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SPECTRAL THEORY OF FINITE-ZONE NONSTATIONARY SCHRÖDINGER OPERATORS. A NONSTATIONARY PEIERLS MODEL

I. M. Krichever

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In [1, 2] there has been exposed a general scheme of the construction of periodic and quasiperiodic solutions of nonlinear equations which allow commutational representations. This scheme has been worked out by the author on the basis of the analysis of algebraically geometrical ideas, which were used earlier in the cycle of papers by S. P. Novikov, B. A. Dubrovin, V. B. Matveev, and A. R. Its. Those works have been devoted to an effective spectral theory of finite-zone Schrödinger operators (Sturm–Liouville), and to the construction, based on this theory, of a wide class of quasiperiodic solutions of the Korteweg–de Vries equation (KdV). (The first step of the development of the theory of finite-zone integration is presented in [3, 4].)

The scheme given in [1, 2] allows one to obtain at first not only periodic and quasiperiodic solutions of spatially two-dimensional equations of the Kadomtsev–Petviashvili type (KP), but also a natural and methodologically convenient presentation of the results of the previous finite-zone theory of one-dimensional equations (KdV, nonlinear Schrödinger, sine-Gordon). It turns out that within such an approach the algebraic-geometrical language permits (if only the problem of the construction of solutions of these equations is considered) to exclude from the considerations, practically completely, the spectral theory of auxiliary linear operators. Evidently, for this reason, an insufficient attention has been paid to the finite-zone spectral theory in the later development of the methods of "finite-zone integration" of nonlinear equations (cf. reviews [5-11]).

The theory has attracted growing interest since its effective applications to problems of the Peierls–Frolich type had been presented in [12, 14]. On the basis of the Peierls model there are usually formulated theories describing the characteristic singularities of quasi-one-dimensional conductors. In [12-14] "single-zone" extremals have been found in various continuous approximations of this theory. Those investigations have been continued in [15-18], where a discrete model was integrated whose limits were all of its previously studied continuous variants. It has been proved in [15] that in this model the "multizone" extremals do not occur, and thus at first it has been demonstrated that the "single-zone" extremals are the ground states of the system. In [16-18] the spectrum of excitations of the ground state has been found, and also the nonintegrable disturbations of the model have been studied.

In all the cited papers only the stationary states of the Peierls model have been considered. The study of nonstationary solutions needed a further development of a "spectral" theory for the nonstationary Schrödinger operator with a biperiodic potential, $u(x, t) = u(x + L, t) = u(x, t + T)$.

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The corresponding operator

$$M = i\partial_t - \partial_x^2 + u(x,t), \quad \partial_t = \partial/\partial t, \partial_x = \partial/\partial x \tag{1}$$

appears in the commutational representation for the KP equation, and therefore it has been actively considered in [1, 2], where also the integrable potentials u(x, t) have been constructed. Neither in those papers, nor in the others (to the author's best knowledge) has any effort been made to formulate the problems connected with the spectral interpretation of the obtained results. Many of such problems became clear to the author during his numerous discussions with I. E. Dzyaloshinskii and S. A. Brazovskii on the nonstationary Peierls problem.

It seems that the "spectral" theory of the nonstationary Schrödinger operator is interesting for its own sake also from the mathematical point of view.

In the first section a short formulation of the construction of algebraic-geometrical potentials of the nonstationary Schrödinger operator is given. There is found a "spectral measure" for those potentials, defining the resolution of identity with respect to the Beiker—Akhiezer functions (Bloch functions), corresponding to the "real Fermi curves."

In the second section there are derived nonlinear relations between the solutions of the nonstationary Schrödinger equation, $M\psi = 0$, and the potential u(x, t). In the stationary case the corresponding relations are well known, and they are used as deriving ones in the proof of the "traces formulas." In the final part of the section these relations have been used in the construction of solutions of the equations presented in [30, 31], describing in various approximations the interactions between the long and short waves in plasma.

The concluding section contains a brief formulation of the nonstationary Peierls problem and some constructions of solutions of its integrable variants. More detailed investigations of the solutions, and of the conditions of their physical applicability have been carried out in the joint paper by I. E. Dzyaloshinskii and the author (to appear). In that paper an approximation in the Peierls model has been studied, which corresponds to the "smallness" of the forbidden zone. Under such an approximation the nonstationary Schrödinger operator is transformed into the nonstationary Dirac operator.

1. SPECTRAL DECOMPOSITION OF ALGEBRAIC-GEOMETRICAL OPERATORS OF RANK 1

The construction of complex quasiperiodic "finite-zone" potentials of the nonstationary Schrödinger operator has been presented in [1] (for details see [2]). The conditions for the reality and the nonsingularity of u(x, t) have been found in [7, 10] (originating from the ideas of [19]). We begin with the survey of these results and of some of their generalizations (containing rational and soliton potentials), which are useful in the further applications to the Peierls model.

Let Γ be a nonsingular algebraic curve of the type g with a fixed point P_0 in whose neighborhood there is chosen a local parameter $k^{-1}(P)$, $k^{-1}(P_0) = 0$. Moreover, let additional data be given on Γ: the pairs of points $\varkappa_i^{\pm}$, $i = 1,\ldots,N$, and a collection of points λ_j, $j = 1,\ldots,M$.

In a standard way one can prove that for any fixed choice of points $\gamma_1,\ldots,\gamma_G$, $G = g + N + M$ in generic position there exists a unique function $\psi(x, t, P)$, called the Beiker—Akhiezer function, such that:

1°. It is meromorphic beside P_0 and has poles only at the points γ_s.

2°. $\psi(x, t, P)$ satisfies the conditions

$$\psi(x,t,\varkappa_i^+) = \psi(x,t,\varkappa_i^-), \tag{2}$$

$$d\psi(x,t,P)|_{P=\lambda_j} = 0. \tag{3}$$

3°. In the neighborhood of P_0 the function $\psi(x, t, P)$ has the form:

$$\psi(x,t,P) = \exp(ikx + ik^2t + i\Phi(k))\left(1 + \sum_{s=1}^{\infty} \xi_s(x,t)k^{-s}\right), \quad k = k(P), \tag{4}$$

where $\Phi = \sum_{j=1}^{G} \varphi_j k^j$ (for brevity, in the explicit expression for ψ the dependence on the auxiliary parameters φ_j has been omitted. Their role will be discussed below).

As it follows from the results of [2] (we preserve the notation taken from [2]) the function $\psi(x, t, P)$ can be sought in the form

$$\psi(x,t,P)=\exp\Big(i\int^{P} x\Omega^{(2)}+t\Omega^{(3)}+\sum_j \varphi_j\Omega^{(j)}\Big)\sum_{m=1}^{N+M}\alpha_m(x,t)\,\frac{\Theta(A(P)+xU^{(2)}+tU^{(3)}+\sum\varphi_jU^{(j)}+\zeta_m)}{\Theta(A(P)+\zeta_m)}, \tag{5}$$

where $\Theta(v_1,\ldots,v_g)$ is the theta-Riemann function; $2\pi U^{(j)}$ are the vectors of the b-periods of Abelian differentials of the second kind with null a-periods, which have singularities of the form $(d(k^{j-1}))$ in the neighborhood of P_0; the vectors ζ_m are equal to $K-A(\gamma_1)-\ldots-A(\gamma_{g-1})-A(\gamma_{g-1+m})$, where $A\colon \Gamma\to J(\Gamma)$ is the Abel mapping, and K is the vector of the Riemann constants.

The coefficients $\alpha_m(x, t)$ in (5) are determined by a system of linear equations resulting from the substitution of (5) into (2) and (3).

THEOREM 1. The function $\psi(x, t, P)$ satisfies the equation

$$(i\partial_t-\partial_x^2+u(x,t))\,\psi(x,t,P)=0, \tag{6}$$

where

$$u(x,t)=2i\partial_x\xi_1(x,t). \tag{7}$$

Proof. The proof is fully analogous to that in [1] (where N = M = 0), and exploits only the uniqueness of $\psi(x, t, P)$.

Remark. For N = M = 0 we obtain the quasiperiodic potential [2]

$$u(x,t)=-2\partial_x^2\ln\Theta(U^{(2)}x+U^{(3)}t+\zeta+\Phi),\quad \Phi=\sum\varphi_jU^{(j)}. \tag{8}$$

For g = 0, and M = 0 the construction gives (after the replacement of t by y, and φ_3 by t) the soliton wave fronts of the Kadomtsev–Petviashvili equation, obtained in [20]. For g = 0, and N = 0 we obtain the rational solutions of KP (see [21]), which for a proper choice of parameters coincide with the rational solitons of KP, decreasing in all directions, and which we have obtained in [22] with the help of the inverse scattering problem method.

In the general case the conditions (2) and (3) can be replaced by an arbitrary system of N + M linear conditions on the values of ψ and their derivatives with respect to P (of arbitrary order) at different points and with constant coefficients. (For further generalizations see [23].)

Remark 2. The parameters γ_s and φ_j are not independent. Usually, γ_s are chosen as the independent parameters, while φ_j are assumed to be zero. It will be clear further that the passage to the parameters φ_j (for a particular choice of γ_s) is essentially more efficient in the description of real and nonsingular potentials u(x, t).

Let on the curve Γ be defined an anti-involution $\tau\colon \Gamma\to\Gamma$ of the separating type, i.e., such that its invariant ovals $\sigma_1,\ldots,\sigma_l$ divide Γ into two segments $\Gamma^{\pm}$. Moreover, let τ leave the point P_0 invariant and transform the local parameter in its neighborhood into $\bar{k}=\tau^*(k)$. The collections of the points $\varkappa_i^{\pm},\lambda_j$ should satisfy the conditions: $\tau(\lambda_i)=\lambda_j$; $\tau(\varkappa_i^+)=\varkappa_i^-$ and $\varkappa_i^{\pm}\in\Gamma^{\pm}$, or $\tau(\varkappa_i^{\pm})=\varkappa_j^{\pm}$, $i\neq j$, and in the latter case both points $\varkappa_i^{\pm}$ belong to only one of the segments $\Gamma^{\pm}$ (the pair $\varkappa_i^{\pm}$ will be called of the first or of the second type, respectively). We shall index the pairs $\varkappa_i^{\pm}$ of the first type with the numbers $i = 1,\ldots,N_1$, $N_1\leqslant N$.

A divisor $D=\{\gamma_s\}$ is called admissible if $\gamma_1,\ldots,\gamma_G$, $\tau(\gamma_1),\ldots,\tau(\gamma_G)$ are the zeros of a certain differential Ω which has the second-order poles with null residua at the points λ_j and P_0, and the simple poles at the points $\varkappa_j^{\pm}$, with the residua $\pm\alpha_j i$. If $\varkappa_j^{\pm}$ is a pair of the first type, then $\alpha_j=\bar{\alpha}_j>0$. Fixing on σ_i the natural orientation induced from the domain Γ^+ we assume that $\Omega|_\sigma\geqslant 0$.

LEMMA 1. If the above assumptions are satisfied and $\varphi_j=\bar{\varphi}_j$, then by virtue of Theorem 1 the potential u(x, t) defined by them is real and nonsingular.

(For N = M = 0 the sufficiency of these conditions has been shown in [7, 10].)

Proof. Denote by $\psi^+(x, t, P)$ the function

$$\psi^+(x,t,P)=\bar{\psi}(x,t,\tau(P)). \tag{9}$$

From the assumptions of the lemma it follows that the differential

$$\tilde{\Omega}=\psi(x,t,P)\psi^+(x,t,P)\,\Omega \tag{10}$$

is meromorphic with the null residua at the points λ_j, and with the residua at the points $\varkappa_i^{\pm}$, with opposite signs. Since the sum of all residua of a meromorphic differential is null, then also the residuum of $\tilde{\Omega}$ at the point P_0 is null. From this we have $\xi_1 + \bar{\xi}_1 = 0$, and thus by virtue of (7) the reality of u(x, t) has been proved.

A singularity of the potential may occur only for those values of x_0, and t_0 for which one of the zeros of $\psi(x_0, t_0, P)$ appears at P_0. But in this case the differential $\tilde{\Omega}(x_0, t_0, P)$ is regular at P_0, and thus we can consider its integral over the invariant ovals. By the positivity of Ω on σ_i, and since we have $\psi^+ = \bar{\psi}$ on σ_i, the integral

$$\int_{\{\sigma_i\}} \tilde{\Omega}(x_0, t_0, P) > 0 \tag{11}$$

is positive. On the other hand, it is equal to the sum of residua at the points $\varkappa_i^+$, where $\varkappa_i^{\pm}$ is the pair of the first type, and moreover it should be negative by the assumptions of the lemma. The resulting contradiction ends the proof of the lemma.

The condition imposed on $\{\gamma_s\}$ is rather complicated. However, we recall again that it is enough to find only one of such collections, and afterwards to find all real nonsingular potentials corresponding to this curve and to the data $\lambda_j, \varkappa_i^{\pm}$, with the help of the variation of the real parameters φ_j, with a fixed set of poles.

We omit the proof of the following simple result.

LEMMA 2. There exist admissible divisors on every curve Γ with an involution of the separable type.

The real ovals $\{\sigma_i\}$ and the pairs $\varkappa_i^{\pm}$, $i = 1,\ldots,N_1$ of the first type will be called the spectrum of the corresponding nonstationary Schrödinger operator. The differential Ω entering the definition of an admissible collection of poles of the Beiker–Akhiezer function will be called the spectral density. This terminology is justified by the following theorem.

THEOREM 2. Let the parameters of Γ, P_0, λ_j, $\varkappa_i^{\pm}$, $\gamma_1, \ldots, \gamma_G$, defining the Beiker–Akhiezer function satisfy all of the above limitations (which guarantee the reality and the nonsingularity of the potential u(x, t)). Then

$$\delta(x-y) = \int_{\{\sigma_i\}\setminus P_0} \psi(x, t, P)\psi^+(y, t, P)\Omega - \sum_{i=1}^{N_1} r_i \psi(x, t, \varkappa_i^+)\psi^+(y, t, \varkappa_i^+), \tag{12}$$

$$r_j = 2\pi i \operatorname{res}_{\varkappa_j^+}\Omega = -2\pi\alpha_j. \tag{13}$$

Proof. Denote by $\partial\Gamma^+$ the boundary Γ^+ (given by the collection of σ_i's), and by $\partial\Gamma_\varepsilon^+$ the cycle obtained by the deformation of $\partial\Gamma^+$ corresponding to the path of radius ε around the point P_0 inside Γ^+.

The differential $\psi(x, t, P)\psi^+(y, t, P)\Omega$ is regular on $\partial\Gamma_\varepsilon^+$. From the definition of Ω it follows that

$$\int_{\partial\Gamma_\varepsilon^+} \psi(x, t, P)\psi^+(y, t, P)\Omega = \sum r_i \psi(x, t, \varkappa_i^+)\psi^+(y, t, \varkappa_i^+). \tag{14}$$

The difference of the cycles $\partial\Gamma_\varepsilon^+ \setminus \partial\Gamma^+$ is the cycle C_ε, which is the boundary of the upper semicircle of the ε-neighborhood of the point P_0. In the neighborhood of P_0, as it follows from (4), the function $\psi(x, t, P) \times \psi^+(y, t, P)$ has the form

$$e^{ik(x-y)}\left(1 + \sum_{s=2}^{\infty} J_s(x, y, t) k^{-s}\right). \tag{15}$$

From this it follows that

$$\int_{C_\varepsilon\setminus P_0} \psi(x, t, P)\psi^+(y, t, P)\Omega = \delta(x-y). \tag{16}$$

Summing up (14) and (16) we obtain the desired equality (12).

As we mentioned before, for M = 0, and N = 0, the potentials u(x, t), obtained in the present construction, are quasiperiodic.

The conditions, which distinguish the potentials periodic in x and t, can be formulated in the following way.

A meromorphic differential on a smooth curve Γ will be called absolutely normed if its integrals with respect to all cycles are real. The differentials of the quasienergy dE and the quasimomentum dp are defined as absolutely normed potentials, which are holomorphic besides P_0, and in the neighborhood of this point have the following form

$$dp = dk\,(1 + O(k^{-2})), \quad dE = dk^2\,(1 + O(k^{-3})). \tag{17}$$

It is easy to see that such differentials exist and are unique.

LEMMA 3. If for any cycle on Γ there are satisfied the conditions

$$\text{a)} \oint dp = \frac{2\pi l}{L}, \quad \text{b)} \oint dE = \frac{2\pi m}{T}, \quad m, l \text{ are integers,} \tag{18}$$

then (for $M = N = 0$) the finite-zone potentials $u(x, t)$, corresponding to this curve, are periodic, and the Beiker–Akhiezer function is the Bloch function

$$\psi(x + L, t, P) = w_1(P)\,\psi(x, t, P), \tag{19}$$

$$\psi(x, t + T, P) = w_2(P)\,\psi(x, t, P). \tag{20}$$

Proof. From conditions (18) it follows that the functions

$$w_1(P) = \exp\left(iL\int_Q^P dp\right), \; w_2(P) = \exp\left(iT\int_Q^P dE\right) \tag{21}$$

are well defined on Γ (i.e., they do not depend on the choice of the integration path between the initial point Q and P). Beside the point P_0 they are holomorphic.

Thus the equalities (19) and (20) follow from the fact that their left- and right-hand sides have the same analytic properties. Since $w_1(P)$ does not depend on x and t, then, by (19) and (20), $u(x, t)$ is periodic.

Remark. Up to now the parameter k^{-1} has not been fixed in the neighborhood of P_0, because the Beiker–Akhiezer function and the potential $u(x, t)$ depend on its choice in a very simple way. With the change of k into $k + \alpha_0 + \alpha_1 k^{-1} + \ldots$, the functions ψ and u are transformed into

$$\psi \to \psi(x + 2\alpha_0 t, t, P)\, e^{i(\alpha_0 x + \alpha_0^2 t + 2\alpha_1 t)}, \quad u \to u\,(x + 2\alpha_0 t, t) - 2\alpha_1.$$

Nevertheless, for the further goals, it is convenient to choose the local parameter so that $dp = dk$. This fixes it uniquely up to proportionality, which is equivalent to the scaling group $x, t \to \lambda x, \lambda^2 t$, and to the shifts $k \to k + \alpha_0$.

As it follows from the proved statement the coefficients ξ_s of the decomposition of (4) with respect to the local parameter are quasiperiodic functions of x and t with ($M = N = 0$). In the general case ξ_s are uniformly bounded in x on the whole direct line.

Let the curve Γ satisfy conditions (18a). Denote by $P_n(w)$ the points on Γ at which the function $w_1(P_n) = w$. Notice that $P_n \to P_0$, $n \to \infty$.

For any function $f(x)$ satisfying the condition

$$f(x + L) = w\,f(x), \tag{22}$$

we consider the series

$$\sum_n c_n(t)\,\psi(x, t, P_n(w)) = \sum c_n \psi_n, \tag{23}$$

where

$$c_n = \frac{1}{L}\int_0^L f(y)\,\frac{\psi^+(y, t, P_n(w))}{\langle \psi_n \psi_n^+ \rangle_x}\,dy, \tag{24}$$

and $\langle\cdot\rangle_x$ are the averages over the period.

THEOREM 3. If $f(x)$ is continuously differentiable, then the series (23) converges to $f(x)$.

(In general the conditions of convergence of the series (23) are the same as for the usual Fourier series.)

Proof. Let R_N (for sufficiently large N's) be the circle of radius $((2N+1)\pi L)^{-1}$ in the neighborhood of P_0, and N is an integer.

Let us consider the integral

$$S_N = \int\limits_{R_N} \int\limits_0^L f(y) \frac{\psi(x,t,P)\,\psi^+(y,t,P)}{(1-ww_1^{-1}(P))} \Omega\, dy. \tag{25}$$

On the one hand, it is equal to the sum of the residua of the integrand inside the contour R_N, which, by (30) and because $iLdp = d\ln w_1$, coincides with the sum of those terms in the series (23) for which P_n lies beyond R_N. On the other hand, using the decomposition (4) for ψ, and the fact that in the neighborhood of P_0 we have $w_1 = e^{ikL}$, it is easy to show that S_N coincides up to $O(N^{-1})$ with the sum of the first N terms of the usual Fourier series for f(x). From this, letting N go to ∞, we obtain the assertion of the theorem.

LEMMA 4. If M = 0 and all the pairs $\varkappa_i^{\pm}$, occurring in conditions (2), are of the first type, then the potential u(x, t) corresponding to these data, with $x + Vt \to \infty$, where

$$V \neq \frac{\operatorname{Im} E(\varkappa_i^+)}{\operatorname{Im} p(\varkappa_i^+)}, \tag{26}$$

has the form

$$u(x,t) = -2\partial_x^2 \ln \Theta(U^{(2)}x + U^{(3)}t + \zeta_\pm + \Phi), \tag{27}$$

$$\zeta_\pm = K - \sum_{s=1}^{g+N} A(\gamma_s) + \sum_{i=1}^{N} A(\varkappa_i^{\mp}). \tag{28}$$

From conditions (2) it follows that $\psi(x, t, P_0)$, for $x + Vt \to \pm\infty$, tends to the functions $\psi_\pm(x, t, P)$, which have at P_0 the form (4). Beside P_0 they have poles at the points γ_s and zeros at the points $\varkappa_i^{\pm}$. By the Riemann–Roch theorem there exist unique meromorphic functions $h_\pm(P)$ with poles at γ_s and zeros at $\varkappa_i^{\pm}$, $h_\pm(P_0) = 1$. Beside $\varkappa_i^{\pm}$, the functions $\psi_\pm$ have yet g zeros. Let us denote them by $\gamma_1^{\pm}, \ldots, \gamma_g^{\pm}$. By the Abel theorem

$$\sum_{i=1}^{g} A(\gamma_i^{\pm}) = \sum_s A(\gamma_s) - \sum_j A(\varkappa_j^{\pm}). \tag{29}$$

The functions $\psi_\pm(x, t, P)h_\pm^{-1}(P)$ are the Beiker–Akhiezer functions with g poles at $\gamma_i^{\pm}$. Therefore formula (27) follows from (29) and (8).

Remark. It is natural to call the above potentials "multisoliton with finite-zone background." Another form of the "dressing" of finite-zone potentials with soliton potentials, based upon the Backlund–Darboux transform, has been presented in [25].

THEOREM 4. If on the curve Γ the differentials dp and dE do not have joint zeros (this condition is satisfied for curves in general position), then the spectral measure Ω, corresponding to a multisoliton potential with finite-zone background is equal to

$$\Omega = \frac{dp}{\langle\psi\psi^+\rangle_x} = i\frac{dE}{\langle\psi'\psi^+ - \psi\psi^{+'}\rangle_t}, \quad \psi' = \partial_x\psi \tag{30}$$

(where $\langle\cdot\rangle_x$ and $\langle\cdot\rangle_t$ denote the averages over x and t, respectively).

Proof. Let $\psi = \psi(x, t, P)$ and $\tilde\psi = \psi(x, t, \tilde P)$, where P and $\tilde P$ are arbitrary points of Γ. Then, from (6) and from the reality of u(x, t) it follows that

$$i\partial_t(\psi\tilde\psi^+) = \partial_x(\psi'\tilde\psi^+ - \psi\tilde\psi^{+'}). \tag{31}$$

Averaging this equality with respect to (x, t) and letting $\tilde P \to P$, we obtain the following equality:

$$i\,dE\,\langle\psi\psi^+\rangle_x = dp\,\langle\psi'\psi^+ - \psi\psi^{+'}\rangle_t. \tag{32}$$

Notice that, as it has been shown in the proof of the preceding lemma, the points $\varkappa_i^{\pm}$ correspond to bound states and

$$\langle\psi\psi^+\rangle_x\big|_{P=\varkappa_i^{\pm}} = \langle\psi'\psi^+ - \psi\psi^{+'}\rangle_t\big|_{P=\varkappa_i^{\pm}} = 0. \tag{33}$$

In addition to these N zeros, $\langle\psi\psi^+\rangle_x$ has also 2g zeros more, which, by (32) and because dp and dE do not have common zeros, must coincide with the zeros of dp. Consequently, the differential $dp/\langle\psi\psi^+\rangle_x$ has poles at the points $\varkappa_i^{\pm}$, zeros at the points $\gamma_1, \ldots, \gamma_{g+N}, \tau(\gamma_1), \ldots, \tau(\gamma_{g+N})$, and the desired form in the neighborhood of P_0.

The author has been informed by B. A. Dubrovin and S. M. Natanson that they have recently obtained another proof of the sufficiency of the conditions on the divisor of the poles of ψ leading to real nonsingular potentials of the Schrödinger operator. Moreover, they have proved the necessity of the existence of an antiinvolution τ.

The values of the quasimomentum $p = p(E)$ for each value of the quasienergy (for a given local choice of branches of these multivalued functions) determine functionals over the set of finite-zone potentials. In the stationary case the coefficients of the expansion of p with respect to powers of $k^{-1} = E^{-1/2}$ in the neighborhood of P_0, $p=k+\sum_s p_s k^{-s}$ possess local densities, i.e., $p_s=\int I_s([u])\,dx$, where $I_s([u]')$ are differential polynomials. In the nonstationary case the situation is different.

THEOREM 5. The variation of the quasimomentum δp under the variation $\delta u(x, t)$ of

$$\delta p=\frac{1}{TL}\iint\frac{\delta u(x,t)\,\psi(x,t,P)\,\psi^+(x,t,P)}{\langle\psi'\psi^+-\psi\psi^{+'}\rangle_t}\,dx\,dt. \tag{34}$$

We omit the proof of this theorem because it is completely parallel to its standard variant [27].

2. NONLINEAR RELATIONS IN THE SPECTRAL THEORY OF THE FINITE-ZONE NONSTATIONARY SCHRÖDINGER OPERATORS

In this section we shall derive nonlinear relations between the algebraic-geometrical potentials of the linear Schrödinger equation and its solutions. As it will be seen further, these relations can be used for the construction of the solutions of some nonlinear equations of mathematical physics (a similar approach to the construction of the finite-zone solutions of the nonlinear Schrödinger equation has been earlier used in [29]).

Let $\psi(x, t, k)$ be a formal solution of Eq. (6) of the form (4). Equation (6) is equivalent to the following system of equations on the coefficients $\xi_s(x, t)$:

$$i\dot{\xi}_s-2i\xi_{s+1}'-\xi_s''+u\xi_s=0. \tag{35}$$

Consider the series

$$\psi(x,t,k)\,\overline{\psi}(x,t,\overline{k})=1+\sum_{s=2}^{\infty}J_s(x,t)\,k^{-s}. \tag{36}$$

The coefficients J_s of this series are the following polynomials of ξ_l, $\bar{\xi}_l$:

$$J_2=\xi_2+\bar{\xi}_2+|\xi_1|^2,\quad J_3=\xi_3+\bar{\xi}_3+\xi_1\bar{\xi}_2+\bar{\xi}_1\xi_2,\\ J_4=\xi_4+\bar{\xi}_4+\xi_1\bar{\xi}_3+\bar{\xi}_1\xi_3+|\xi_2|^2 \tag{37}$$

etc.

In the stationary case these coefficients are equal to $J_s = \delta I_s/\delta u$, the variational derivatives of the KdV integrals, and they are differential polynomials of $u(x)$ [3, 4].

We shall find analogous expressions for J_s in the nonstationary case. Since $\xi_1 + \bar{\xi}_1 = 0$, then from Eq. (35) for $s = 1$ we get

$$\dot{\xi}_1=\xi_2-\bar{\xi}_2,\quad i(\xi_2'+\bar{\xi}_2')+\xi_1''-i(\xi_1^2)'=0. \tag{38}$$

Consequently,

$$J_2=i\xi_1'+A(t)=\frac{u}{2}+A(t). \tag{39}$$

Similarly, from Eq. (42) for $s = 2$ we derive the equality

$$\partial_x J_3=\frac{1}{2}\dot{A}+i(\dot{\xi}_1)'. \tag{40}$$

We assume also that ξ_s are bounded on the whole axis in x (which always holds in our constructions). Then from (40) we have $\dot{A} = 0$ and $A = \text{const}$. Integrating (40), we get

$$J_3 = i\dot{\xi}_1 + B(t). \tag{41}$$

Finally, for s = 3, from (42), after a simple computation and using the previous equalities we obtain

$$\partial_x J_4 = \frac{3}{4} i\dot{\xi}_1 - \frac{i}{4}\xi_1'''' - 3\xi_1'\xi_1'' + \frac{1}{2}\dot{B}. \tag{42}$$

For the further use we need only the above expressions. Considering the remaining coefficients J_s, it is natural to assume that $\partial_x^{s-2} J_s$ is a differential polynomial of u. (Perhaps, in the proof of this statement it might be necessary to apply the hierarchy of the KP equation, based on the introduction of the τ-function, presented in [35].)

The algebraic-geometrical potentials u(x, t) are characterized among all potentials by the fact that only a finite number of the coefficients J_s is linearly independent.

Denote by $\mathscr{L} = \mathscr{L}(\Gamma, \lambda_j, \varkappa_i^{\pm})$ the linear space of meromorphic differentials, which have poles only of at most second order with null residua at the points λ_j, and the simple poles at the points $\varkappa_i^{\pm}$ with opposite residua. The dimension of this space is g + N + M = G.

Since

$$\psi(x, t, P)\psi^+(x, t, P)\,\Omega - dp \in \mathscr{L}(\Gamma, \lambda_j, \varkappa_i^{\pm}),$$

then among the coefficients J_s there are at most G linearly independent.

In the case of general position, as for the basic differentials we can take the differentials Λ_s, normalized by the conditions

$$\Lambda_s = dk\,(k^{-s} + O(k^{-G-2})), \quad s = 2, \ldots, G+1. \tag{43}$$

In this basis the decomposition coefficients

$$\psi\psi^+\Omega = dp + \sum_{s=2}^{G+1} \Lambda_s \hat{J}_s(x, t) \tag{44}$$

coincide with the first coefficients of the decomposition

$$\frac{\psi(x, t, k)\,\psi^+(x, t, k)}{\langle\psi\psi^+\rangle_x} = 1 + \sum_{s=2}^{\infty} \hat{J}_s(x, t)\,k^{-s}. \tag{45}$$

Notice, that if $\langle\xi_1\rangle_x = 0$, then

$$\hat{J}_2 = J_2, \quad \hat{J}_3 = J_3, \quad \hat{J}_4 = J_4, \tag{46}$$

where J_2, J_3, J_4 are defined above.

As it will be seen below, by a particular choice of the parameters of Γ, P_0, λ_j, $\varkappa_i^{\pm}$ we can obtain from (46) equalities which together with Eq. (8) give a system of equations describing the interaction between the short and long waves in plasma, in various approximations.

Suppose that there exists on Γ a function h(P), holomorphic beside the points P_0, P_1, $\tau(P_1) = P_1$. At the last point it has a simple pole. Let, moreover, the parameters $\varkappa_i^{\pm}$ and λ_j be chosen in such a way that

$$h(\varkappa_i^+) = h(\varkappa_i^-), \quad dh|_{p=\lambda_j} = 0. \tag{47}$$

Denote by $\Phi(x, t)$ the value of the Beiker–Akhiezer function at the point P_1: $\Phi(x, t) = \psi(x, t, P_1)$.

<u>THEOREM 6.</u> If h(P) has at the point P_0 a pole of the second order

$$h(P) = k^2 + \alpha k + O(1), \tag{48}$$

then $\Phi(x, t)$ satisfies the relation:

$$r(|\Phi|^2)_x = \frac{1}{2}(\dot{u} + \alpha u_x), \tag{49}$$

$$r = \operatorname{res}_{P_1} h\Omega. \tag{50}$$

If h(P) has a third order pole at P_0, and

$$h(P) = k^3 + \beta k + O(1), \tag{51}$$

then

$$r(|\Phi|^2)_{xx}=\frac{3}{8}\ddot{u}-\frac{1}{8}(u_{xxx}-3uu_x)_x-\beta u_{xx}. \tag{52}$$

Proof. It follows from (47) that the sum of the residua of the differential

$$\Lambda = h(P)\,\psi(x,t,P)\,\psi^+(x,t,P)\,\Omega(P) \tag{53}$$

at all points except P_0 and P_1 is null. Therefore

$$\mathrm{res}_{P_1}\Lambda = -\,\mathrm{res}_{P_0}\Lambda. \tag{54}$$

The left-hand side of this equality coincides with $r|\Phi|^2$, and the right-hand side in the first case coincides with $J_3 + \alpha J_2$, and with $J_4 + \beta J_2$ in the latter. From (54) and the formulas (39)-(42) the equalities (49)-(52) follow.

COROLLARY. Formula (8) and the formula

$$\Phi(x,t)=\exp\Big(i\int^{P_1}(x\Omega^{(2)}+t\Omega^{(3)}+\Sigma\varphi_j\Omega^{(j)})\Big)\frac{\theta(A(P_1)+xU^{(2)}+tU^{(3)}+\zeta_0+\Phi)\,\theta(\zeta_0+\Phi)}{\theta(A(P_1)+\zeta_0+\Phi)\,\theta(xU^{(2)}+tU^{(3)}+\zeta_0+\Phi)} \tag{55}$$

give quasiperiodic solutions of the systems (6) and (49), and (6) and (52), in the first and in the latter case, respectively.

The system (6), (49) has been presented in [30], where also its integrability has been shown. The system (6), (52) has been obtained in [31], where also a simpler soliton solution has been found. A family of soliton solutions has been found in [32]. In [33] the L, A, B-triple has been found for the system (6), (52). Periodic, finite-zone solutions of both systems have not been constructed till now.

3. A NONSTATIONARY PEIERLS MODEL

The decomposition theorem proved in Sec. 1 shows that the real ovals of the curve Γ and the pairs $\varkappa_i^{\pm}$ of the first type correspond to the "single-valued" states of electron. Introducing the occupation numbers $c(P), P\in\{\sigma_i\}$; c_j, $j = 1,\dots,N_1$ of these states, we can represent the Lagrangian of the Peierls model in the form

$$\mathcal{L}=\int_0^L dx\Big[\int_{\{\sigma_i\}}c(P)\,\Omega(P)\,\psi^+(i\partial_t-\partial_x^2+w_x)\psi-\sum_{i=1}^{N_1}r_ic_i\psi_i^+(i\partial_t-\partial_x^2+w_x)\psi_i-\frac{M}{2}(w_t)^2+\frac{g}{2}(w_x)^2\Big], \tag{56}$$

where r_i is defined by formula (13), $\psi_i=\psi(x,t,\varkappa_i^+)$.

By virtue of the Pauli principle the occupation numbers should satisfy the condition $0 \leqslant c(P) \leqslant 2$, $0 \leqslant c_i \leqslant 2$. The last two components are the kinetic energy and the energy of the elastic deformation of the ion lattice. In the general case the deformation potential $\Phi(w_x, w_{xx},\dots)$ can be sufficiently arbitrary.

The variation of $\mathcal{L}$ with respect to ψ^+ leads to the nonstationary Schrödinger equation (6) for $\psi(x, t, P)$ with the potential $u(x, t) = w_x(x, t)$. From the variation of $\mathcal{L}$ with respect to w we obtain the self-congruency equation

$$Mw_{tt}-gw_{xx}=\partial_x\Big(\int_{\{\sigma_i\}}|\psi|^2(x,t,P)\,c(P)\,\Omega(P)-\sum_{i=1}^{N_1}r_ic_i|\psi|^2\Big). \tag{57}$$

Equations (6), (57) form a system of equations of the nonstationary Peierls model. We shall look for the solutions of these equations in the form of a propagating wave, weakly modulated in time, i.e., w(x, t) is sought in the form $w = \tilde{w}(x + Vt, t)$, where $\tilde{w}(x, t)$ weakly depends on t. Analogously, $\psi(x, t)$ will be represented in the form

$$\psi(x,t,P)=e^{-i/2Vx-i/4V^2t}\,\tilde{\psi}(x+Vt,t,P).$$

Together with that, the function ψ will satisfy Eq. (6) with the potential $u(x, t) = \tilde{w}_x(x, t)$, and Eq. (57), with the accuracy up to the second-order terms, will be transformed into the equation

$$\varkappa_2\tilde{w}_{xx}-\varkappa_3\tilde{w}_{xt}=\partial_x\Big[\int_{\{\sigma_i\}}|\tilde{\psi}|^2c(P)\,\Omega-\sum_{i=1}^{N_1}r_ic_i|\tilde{\psi}_i|^2\Big], \tag{58}$$

where $\varkappa_2 = MV^2 - g$, $\varkappa_3 = MV$.

Denote by F_s, $s = 2,\ldots,G + 1$ the numbers

$$F_s = \int_{\{\sigma_i\}} c(P)\Lambda_s - \sum_i r_{is}c_i, \tag{59}$$

where Λ_s are the differentials on Γ defined by conditions (43), and

$$r_{is} = 2\pi i \operatorname{res}_{\varkappa_i^+} \Lambda_s. \tag{60}$$

A direct consequence of equality (44) is

THEOREM 7. If the relations

$$F_2 = \varkappa_2,\ F_3 = \varkappa_3,\ F_s = 0,\quad s = 4, \ldots, G + 1, \tag{61}$$

hold, then the Beiker–Akhiezer function $\psi(x, t, P)$, and $w(x, t) = 2i\xi_1(x, t)$ satisfy the system of equations (6) (when $u = w_x$) and (58).

Remark. If the occupation numbers c_i, $c(P)$ assume the values 0 and 2 only, and $c(P)$ is constant on every oval σ_i (i.e., some of the zones and levels are filled up completely, and the rest of them are empty), then, obviously, in the analogy to the standard case, we can show that already the part of Eqs. (61), corresponding to the indices $s \geqslant 4$, does not have solutions for $G \geqslant 6$.

Equation (61) should be completed with the equation which determines the "density of electrons":

$$\int_{\{\sigma_i\}} c(P)\,dp = 2\pi\rho = \text{const.} \tag{62}$$

For $G = 2$ we have altogether 3 equations for the construction parameters: the parameters of the curve Γ (their number for $g = 2$ is 3), and the point P_0.

We shall consider now in detail a physically interesting case, corresponding to the degeneration of the curves of the kind 2 to elliptical curves with self-intersections $g = 1$, $N = 1$ (i.e., the case of a "single-soliton with single-zone background"). For this degenerated case we have 3 parameters, i.e., the periods 2ω, and $2\omega'$ of the elliptic curve Γ, and the point $\varkappa$, entering conditions (2).

A direct computation reduces Eqs. (61) and (62) to the following explicit form:

$$F_2 = -4\omega = \varkappa_2,\quad \rho = \frac{i}{\omega'}, \tag{63}$$

$$F_3 = \frac{2}{\wp(\bar{\varkappa}) - \wp(\varkappa)}\left[(\bar{\varkappa} - \varkappa)\eta + \omega(\zeta(\varkappa) - \zeta(\bar{\varkappa})) + \pi i c_1\right] = \varkappa_3. \tag{64}$$

With this we assume also that the lowest (with respect to the quasienergy) distinguished zone is completely filled up: $c(P) = 2$, $\operatorname{Im} P = -i\omega'$, and that the upper one is empty: $c(P) = 0$, $\operatorname{Im} P = 0$. Equations (63), determining the periods of Γ, coincide with the self-congruency equations for the stationary Peierls model [13]. From (64) we can find $\varkappa$.

The limited volume of this article does not allow us to display the explicit formulas for ψ and w, corresponding to Γ and $\varkappa$. They are obtained by the direct substitution of formula (5) into (2). It follows from Lemma 4 that asymptotically for large x and t the potential $u(x, t) = w_x(x, t)$ has the form

$$u \to 2\wp(ix + \zeta_0 \pm (\varkappa - \bar{\varkappa})) + \text{const.} \tag{65}$$

From the point of view of the solutions of the nonstationary Peierls model only those solutions of Eqs. (6) and (58) are interesting, which depend weakly on t. In the case of an elliptic curve Γ the quasienergy E is $\wp(z)$ – the $\wp$-Weierstrass function [34]. Therefore, as it follows from the proof of Lemma 4, the "smallness" of the velocity of a soliton with the background of canoidal wave is equivalent to the "smallness" of $|\wp(\varkappa) - \wp(\bar{\varkappa})|$. The latter is satisfied if

$$|i \operatorname{Im}\varkappa - \omega'| \ll 1. \tag{66}$$

The self-congruency equation (64) has the solution $\varkappa$ satisfying condition (66) only if

$$c_1 = 1. \tag{67}$$

In such a way, like in the stationary case, on the discrete level there should be localized only one electron. In this case the Peierls model has a solution of the propagating wave type, weakly modulated in time.

In analogy to the previous theorem, from (44) follows

Theorem 8. If the conditions

$$F_2 = -g,\ F_3 = 0,\ F_4 = M,\ F_s = 0;\quad s = 5, \ldots, G+1, \tag{68}$$

are satisfied, then $\psi(x, t, P)$ and $w(x, t) = 2i\xi_1(x, t)$ satisfy Eq. (6) and the self-congruency condition

$$M\left(w_{tt} - \frac{1}{8} w_{xxxx} + \frac{3}{8} w_x w_{xx}\right) - g w_{xx} = \partial_x \left[\int_{\sigma_i} |\psi|^2 c(P)\,\Omega - \sum_i c_i r_i |\psi_i|^2 \right]. \tag{69}$$

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FINITE-DIMENSIONAL PERTURBATIONS OF DISCRETE OPERATORS AND FORMULAS FOR TRACES

V. A. Sadovnichii and V. A. Lyubishkin UDC 517.956

1. Introduction

The theory of regularized traces in problems generated by ordinary differential expressions on a finite interval is practically completed at the present time (cf. [1,2]). This circumstance is mainly connected with the fact discovered in [1] and contained there that obtaining formulas for traces in this case can be reduced to the study of zeros of entire functions with a completely determined asymptotic structure stipulated by the concrete form of the fundamental system of solutions of a differential equation. However, even in this situation, one cannot avoid the fact (cf. [2]) that the zeros of the involved entire functions do not have a sufficiently regular asymptotic behavior which makes it necessary to call upon the methods of perturbation theory in these cases.

The situation becomes significantly more complicated if we consider problems generated by partial differential operators. This is connected, first of all, with the complicated structure of the spectrum, and consequently of functions of the spectral parameter λ, arising in similar problems.

At the present time, the theory of regularized traces of discrete operators has been worked out relatively little. From the results we would like to mention [3], in which trace formulas are obtained in the case of self-adjoint nuclear perturbations, and [4], in which the theory is extended to the case of dissipative perturbations. In [5] an algorithm is constructed for obtaining trace formulas for a large class of discrete operators.

We would like to emphasize that as usual, the methods of the theory of functions have great significance in the study of the spectral problems of abstract operators. Namely, the combination of methods of perturbation theory with those of the theory of functions leads to

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