \eta, 0)=\left(\frac{1 \mid 0}{\alpha \mid 1}\right). \tag{1.38}$$

These are three linear conditions on Φ_∞. We impose still another condition by requiring that $R_\infty(\xi, \eta, \infty)$ be an upper triangular matrix. Since $\det\Psi = 1$, it follows that $R_\infty(\xi, \eta, \infty)$ has the form

$$R_\infty(\xi, \eta, \infty)=\left(\frac{g \mid g_1}{0 \mid g^{-1}}\right). \tag{1.39}$$

We consider the logarithmic derivatives of Φ_∞. It follows from (1.37) that in a neighborhood of $\lambda = \infty$

$$(\partial_\xi\Phi_\infty)\Phi_\infty^{-1}=\partial_\xi R_\infty R_\infty^{-1}+R_\infty\left(\begin{array}{c|c}\frac{iu_\xi}{2} & 1 \\ \hline \lambda^{-1} & -\frac{iu_\xi}{2}\end{array}\right)R_\infty^{-1}. \tag{1.40}$$

Hence $\partial_\xi\Phi_\infty\cdot\Phi_\infty^{-1}$ is regular in a neighborhood of $\lambda = \infty$, and its value at this point is an upper triangular matrix. Since $\partial_\xi\Phi_\infty\Phi_\infty^{-1}$ is regular everywhere, it is constant. Moreover, it follows from (1,38) that

$$\partial_\xi\Phi_\infty\cdot\Phi_\infty^{-1}|_{\lambda=0}=\begin{pmatrix}0 & 0\\ \alpha_\xi & 0\end{pmatrix}. \tag{1.41}$$

Hence $\partial_\xi\Phi_\infty = 0$.

Since Φ_∞ does not depend on ξ, it follows that

$$\Phi_\infty(\eta, \lambda)=R_\infty(0, \eta, \lambda)\Psi(0, \eta, \lambda). \tag{1.42}$$

The function $\Psi(0, \eta, \lambda)$ has only one essential singular point $\lambda = \infty$, and it satisfies condition (1.38). From the uniqueness of Φ_∞ it follows that

$$\Phi_\infty(\eta, \lambda)=\Psi(0, \eta, \lambda). \tag{1.43}$$

It follows from (1.36) that $\Phi_\infty(\eta, \lambda)$ satisfies the equation

$$\partial_\eta\Phi_\infty=\begin{pmatrix}0, & \lambda e^{-iu_1}\\ e^{iu_1}, & 0\end{pmatrix}\Phi_\infty, \tag{1.44}$$

where $u_1(\eta) = u(0, \eta)$.

We consider in a similar way the function $\Phi_0(\xi, \eta, \lambda)$ regular in the entire extended complex plane except at the point $\lambda = 0$; in a neighborhood of this point it can be represented in the form

$$\Phi_0(\xi, \eta, \lambda)=R_0(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \tag{1.45}$$

where R_0 is regular in this neighborhood.

We choose the following normalization conditions uniquely determining Φ_0. The matrix $\Phi_0(\xi, \eta, \infty)$ is upper triangular, and $R_0(\xi, \eta, 0)$ has the form (1.38).

In a neighborhood of $\lambda = 0$ we have, according to (1.45),

$$\partial_\eta\Phi_0\Phi_0^{-1}=\partial_\eta R_0R_0^{-1}+R_0\begin{pmatrix}0 & \lambda e^{-iu}\\ e^{iu} & 0\end{pmatrix}R_0^{-1}. \tag{1.46}$$

This implies that $\partial_\eta\Phi_0\Phi_0^{-1}$ is regular everywhere and is hence constant. From the normalization conditions and (1.46) it follows that $\partial_\eta\Phi_0\Phi_0^{-1}|_{\lambda=0}$ can have only the left lower element nonzero. It is equal to zero, since $\Phi_0(\xi, \eta, \infty)$ is upper triangular. Hence $\partial_\eta\Phi_0 = 0$ or $\Phi_0(\xi, \eta, \lambda) = \Phi_0(\xi, \lambda)$. Repeating literally the arguments for Φ_∞, we find that $\Phi_0(\xi, \lambda) = \Psi(\xi, 0, \lambda)$ and

$$\partial_\xi\Phi_0=\begin{pmatrix}w & 1\\ \lambda^{-1} & -w\end{pmatrix}\Phi_0, \tag{1.47}$$

where $w(\xi)=\frac{i}{2}u_\xi(\xi, 0)$.

We have thus proved the following result.

LEMMA 1.3. The solutions Φ_∞ and Φ_0 of the Riemann problems posed depend only on η and ξ, respectively, $\Phi_\infty = \Phi_\infty(\eta, \lambda)$, $\Phi_0 = \Phi_0(\xi, \lambda)$, and satisfy Eqs. (1.44), (1.47).

We now consider the inverse problem. Suppose there are given two arbitrary functions $u_1(\eta)$ and $w(\xi)$. We define $\Phi_\infty(\eta, \lambda)$ and $\Phi_0(\xi, \lambda)$ as solutions of Eqs. (1.44) and (1.47), respectively, with the initial conditions $\Phi_\infty(0, \lambda) = \Phi_0(0, \lambda) = 1$.

Suppose that $\Psi(\xi, \eta, \lambda)$ is a regular function of λ away from the points $\lambda = 0$ and $\lambda = \infty$ in neighborhoods of which it has the form

$$\begin{gathered}\Psi(\xi, \eta, \lambda) = \tilde{R}_0(\xi, \eta, \lambda)\Phi_0(\xi, \lambda),\\ \Psi(\xi, \eta, \lambda) = \tilde{R}_\infty(\xi, \eta, \lambda)\Phi_\infty(\eta, \lambda),\end{gathered} \tag{1.48}$$

where $\tilde{R}_0$ and $\tilde{R}_\infty$ are regular matrix functions in the corresponding neighborhoods.

The function Ψ exists and is unique if we additionally require that $\tilde{R}_\infty|_{\lambda=\infty}$ and $\tilde{R}_0|_{\lambda=0}$ have the form

$$\tilde{R}_\infty(\xi, \eta, \infty) = \begin{pmatrix} g & g_1 \\ 0 & g^{-1}\end{pmatrix}, \quad \tilde{R}_0(\xi, \eta, 0) = \begin{pmatrix} 1 & 0 \\ \alpha & 1\end{pmatrix}. \tag{1.49}$$

THEOREM 1.3. The function $\Psi(\xi, \eta, \lambda)$ satisfies Eqs. (1.35), (1.36) where $u(\xi, \eta) = u_1(\eta) - 2i \ln g(\xi, \eta)$. Here $u(\xi, \eta)$ is a solution of Eq. (1.34). All solutions of this equation are given by the proposed construction.

The proof of the theorem follows from considerations connected with the analysis of the logarithmic derivatives of Ψ which have already been used repeatedly.

It should be noted that construction of local solutions of the sine-Gordon equation depending on two arbitrary functions was proposed in the work [61]. The solutions are represented in the form of series whose convergence is proved by means of the theory of infinite-dimensional Lie algebras. Here there was originally no connection of the functional parameters figuring in this construction with the initial data of problems of Goursat type. Recently [62] this gap was partially filled.

2. "Finite-Zone Solutions" of Equations Admitting a Representation of Zero Curvature

The purpose of the present section is to distinguish algebrogeometric or "finite-zone" solutions of general equations admitting a representation of zero curvature. As in the preceding section, we consider Eqs. (1.17) without specifying the gauge and without carrying out reduction of the general system to some invariant submanifolds.

The restriction of system (1.17) to equations describing finite-zone solutions is carried out by means of an additional condition which is equivalent to the imbedding of Eqs. (1.17) in an extended system.

Definition. "Finite-zone solutions" of Eqs. (1.17) are solutions $U(x, t, \lambda)$ and $V(x, t, \lambda)$ of these equations for which there exists a matrix function $W(x, t, \lambda)$ meromorphic in λ such that the following equations are satisfied:

$$[\partial_x - U, W] = 0, \quad [\partial_t - V, W] = 0. \tag{1.50}$$

As previously, we consider a solution $\Psi(x, t, \lambda)$ of Eqs. (1.15), (1.16) normalized by the condition $\Psi(0, 0, \lambda) = 1$.

It follows from (1.50) that $W(x, t, \lambda)\Psi(x, t, \lambda)$ also satisfies Eqs. (1.15), (1.16). Since a solution of the latter system is uniquely determined by the initial condition, it follows that

$$W(x, t, \lambda)\Psi(x, t, \lambda) = \Psi(x, t, \lambda) W(0, 0, \lambda). \tag{1.51}$$

Hence, the coefficients of the polynomial

$$Q(\lambda, \mu) = \det(W(x, t, \lambda) - \mu \cdot 1) \tag{1.52}$$

do not depend on x and t. They are integrals of Eqs. (1.50).

It will henceforth be assumed that for almost all λ the matrix $W(0, 0, \lambda)$ has distinct eigenvalues, i.e., the equation

$$Q(\lambda, \mu)=0 \tag{1.53}$$

defines in C^2 the affine part of an algebraic curve Γ which is ramified in l-sheeted fashion over the λ plane where l is the dimension of the matrices U, V, W. The corresponding solutions are called finite-zone solutions of rank 1.

Remark. A description of finite-zone solutions of ranks higher than 1 [here the polynomial $Q(\lambda, \mu) = \tilde{Q}^r(\lambda, \mu)$, where r is the rank of the solution] for the Kadomtsev–Petviashvili equation and the theory connected with this which is based on the possibility of applying the apparatus and language of the theory of multidimensional holomorphic bundles over algebraic curves can be found in [35-37].

If the roots of Eq. (1.53) are simple for almost all λ, then to each such root, i.e., each point $\gamma = (\lambda, \mu)$ of the curve Γ, there corresponds a unique eigenvector $h(\gamma)$,

$$W(0, 0, \lambda)\, h(\gamma)=\mu h(\gamma), \tag{1.54}$$

which is normalized since its first coordinate $h_1(\gamma) \equiv 1$. The remaining coordinates $h_i(\gamma)$ are hereby meromorphic functions on Γ.

Suppose that $\Psi^i(x, t, \lambda)$ is the i-th column vector of the matrix $\Psi(x, t, \lambda)$. We consider the solution $\psi(x, t, \gamma)$ of Eqs. (1.15), (1.16) given by the formula

$$\psi(x, t, \gamma)=\sum_{i=1}^{l} h_i(\gamma)\Psi^i(x, t, \lambda). \tag{1.55}$$

We denote by P_α, $\alpha = 1,\ldots,l(n + m)$, the preimages on Γ of the points μ_r and λ_k — the poles of V and U. Then, since $\Psi(x, t, \lambda)$ is analytic away from the points λ_k, μ_r, it follows that $\psi(x, t, \gamma)$ is meromorphic on Γ away from the points P_α. The divisor of its poles D coincides with the divisor of poles of $h(\gamma)$.

(Here and below a divisor is simply a collection of points with multiplicities.)

In order to find the form of the singularities of $\psi(x, t, \gamma)$ at the points P_α, we use the fact that by (1.51)

$$W(x, t, \lambda)\psi(x, t, \gamma)=\mu\psi(x, t, \gamma). \tag{1.56}$$

Hence,

$$\psi(x, t, \gamma)=f(x, t, \gamma)\, h(x, t, \gamma), \tag{1.57}$$

where $f(x, t, \gamma)$ is a scalar function, and $h(x, t, \gamma)$ is an eigenvector of the matrix W normalized by the condition $h_1(x, t, \gamma) = 1$. As in the case of $h(\gamma)$, all the remaining coordinates $h_i(x, t, \gamma)$ are meromorphic functions on Γ.

We consider the matrix $\tilde{\Psi}(x, t, \lambda)$ whose columns are the vectors $\psi(x, t, \gamma_j)$, where $\gamma_j = (\lambda, \mu_j)$ are the preimages of the point λ on the curve Γ. This matrix is uniquely determined up to a permutation of columns. From (1.57) we have

$$\tilde{\Psi}(x, t, \lambda)=\tilde{H}(x, t, \lambda)\tilde{F}(x, t, \lambda), \tag{1.58}$$

where the matrix $\tilde{H}$ is constructed from the vectors $h(x, t, \gamma_j)$ in the same way as $\tilde{\Psi}$, the matrix $\tilde{F}$ is diagonal, and its elements are equal to $f(x, t, \gamma_j)\delta_{ij}$.

Since $\psi(x, t, \gamma)$ satisfies Eqs. (1.15), (1.16), it follows that

$$U(x, t, \lambda)=\tilde{\Psi}_x\tilde{\Psi}^{-1}=\tilde{H}_x\tilde{H}^{-1}+\tilde{H}\tilde{F}_x\tilde{F}^{-1}\tilde{H}^{-1}, \tag{1.59}$$

$$V(x, t, \lambda)=\tilde{\Psi}_t\tilde{\Psi}^{-1}=\tilde{H}_t\tilde{H}^{-1}+\tilde{H}\tilde{F}_t\tilde{F}^{-1}\tilde{H}^{-1}. \tag{1.60}$$

Hence, in a neighborhood of the points P_α the function $f(x, t, \gamma)$ has the form

$$f(x, t, \gamma)=\exp(q_\alpha(x, t, k_\alpha))\, f_\alpha(x, t, \gamma), \tag{1.61}$$

where $q(x, t, k)$ is a polynomial in k, $f_\alpha(x, t, \gamma)$ is a meromorphic function in a neighborhood of P_α, and $k_\alpha^{-1}(\gamma)$ is the local parameter in neighborhoods of the points $k_\alpha^{-1}(P_\alpha) = 0$.

Summarizing, we arrive at the following assertion.

THEOREM 1.4. The vector-valued function $\psi(x, t, \gamma)$.

1) It is meromorphic on Γ away from the points P_α. Its divisor of poles does not depend on x, t. If W is nondegenerate, then in general position it may be assumed that the curve Γ is nonsingular. Here the degree of the divisor of poles of ψ is equal to $g + l - 1$, where g is the genus of the curve Γ.

2) In a neighborhood of the point $P_\alpha \psi(x, t, \gamma)$ has the form

$$\psi(x, t, \gamma) = \left(\sum_{s=N}^{\infty} \xi_{s\alpha}(x, t) k_\alpha^{-1}\right) \exp(q_\alpha(x, t, k_\alpha), \tag{1.62}$$

where the first factor is the expansion in a neighborhood of P_α in terms of the local parameter $k_\alpha^{-1} = k_\alpha^{-1}(\gamma)$ of some meromorphic vector, and $q_\alpha(x, t, k)$ is a polynomial in k. If U (or V) has no pole at the point λ which is the projection of P_α onto the λ plane, then q_α does not depend on x (or t).

The only assertion of the theorem not proved above regarding the number of poles of ψ follows from the fact that $(\det \tilde{H})^2$ is a well-defined meromorphic function of λ. The poles of this function coincide with the projections of the poles of $h(\gamma)$, i.e., with the projections of the poles of ψ, while the zeros coincide with the images of the branch points of Γ, i.e., with points at which eigenvalues of $W(0, 0, \lambda)$ coalesce. We have $2N = \nu$, where N is the number of poles of ψ and ν is the number of branch points of Γ (both counting multiplicities). Using the formula [46]

$$2g-2 = \nu - 2l, \tag{1.63}$$

connecting the genus of an l-sheeted covering of the plane with the number of branch points, we obtain

$$N = g + l - 1. \tag{1.64}$$

Axiomatization of the analytic properties of $\psi(x, t, \gamma)$ established in Theorem 1.4 forms the basis for the concept of the Baker–Akhiezer function which is perhaps the central concept in the algebrogeometric version of the method of the inverse problem.

A general definition of such functions was given in [26].

In a neighborhood of points $P_1, \ldots, P_M$ of a nonsingular curve Γ we fix local parameters $k_\alpha^{-1}(\gamma)$, $k_\alpha^{-1}(P_\alpha) = 0$. In analogy with the space $\mathcal{L}(D)$ of meromorphic functions on Γ associated with a divisor D, $f(\gamma) \in \mathcal{L}(D)$, if $D + D_f \geqslant 0$, where D_f is the principal divisor of f, we introduce the space $\Lambda(\mathbf{q}, D)$, where $\mathbf{q}$ is a collection of polynomials $q_\alpha(k)$.

A function $\Phi(\mathbf{q}, \gamma)$ belongs to $\Lambda(\mathbf{q}, \gamma)$ if

1) away from the points P_α it is meromorphic, while for the divisor of its poles D_Φ (the multiplicity with which the point γ_s is contained in D is equal with a minus sign to the multiplicity of the pole of the function at this point) we have $D_\Phi + D \geqslant 0$;

2) in a neighborhood of P_α the function $\Phi(\mathbf{q}, \gamma) \exp(-q_\alpha(k_\alpha(\gamma)))$ is analtyic, $k_\alpha^{-1}(P_\alpha) = 0$.

THEOREM 1.5. For a nonspecial divisor $D \geqslant 0$ of degree $N \geqslant g \dim \Lambda(\mathbf{q}, D) = N - g + 1$.

We recall that those divisors for which $\dim \mathcal{L}(D) = N - g + 1$ are called nonspecial divisors forming an open set among all divisors.

In some special cases this assertion was first proved by Baker and Akhiezer [1, 52]. Therefore, functions of this type are called Baker–Akhiezer functions. The method of proof of this assertion used in the work [1] (see also [14, 23]) is based on the fact that $d\Phi/\Phi$ is an Abelian differential on Γ. To considerable extent the proof repeated the course of the proof of Abel's theorem and the solution of the Jacobi inversion problem [21].

The explicit construction of $\Phi(\mathbf{q}, D)$ is the simplest and at the same time most effective means of proving the theorem.

On a nonsingular algebraic curve Γ of genus g we fix a basis of cycles

$$a_1, \ldots, a_g; \quad b_1, \ldots, b_g$$

with intersection matrix $a_i \circ a_j = b_i \circ b_j$, $a_i \circ b_j = \delta_{ij}$. We introduce the basis of holomorphic differentials Ω_k on Γ normalized by the conditions $\oint_{a_i} \Omega_k = \delta_{ik}$. We denote by B the matrix of b periods $B_{lk}=\oint_{b_l}\Omega_k$. It is known that it is symmetric and has positive-definite imaginary part.

Integral combinations of vectors in C^g with coordinates δ_{jk} and B_{ik} form a lattice defining a complex torus $J(\Gamma)$ called the Jacobi manifold of the curve.

Let P_0 be a distinguished point on Γ; then the mapping $A:\Gamma \to J(\Gamma)$ is defined. The coordinates of the vector $A(\gamma)$ are equal to $\int_{P_0}^{\gamma}\Omega_k$.

On the basis of the matrix of b periods, just as for any matrix with positive-definite imaginary part, it is possible to construct an entire function of g complex variables

$$\theta(u_1,\dots,u_g)=\sum_{k\in Z^g}\exp(\pi i(Bk,k)+2\pi i(k,u)),$$

where $(k, u) = k_1u_1 + \dots + k_gu_g$.

It possesses the following easily verified properties:

$$\theta(u_1,\dots,u_j+1,\ u_{j+1},\dots u_g)=\theta(u_1,\dots,u_j,\dots u_g),$$
$$\theta(u_1+B_{1k},\dots,u_g+B_{gk})=\exp(-\pi i(B_{kk}+2u_k))\,\theta(u_1,\dots,u_g). \tag{1.65}$$

Moreover, for any nonspecial effect divisor $\tilde{D} = \gamma_1 + \dots + \gamma_g$ of degree g there exists a vector $Z(\tilde{D})$ such that the function $\theta(A(\gamma) + Z(\tilde{D}))$ defined on Γ dissected along the cycles a_i, b_j has exactly g zeros coinciding with the points γ_i (see [21]),

$$Z_k(\tilde{D})=-\sum_{s=1}^{g}A_k(\gamma_s)+\frac{1}{2}-\frac{1}{2}B_{kk}+\sum_{j\neq k}\oint_{a_j}\left(\int_{P_0}^{t}\Omega_k\right)\Omega_j,\quad t\in a_j.$$

For any collection of polynomials $q_\alpha(k)$ there exists a unique Abelian differential of second kind (see [44]) ω (the index of ω_q we omit to simplify notation) having a singularity at the distinguished point P_α on Γ of the form $dq_\alpha(k_\alpha)$ in the local parameter k_α^{-1} and normalized by the conditions $\oint_{a_l}\omega=0$.

LEMMA 1.4. Let $\tilde{D}$ be an arbitrary effective, nonspecial divisor of degree g; then the function

$$\psi(\mathbf{q},\gamma)=\exp\left(\int_{P_0}^{\gamma}\omega\right)\frac{\theta(A(\gamma)+Z(\tilde{D})+\mathbf{U})}{\theta(A(\gamma)+Z(\tilde{D}))}, \tag{1.66}$$

where $\mathbf{U} = (U_1,\dots,U_g)$ and $U_k=\frac{1}{2\pi i}\oint_{b_k}\omega$ is a generator of a one-dimensional space $\Lambda(\mathbf{q}, D)$.

The proof of the lemma follows from a simple verification of the properties of the function $\psi(q, \gamma)$. It follows directly from the properties (1.65) that the right side of Eq. (1.66) gives a well-defined function on Γ, i.e., its values do not change on passing around the cycles a_i, b_j. In a neighborhood of P_α the funtion ψ, as follows from the definition of ω, has the required essential singularity (a formula of this type for the Bloch function of the finite-zone Schrödinger operator was first obtained by Its [23]).

The fact that $\Lambda(q, \tilde{D})$ is one-dimensional follows from the fact that if $\psi_1\in\Lambda(\mathbf{q},\tilde{D})$, then ψ_1/ψ is a meromorphic function on Γ with g poles. By the Riemann–Roch theorem and the fact that the divisor is nonspecial we find that ψ_1/ψ = const.

To complete the proof of the theorem it suffices to note that functions $\psi_i(q, \gamma)$ of the form (1.66) corresponding to divisors $\tilde{D}_i = \gamma_1 + \dots + \gamma_{g-1} + \gamma_{g+i}$ (where $D = \gamma_1 + \dots + \gamma_N$) form a basis of the space $\Lambda(\mathbf{q}, D)$.

By Theorem 1.4 to each finite-zone solutions of Eqs. (1.17) of rank 1 there correspond a curve Γ, which can be assumed nonsingular in general position, a collection of polynomials $q_\alpha(x, t, k)$, and a nonspecial divisor of degree $g + l - 1$, where g is the genus of the curve. We shall use the preceding theorem to construct an inverse mapping.

Thus, suppose that the collection of data listed above is given. By Theorem 1.5 dim × $\Lambda(\mathbf{q}, D) = l$. In this space we choose an arbitrary basis $\psi_i(x, t, \gamma)$ (the polynomials q_α depend on x and t as parameters; ψ will obviously also depend on these parameters).

THEOREM 1.6. Let $\psi(x, t, \gamma)$ be a vector-valued function with coordinates the functions $\psi_i(x, t, \gamma)$ constructed above. There exist unique matrix functions $U(x, t, \lambda)$, $V(x, t, \lambda)$, $W(x, t, \lambda)$ rational in λ such that

$$\partial_x\psi = U\psi, \quad \partial_t\psi = V\psi, \quad W\psi = \mu\psi, \quad \gamma = (\lambda, \mu) \in \Gamma. \tag{1.67}$$

As above, to prove the theorem we consider the matrix $\tilde{\Psi}(x, t, \lambda)$ whose columns are the vectors $\psi(x, t, \gamma_j)$, where $\gamma_j = (\lambda, \mu_j)$ are the preimages of the point λ on Γ. This matrix, as a function of λ, is defined up to a permutation of columns. It is easy to see that the matrices

$$(\partial_x\tilde{\Psi})\tilde{\Psi}^{-1}, \quad (\partial_t\tilde{\Psi})\tilde{\Psi}^{-1}, \quad \tilde{\Psi}\hat{\mu}\tilde{\Psi}^{-1} \tag{1.68}$$

are well defined and because of the analytic properties of ψ are rational functions of λ. They are denoted by U, V, W, respectively. Here $\hat{\mu}$ is the diagonal matrix given by $\mu_{ij} = \mu_j\delta_{ij}$.

Using the course of the proof of equality (1.64) in the reverse direction, we find that $\det\tilde{\Psi} \neq 0$ if λ is not a branch point of the covering $\Gamma \to C^1$. A corollary of this is that U has poles only at the projections of the points P_α (and only in the case where the dependence of the corresponding polynomial q_α on x is nontrivial). An analogous assertion also holds for V. The degree of the poles of U and V at the point P_α is equal to the maximal degree of q_α with coefficient depending nontrivially on x and t, respectively. As a function of λ the matrix W has poles at the images of the poles μ.

COROLLARY. The matrices U, V, W constructed according to formulas (1.68) satisfy equations (1.17) and (1.50).

In the construction of the vector $\psi(x, t, \gamma)$ on the basis of the collection of data presented before Theorem 1.6 there is some ambiguity connected with the possibility of choosing various bases in the space $\Lambda(\mathbf{q}, D)$.

To this ambiguity, under which $\psi(x, t, \gamma)$ goes over into $g(x, t)\psi(x, t, \gamma)$, where g is a nondegenerate matrix, there corresponds the gauge symmetry (1.18) of Eqs. (1.17) and (1.50); under this transformation the matrix W goes over into

$$W \to gWg^{-1}. \tag{1.69}$$

We now consider two vector-valued Baker–Akhiezer functions $\psi(x, t, \gamma)$, $\tilde{\psi}(x, t, \gamma)$ corresponding to two equivalent divisors D and $\tilde{D}$. Equivalence of these divisors means that there exists a meromorphic function $f(\gamma)$ such that its poles coincide with D and its zeros with $\tilde{D}$. From the definition of the Baker–Akhiezer functions it follows that $f\cdot\tilde{\psi}\in\Lambda(\mathbf{q}, D)$. Hence,

$$\psi(x, t, \gamma) = g(x, t)\, f(\gamma)\, \tilde{\psi}(x, t, \gamma), \tag{1.70}$$

and the functions ψ and $\tilde{\psi}$ define gauge-equivalent solutions of the gauge-invariant equations.

We shall consider both Eqs. (1.17), (1.50) and their solutions up to transformations (1.18), (1.69). From (1.69) it follows that gauge transformations leave the curves Γ [i.e., Eqs. (1.52), (1.53)] invariant.

THEOREM 1.7. The set of finite-zone solutions (defined up to gauge equivalence) corresponding to a nonsingular curve Γ is isomorphic to a torus — the Jacobian of the curve $J(\Gamma)$.

The assertion of the theorem follows from the known [44] isomorphism between equivalence classes of divisors and the Jacobian $J(\Gamma)$. Since the coefficients of the polynomial $Q(\lambda, \mu)$ are integrals of Eqs. (1.17), (1.50), this theorem implies that in general position the level set of these integrals is a torus.

For special values of the integrals for which the curve Γ has singularities the corresponding level manifold is isomorphic to the generalized Jacobian of such a curve. Without going into the details of this assertion (a definition of generalized Jacobians can be found in [42]), we note that to multisoliton and rational solutions of Eqs. (1.17) there correspond rational curves Γ. Moreover, to different types of singularities there correspond different types of solitons. For example, in the case of singularities of intersection type we obtain multisoliton solutions (see, for example, Sec. 10 of [17] for the KdV equation), while in the case of singularities of "beak" type we obtain rational solutions [31].

So far we have discussed the general equations (1.17). Conditions imposed on U and V to distinguish invariant submanifolds of these equations lead to corresponding conditions on the parameters of the construction of finite-zone solutions of these equations.

We shall consider these conditions for the example of the sine-Gordon equation. Finite-zone solutions of this equation, which has, in addition to the representation (1.17) with matrices (1.35), (1.36) also the ordinary representation (1.2) with (4×4)-matrix operators, were first constructed in [24]. Subsequently, application of the general construction of finite-zone solutions proposed by the author [32] was carried out in application to the sine-Gordon equation in [22] (see also [45, 46]).

It may be assumed with no loss of generality that $W(\xi, \eta, \lambda)$ does not have poles at the points $\lambda = 0$ and $\lambda = \infty$, since this can always be achieved by multiplying W by a constant rational function of λ.

Since the left sides of the equations

$$W_\xi = [W, U], \quad W_\eta = [W, V], \tag{1.71}$$

which coincide with (1.50), do not have poles for $\lambda = 0$, $\lambda = \infty$, it follows from the form of U and V (1.35), (1.36) that

$$\left[W(\xi, \eta, 0), \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right)\right] = 0; \quad \left[W(\xi, \eta, \infty), \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right)\right] = 0.$$

Hence, $W(\xi, \eta, \lambda)$ in a neighborhood of these points has the form

$$W(\xi, \eta, \lambda) = \left(\begin{array}{c|c} w_1 & 0 \\ \hline w_2 & w_1 \end{array}\right) + O(\lambda), \tag{1.72}$$

$$W(\xi, \eta, \lambda) = \left(\begin{array}{c|c} w_1' & w_2' \\ \hline 0 & w_1' \end{array}\right) + O(\lambda^{-1}). \tag{1.73}$$

Hence, the curve Γ defined by the characteristic equation (1.53)

$$\mu^2 - r_1(\lambda)\mu + r_2(\lambda) = 0, \tag{1.74}$$

$r_1 = \mathrm{Sp}\, W$, $r_2 = \det W$, branches at the points $\lambda = 0$ and $\lambda = \infty$. It also follows from (1.72), (1.73) that

$$h_2(\xi, \eta, \infty) = 0; \quad h_2(\xi, \eta, \lambda) = 0(\lambda^{-1/2}), \quad \lambda \to 0, \tag{1.75}$$

where, as previously, $h_i(\xi, \eta, \lambda)$ are coordinates of an eigenvector of $W(\xi, \eta, \lambda)$ normalized by the condition $h_1 = 1$.

With consideration of these remarks Theorem 1.4 assigns to each finite-zone solution of the sine-Gordon equation of hyperelliptic curve Γ (1.74) with two distinguished branch points P_0 and P_∞ situated over $\lambda = 0$ and $\lambda = \infty$ and also a divisor D_0 of degree g.

Away from the points P_0 and P_∞ the corresponding Baker–Akhiezer functions have g poles at points of the divisor D_0. In a neighborhood of P_0 they have the form

$$\begin{aligned} \psi_1(\xi, \eta, k) &= e^{k\xi}\left(1 + \sum_{s=1}^{\infty} \chi_{s1}(\xi, \eta) k^{-s}\right), \\ \psi_2(\xi, \eta, k) &= e^{k\xi} k \left(1 + \sum_{s=1}^{\infty} \chi_{s2}(\xi, \eta) k^{-s}\right), \end{aligned} \tag{1.76}$$

where $k = \lambda^{-1/2}$.

In a neighborhood of P_∞ we have

$$\begin{aligned} \psi_1(\xi, \eta, k) &= c_1 e^{k\eta}\left(1 + \sum_{s=1}^{\infty} \zeta_{s1}(\xi, \eta) k^{-s}\right), \\ \psi_2(\xi, \eta, k) &= c_2 e^{k\eta} k^{-1} \left(1 + \sum_{s=1}^{\infty} \zeta_{s2}(\xi, \eta) k^{-s}\right), \end{aligned} \tag{1.77}$$

where $k = \lambda^{1/2}$, $c_i = c_i(\xi, \eta)$.

The divisor D of degree g + 1 which figured in Theorem 1.4 is simply $D_0 + P_0$.

By Theorem 1.5 the curve Γ and divisor D_0 uniquely determine ψ_1 and ψ_2. Computing the logarithmic derivatives of the corresponding matrix $\tilde{\Psi}$, it can be seen that the matrices U and V have the required form (1.35), (1.36). This can also be seen from the following arguments.

It follows from the definitions of ψ_1 and ψ_2 that the functions $\partial_\eta\psi_1$ and $\lambda\psi_2$ possess identical analytic properties. They are thus proportional. To compute the constant of proportionality it is necessary to compare the coefficients of the term $\lambda^{1/2}$ in the expansion of these functions at P_∞. We have

$$\partial_\eta\psi_1 = e^{-iu}\lambda\psi_2, \quad e^{-iu} = c_1 c_2^{-1}. \tag{1.78}$$

Similarly,

$$\partial_\eta\psi_2 = e^{iu}\psi_2. \tag{1.79}$$

It can thus be proved that

$$\partial_\xi\psi_1 = \frac{iu_\xi}{2}\psi_1 + \psi_2, \tag{1.80}$$

$$\partial_\eta\psi_2 = \lambda^{-1}\psi_1 - \frac{i}{2}u_\xi\psi_2. \tag{1.81}$$

Equalities (1.78)-(1.81) are the coordinate notation for Eqs. (1.35), (1.36).

COROLLARY. The function u defined from (1.77), (1.78) is a solution of the sine-Gordon equation.

We shall find the explicit form of finite-zone solutions of the sine-Gordon equation.

Let ω_0 and ω_∞ be normalized Abelian differentials on Γ of second kind with the sole singularities of the form $d\lambda^{-1/2}$ and $d\lambda^{1/2}$, respectively. By Theorem 1.5

$$\psi_1(\xi, \eta, \gamma) = r_1(\xi, \eta)\exp\left(\xi\int_{\gamma_0}^{\gamma}\omega_0 + \eta\int_{\gamma_0}^{\gamma}\omega_\infty\right)\frac{\theta(U_0\xi + U_\infty\eta + A(\gamma) + Z)}{\theta(A(\gamma) + Z)}, \tag{1.82}$$

where $2\pi iU_0$ and $2\pi iU_\infty$ are the vectors of b periods of the differentials ω_0, ω_∞; $Z = Z(D_0)$.

The normalizing function $r_1(\xi, \eta)$ is determined from condition (1.76) according to which the value of the regular factor of the exponential function at the point P_0 is equal to 1. We have

$$r_1(\xi, \eta) = \frac{\theta(A(P_0) + Z)}{\theta(A(P_0) + U_0\xi + U_\infty\eta + Z)}. \tag{1.83}$$

Expanding (1.82) in a neighborhood of P_∞ we find that

$$c_1(\xi, \eta) = \frac{\theta(A(P_\infty) + U_0\xi + U_\infty\eta + Z)\,\theta(A(P_0) + Z)}{\theta(A(P_0) + U_0\xi + U_\infty\eta + Z)\,\theta(A(P_\infty) + Z)}. \tag{1.84}$$

The explicit form of $\psi_2(\xi, \eta, \gamma)$ could be found according to the general recipe given in the course of the proof of Theorem 1.5. However, in the present case this rule and the corresponding formulas can be simplified.

We denote by $f(\gamma)$ a meromorphic function on Γ having poles at points of the divisor D_0 and at the point P_0. The condition $f(P_\infty) = 0$ determines this function up to proportionality. We normalize it so that the coefficient of $\lambda^{-1/2}$ in its expansion in a neighborhood of P_0 is equal to 1. The constant e^{I_0} equal to the coefficient of $\lambda^{-1/2}$ in the expansion of f in a neighborhood of infinity is then determined.

The function $f^{-1}\psi_2(\xi, \eta, \gamma)$ possesses the same analytic properties as ψ_1 up to replacement of the divisor D_0 by the divisor $\tilde{D}$ coinciding with the zeros of $f(\gamma)$ distinct from P_∞. Hence,

$$c_2(\xi, \eta) = e^{-I_0}\frac{\theta(A(P_\infty) + U_0\xi + U_\infty\eta + \tilde{Z})\,\theta(A(P_0) + \tilde{Z})}{\theta(A(P_0) + U_0\xi + U_\infty\eta + \tilde{Z})\,\theta(A(P_\infty) + \tilde{Z})},$$

where $\tilde{Z} = Z(\tilde{D})$.

The divisors $D_0 + P_0$ and $\tilde{D} + P_\infty$ are equivalent. Therefore, by Abel's theorem [44]

$$Z+A(P_0)=\tilde{Z}+A(P_\infty).$$

It follows from this theorem that the vector $\Delta = A(P_\infty) - A(P_0)$ is a half period, since $2P_0$ and $2P_\infty$ are equivalent (the function λ has a pole of second order at P_∞ and a zero of multiplicity 2 at P_0).

We arrive finally at the formula

$$e^{iu}=e^{-I_0}\frac{\theta^2(W+U_0\xi+U_\infty\eta)\,\theta(W-\Delta)\,\theta(W+\Delta)}{\theta(W-\Delta+U_0\xi+U_\infty\eta)\,\theta(W+\Delta+U_0\xi+U_\infty\eta)\,\theta^2(W)} \tag{1.85}$$

for finite-zone solutions of the sine-Gordon equation. In this formula $W = Z(D_0) + A(P_0)$. However, this vector may be assumed arbitrary, since as D_0 varies the vectors W fill out the entire Jacobian.

In this chapter we have not touched on the important question concerning distinguishing real finite-zone solutions not having singularities. It is rather trivial to distinguish real solutions in those cases where the operators whose commutativity condition is equivalent to the equation considered are self-adjoint. In this case an antiinvolution is defined naturally on the curve Γ, and real solutions correspond to data invariant under this antiinvolution.

For non-self-adjoint operators the problem is considerably more complicated. Conditions that solutions (1.85) of the sine-Gordon equation be real were described in [45, 46]. Effectivization of these conditions and completion of the question regarding realness of finite-zone solutions (1.85) were accomplished in [16].

3. Representations of "Zero Curvature" and Elliptic Curves

Recently active attempts have been made to generalize Eqs. (1.17) to the case of pencils in which the matrices U and V are meromorphic functions of a parameter λ defined on an algebraic curve Γ of genus greater than zero. (The case of rational pencils corresponds to $g = 0$.) We note that the Riemann–Roch theorem impedes the automatic transfer of Eqs. (1.17) to a curve of genus $g > 0$.

Indeed, suppose that $U(x, t, \lambda)$ and $V(x, t, \lambda)$ are meromorphic functions on Γ, $\lambda \in \Gamma$, having divisors of poles of multiplicity N and M. Then by the Riemann–Roch theorem [46] the number of independent variables is equal to $l^2(N - g + 1)$ for U and $l^2(M - g + 1)$ for V (where l is the dimension of the matrices). The commutator [U, V] has poles of total multiplicity N + M. Therefore, Eqs. (1.17) are equivalent to $l^2(N + M - g + 1)$ equations for the unknown functions. With consideration of gauge symmetry (1.18) for $g \geqslant 1$ the number of equations is always greater than the number of unknowns.

There are two ways to circumvent this obstacle. One way is proposed in the work [35] where, in addition to poles fixed relative to x and t, the matrices U and V admit poles depending on x, t in a particular manner. It was shown that the number of equations [with consideration of (1.18)] hereby coincides with the number of independent variables which are the singular parts of U and V at the fixed poles. Since no physically interesting equations have so far been found in this scheme, we shall not consider it in more detail.

The second way is based on a choice of a special form of the matrices U and V and has been successfully realized only in some examples on elliptic curves Γ ($g = 1$). The physically most interesting example of such equations is the Landau–Lifshits equation

$$\vec{S}_t=\vec{S}\times\vec{S}_{xx}+\vec{S}\times I\vec{S}, \tag{1.86}$$

where $\vec{S}$ is a three-dimensional vector of unit length, $|\vec{S}| = 1$, and $I_{\alpha\beta} = I_\alpha\delta_{\alpha\beta}$ is a diagonal matrix. It was shown in the work [67] that Eq. (1.86) is the compatibility condition for linear equations (1.15), (1.16) where the 2×2 matrices U and V are

$$U=-i\sum_{\alpha=1}^{3}w_\alpha(\lambda)\,S_\alpha(x,t)\,\sigma_\alpha, \tag{1.87}$$

$$V=-i\sum_{\alpha,\beta,\gamma}b_\alpha(\lambda)\,\sigma_\alpha S_\beta S_{\gamma x}e^{\alpha\beta\gamma}-2i\sum_{\alpha=1}^{3}a_\alpha(\lambda)\,S_\alpha\sigma_\alpha, \tag{1.88}$$

where

$$w_1=b_1=\frac{\rho}{\operatorname{sn}(\lambda, k)};\quad w_2=b_2=\rho\frac{\operatorname{dn}(\lambda, k)}{\operatorname{sn}(\lambda, k)};$$
$$w_3=b_3=\rho\frac{\operatorname{cn}(\lambda, k)}{\operatorname{sn}(\lambda, k)};\quad a_1=-w_2w_3,\quad a_2=-w_3w_1, \tag{1.89}$$
$$a_3=-w_1w_2;\quad \operatorname{sn}(\lambda, k),\quad \operatorname{cn}(\lambda, k),\quad \operatorname{dn}(\lambda, k)$$

are Jacobi elliptic functions [2], and σ_α are the Pauli matrices.

The parameters I_α are given by the relations

$$k=\sqrt{\frac{I_2-I_1}{I_3-I_1}},\quad \rho=\frac{1}{2}\sqrt{I_3-I_1},\quad 0<k<1. \tag{1.90}$$

In [47, 64] multisoliton solutions of these equations were constructed, and attempts were made to construct finite-zone solutions; these attempts have so far not given an effective answer.

The pair (1.87), (1.88) has 4 poles on the curve Γ. It turns out that Eq. (1.86) can be represented in the form of a commutation condition of "single-pole pencils" if in U and V we give up the condition that the pencils be meromorphic on the entire curve Γ.

We fix an elliptic curve Γ and an l-dimensional vector $z = (z_1,\dots,z_l)$, $z_i \neq z_j$. We denote by $G(\Gamma, z)$ the infinite-dimensional algebra of matrix-valued functions $U(\lambda)$ such that

1) they are meromorphic away from the point $\lambda = 0$;

2) in a neighborhood of $\lambda = 0$ the matrix element of U has the form

$$U_{ij}(\lambda)=\exp\left(\frac{z_i-z_j}{\lambda}\right)\left(\sum_{s=N}^{\infty}\xi_s\lambda^{-s}\right) \tag{1.91}$$

(henceforth $z_i - z_j$ is denoted by z_{ij}). Conditions 1) and 2) mean that the matrix elements are functions of Baker–Akhiezer type.

For any divisor of degree N the dimension of the linear space of matrix functions of the type described having poles at the points of this divisor is equal to Nl^2. As in the case of rational pencils, it is possible to take as independent parameters the singular terms of $U(\lambda)$.

For example, suppose that $U(\lambda)$ has simple poles at the points $\lambda_1,\dots,\lambda_N$; then

$$U_{ij}(\lambda)=\sum_{k=1}^{N}u_{ij}^{k}\Phi(z_{ij}, \lambda, \lambda_k), \tag{1.92}$$

where

$$\Phi(z, \lambda, \lambda_k)=\frac{\sigma(z-\lambda+\lambda_k)}{\sigma(\lambda-\lambda_k)\sigma(z)}e^{\zeta(\lambda)z}=\left(\frac{1}{\lambda-\lambda_k}+\dots\right)e^{\zeta(\lambda)z} \tag{1.93}$$

is the simplest function of Baker–Akhiezer type (here σ, ζ are the Weierstrass functions [2]).

For the diagonal elements we have

$$U_{ij}(\lambda)=u_i^0+\sum_{k=1}^{N}u_i^k\zeta(\lambda-\lambda_k),\quad \sum_{k=1}^{N}u_i^k=0. \tag{1.94}$$

Assertion. Suppose that $U(\xi, \eta, \lambda)$ and $V(\xi, \eta, \lambda)$ are matrix-valued functions belonging for all ξ and η to $G(\Gamma, z(\xi, \eta))$ and having poles in the divisors of degree N and M, respectively, which contain the point $\lambda = 0$; then the equation

$$U_\xi-V_\eta+[U, V]=0 \tag{1.95}$$

is equivalent to a system of equations for the independent parameters determining U and V and for the functions $z_i(\xi, \eta)$. This system is equivalent to the vanishing of the singular terms in (1.95). With consideration of gauge symmetry (1.95) the number of equations in it is equal to the number of unknown functions.

Example 1. We consider the simplest case: U has a simple role at the point $\lambda = 0$, while V has a pole of second order:

$$U_{ij}=S_{ij}\Phi(z_{ij},\lambda), \tag{1.96}$$

$$V_{ij}=S_{ij}\tilde{\Phi}(z_{ij},\lambda)+v_{ij}\Phi(z_{ij},\lambda), \tag{1.97}$$

where $\Phi(z, \lambda) = \Phi(z, \lambda; 0)$ is the same as in (1.93), and

$$\tilde{\Phi}(z,\gamma)=\frac{\sigma(\lambda-z+a)\sigma(\lambda-a)}{\sigma^2(\lambda)\sigma(z-a)\sigma(a)}e^{\zeta(\lambda)z}=\left(\frac{1}{\lambda^2}+O(1)\right)e^{\zeta(\lambda)z},$$

if $\zeta(z - a) + \zeta(a) = 0$.

Let $z_i(\xi, \eta) = z_i$ be the half periods of the curve Γ (i.e., $l = 3$); then Eqs. (1.95) are equivalent to the Landau–Lifshits equation (1.86) where S_{ij} is the skew-symmetric matrix corresponding to the vector S_α, and I_α are given by the equality

$$I_\alpha=R_{ij}=\wp(x_i-x_j). \tag{1.98}$$

Example 2. As a second example we consider the problem of constructing elliptic solutions of the Kadomtsev–Petviashvili equation and of constructing variables of "action-angle" type for a system of particles on the line with pair potential interaction whose Hamiltonian has the form

$$H=\frac{1}{2}\sum_{i=1}^{N}p_i^2-2\sum_{i\neq j}\wp(x_i-x_j), \tag{1.99}$$

where $\wp(x)$ is the Weierstrass $\wp$-function. This problem was solved in [34] where the representation (1.95) with matrices whose elements are Baker–Akhiezer functions first arose.

Without going into the details of [34], we formulate its basic assertions.

For the equation of motion of system (1.99)

$$\ddot{x}_i=4\sum_{i\neq j}\wp'(x_i-x_j) \tag{1.100}$$

a Lax representation was known $\dot{L} = [M, L]$ not containing any spectral parameter [54]. It was shown in [65] that the integrals $I_k = \operatorname{tr} L^k/k$ are independent and in involution. Thus, system (1.99) is completely integrable by Liouville's theorem. Introduction of a spectral parameter in the Lax representation for (1.100) makes it possible to move forward in the construction of variables of angle type.

We define the matrices

$$U_{ij}=\dot{x}_i\delta_{ij}+2(1-\delta_{ij})\Phi(x_{ij},\lambda), \tag{1.101}$$

$$V_{ij}=\delta_{ij}\left(-\wp(\lambda)+2\sum_{k\neq i}\wp(x_{ik})\right)+2(1-\delta_{ij})\Phi'(x_{ij},\lambda), \tag{1.102}$$

where $\Phi(x, \lambda)$ is the same as in (1.93), and $\Phi' = \partial\Phi(x, \lambda)/\partial x$.

LEMMA 1.5. Equations (1.100) are equivalent to the equation

$$U_t+[U,V]=0 \tag{1.103}$$

[i.e., Eq. (1.95) in which there is no dependence on η and ξ is replaced by t] where U and V have the form (1.101), (1.102).

The assertion of the lemma follows by direct verification.

It follows from (1.103) that the function

$$R(k,\lambda)=\det(2k+U(\lambda,t)) \tag{1.104}$$

does not depend on t. The matrix U, having essential singularities at $\lambda = 0$, can be represented in the form

$$L(t,\lambda)=g(t,\lambda)\tilde{L}(t,\lambda)g^{-1}(t,\lambda),$$

where $\tilde{L}$ has no essential singularity at $\lambda = 0$, and g is a diagonal matrix, $g_{ij} = \delta_{ij}\exp\times(\zeta(\lambda)x_i)$. Hence, $r_i(\lambda)$, the coefficients of the expression

$$R(k,\ \lambda)=\sum_{l=0}^{n} r_l(\lambda)\,k^l,$$

are elliptic functions with poles at the point $\lambda = 0$. The functions $r_i(\lambda)$ can be represented as a linear combination of the $\wp$-function and its derivatives. The coefficients of this expansion are integrals of the system (1.99). Each collection of fixed values of these integrals gives by the equation $R(k, \lambda) = 0$ an algebraic curve Γ_n which covers the original elliptic curve Γ in n-sheeted fashion.

As shown in [34], the genus of Γ_n is equal to n in general position. The Jacobian of the curve Γ_n is isomorphic to the level manifold of the integrals r_i, and the variables on it are variables of angle type.

Further effectivization of the solution of Eqs. (1.100) used the connection of Eqs. (1.103) with solutions of special type for the nonstationary Schrödinger equation with an elliptic potential.

THEOREM 1.8. The equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}+2\sum_{i=1}^{n}\wp(x-x_i(t))\right)\psi=0 \tag{1.105}$$

has a solution ψ of the form

$$\psi=\sum_{i=1}^{n} a_i(t,\ k,\ \lambda)\,\Phi(x-x_i,\ \lambda)\,e^{kx+k^2t}, \tag{1.106}$$

where

$$\Phi(x,\ \lambda)=\frac{\sigma(\lambda-x)}{\sigma(\lambda)\,\sigma(x)}e^{\zeta(\lambda)x},$$

if and only if $x_i(t)$ satisfy Eqs. (1.100).

A function ψ of the form (1.106), as a function of the variable x, has simple poles at the points $x_i(t)$. Substituting it into (1.106) and equating the coefficients of $(x - x_i)^{-2}$ and $(x - x_i)^{-1}$ to zero, we find that ψ satisfies (1.105) if and only if $a = (a_1,\dots,a_n)$ satisfies the equation

$$U(t,\ \lambda)\,a(t,\ \lambda,\ k)=-2ka, \tag{1.107}$$

$$\left(\frac{\partial}{\partial t}+V(t,\ \lambda)\right)a(t,\ \lambda,\ k)=0, \tag{1.108}$$

where U and V are the same as in (1.101), (1.102).

The analytic properties of $a(t, k, \lambda)$ on the Riemann surface Γ can be established in a manner similar to the way in which this was done in Sec. 2.

We formulate the final assertion.

THEOREM 1.9. The eigenfunction of the nonstationary Schrödinger equation (1.105) $\psi(x, t, \gamma)$ is defined on an n-sheeted covering Γ_n of the original elliptic curve. The function $\psi(x, t, \gamma)$ is a Baker–Akhiezer function with the single essential singularity of the form

$$\exp\left(n\lambda^{-1}(x-x_1(0))+n^2\lambda^{-2}t\right)$$

at the distinguished preimage P_0 on Γ_n of the point $\lambda = 0$.

The coordinates $x_i(t)$ of the system of particles (1.99) are given by the equation

$$\theta(\mathbf{U}x+\mathbf{V}t+Z)=0=\prod_{i=1}^{n}\sigma(x-x_i(t)). \tag{1.109}$$

Here θ is the theta function constructed on the basis of the matrix of b periods of the curve Γ_n; the vectors $\mathbf{U}$ and $\mathbf{V}$ are the vectors of b periods of Abelian differentials with poles at P_0 of second and third orders, respectively.

The examples presented demonstrate the broad possibilities of representations of zero curvature in matrices whose elements are functions of Baker–Akhiezer type, although in full measure these possibilities have not been analytized in detail.

CHAPTER 2

SPECTRAL THEORY OF THE PERIODIC SCHRÖDINGER DIFFERENCE OPERATOR AND THE PEIERLS MODEL

The original approach to the construction of finite-zone solutions of the KdV equation, the nonlinear Schrödinger equation, and a number of others was based on the spectral theory of linear operators with periodic coefficients. We shall briefly indicate the connection of this approach with the algebrogeometric approach expounded in the preceding chapter.

Let $U(x, t, \lambda)$ and $V(x, t, \lambda)$ be solutions of the equations of zero curvature (1.17) which depend periodically on x. We consider the matrix

$$W(x, t, \lambda)=\Psi(x+T, t, \lambda)\Psi^{-1}(x, t, \lambda), \tag{2.1}$$

where T is the period and Ψ is a solution of Eqs. (1.15), (1.16); this matrix is called the monodromy matrix [which describes translation by a period of solutions of the linear equations (1.15), (1.16)].

From the fact that $\Psi(x + T, t, \lambda)$ is also a solution of Eqs. (1.15), (1.16) it follows that

$$[\partial_x-U, W]=0; \quad [\partial_t-V, W]=0,$$

and we arrive at Eqs. (1.50).

The matrix $W(x, t, \lambda)$ is analytic away from the poles of U and V where it has essential singularities.

The vector-valued function $\psi(x, t, \gamma)$ defined by equalities (1.51)-(1.55) is an eigenfunction of the operator of translation by a period and is called a Bloch function. The curve Γ on which the Bloch function becomes single-valued has infinite genus in the general case (its branch points accumulate at the poles of U and V).

Finite-zone periodic solutions are distinguished by the condition of finiteness of the genus of the surface Γ which is equivalent to rationality in λ of the monodromy matrix $W(x, t, \lambda)$.

Thus, periodic solutions of Eqs. (1.17), (1.50) possess the property that the Bloch corresponding to them is defined on a curve of finite genus and coincides with a Baker–Akhiezer function.

It is clear that the finite-zone concept carries over to any linear operator $\partial_x - U(x, \lambda)$ without reference to nonlinear equations. The corresponding matrices U are called finite-zone potentials.

As an example we consider the Sturm–Liouville operator with a periodic real potential $u(x)$, $L = -\partial_x^2 + u(x)$.

The spectrum of this operator in $L_2(R)$ consists of segments of the real axis called permitted zones. The lacunae in the spectrum are called forbidden zones. The end points of the zones E_i are simple points of the spectrum of the periodic and antiperiodic problems for the operator L. They are branch points of the Riemann surface Γ which is situated over the E plane in two-sheeted fashion and on which the Bloch function is meromorphic way from the infinitely distant point. There is one pole of ψ in each of the forbidden zones.

The condition that the potential u(x) be a finite-zone potential means that all forbidden zones vanish from some index onward, and all points of the spectrum of the periodic and antiperiodic problems for L with the exception of a finite number $E_1 < \ldots < E_{2n+1}$ are degenerate. The curve Γ is given in C^2 by the equation

$$y^2=\prod_{i=1}^{2n+1}(E-E_i). \tag{2.2}$$

The function ψ is a Baker–Akhiezer function and can hence be expressed by the formula

$$\psi(x, \gamma)=\exp\left(x\int_{E_1}^{\gamma}\omega\right)\frac{\theta(A(\gamma)+Ux+Z)\theta(Z)}{\theta(A(\gamma)+Z)\theta(Ux+Z)}. \tag{2.3}$$

Exanding (2.3) in a neighborhood of infinity, we obtain a formula [23] for finite-zone potentials of the Sturm–Liouville operator

$$u(x)=-2\frac{\partial^2}{\partial x^2}\ln\theta(Ux+Z)+\text{const}. \tag{2.4}$$

An exposition of these results and of the spectral theory of finite-zone Sturm–Liouville operators globally can be found in [17].

1. Periodic Schrödinger Difference Operator

The method of the inverse problem is applicable not only to partial differential equations but also to some differential-difference systems. For example, the equations of the Toda lattice [69]

$$\ddot{x}_n=e^{-x_n+x_{n+1}}-e^{-x_{n-1}+x_n} \tag{2.5}$$

admit the Lax representation $\dot{L} = [A, L]$, where

$$L\psi_n=c_n\psi_{n+1}+v_n\psi_n+c_{n-1}\psi_{n-1}, \tag{2.6}$$

$$A\psi_n=\frac{c_n}{2}\psi_{n+1}-\frac{c_{n-1}}{2}\psi_{n-1}, \tag{2.7}$$

$v_n = \dot{x}_n$, $c_n^2 = e^{x_{n+1}-x_n}$. This representation was found in [38, 58].

An important difference of such systems from continuous systems is that all periodic solutions of differential-difference equations admitting representations of Lax type or of the more general difference analogue of the equations of zero curvature are finite-zone solutions.

Explicit expressions for periodic solutions of the equations of the Toda lattice were obtained in [33] (see also the author's appendix to [13]).

The purpose of the present section is an exposition of the algebrogeometric approach to the spectral theory of periodic difference operators for the example of the Schrödinger difference operator (2.6). As already mentioned in the introduction, this theory plays an essential role not only in the construction of solutions of differential-difference systems but also in investigations of the Peierls model.

We consider operator (2.6) with periodic coefficients $c_n = c_{n+N}$, $v_{n+N} = v_n$.

The basis of the modern approach to spectral problems for periodic operators is the investigation of the analytic properties of solutions of the equation

$$L\psi_n=E\psi_n \tag{2.8}$$

for all values of the parameter E including complex values.

For any E the space of solutions of Eq. (2.8) is two-dimensional. Having given arbitrary values ψ_0 and ψ_1, all the remaining values ψ_n are found from (2.8) in recurrent fashion. The standard basis $\varphi_n(E)$ and $\theta_n(E)$ is given by the conditions $\varphi_0=1$, $\varphi_1=0$, $\theta_0=0$, $\theta_1=1$. From the recurrent procedure for computing $\varphi_n(E)$ and $\theta_n(E)$ it follows that they are polynomials in E,

$$\varphi_n(E)=\frac{c_0}{c_1\dots c_{n-1}}\left(E^{n-2}-\left(\sum_{k=2}^{n-1}v_k\right)E^{n-3}+\dots\right),$$

$$\theta_n(E)=\frac{1}{c_1\dots c_{n-1}}\left(E^{n-1}-\left(\sum_{k=1}^{n-1}v_k\right)E^{n-2}+\left(\sum_{0<i<j}^{n-1}v_iv_j-\sum_{k=1}^{n-2}c_k^2\right)E^{n-3}+\dots\right). \tag{2.9}$$

The matrix T(E) of the monodromy operator $\hat{T}:y_n \to y_{n+N}$ in the basis φ and θ has the form

$$\hat{T}(E)=\begin{pmatrix}\varphi_N(E), & \theta_N(E)\\ \varphi_{N+1}(E), & \theta_{N+1}(E)\end{pmatrix}. \tag{2.10}$$

From (2.8) it follows easily that for any two solutions of this equation, in particular, for φ and θ, the expression (the analogue of the Wronskian)

$$c_n(\varphi_n\theta_{n+1}-\varphi_{n+1}\theta_n)$$

does not depend on n. Since $c_0 = c_N$, it follows that

$$\det\hat{T}=\varphi_N\theta_{N+1}-\theta_N\varphi_{N+1}=\varphi_0\theta_1-\varphi_1\theta_0=1.$$

The eigenvalues w of the monodromy operator are found from the characteristic equation

$$w^2-2Q(E)w+1=0;\ Q(E)=\frac{1}{2}(\varphi_N(E)+\theta_{N+1}(E)). \tag{2.11}$$

The polynomial Q has degree N, and its leading terms have the form

$$2Q(E)=\frac{1}{c_0\ldots c_{N-1}}\left(E^N-\left(\sum_{k=0}^{N-1}v_k\right)E^{N-1}+\left(\sum_{0<i<j}^{N-1}v_iv_j-\sum_{k=0}^{N-1}c_k^2\right)E^{N-2}+\ldots\right). \tag{2.12}$$

The spectra $E_i^{\pm}$ of the periodic and antiperiodic problems for L are determined from the equation $Q(E_i^{\pm})=\pm1$, since in this case $w=\pm1$.

We denote by E_i, $i=1,\ldots,2q+2$, $q\leqslant N-1$ the simple points of the spectrum of the periodic and antiperiodic problems for L, i.e., the simple roots of the equation

$$Q^2(E)=1. \tag{2.13}$$

For a point E of general position Eq. (2.11) has two roots w and w^{-1}. To each root there corresponds a unique eigenvector $\{\psi_n\}$ of the monodromy matrix normalized by the condition $\psi_0 = 1$,

$$L\psi_n=E\psi_n,\ \psi_{n+N}=w\psi_n.$$

This solution is called a Bloch solution.

THEOREM 2.1. The two-valued function $\psi_n^{\pm}(E)$ is a single-valued meromorphic function $\psi_n(\gamma)$ on the hyperelliptic curve Γ, $\gamma\in\Gamma$, corresponding to the Riemann surface of the function $\sqrt{R(E)}$,

$$R(E)=\prod_{i=1}^{2q+2}(E-E_i). \tag{2.14}$$

Away from the infinitely distant points it has q poles $\gamma_1,\ldots,\gamma_q$. In a neighborhood of the infinitely distant points

$$\psi_n^{\pm}(E)=e^{\pm x_n}E^{\pm n}\left(1+\sum_{s=1}^{\infty}\xi_s^{\pm}(n)E^{-s}\right). \tag{2.15}$$

Here the signs ± correspond to the upper and lower sheets of the surface Γ (the upper sheet is the one on which at infinity $\sqrt{R}\sim E^{q+1}$).

The proof of this theorem repeats to considerable extent the analogous assertions from Sturm–Liouville theory [17].

The Bloch solution, just as any other solution of Eq. (2.8), has the form $\psi_n=\psi_0\varphi_n+\psi_1\theta_n$. The vector (ψ_0, ψ_1) is an eigenvector for the matrix $\hat{T}$. Hence, $\psi_0=1$, $\psi_1=\frac{w-\varphi_N}{\theta_N}$ or

$$\psi_n=\varphi_n(E)+\frac{w-\varphi_N}{\theta_N}\theta_n(E). \tag{2.16}$$

Suppose that e_j, $j=1,\ldots,N-q-1$, are twofold roots of the equation $Q^2(E)=1$, i.e.,

$$Q^2(E)-1=C^2r^2(E)R(E),\quad r(E)=\prod_{j=1}^{N-q-1}(E-e_j),\qquad C^{-1}=c_0\ldots c_{N-1}. \tag{2.17}$$

At the points e_j in the Bloch basis the matrix of the operator $\hat{T}$ is equal to ±1. Hence, it is equal to ±1 in any other basis. Thus,

$$\theta_N(E)=r(E)\tilde{\theta}_N(E),\quad \varphi_{N+1}(E)=r(E)\tilde{\varphi}_{N+1}(E), \tag{2.18}$$

$$\varphi_N(e_j)=\theta_{N+1}(e_j)=w(e_j)=\pm1. \tag{2.19}$$

From (2.19) it follows that $Q(E)-\varphi_N(E)=r(E)\tilde{Q}(E)$.

Here $\tilde{\theta}_N$, $\tilde{\varphi}_{N+1}$, $\tilde{Q}$ are polynomials in E. Substituting into (2.16) $w = Q + Cr\sqrt{R}$ and using the preceding equalities, we obtain

$$\psi_n^{\pm}=\varphi_n(E)+\frac{\tilde{Q}(E)\pm C\sqrt{R(E)}}{\tilde{\theta}_N(E)}\theta_n(E). \tag{2.20}$$

This equality means that the two-valued function $\psi_n^{\pm}(E)$ is a single-valued, meromorphic function of the point $\gamma\in\Gamma$. The poles of ψ lie at the points $\gamma_1,\ldots,\gamma_q$ each over a root of the polynomial $\tilde{\theta}_N(E)$. Indeed, if $\tilde{\theta}_N(E) = 0$, then the two roots $w_{1,2}$ are equal to $\varphi_N(E)$ and $\theta_{N+1}(E)$. Here $\varphi_N(E)\neq\theta_{N+1}(E)$. Hence, for one of the roots w [i.e., on one of the sheets of Γ over the root $\tilde{\theta}_N(E) = 0$] the numerator of the fraction in (2.20) vanishes. The pole of ψ_n lies on the second sheet.

To complete the proof it remains to consider the behavior of $\psi_n^{\pm}(E)$ as $E \to \infty$. From (2.16) it follows that at P^+ $\psi_1(E)$ has a simple pole. We find directly from (2.8) that ψ_n has a pole at P^+ of n-th order, $n > 0$. Similarly, $\psi_{-n}(E)$ has a pole at P^- of n-th order. From this and the fact that w has at P^+ a pole of N-th order while at P^- it has a zero of multiplicity N we obtain equality (2.15) where x_n are such that $x_0 = 0$, $\exp(x_n - x_{n+1}) = c_n$.

The parameters $\gamma_1,\ldots,\gamma_q$ or, more precisely, their projections onto the E plane (which we henceforth denote in the same way for brevity) have a natural spectral meaning.

LEMMA 2.1. The collection of points e_j (twofold degenerate points of the spectrum of the periodic and antiperiodic problems for L) and $\gamma_1,\ldots,\gamma_q$ forms the spectrum of problem (2.8) with zero boundary conditions $\tilde{\psi}_0 = \tilde{\psi}_N = 0$.

Proof. The curve Γ over the points e_j has two sheets on each of which the function w(E) assumes the same value 1 or −1. For $\tilde{\psi}_n$ it is possible to take

$$\tilde{\psi}_n(e_j)=\psi_n^+(e_j)-\psi_n^-(e_j)=\frac{2C\sqrt{R(e_j)}}{\tilde{\theta}_N(e_j)}\theta_n(e_j). \tag{2.21}$$

From (2.18) and the fact that $r(e_j) = 0$ we find that $\tilde{\psi}_0(e_j) = \tilde{\psi}_N(e_j) = 0$.

The points γ_i are the zeros of $\tilde{\theta}_N(E)$. As already mentioned above, for $E = \gamma_i$ and for one of the signs in front of $\sqrt{R}$ in (2.20) the numerator of the second term vanishes. Hence for the second it is nonzero. Suppose, for example, this is a plus sign. Then

$$\tilde{\psi}_n(\gamma_j)=(\tilde{Q}+C\sqrt{R})\theta_n(\gamma_j)$$

is a nontrivial solution of Eq. (2.8), $E = \gamma_j$, with zero boundary conditions.

We consider the inverse problem. Suppose there are given arbitrary distinct points E_i, $i = 1,\ldots,2q + 2$ and points $\gamma_1,\ldots,\gamma_q$ on the Riemann surface Γ of the function $\sqrt{R(E)}$ whose projections onto the E plane are distinct. In difference problems the analogue of Theorem 1.5 is the Riemann–Roch theorem [44]. In the present case it asserts that there exists a meromorphic function $\psi_n(\gamma)$ on Γ which is unique up to proportionality and has poles at the points $\gamma_1,\ldots,\gamma_q$, a pole of n-th order at P^+, and a zero of n-th order at the point P^-. The function $\psi_n(\gamma)$ can be normalized up to sign by requiring that the coefficients of $E^{\pm n}$ on the upper and lower sheets at infinity be mutual inverses. Fixing the sign in arbitrary fashion, we denote the corresponding coefficients by $e^{\pm x_n}$. Here $\psi_n(\gamma)$ will have the form (2.15) in a neighborhood of infinity.

LEMMA 2.2. The function $\psi_n(\gamma)$ constructed satisfy Eq. (2.8) where the coefficients of the operator L are

$$c_n=\exp(x_n-x_{n+1}),\quad v_n=\xi_1^+(n)-\xi_1^+(n+1). \tag{2.22}$$

Proof. We consider the function $\varphi_n=L\psi_n-E\psi_n$. It has poles at the points $\gamma_1,\ldots,\gamma_q$. It follows from (2.15), (1.22) that φ_n has a pole of order $n - 1$ at P^+ and a zero of order n at the point P^-. By the Riemann–Roch theorem $\varphi_n=0$.

The method of obtaining explicit formulas for ψ_n and the coefficients of L is completely analogous to the continuous case which was treated in the first chapter. As before, on Γ we fix a canonical collection of cycles. We denote by dp the normalized Abelian differential of third kind with sole singularities at infinity

$$idp = \frac{E^q + \sum_{l=1}^{q} \alpha_l E^{q-l}}{\sqrt{R(E)}} dE = \frac{P(E)\,dE}{\sqrt{R(E)}}. \tag{2.23}$$

The coefficients α_i are determined from the normalization conditions

$$\oint_{a_l} dp = \int_{E_{2l}}^{E_{2l+1}} dp = 0. \tag{2.24}$$

LEMMA 2.3. The function $\psi_n(\gamma)$ has the form

$$\psi_n = r_n \exp\left(in \int_{E_1}^{\gamma} dp \right) \frac{\theta(A(\gamma) + n\mathbf{U} + Z(D))}{\theta(A(\gamma) + Z(D))}, \tag{2.25}$$

where $2\pi i U_k = \oint_{b_k} dp$, and r_n is a constant.

In a neighborhood of the infinitely distant point on the upper sheet we have

$$\exp\left(i \int_{E_L}^{\gamma} dp \right) = Ee^{-I_0}(1 - I_1 E^{-1} + \ldots). \tag{2.26}$$

It follows from (2.15) that $\exp(2x_n + 2I_0 n)$ is equal to the ratio of the values of the factors of the exponential function in (2.25) taken at the images $A(P^{\pm}) = \pm z^0$. From (2.25) and the fact that according to the Riemann bilinear relations $2z^0 = -\mathbf{U}$ we obtain

$$c_n^2 = e^{-2I_0} \frac{\theta(\mathbf{U}(n-1) + Z)\,\theta((n+1)\mathbf{U} + Z)}{\theta^2(n\mathbf{U} + Z)}, \tag{2.27}$$

where $Z = Z(D) - z^0$.

In a neighborhood of P^+ we have

$$A(\gamma) = z^0 + \mathbf{V}E^{-1} + O(E^{-2}),$$

where the coordinates V_k of the vector $\mathbf{V}$ are determined by the equality

$$\Omega_k = dE^{-1}(V_k + O(E^{-1})).$$

Expanding (2.25) in a series in E^{-1}, we obtain from (2.22)

$$v_n = \frac{d}{dt} \ln \frac{\theta((n-1)\mathbf{U} + Z + \mathbf{V}t)}{\theta(n\mathbf{U} + Z + \mathbf{V}t)} \Big|_{t=0} + I_1. \tag{2.28}$$

THEOREM 2.2. Formulas (2.27), (1.28) recover the coefficients of L on the basis of the parameters E_i and γ_j.

It is important to note that in general the formulas (2.27), (1.28) determine quasi-eriodic functions c_n and v_n. In order that c_n and v_n be periodic it is necessary and sufficient that for the corresponding differential dp the following conditions be satisfied:

$$U_k = \frac{1}{2\pi} \oint_{b_k} dp = \frac{m_k}{N}, \quad m_k \text{— integers.} \tag{2.29}$$

It follows from the definition of ψ_n that the parameters E_i, γ_j determine them up to sign. Change of the signs of $\{\psi_n\}$ leads only to a corresponding change of the signs of $\{c_n\}$. It is possible to not distinguish operators differing only by the signs of the c_n, since their eigenfunctions go over into one another trivially. [Formula (2.27) is consistent with the remain made above, since it expresses in terms of E_i and γ_j not c_n themselves but their squares.]

Until now we have considered operators L with arbitrary complex coefficients. Suppose now that v_n and c_n are real; then all the polynomials $\theta_n(E)$, $\varphi_n(E)$, $Q(E)$ introduced above are also real. Moreover, the periodic and antiperiodic problems for L are self-adjoint. This means that there are N real points of the spectrum of each problem, i.e., the polynomial $Q^2 - 1$ has 2N real roots. Hence at the extrema of the polynomial Q(E), $dQ(E) = 0$ we have $|Q(E)| \geqslant 1$.

The segments $[E_{2i-1}, E_{2i}]$ on which $|Q(E)| \leqslant 1$ are called permitted zones. In these segments $|w| = 1$ and the multivalued function $p(E)$ defined from the equality $w = e^{ipN}$ is real. It is called the quasimomentum. Its differential coincides with (2.23) where in (2.24) a_i are cycles situated over the forbidden zones $[E_{2i}, E_{2i+1}]$.

LEMMA 2.4. The poles γ_i of the Bloch function $\psi_n(\gamma)$ of a real operator L are situated such that there is one in each of the finite forbidden zones, $E_{2i} \leqslant \gamma_i \leqslant E_{2i+1}$.

Proof. The poles γ_i are zeros of the polynomial $\theta_N(E)$. At these points

$$1 = \det T = \varphi_N(\gamma_j)\,\theta_{N+1}(\gamma_j) = 1.$$

Since φ_N and θ_{N+1} are real, it follows that

$$|Q(\gamma_j)| = \frac{1}{2}\,|\varphi_N(\gamma_j) + \theta_{N+1}(\gamma_j)| \geqslant 1$$

and γ_j lies either in a forbidden zone or in one of the collapsed zones — the points e_j. In the latter it was shown above that $\psi_n(\gamma)$ has no singularities. We consider the family of operators L_t with coefficients $c_n^t = t + (1 - t)c_n$, $v_n^t = (1 - t)v_n$. It may be assumed with no loss of generality that $c_n > 0$, since this can be achieved by changing the signs of c_n which has no effect on E_i, γ_j. Since $c_n^t \neq 0$, it follows that γ_i^t, which depend continuously on t (just as E_i^t), lie in forbidden zones for all t. For $t = 1$ all forbidden zones close up, i.e., all roots of the polynomial $Q^2 - 1$ except the first and last are twofold degenerate. Hence, γ_i^1 lie at two fold roots of $Q^2 - 1$. Deformation with respect to t leads to the situation that part of the twofold roots decompose into pairs of simple roots which are end points of a forbidden zone. Inside each of them there lies one point γ_i by the continuity in t. For $t = 0$ we obtain the required assertion.

THEOREM 2.3. If the points $E_1, \ldots, E_{2q+2}$ are real and there is one of the points $\gamma_1, \ldots, \gamma_q$ of the corresponding Riemann surface over each of the forbidden zones $[E_{2i}, E_{2i+1}]$, then the coefficients c_n and v_n of the operator L determined by them by Theorem 2.2 are real.

Proof. The necessity of the conditions of the theorem in the class of periodic operators is given by Lemma 2.4.

Suppose that the E_i are real. Complex conjugation induces an antiinvolution τ of the curve $\Gamma: (E, \sqrt{R}) \to (\bar{E}, \sqrt{\bar{R}})$. The fixed ovals of this antiinvolution are cycles situated over the segments $[E_{2i}, E_{2i+1}]$ and over the infinite zone joining the points E_{2q+2}, E_1 through infinity.

We consider $\bar{\psi}_n(\tau(\gamma))$. This function posesses all the analytic properties of ψ_n. Since ψ_n is determined by these properties up to sign, it follows that

$$\bar{\psi}_n(\tau(\gamma)) = \pm\psi_n(\gamma). \tag{2.30}$$

It follows from (2.22) that the v_n are real, while the c_n are either real or pure imaginary (i.e., c_n^2 is real).

We shall prove that under the assumptions of the theorem $c_n \neq 0$, $c_n \neq \infty$. The negation of this assertion is equivalent to the fact that one or several of the zeros $\gamma_i(n)$ of the function $\psi_n(\gamma)$ is at infinity on the upper or lower sheet of Γ. From (2.30) it follows that on cycles situated over $[E_{2i}, E_{2i+1}]$, ψ_n is either real or pure imaginary. On each cycle there is one pole γ_i; therefore, there is at least one zero. Since there is a total of q zeros, it follows that $\gamma_i(n)$, like γ_i, are situated one over each $[E_{2i}, E_{2i+1}]$ and are hence bounded away from infinity.

By what has been proved, the sign of c_n^2 does not change under continuous deformation of E_i and γ_i for which the conditions of the theorem are satisfied. We deform them so that all the forbidden zones close. Here it is easy to verify that the operator L deforms into an operator L_0 for which $v_n = 0$ and $c_n^2 = \text{const} > 0$. The proof of the theorem is complete.

To conclude the section we consider conditions distinguishing operators L for which $v_n = 0$, i.e.,

$$L\psi_n = c_n\psi_{n+1} + c_{n-1}\psi_{n-1}. \tag{2.31}$$

Connected with this operator is the problem of integrating the difference analogue of the KdV equation

$$\dot{c}_n' = c_n'(c_{n+1}' - c_{n-1}'), \quad c_n' = c_n^2. \tag{2.32}$$

THEOREM 2.4. The necessary and sufficient conditions that the operator L (constructed by Theorem 2.2 on the basis of the data E_i and γ_j) has the form (2.31), i.e., $v_n = 0$, is the symmetry of the points E_i relative to zero and the invariance of the points γ_j relative to the involution on Γ

$$(E, \sqrt{R}) \to (-E, \sqrt{R}), \quad R(E) = \prod_{l=1}^{q+1} (E^2 - E_l^2).$$

Necessity of the conditions of the theorem follows from the fact that if $\psi_n(\gamma)$ is the Bloch function for the operator L of (2.31), then for $\tilde{\psi}_n = (-1)^n \psi_n$

$$L\tilde{\psi}_n = -E\tilde{\psi}_n, \quad \tilde{\psi}_{n+N} = (-1)^N w \tilde{\psi}_n.$$

Sufficiency of the conditions is proved in analogy to the proof of Theorem 2.3.

2. "Finite-Zone Potentials" and Variations of Kruskal's Integrals

As mentioned above, periodic operators are all finite-zone operators. Here the number of zones in the general case has the order of the period N of the operator. Distinguishing q-zone difference operators $q < N$ can be carried out with the help of the same algebraic Ansatz as in the continuous case. Thus, for the Schrödinger difference operator considered this Ansatz has the form: the operator L is a q-zone operator if and only if there is an operator

$$L_1 \psi_n = \sum_{k=-q-1}^{q+1} d_{n,k} \psi_{n+k} \tag{2.33}$$

such that

$$[L_1, L] = 0. \tag{2.34}$$

As in the continuous case, Eqs. (2.34) constitute essentially a pencil of nonlinear equations for the coefficients v_n and c_n of the operator L.

All solutions of these equations are given by formulas (2.27), (1.28) and are in general quasiperiodic functions of n.

The purpose of the present section is to present for this example still another aspect of the theory of finite-zone integration — its connection with variational principles for functionals of Kruskal type.

We define functionals $I_k = I_k\{c_n, v_n\}$ by the formula

$$ip = \ln E - \sum_{k=0}^{\infty} I_k E^{-k}, \tag{2.35}$$

where p(E) is the quasimomentum. These functionals have the form

$$I_k\{c_n, v_n\} = \frac{1}{N} \sum_{n=0}^{N-1} r_k(c_n, v_n),$$

where the local densities r_k are polynomials. From (2.12) we have

$$I_0 = \frac{1}{N} \sum_{n=0}^{N-1} \ln c_n,$$

$$I_1 = \frac{1}{N} \sum_{n=0}^{N-1} v_n,$$

$$I_2 = \frac{1}{N} \sum_{n=0}^{N-1} \left(c_n^2 + \frac{v_n^2}{2}\right) \text{ etc.}$$

THEOREM 2.5. The operator L is a q-zone operator if and only if its coefficients are extremals of the functional $H = I_{q+2} + \sum_{k=0}^{q+1} \alpha_k I_k$, where α_k are certain constants.

Because of this theorem, Eqs. (2.34) give a commutation representation of the pencil of equations

$$\delta H=\delta I_{q+2}+\sum_{k=0}^{q+1}\alpha_k\delta I_k=0;\quad \delta=\sum_n\left(\frac{\partial}{\partial c_n}\delta c_n+\frac{\partial}{\partial v_n}\delta v_n\right) \tag{2.36}$$

parametrized by the constants α_k.

Proof. We shall first prove necessity of the conditions of the theorem.

LEMMA 2.5. There is the equality

$$i\delta p=\frac{l_0E^{q+1}+l_1E^q+\ldots+l_{q+1}}{\sqrt{R(E)}};\quad l_i=l_i(\delta c_n,\ \delta v_n). \tag{2.37}$$

The proof of this relation can be obtained in complete analogy to the proof of its continuous variant [15].

We shall derive it from other considerations "which generalize easily to the case of variations not preserving the group of periods." (For the meaning of the last assertion see below.)

We consider an arbitrary variation δc_n, δv_n of the operator L in the class of operators with the same period N. Under such variation the spectrum of the operator L is perturbed so that the end points of the old zones E_i are shifted (we denote them by E_i') and the twofold points of the spectrum e_j possibly decompose into simple points $e_j^{\pm}$ (at the site of the e_j new lacunae are formed).

The quasimomentum is a multivalued function on Γ which on passing about the b-cycles changes by a integral multiple of $2\pi/N$. Hence δp is a single-valued meromorphic function on Γ. From (2.23) it follows that δdp has poles of second order at the points E_i and possibly simple poles over the points e_j. From the single-valuedness of δp it follows that the residues of δdp over e_j are equal to zero. Hence, δp has simple poles at the end points of the zones, and, since it is odd, equality (2.37) holds.

Expanding equality (2.37) in a neighborhood of infinity, we obtain

$$\begin{gathered}l_0=-\delta I_0,\quad l_1=-\delta I_1+\frac{s_1}{2}\delta I_0,\quad s_1=\sum_i E_i,\\ l_2=-\delta I_2+\frac{s_1}{2}\delta I_1+\left(\frac{s_1^2}{8}-\frac{s_2}{2}\right)\delta I_0,\quad s_2=\sum E_iE_j,\\ l_k=-\delta I_k+\sum_{i=0}^{k-1}\beta_{ik}\delta I_i.\end{gathered} \tag{2.38}$$

From the first q + 1 equalities the coefficients l_k are expressed in terms of δI_k, $k\leqslant q+1$. Equating the coefficients of the expansion (2.37) and (2.35) for E^{-q-2}, we obtain (2.36) where α_k are symmetric polynomials in E_i.

The necessity of the conditions of the theorem follows from the fact that according to formula (2.37) for q-zone operators the coefficients l_k are independent and hence the differentials δI_k, $k\leqslant q+2$ are independent.

Let M^q be the manifold of ordered collections $E_1<\ldots<E_{2q+2}$. The closure $\hat{M}^N$ is a stratified manifold containing $M^q\in\hat{M}^N$ for all $q\leqslant N$. For any collection $E_1,\ldots,E_{2q+2}$ formula (2.23) defines a differential dp, and hence by (2.35) functionals $I_k=I_k(\{E_i\})$.

We now consider the variation of $(E_1,\ldots,E_{2q+2})$ in $\hat{M}^N$. Under such variation, in addition to the variation E_i, there appear new lacunae $e_j^-<e_j^+$, $j=1,\ldots,N-q$.

LEMMA 2.6. Let δU_k be the variation of the group of periods, i.e.,

$$\delta U_k=\frac{1}{2\pi}\oint_{b_k}\delta dp.$$

Then

$$i\delta p=\frac{\sum_{k=0}^{q+1}l_kE^{q+1-k}}{\sqrt{R(E)}}+\sum_{k=1}^{q}\delta U_k\oint_{a_k}\frac{dt\sqrt{R(t)}}{\sqrt{R(E)}(E-t)}. \tag{2.39}$$

The proof of the lemma is carried out as in the proof of the preceding lemma with the help of comparing the analytic properties of the right and left sides.

We note that (2.39) coincides with (2.37) if $\delta U_k = 0$ (the corresponding variations are called variations preserving the group of periods of the differential dp).

In the next section we shall need formulas for the second variation of the quasimomentum.

THEOREM 2.6. For an arbitrary variation $(E_1,\ldots,E_{2q+2})$ in $\hat{M}^N$ we have

$$i\delta^2 p=\frac{\sum_{k=0}^{q+1}\tilde{l}_k E^{q+1-k}}{\sqrt{R}}+\sum_{l=1}^{q}\delta^2 U_l\oint_{a_l}\frac{dt\sqrt{R(t)}}{\sqrt{R(E)}\,(E-t)}+\frac{1}{\sqrt{R}}\left(\sum_{l=1}^{2q+2}\frac{P(E_l)}{2(E-E_l)}(\delta E_l)^2+\sum_{j=1}^{N-q}\frac{P(e_j)}{4(E-e_j)}(\delta e_j)^2\right), \tag{2.40}$$

where $\delta E_i = E_i' - E_i$, $\delta e_j = e_j^+ - e_j^-$; $P(E)$ is the same as in (2.23).

Proof. Let $(E_1',\ldots,E_{2q+2}', e_j^{\pm})$ be a variation of $(E_1,\ldots,E_{2q+2})$. The quasimomentum corresponding to this collection has the form

$$idp'=\frac{P_N(E)}{\sqrt{R'(E)}\sqrt{\prod_{j=1}^{N-q}(E-e_j^+)(E-e_j^-)}}\,dE. \tag{2.41}$$

The coefficients of the polynomial $P_N(E)$ are determined from the equations

$$\int_{E_{2i}'}^{E_{2i+1}'} dp'=0,\quad i=1,\ldots,q, \tag{2.42}$$

$$\int_{e_j^-}^{e_j^+} dp'=0,\quad j=1,\ldots,N-q. \tag{2.43}$$

For $e_j^+ = e_j^- = e_j$, $E_i' = E_i$ the polynomial $P_N(E)$ is equal to $P(E)\prod_j(E-e_j)$. From this and the fact that δp has no poles over e_j we find by expanding (2.41) that $i\delta^2 p$ has the form (2.41) with some undermined coefficients of $(\delta E_i)^2$ and $(\delta e_j)^2$. To determine the latter it is necessary to compare the leading terms of the second variation (2.41) and the differential of (2.40) at the points E_i and e_j.

We note that in the case of variation of periodic operators the corresponding second variation $\delta^2 p$ is given by the formula (2.40) in which $\delta^2 U_k = 0$. The coefficients of $\tilde{l}_k$ are connected with $\delta^2 I_k$ by the same formulas as l_k and δI_k (2.28).

COROLLARY. Let (c_n, v_n) be an extremal of the functional H of (2.36). Then the value of $\delta^2 H$ on the variation $(\delta c_n, \delta v_n)$ is

$$\delta^2 H=\sum_{l=1}^{2q+2}\frac{P(E_l)}{2}(\delta E_l)^2+\sum_{j=1}^{N-q}\frac{P(e_j)}{4}(\delta e_j)^2. \tag{2.44}$$

CHAPTER 3

THE PEIERLS MODEL

1. Integrable Cases in the Peierls Model

In this section we consider the special cases of the Peierls model briefly described in the introduction.

We choose the deformation energy in the form of a linear combination of integrals of the Toda lattice (2.35). The Peierls functional then takes the form

$$H=\frac{1}{N}\sum_{l=1}^{m}E_l^+ +\sum_{k=0}^{l}\varkappa_k I_k, \tag{3.1}$$

where E_i^+ is the spectrum ordered in increasing order of the periodic problem for the Schrödinger operator.

Together with this model ($l = 2$), in [7] a model in which $v_n = 0$ (absence of internal freedom) was considered. The latter model in the linearized limit

$$x_n = na + u_n, \quad |u_n| \ll 1, \quad c_n = c_0(1 - \alpha(u_{n+1} - u_n))$$

coincides with the known lattice model of Su, Schriffer, and Heeger [68] which has been intensely investigated numerically.

In this limit the model admits a path-integral approximation [6, 7, 3, 4]. Thus, if $\rho \ll 1$, where $\rho = m/N$ is the density of the electrons, then the path-integral limit of H has the form ($v_n = 0$)

$$H = \frac{1}{2\pi} \oint_{E_1}^{\mu} E\,dp + g \int_0^T u^2(x)\,dx, \tag{3.2}$$

where dp is the differential of quasimomentum of the Sturm–Liouville operator $L = -d^2/dx^2 + u(x)$.

THEOREM 3.1. The extremals of the functional H of (3.1) have no more than $2l - 2$ forbidden zones in the spectrum of the corresponding Schrödinger operator.

Proof. We first consider the variation of the first part of the functional (3.1). Since E_i^+ are roots of the polynomial $Q(E) - 1$ of (2.12), it follows that

$$\sum_{l=1}^{m} \delta E_l^+ = -\sum_{l=1}^{m} \operatorname{res}_{E_l} \frac{\delta Q}{Q-1}. \tag{3.3}$$

We introduce the notation

$$Q \mp 1 = CR_{\pm} r_{\pm}^2, \tag{3.4}$$

where $R_\pm(E)$ are polynomials whose roots are simple points of the spectrum of the periodic (respectively, antiperiodic) problem for L. The polynomials $r_\pm(E)$ have roots at the twofold points of the spectrum of these same problems.

From the definition of the quasimomentum we have

$$i\delta p = \frac{1}{N} \frac{\delta Q}{\sqrt{Q^2-1}} = \frac{1}{NC} \frac{\delta Q}{\sqrt{R}\, r_+ r_-}. \tag{3.5}$$

Comparing this formula with (2.37), we obtain

$$\frac{1}{NC} \delta Q = r_+ r_- \left(\sum_{k=0}^{q+1} l_k E^{q+1-k} \right). \tag{3.6}$$

With the help of (3.3) and (3.6) the equations of the extremals assume the form

$$0 = \delta H = \sum_{k=0}^{l} \varkappa_k \delta I_k - \sum_{l=1}^{m} \operatorname{res}_{E_l} \frac{\left(\sum_{k=0}^{q+1} l_k E^{q+1-k} \right) r_-(E)}{R_+(E)\, r_+(E)}. \tag{3.7}$$

We suppose further that $q + 1 \geqslant l$. Then

$$l_k = \sum_{j=0}^{k} \beta_{jk} \delta I_j. \tag{3.8}$$

Under the assumption made the variations δI_k, $k \leqslant q + 1$, are independent. Hence the coefficients of δI_k in (3.7) must be equal to zero. Equating to zero the coefficients of δI_k, $k = l + 1, \ldots, q + 1$, gives

$$\sum_{l=1}^{m} \operatorname{res}_{E_l} \frac{E^{q+1-k} r_-(E)}{R_+(E)\, r_+(E)} = 0, \quad k = q+1, \ldots, l+1. \tag{3.9}$$

Similarly, equating to zero the coefficients of δI_j, $j = 0, \ldots, l$, implies that

$$\varkappa_j=\sum_{k=0}^{q+1}\beta_{jk}\sum_{i=1}^{m}\operatorname{res}_{E_i}\frac{E^{q+1-k}r_-(E)}{R_+(E)r_+(E)}. \tag{3.10}$$

We shall show that for $q > 2l - 2$ Eqs. (3.9) have no solutions. It may be assumed with no loss of generality that for the left of $\mu = E_m$ there are no more than $[q/2]$ forbidden zones [since otherwise it is possible to pass from Eqs. (3.9) to equivalent equations in which the summation goes over $i > m$].

Let μ_s be arbitrary points (one in each of the forbidden zones to the left of μ). Then

$$\sum_{i=1}^{m}\operatorname{res}_{E_i}\frac{\prod_s (E-\mu_s)r_-(E)}{R_+(E)r_+(E)}\neq 0. \tag{3.11}$$

Indeed, the polynomial R_+r_+ has simple roots. Hence, at neighboring points the signs of the residues of $(R_+r_+)^{-1}$ are opposite. From the definition of the permitted zones it is evident that between neighboring roots of R_+r_+ lying in one zone there must be a twofold root of the polynomial $Q + 1$ or a root of the polynomial r_-. Thus, the sign of the residues of $r_-(r_+R_+)^{-1}$ is constant within each permitted zone and changes on passing to a neighboring zone. Since the μ_s lie in forbidden zones, all the residues in (3.11) have constant sign, and their sum is nonzero.

If the degree of the polynomial $\prod_s (E-\mu_s)$ (which does not exceed $[q/2]$) is less than or equal to $q - l$, then (3.11) contradicts (3.9). Hence $[q/2] > q - l$ or $q \leqslant 2l - 2$.

The thermodynamic limit in this model, $N \to \infty$, $m \to \infty$, $m/N \to \rho$ — the density of electrons, is of basic interest.

By the theorem proved, as $N \to \infty$ the number of permitted zones for the extremals of H does does not exceed $2l - 2$. Hence, the points of the spectrum of the periodic problem for L densely fill out the permitted zones which makes it possible to pass from summation to integration in Eqs. (3.10). From the definition of the quasimomentum it follows that the differential (2.23)

$$i\,dp=\frac{P(E)\,dE}{\sqrt{R(E)}}$$

determines the density with which the points $E_i^{\pm}$ occur in the permitted zones.

The residues of the differential

$$\frac{dQ}{Q-1}=\frac{dE\,Pr_+r_-}{CR_+r_+^2}=\frac{Pr_-}{CR_+r_+}\,dE$$

at all twofold points of the spectrum are equal to 2. Hence, at all (except no more than $4l$) points the residue in (3.10) is equal to $E^{q+1-l}/2P(E)$.

COROLLARY. In the thermodynamic limit the end points of the zones E_i of the Peierls functional are determined from the equations

$$\oint_{E_1}^{\mu}\frac{E^{q+1-k}}{\sqrt{R}}\,dE=0,\quad l<k\leqslant q+1<2l-1,$$
$$\sum_{k=0}^{q+1}\beta_{kj}\frac{1}{2\pi i}\oint_{E_1}^{\mu}E^{q+1-k}\frac{dE}{\sqrt{R}}=\varkappa_j,\quad j=0,\ldots,l. \tag{3.12}$$

Here μ, the so-called chempotential, is the last energy level before which there is summation in (3.1). For determining μ there is the equation

$$\frac{1}{2\pi}\oint_{E_1}^{\mu}\frac{P(E)\,dE}{i\sqrt{R}}=\rho. \tag{3.13}$$

(The sign $\oint_a^b$ means that integration goes over a contour integrating the real axis at the points a and b.)

Equations (3.12), (3.13) were obtained another way in [7]. Instead of passing to the limit in Eqs. (3.10), in the work [7] a limit functional $\mathscr{H}$ was defined on the set of all finite-zone potentials which was then varied.

With a view to investigating the thermodynamic limit for the Peierls model, we determine this functional $\mathscr{H}$ more precisely. Let $E_1,\ldots,E_{2q+2}$ be an ordered collection of distinct real points. Formulas (2.23), (2.24) define the differential dp which, in turn, gives the functionals $I_k\{E_i\}$.

The functional $\mathscr{H}$ is given by the formula

$$\mathscr{H}\{E_i\}=\frac{1}{2\pi}\oint_{E_1}^{\mu}E\,dp+\sum_{k=0}^{l}\varkappa_k I_k, \tag{3.14}$$

where μ is determined from condition (3.13).

Equations (3.9), (3.10) are the equations for the extremals of H in the class of periodic potentials. Periodicity of the potentials $\{c_n, v_n\}$ corresponding to the collection $\{E_i\}$ requires that relations (2.29) be satisfied which do not change under periodic variations. Passing to the limit $N \to \infty$ in them, we obtain the assertion.

THEOREM 3.2. Equations (3.12) are the necessary and sufficient condition that $\mathscr{H}$ be extremal (under the condition ρ = const) relative to variations such that $\delta U_k = 0$, where

$$U_k=\frac{1}{2\pi}\oint_{E_1}^{E_{2k}}dp. \tag{3.15}$$

Here, as in the derivation of Eqs. (3.9), (3.10), it is assumed that $q + 1 \geqslant l$. Under this assumption from formula (2.39) and (2.35) it follows that

$$\delta U_k=\sum_{s=q+2}^{2q}a_{ks}\delta I_s+\sum_{l=0}^{q+1}b_{kl}\delta I_l.$$

In analogy to the derivation of (3.9), (3.10) we obtain

THEOREM 3.3. A necessary and sufficient condition that $\mathscr{H}$ be extremal (under the condition ρ = const) on the set of all finite-zone states are Eqs. (3.12) and the equations

$$\oint_{E_1}^{\mu}\frac{dE}{\sqrt{R(E)}}\oint_{E_{2k}}^{E_{2k+1}}\frac{\sqrt{R(t)}\,dt}{E-t}=0,\quad k=1,\ldots,q. \tag{3.16}$$

We now consider the stability of extremals of the functional $\mathscr{H}$. By Theorem 3.1 the number of zones for an extremal of $\mathscr{H}$ does not exceed $2l - 2$.

THEOREM 3.4. At minima of $\mathscr{H}$ the chempotential μ lies in a forbidden zone. The number of forbidden zones does not exceed $l - 1$.

Proof. Suppose that μ lies in a permitted zone

$$\delta^2\mathscr{H}=\frac{1}{2\pi}\oint_{E_1}^{\mu}(E-\mu)\,d\delta^2p+\sum_{k=0}^{l}\varkappa_k\delta^2I_k+(\delta\mu)^2\frac{dp}{dE}(\mu)=-\frac{1}{2\pi}\oint_{E_1}^{\mu}\delta^2p\,dE+\sum_{k=0}^{l}\varkappa_k\delta^2I_k+(\delta\mu)^2\frac{dp}{dE}(\mu).$$

We consider a variation under which a gap opens at the point μ. The coefficient of $(\delta\mu)^2$ in δ^2p has the form

$$-\frac{1}{2\pi}\oint_{E_1}^{\mu}\frac{P(\mu)\,dE}{\sqrt{R}\,(E-\mu)}+O(1).$$

Since $P(\mu)(R(\mu))^{-1/2} > 0$, the first term is singular and becomes $-\infty$ which proves the instability of any extremal for which μ lies in a permitted zone.

If μ lies in a forbidden zones, then $U_m = \rho$ (where $E_{2m} \leqslant \mu \leqslant E_{2m+1}$). Moreover, relative to admissible variations $U_m = \rho$ = const. Hence, the corresponding extremals are determined from Eqs. (3.12) and (3.16). In the last system the equation with index m must be omitted.

We shall prove the last assertion of the theorem. Suppose that $q \geqslant l$.

LEMMA 3.1. The second variation of $\mathscr{H}$ is equal to

$$\delta^2\mathscr{H}=\sum_{k=1}^{2q+2}\Lambda(E_k)(\delta E_k)^2+\frac{1}{2}\sum_{j=1}^{N-q}\Lambda(e_j)(\delta e_j)^2, \tag{3.17}$$

where

$$\Lambda(e)=\frac{P(e)}{8\pi i}\oint_{E_1}^{E_{2m}}\frac{dE}{\sqrt{R(E)}(E-e)}. \tag{3.18}$$

Proof. As in the proof of the corollary of Theorem 2.6, we note that the coefficients of the polynomial $\tilde{l}$ and the quantity $\delta^2 U_k$ figuring in formula (2.40) are connected with $\delta^2 I_j$ by the same formulas as l_k and δU_k in (2.39) are connected with the first variations δI_j. Hence, by Eqs. (3.12), (3.16) we have

$$\sum_{k=0}^{l}\varkappa_k\delta^2 I_k-\frac{1}{2\pi}\oint_{E_1}^{E_{2m}}\frac{\tilde{l}(E)\,dE}{i\sqrt{R(E)}}+\frac{1}{2\pi i}\sum_{k\neq m}\delta^2 U_k\oint_{E_1}^{E_{2m}}dE\oint_{E_{2k}}^{E_{2k+1}}\frac{dt\sqrt{R(t)}}{\sqrt{R(E)}(E-t)}=0. \tag{3.19}$$

Substituting the expression (2.40) into

$$\delta^2\mathscr{H}=-\frac{1}{2\pi}\oint_{E_1}^{E_{2m}}\delta^2 p\,dE+\sum_{k=0}^{l}\varkappa_k\delta^2 I_k$$

and considering (3.19), we obtain the assertion of the lemma.

Since at minima all $\Lambda(e)$ must be positive while $P(E)$ changes sign on passing across a forbidden zone, the second term must also change sign (if the contour of integration does not intersect this zone).

Hence, there exist points $E_{2k}\leqslant\mu_k\leqslant E_{2k+1}$, $k\neq m$ such that

$$\oint_{E_1}^{E_{2m}}\frac{dE}{\sqrt{R(E)}(E-\mu_s)}=0. \tag{3.20}$$

The residue at infinity of the integrand in (3.20) is equal to zero. Therefore,

$$\oint_{E_{2m}}^{E_{2q+2}}\frac{dE}{\sqrt{R(E)}(E-\mu_s)}=0. \tag{3.21}$$

It may be assumed with no loss of generality that $m\leqslant q+1-m$, since otherwise in all subsequent arguments it is necessary to pass from the integrals (3.20) to (3.21).

Let ν_s, $s<m$, be arbitrary points lying one each in each forbidden zone with indices less than m. Then

$$\oint_{E_1}^{E_{2m}}\frac{dE}{i\sqrt{R(E)}}\cdot\frac{\prod_{k<m}(E-\nu_k)}{\prod_{s>m}(E-\mu_s)}\neq 0, \tag{3.22}$$

since the integrand along all permitted zones located within the contour has constant sign (on passing from zone to zone the sign of $\sqrt{R}$ changes, but the sign of the second factor in the integrand also changes). Expanding the integrand in simplest fractions, we find that if $m<g+1-m$, then (3.22) is the sum of expressions of the form (3.20), and hence it must be equal to zero. If $m=q+1-m$, then (3.22) reduces in similar fashion to $\oint_{E_1}^{E_{2m}}\frac{dE}{i\sqrt{R(E)}}$, which is also equal to zero, because of the consistency equations (3.12) for $q\geqslant l$. The contradiction obtained between (3.22) and (3.12), (3.20) proves the theorem.

We consider as a basic example the model investigated in [7] in which $l=2$. The results presented above in this special case imply that the minimum of $\mathscr{H}$ is realized on a single-zone state. The end points of the zones E_1, E_2, E_3, E_4 corresponding to the minimum are determined from the equations

$$\varkappa_2=\frac{i}{2\pi}\oint_{E_1}^{\mu}\frac{dE}{\sqrt{R}}, \tag{3.23}$$

$$0=\oint_{E_1}^{\mu}(2E-s_1)\frac{dE}{\sqrt{R}}, \quad s_1=\sum_{i=1}^{4}E_i, \tag{3.24}$$

$$\varkappa_0=\frac{i}{2\pi}\oint_{E_1}^{\mu}\frac{\left(E^2-\frac{s_1}{2}E+\frac{s_2}{2}-\frac{s_1^2}{8}\right)}{\sqrt{R}}dE, \quad s_2=\sum E_iE_j, \tag{3.25}$$

$$\rho=\frac{1}{2\pi i}\oint_{E_1}^{\mu}\frac{(E+a)\,dE}{\sqrt{R}}. \tag{3.26}$$

The constant a is found from the normalization conditions (2.25)

$$\oint_{E_2}^{E_3}\frac{E+a}{\sqrt{R}}dE=0. \tag{3.27}$$

As shown in [7], Eqs. (3.23)-(3.26) are considerably simplified after passing to an elliptic parametrization.

The function

$$z=\int_{E_1}^{E}\frac{dE'}{\sqrt{R(E')}}. \tag{3.28}$$

maps the elliptic curve Γ of the function $\sqrt{R}$, $R=\prod_{i=1}^{4}(E-E_i)$ onto a torus with periods 2ω, $2\omega'$,

$$\omega=\int_{E_1}^{E_2}\frac{dE}{\sqrt{R(E)}}, \quad \omega'=\int_{E_2}^{E_3}\frac{dE}{\sqrt{R(E)}}. \tag{3.29}$$

Inversion of the elliptic integral (3.29) is given by the formula

$$E(z)=\zeta(z+z_0)-\zeta(z-z_0)+h, \tag{3.30}$$

where $\zeta(z) = \zeta(z|\omega, \omega')$ is the Weierstrass function. (All the necessary information regarding elliptic functions can be found in [2].) The parameters ω, ω', z_0, h replace E_1, E_2, E_3, E_4. In these parameters Eqs. (3.23)-(3.26) acquire the form

$$z_0=(\rho-1)\omega', \tag{3.31}$$

$$\varkappa_2=\frac{i}{\pi}\omega, \tag{3.32}$$

$$\pi i+2\eta z_0=\omega(2\zeta(2z_0)+h), \tag{3.33}$$

$$-\varkappa_0=\frac{2i}{\pi}(\eta+\wp(2z_0))\omega. \tag{3.34}$$

The corresponding values of c_n and v_n are given by formulas (2.27) and (2.28) in which the theta function is determined by one parameter $B_{11} = \tau = -\omega/\omega'$, $U = \rho$, $V = 1$. In [7] a formula is also presented for the energy of the base state $\mathscr{H}(\rho)=\min\mathscr{H}$, which we shall omit here because of its complexity. The model described in the introduction (in which $v_n = 0$) is also investigated there.

2. General Peierls Model

The next two assertions, which follow from the results of the preceding section, are important from a physical point of view. First of all, the base state of the system is degenerate. Translation by the vector Z in formulas (2.27), (2.28) does not change the value of $\mathscr{H}$, since $\hat{\mathscr{H}}$ depends only on the end points of the zones E_i and does not depend on γ_j. Such degeneracy is responsible for the so-called Frölich conductivity and occurs in the presence of additional sound (gapless) branches of the spectrum of excitations over the base state of the system. A second consequence is that $\mathscr{H}(\rho)$ — the energy of the base state — is a smooth function of the density of electrons.

Below we shall consider the general Peierls model

$$\mathscr{H}=\frac{1}{2}\oint_{E_1}^{\mu} E\,dp+\varkappa_2 I_2-\varkappa_0 I_0+gW, \tag{3.35}$$

where

$$W=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(\Phi_1(v_n)+\Phi_2(c_n^2)). \tag{3.36}$$

It is found that in the general case the energy of the base state $\mathscr{H}_*(\rho)$ for $g \neq 0$ becomes discontinuous at rational points $\rho = r/q$. The magnitude of the jump has order $\sim g\exp(-\alpha q)$. A number of physical considerations and results of machine computations [9, 50] indicate that degeneracy of the base state is also connected with questions of commensurability (to which a great deal of attention has recently been devoted in the physics literature (see, for example, the survey [51])). We shall prove that for irrational ρ and $0 \leqslant g \leqslant g_\rho$ the base state is indeed degenerate.

Below the functional (3.35) with $g = 0$ (which was considered as the basic example in the preceding section) will be denoted by $\mathscr{H}_0$.

We point out that in a neighborhood of the base state $\mathscr{H}_0$ is not a smooth function, since in this state $E_2 \leqslant \mu \leqslant E_3$. The functional $\mathscr{H}_0$ is smooth relative to variations for which

$$\frac{1}{2\pi}\int_{E_1}^{E_2} dp=U=\text{const}. \tag{3.37}$$

We denote by $\mathscr{H}_0(\rho, U)$ the minimum of $\mathscr{H}_0$ on the set of single-zone states for fixed ρ and U in (3.37). The end points of the zones corresponding to this minimum are determined from Eqs. (3.23)-(3.26). Here if $U > \rho$, then $E_1 \leqslant \mu < E_2$, while if $U < \rho$ then $E_3 < \mu \leqslant E_4$. The derivative $\frac{\partial\mathscr{H}_0(U,\rho)}{\partial U}$ is discontinuous at the point $U = \rho$. It follows from (2.39) that it has right and left limits $h(E_3) > 0$ and $h(E_2) < 0$, respectively, where

$$h(e)=\frac{1}{2\pi i}\oint_{E_1}^{e}\frac{dE}{\sqrt{R(E)}}\oint_{E_2}^{E_3}\frac{\sqrt{R(t)}\,dt}{E-t}. \tag{3.38}$$

Function $h(e)$ decreases monotonically on the segments $[E_1, E_2]$ and $[E_3, E_4]$. Here $h(E_1) = h(E_4) = 0$. On the segment $[E_2, E_3]$ it increases linearly from $h(E_2) < 0$ to $h(E_3) > 0$. Hence, in a neighborhood of $U = \rho$.

$$\begin{gathered}\mathscr{H}_0(U,\rho)=\mathscr{H}_0(\rho)+h_1|U-\rho|+h_2(U-\rho)+O((U-\rho)^2),\\ h_1+h_2=h(E_3),\quad h_2-h_1=h(E_2).\end{gathered} \tag{3.38'}$$

Remark. In the definition of the chempotential given above its value inside a forbidden zone was ambiguous. If we define μ in this case by means of the relation

$$\mu=\frac{\partial\mathscr{H}(\rho)}{\partial\rho},$$

then from (3.38) μ will satisfy the equality $h(\mu) = 0$.

For small g the system (3.35) can be considered as a perturbation of the integrable model with $g = 0$. In order to apply considerations usual in such cases, in proving the main theorem of this section, we shall need formulas for the second variation of $\mathscr{H}_0$ and for the first variation of W.

Substituting (2.27) and (2.28) into (3.36), it may be assumed that W is a function

$$W=W(\{E_i\},\ Z)$$

of the collection $\{E_i\} = (E_1,\dots,E_{2q+2})$, $Z = (z_1,\dots,z_q)$.

We consider the nature of the dependence of W on Z.

Formula (2.29) assigns to each collection $\{E_i\}$ a q-dimensional vector $U(\{E_i\})$ with coordinates $0 < U_1 < \ldots < U_q < 1$. The collection $\{E_i\}$ is called a nonresonance collection if there does not exist an integral vector $r = (r_1, \ldots, r_q)$ such that

$$\langle r, U(\{E_i\}) \rangle = r_0, \tag{3.39}$$

where r_0 is an integer. For resonance collections $R(\{E_i\})$ will denote the group formed by the same $r \in R(\{E_i\})$ for which (3.39) holds for some integer r_0.

LEMMA 3.2. The functional W is equal to

$$W(\{E_i, Z\}) = \sum_{r \in R(\{E_i\})} \mathcal{F}_r \exp(2\pi i < r, Z > + \pi i r_0), \tag{3.40}$$

where

$$\mathcal{F}_m = \int \ldots \int \mathcal{F}(z_1, \ldots, z_q) e^{-2\pi i \langle m, r \rangle} dz_1 \ldots dz_q$$

are the Fourier coefficients of the function

$$\mathcal{F}(Z) = \Phi_1(v(Z)) + \Phi_2(c^2(Z)), \tag{3.41}$$

$$c^2 = \frac{\theta(Z-U)\theta(Z+U)}{\theta^2(Z)} e^{-2I_0}, \quad U = U(\{E_i\}),$$

$$v(Z) = \frac{d}{dt} \ln \frac{\theta(Z+U)}{\theta(Z+U+Vt)}\bigg|_{t=0} + I_1. \tag{3.42}$$

From (2.27), (2.28) it follows that

$$\Phi_1(v_n) + \Phi_2(c_n^2) = \mathcal{F}\left(nU - \frac{U}{2} + Z\right).$$

The limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{F}\left(nU - \frac{U}{2} + Z\right)$$

is easily found by using the Fourier expansion for $\mathcal{F}(z)$.

COROLLARY 1. For nonresonance collections $\{E_i\}$ W does not depend on Z and is equal to

$$W_0(\{E_i\}) = \int \ldots \int \mathcal{F}(Z) dz_1 \ldots dz_q \tag{3.43}$$

COROLLARY 2. If not all the frequencies U_k are rational, then the corresponding level of the functional W is degenerate.

Proof. Under the assumption of the corollary all the resonance relations are dependent. Hence, there is a vector Z_* such that $\langle r, Z_* \rangle = 0$. It then follows from (3.40) that $W(\{E_i\}, Z + tZ_*)$ does not depend on t.

Formula (3.43) defines the "continuous" part of the functional W which for general Φ_1 and Φ_2 is discontinuous at all resonance collections. If $|r|$ is the minimal order of resonance, $|r| = |r_1| + \ldots + |r_q|$, $r \in R(\{E_i\})$, then the magnitude of the jump

$$|W(\{E_i\}, Z) - W_0(\{E_i\})|$$

has order $\mathcal{F}_r$ and decreases with increasing $|r|$ for analytic Φ_i like $\exp(-a|r|)$, where a is a constant.

We denote by $W_*(\{E_i\}) = \min_Z W(\{E_i\}, Z)$. By Corollary 1 it is equal to W_0 almost everywhere. It follows from (3.40) that

$$\int \ldots \int W(\{E_i\}, Z) dz_1 \ldots dz_q = W_0(\{E_i\}).$$

Hence,

$$W_*(\{E_i\}) \leqslant W_0(\{E_i\}).$$

We consider an arbitrary variation ($E_1',\ldots,E_4'$, $e_j^{\pm}$) of the single-zone state $E_1,\ldots,E_4$.

LEMMA 3.3. The variation of W is equal to

$$|\delta W| \leqslant |W_*(\{E_i'\}) - W_*(\{E_i\})| + \sum_{j=1}^{N} f(e_j)\delta e_j, \tag{3.44}$$

$$f(e_j) \leqslant C\exp(-r_j a_1),\ C-\text{const}, \tag{3.45}$$

where r_j is the minimal (positive) integer such that

$$\frac{1}{2\pi}\oint_{E_1}^{e_j} dp = r_j U + r_j',\quad U = \frac{1}{2\pi}\oint_{E_1}^{E_2} dp \tag{3.46}$$

for some integer r_j'.

Proof. Let Ω be the normalized differential on the surface Γ of the function $\sqrt{R(E)}$

$$\Omega = \alpha\frac{dE}{\sqrt{R}},\quad -\oint_{E_2}^{E_3}\Omega = 1,\quad B_{11} = \oint_{E_1}^{E_2}\Omega.$$

The normalized Abelian differentials Ω_1 and Ω_2 on the surface Γ' with branch points ($E_1,\ldots,$ E_4, e^-, e^+) up to terms of order $(\delta e)^2 = (e^+ - e^-)^2$ are

$$\Omega_1 = \Omega,\quad \Omega_2 = \frac{\alpha_1 E + \alpha_2}{\sqrt{R(E)}\,(E-e)}\,dE. \tag{3.47}$$

The constants α_i are determined by the condition that the residue of Ω_2 at the point $E = e$ is equal to 1 (equal to the integral over the contracted cycle a_2), while $\int_{E_2}^{E_3}\Omega_2 = 0$. The matrix of b-periods of the curve Γ' is

$$B_{11}' = B_{11} + O((\delta e)^2), \tag{3.48}$$

$$B_{12}' = 2\int_{E_1}^{e}\Omega, \tag{3.49}$$

$$\exp(\pi i B_{22}) = \tilde{A}(e)\,\delta e + O((\delta e)^2). \tag{3.50}$$

Expanding the corresponding θ-function, we obtain

$$\theta(z_1, z_2) = \theta(z_1) + e^{\pi i B_{22}}\left\{\sum_{m=-\infty}^{\infty}{}' [\exp(2\pi i(mz_1 + z_2) + \pi i(2B_{12}m + B_{11}m^2)) + \exp(2\pi i(mz_1 - z_2) + \pi i(B_{11}m^2 - 2B_{12}m))]\right\} + O((\delta e)^2) = \theta(z_1) + \tilde{A}(e)\,\delta e\,[\theta(z_1 + B_{12})\,e^{2\pi i z_2} + \theta(z_1 - B_{12})\,e^{-2\pi i z_2}] + O((\delta e)^2). \tag{3.51}$$

Hence,

$$\mathscr{F}(z_1, z_2) = \mathscr{F}(z_1) + \delta e\,(F_+(z_1)\,e^{2\pi i z_2} + F_-(z_1)\,e^{-2\pi i z_2}). \tag{3.52}$$

Here $F_\pm(z_1)$ are periodic functions of z_1 depending analytically on $B_{12}(e)$. Substituting (3.52) into (3.40), we find that

$$|\delta W| \leqslant |W_*(\{E_i'\}) - W_*(\{E_i\})| + \sum_{i=1}^{N-1}\delta e_i\left(\sum_{k=1}^{\infty}|F_{kr(e)}^{+}| + |F_{kr(e)}^{-}|\right) + O((\delta e)^2), \tag{3.53}$$

where r(e) is an integer such that

$$\oint_{E_1}^{r(e)} dp \equiv 2\pi r(e)\,U \pmod 1,$$

and $F_m^{\pm}$ are the Fourier coefficients of the functions $F_\pm(z_1)$. The proof of the lemma is complete.

Combining the results obtained, we arrive at the following basic assertion.

THEOREM 3.5. Suppose that Φ_1 and Φ_2 are analytic in some neighborhood of the real axis on which they are positive, and suppose that ρ satisfies the condition $|\rho - m/n| > \alpha/n^2$ for $n > n_0$, where n_0 and α are some constants. Then there exists $g_\rho > 0$ such that for $g < g_\rho$ the energy of the base state $\mathcal{H}_*(\rho) = \min \mathcal{H}$ satisfies the inequality

$$|\mathcal{H}_*(\rho) - \mathcal{H}_*(\{E_i^*\})| < \frac{g^2C^2}{2\Lambda},$$

where E_i^* is the base state for the unperturbed functional $\mathcal{H}_0$, which is given by (3.23)-(3.26) and $\Lambda > 0$ is equal to $\min \Lambda(e)$ in (3.18).

Furthermore,

1. The spectrum of the operator L corresponding to the base state of the system has gaps at the points e_s defined from the condition

$$\frac{1}{2\pi}\int_{E_1}^{e_s} dp \equiv s\rho \pmod 1).$$

2. The width of the gap has order

$$|\delta e_s| \leqslant \frac{gC\exp(-sa_1)}{2\Lambda} + O(g^2).$$

3. The base state is given by formulas (2.27) and (2.28) in which all frequencies have the form

$$U_k = r_k\rho + r_k',$$

where r_k, r_k' are integers.

4. If ρ is irrational, then the base state is degenerate.

Proof. We denote by $V_h \subset \hat{M}$ a neighborhood of $\{E_i^*\}$ consisting of collections $(E_i', e_j^{\pm})$ such that

$$\sum_{j=1}^{N-1}(\delta e_j) + \sum_{l=1}^{4}|\delta E_l| = \varepsilon \leqslant h, \tag{3.54}$$

and we denote by $\overline{V}_h$ the complement of this neighborhood.

Because $\mathcal{H}_0$ has no extremals besides $\{E_i^*\}$ and Φ_1 and Φ_2 are positive, for sufficiently small h we have

$$\min_{\overline{V}_h} \mathcal{H}_* \geqslant \min_{\overline{V}_h} \mathcal{H}_0 \geqslant \mathcal{H}_0(\rho) + \frac{\Lambda}{2}h^2.$$

If $gW_*(\{E_i\}) \leqslant \frac{\Lambda}{2}h^2$, then the minimum of $\mathcal{H}$

$$\mathcal{H}_*(\rho) \leqslant \mathcal{H}_0(\rho) + gW(\{E_i\})$$

is achieved in the neighborhood V_h.

Let ρ be as in the condition of the theorem; then W is differentiable at $\{E_i^*\}$ relative to all variations including those changing the period

$$\frac{1}{2\pi}\oint_{E_1}^{E_2} dp = U.$$

Indeed, if $|U - \rho| < \varepsilon$, $U = m/n$, then $n > \sqrt{\alpha/\varepsilon}$.

Hence

$$|W_*(\{E_i\}) - W_0(\{E_i\})| < C_1 \exp\left(-\sqrt{\frac{\alpha}{\varepsilon}}\,a_1\right)$$

and $W(\{E_i\})$ has a derivative with respect to U equal to the derivative of W_0.

Suppose that g satisfies the condition

$$g \left.\frac{\partial W}{\partial U}\right|_{E_l=E_l^*} \leqslant \min(h_1 \pm h_2),$$

where the h_i are defined in (3.38'). From (3.38') it then follows that $\min \mathscr{H}$ is achieved for $U = \rho$.

Let $(E_i^!, e_j^{\pm})$ be the base state of the system. Since it belongs to V_h, it is possible to use the results of Lemmas 3.1 and 3.3. We have

$$\mathscr{H}_0 + gW \geqslant \mathscr{H}_0(\rho) + \frac{\Lambda}{2}\varepsilon^2 + gW_*(\{E_i^*\}) - gC\varepsilon \geqslant \mathscr{H}_0(\rho) + gW_*(\{E_i^*\}) - \frac{g^2C^2}{2\Lambda},$$

where ε is defined in (3.54). From the "diagonality" of $\delta^2\mathscr{H}_0$ it is possible to obtain a more precise estimate for the width of each new gap

$$\delta e_j \sim \frac{gf(e_j)}{2\Lambda(e_j)} \leqslant g\frac{f(e_j)}{2\Lambda} < g\frac{C\exp(-r_j a_1)}{2\Lambda}.$$

Thus, all parts of the theorem except the last have been proved. The degeneracy of the base state for irrational ρ is given by Corollary 2 of Lemma 3.2.

We note that the zone structure of the Schrödinger difference operator corresponding to the base state of the Peierls model (which, by what has been proved, is quasiperiodic with the two periods ρ and 1) is altogether analogous to the structure of the spectrum of the Sturm–Liouville operator with an almost periodic potential which was obtained in [56] (for application of the results of [56] in path-integral approximations in the Peirls problem see [51]).

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