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WESS-ZUMINO LAGRANGIANS IN CHIRAL MODELS AND QUANTIZATION OF THEIR CONSTANTS

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A general method of constructing the Wess-Zumino type lagrangians is proposed. The corresponding constants are shown to be quantized. Some examples in 1d, 3d and 4d are considered.

1. Introduction

Recently [1, 2] Witten has considered the Skyrme model [3] with the additional term - the so-called Wess-Zumino lagrangian (WZL), which resulted from anomalies in the current algebra [4]. This term plays a crucial role in the theory; it determines the baryon charge of the soliton solution. The whole lagrangian with the WZ term reproduces correctly the discrete symmetries of QCD.

The WZL in [1, 2] is represented in the form of an integral over a 5d manifold which has an additional variable apart from space-time variables. In this form its topological sense and quantization of the constant become clear. The latter fact leads to the remarkable conclusion that in the low-energy limit the properly normalized amplitudes of some reactions must be integers.

WZL is a particular case of multivalued functionals defined by Novikov [5, 6]. According to [5, 6] multivalued functionals for chiral models which describe the fields mapping d-dimensional space-time M^d into the manifold X are defined by the closed $(d+1)$-form Ω on X and by fixing in open subsets U_j of X the d-forms of ω_j such that

$$\Omega(y) = d\omega_j(y)\,, \qquad y \in \mathrm{U}_j\,. \tag{1}$$

If the image of mapping φ lies completely in U_j then the multivalued functional is defined by the integral

$$\Gamma = \int_{\mathrm{M}^d} \omega_j(\varphi)\,. \tag{2}$$

This definition is the direct generalization of the lagrangian for a charge in a magnetic field. In this particular case Ω corresponds to the magnetic field and ω_j corresponds to the vector potential. Relation (1) takes the usual form $\boldsymbol{F} = \mathrm{rot}\,\boldsymbol{A}_j$. If $\boldsymbol{F}$ is the field of a magnetic monopole, then, as a result of the Dirac string, $\boldsymbol{A}_j$ can be defined only locally.

This approach, in contrast with the construction proposed in [1, 2], is free of restrictions on the topological type of chiral fields. As it does not require additional variables it may be called "internal".

Though "external" construction [1, 2] uses an artificial surface with an additional degree of freedom, it has its own advantages. Firstly, it is geometrically apparent and the quantization of the WZL constant becomes evident. Moreover, the calculation of the normalizing factor is more feasible in comparison with the internal approach. Unfortunately, Witten's approach encounters topological obstacles so here we generalize his construction in such a way that topological restrictions disappear and the method may be applied to a wide class of chiral models.

2. Witten's construction of WZL

First of all, let us recall the topological conditions necessary for the existence of WZL and quantization of its constant in Witten's construction [1, 2].

We consider the euclidian version of the theory and suppose that fields vanish at infinity. Thus, space-time is identified with a large sphere S^d. The first condition necessary for the construction [1, 2] is that any field $\varphi(x)$ defined on S^d can be extended onto the disk D^{d+1} with the boundary S^d. This implies that the homotopic group of order d of X is trivial

$$\pi_d(X) = 0\,. \tag{3}$$

Next it is required that there should exist an antisymmetric tensor of rank $d+1$, $\Omega_{i_1\cdots i_{d+1}}$, on X with a vanishing curl ($\partial_{\{i}\Omega_{i_1\cdots i_{d+1}\}}=0$). In other words, the external form $\Omega=\Omega_{i_1\cdots i_{d+1}}dy^{i_1}\cdots dy^{i_{d+1}}$ is closed,

$$d\Omega = 0\,. \tag{4}$$

The field $\varphi(x)$ defines the pull-back form $\tilde{\Omega}$ on the disk $D^{d+1}: \tilde{\Omega}=\Omega(\varphi)$. The integral

$$\Gamma = \lambda \int_{D^{d+1}} \tilde{\Omega}_{i_1\cdots i_{d+1}}\, d\Sigma^{i_1\cdots i_{d+1}}\,, \tag{5}$$

(where $d\Sigma^{i_1\cdots i_{d+1}}$ is a measure on D^{d+1}) in WZL. According to Stokes theorem and in view of (1) the integral is reduced to the integral (2). Thus, the definition (5) is independent of an infinitisimal disk mapping deformation.

The multivaluedness of Γ is related to large deformations of the disk mapping. The difference between the two integrals (5) defines the integral over the sphere, glued from two disks $S^{d+1}=D_1^{d+1}\mathrm{U}-D_2^{d+1}$ (minus denotes an opposite orientation of D_1^{d+1} and D_2^{d+1})

$$\Gamma_1-\Gamma_2 = \lambda \int_{S^{d+1}} \tilde{\Omega}_{i_1\cdots i_{d+1}}\, d\Sigma^{i_1\cdots i_{d+1}}\,. \tag{6}$$

If Ω is exact (i.e. (1) holds globally over the whole X) or can be decomposed into a product of two or more forms, then (6) equals zero. Therefore, Γ is defined unambiguously and λ may be arbitrary. But if Ω is indecomposible and moreover represents an integral cohomology class in the group $H^{d+1}(X)$ then

$$\Gamma_1 = \Gamma_2 + \lambda n\,. \tag{7}$$

Thus the functional integral $\int D\varphi \exp\{i\Gamma\}$ is not well defined unless $\lambda n = 2\pi k$. Therefore, one more condition is necessary for the quantization of λ:

$$\{H^{d+1}(X) \text{ contains only indecomposible elements}\} \tag{8}$$

In [1] the quantization is based on the condition

$$\pi_{d+1}(X) = Z \tag{9}$$

inspite of (8).

The conditions (3), (8) are so restrictive that only a few examples among homogeneous spaces X satisfy them (the most interesting cases have been considered in [2]).

3. General approach

In fact it is unnecessary to extend $\varphi(x)$ from S^d on a disk D^{d+1}. One may extend $\varphi(x)$ on a manifold N^{d+1} with a more sophisticated topology provided that N^{d+1} has S^d as the boundary. Then WZL is represented as the integral over N^{d+1}:

$$\Gamma = \lambda \int_{N^{d+1}} \tilde{\Omega}_{i_1\cdots i_{d+1}}\, \mathrm{d}\Sigma^{i_1\cdots i_{d+1}}\,. \tag{10}$$

The difference between two such integrals corresponding to two extensions is equal to the integral

$$\Gamma_1 - \Gamma_2 = \lambda \int_{W^{d+1}} \tilde{\Omega}_{i_1\cdots i_{d+1}}\, \mathrm{d}\Sigma^{i_1\cdots i_{d+1}}\,. \tag{11}$$

Here W^{d+1} is a closed manifold "glued" from the two "pieces" N_1^{d+1} and N_2^{d+1}.

Remark 1. Consider a set of mappings of different manifolds into a fixed manifold X. Two mappings of manifolds N_1 and N_2 are assumed to be equivalent (bordant) if there exist a manifold W connecting N_1 and N_2 (i.e. the boundary of W consists of two components N_1 and N_2) and an extension of the mappings on W. Analysis of such mapping extensions from boundaries on the whole manifold is the concept of the so-called *bordism theory.* For the sake of simplicity we shall not introduce accurate definitions of this theory. Some statements of this theory which we shall use below can be extracted from [7]. Unfortunately, the corresponding mathematical formalism is represented in a form incomprehensible to most physicists.

The first step of our construction is feasible if

$$\{\text{the group } H^d(X) \text{ contains only decomposible elements}\}\,. \tag{12}$$

In particular, $H^d(X)$ may be trivial. If this single topological condition is satisfied, then any cohomology class of $H^{d+1}(X)$ defines WZL according to (10), where Ω

corresponds to the chosen class. Γ undergoes a nonessential shift if Ω' is another tensor of the same cohomology class.

Since any rational homology class may be represented as an image of some closed manifold, it follows from (11) that Γ is defined up to adding to it the integrals of Ω over all $(d+1)$ cycles. Thus the functional integral for the theory with additional WZL is defined correctly if the constant λ is quantized.

Remark 2. The single topological obstruction (12) can be easily eliminated by a nonessential complication of the construction (see examples 4, 6 below).

4. Examples

In the following examples Witten's approach is inapplicable directly. Nevertheless, we construct WZL with quantized constants using the method developed above.

Example 1. Let us consider two noninteracting fields u and v taking the values in $X = S^3 \times S^2$

$$u: \qquad S^4 \to SU_2 \sim S^3 ,$$

$$v: \qquad S^4 \to SU_2/U(1) \sim S^2 ,$$

with the lagrangian

$$\mathscr{L} = \mathscr{L}_{\text{kin}}(u) + \mathscr{L}_{\text{kin}}(v) + \mathscr{L}_{\text{Skyrme}}(u) + \mathscr{L}_{\text{Skyrme}}(v) ,$$

$$\mathscr{L}_{\text{kin}}(u) = \operatorname{tr}(R_\alpha R^\alpha) , \qquad R_\alpha = u^+ \partial_\alpha u ,$$

$$\mathscr{L}_{\text{kin}}(v) = \tfrac{1}{2} B_\mu \bar{B}_\mu , \qquad B_\mu = L_\mu^{(1)} + i L_\mu^{(2)} ,$$

$$L_\mu = v^+ \partial_\mu v , \qquad L_\mu^a = \tfrac{1}{2} \operatorname{tr}(L_\mu \sigma^a). \tag{13}$$

$\mathscr{L}_{\text{skyrme}}$ is the term providing the solitons stability.
There are two topological charges:

$$q_1(u) = \frac{1}{48\pi} \int_{S^3} d^3x \, \varepsilon^{\alpha\beta\gamma} \operatorname{tr}([R_\alpha, R_\beta] R_\gamma) , \tag{14}$$

$$q_2(v) = \frac{1}{4\pi} \int_{S^2} d^2x \, \varepsilon^{\mu\nu} \partial_\mu A_\nu , \qquad A_\nu = L_\nu^3 . \tag{15}$$

The first integral defines the charge of skyrmion configurations and the second one the charge of flux tube configurations, which are independent of one of the coordinates. This charge possesses a nontrivial topology in a perpendicular plane. Moreover, v possesses an additional topological charge - the Hopf invariant. We shall not exploit its explicit form.

Since $\pi_4(X) = \pi_4(S^3) + \pi_4(S^2) = Z_2 + Z_2$, the condition (3) is broken. Meanwhile $H^4(X) = \sum_{p+q=4} H^p(S^3) \otimes H^q(S^2) \cong 0$, and we can use our method (see the remark after (11)). The group $H^5(X)$ is generated by the form which is an exterior product of charge densities (14), (15)

$$H^5(X) = H^3(S^3) \otimes H^2(S^2) \simeq Z\,,$$

$$\Omega_{\alpha\beta\gamma\mu\nu} = \varepsilon^{\alpha\beta\gamma\mu\nu}\,\mathrm{tr}\,([R_\alpha, R_\beta]R_\gamma)\partial_\mu A_\nu\,. \tag{16}$$

Then we have the following representation of WZL:

$$\Gamma = \lambda \int_{N^5} \tilde{\Omega}_{\alpha\beta\gamma\mu\nu}\, \mathrm{d}\Sigma^{\alpha\beta\gamma\mu\nu}\,, \tag{17}$$

where $\tilde{\Omega}$ is the pull-back of Ω. Here N^5 is a manifold with the boundary S^4. For instance, it may be bordant to the space X without a disk D^5. In view of (16), λ is quantized.

By means of WZL the nontrivial configurations of the fields u and v (skyrmions and flux tubes) interact. Perhaps this theory can be considered as a low-energy approximation of the gauge theory with two types of gluons and fermions with an anomalous interaction giving rise to WZL (17). It can be evidently generalized to a theory of the type $S^4 \to X \simeq G \times \mathbb{C}P^n$, where G is a simple Lie group.

Example 2. Consider the theory in the 3d space

$$\varphi: \qquad S^3 \to X \simeq \mathbb{C}P^2 = SU_3/SU_2 \otimes U_1\,.$$

Here the group SU_3 is spontaneously broken in the current algebra to the subgroup $SU_2 \otimes U_1$. The complex fields $\varphi = (\varphi_1, \varphi_2)$ are similar to the K-mesons. The kinetic term has the following form

$$\mathscr{L}_{\mathrm{kin}} = \int_{S^3} \mathrm{d}^3x\, g_{\alpha\beta}(\varphi)\partial_\mu \varphi^\alpha \partial^\mu \bar{\varphi}^\beta\,, \tag{18}$$

where the metric $g_{\alpha\beta}$ is generated by the Kähler potential

$$g_{\alpha\beta} = \frac{\partial^2}{\partial\varphi^\alpha \partial\bar{\varphi}^\beta} \ln(1 + |\varphi^1|^2 + |\varphi^2|^2)\,. \tag{19}$$

For this model $\pi_3(X) = 0$ and (3) is satisfied. Thus it is possible to extend the map $\varphi: S^3 \to X$ to $D^4 \to X$. Because for

$$H^4(\mathbb{C}P^2) \simeq \mathbb{Z} \tag{20}$$

there exists WZL. But integration over disks does not lead to quantizing of the WZL constant since $\pi_4(CP^2) = 0$ (see (9)).

In fact, the constant is quantized though it is not evident in Witten's approach. To see this one should consider extentions of some fields on the manifolds like the manifold $\mathbb{C}P^2$ without disk, $N^4 \simeq \mathbb{C}P^2 \backslash D^4$.

The generic element of $H^4(\mathbb{C}P^2)$ may be defined by means of the tensor $g_{\alpha\beta}(19)$. Then

$$\Gamma = \lambda \int_{N^4} \tilde{\Omega}_{\alpha\beta\gamma\delta}\, d\Sigma^{\alpha\beta\gamma\delta}\,,$$

where

$$\Omega_{\alpha\beta\gamma\delta} = \varepsilon^{\alpha\beta\gamma\delta}\omega_{\alpha\beta}\omega_{\gamma\delta}\,, \qquad \tilde{\Omega} = \Omega(\varphi)\,, \qquad \omega_{\alpha\beta} = \mathrm{Im}\, g_{\alpha\beta}\,.$$

In view of (20) λ is quantized.

Example 3. As in the previous example, let us consider the group SU_3 but now let the symmetry be broken to subgroup $U_1 \times U_1$. Then we have a theory with three complex fields

$$\varphi = (\varphi^1, \varphi^2, \varphi^3): \qquad S^3 \to X = SU_3/U_1 \times U_1\,,$$

which can be identified with massless mesons with nonzero charge or hypercharge. The kinetic part of the lagrangian is:

$$\mathscr{L}_{\mathrm{kin}} = \int_{S^3} d^3x\, (g^{(1)}_{\alpha\beta} + g^{(2)}_{\alpha\beta})\partial_\mu\varphi^\alpha\partial^\mu\bar{\varphi}^\beta\,, \tag{21}$$

where

$$g^{(j)}_{\alpha\beta} = \frac{\partial^2 K^{(j)}}{\partial\varphi^\alpha\partial\bar{\varphi}^\beta}\,, \qquad (j=1,2)\,,$$

$$K^{(1)} = \ln(1 + |\varphi^1|^2 + |\varphi^2|^2)\,,$$

$$K^{(2)} = \ln(1 + |\varphi^2|^2 + |\varphi^3|^2 + |\varphi^1|^2|\varphi^3|^2 - 2\,\mathrm{Re}\,\varphi_1\bar{\varphi}_2\varphi_3)\,.$$

Here $\pi_3(X) \neq 0$, but $H^3(X) = 0$ and we can apply our method. It is noteworthy that there are two types of topological charges ($H^2(X) = \mathbb{Z}\oplus\mathbb{Z}$) corresponding to the two types of flux tubes

$$q_j(\varphi) = \tfrac{1}{2}i \int_{S^2} \varepsilon^{\alpha\beta} g^{(j)}_{\alpha\beta}(\varphi)\, d^2x\,. \tag{22}$$

Since

$$H^4(X) = \mathbb{Z}\oplus\mathbb{Z}\,, \tag{23}$$

the WZL also contains two terms constructed by means of tensors $g^{(j)}_{\alpha\beta}$ (21):

$$\Gamma = \sum_{j=1}^{2} \lambda_j \int_{N^4} \tilde{\Omega}^{(j)}_{\alpha\beta\gamma\delta}\, d\Sigma^{\alpha\beta\gamma\delta}\,,$$

where

$$\Omega^{(j)}_{\alpha\beta\gamma\delta} = \omega^{(j)}_{\alpha\beta}\omega^{(j)}_{\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta}\,, \qquad \tilde{\Omega} = \Omega(\varphi)\,, \qquad \omega^{(j)}_{\alpha\beta} = \mathrm{Im}\, g^{(j)}_{\alpha\beta}\,.$$

Here we integrate over the manifold with boundary S^3. As always in our construction, the constants λ_j are quantized.

Example 4. Consider the motion of a charged particle constrained on a torus T^2 in the field of a monopole located inside the torus. Here the condition (12) is broken because $H^1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ has only indecomposible elements. In fact, the trajectories which are nontrivial one-cycles of the torus do not admit the attachment of two-dimensional surfaces and at first sight there exists only a local expression for WZL (electromagnetic term)

$$\Gamma = \lambda \int_{S^1} A_\mu \, dx^\mu \tag{24}$$

(see (2)). We propose the following modification of our construction. Consider a fixed mapping x_0 from S_0^1 to T^2 in each homological class. Then WZL has the following form:

$$\Gamma' = \lambda \int_{N^2} \tilde{F}_{\mu\nu} \, d\Sigma^{\mu\nu} . \tag{25}$$

Here N^2 is such a manifold with the boundary $S^1 U - S_0^1$ that there exists a map f from N^2 to the torus satisfying the following conditions:

$$f|_{S^1} = x , \qquad f|_{S_0^1} = x_0 .$$

Clearly in this case we can take N^2 to be a cylinder, $S^1 \times I$, but we prefer to define the construction in a more general form, which can be applicable to other models.

The tensor $\tilde{F}_{\mu\nu}$ in (25) is the pull-back of the field strength $F_{\mu\nu}$. Stokes theorem implies that Γ' in (25) differs from Γ on a fixed constant. The quantization of λ results from $H^2(X) = \mathbb{Z}$.

Example 5. Let X be the direct product:

$$X = K_3 \times T^2 .$$

This manifold describes compactification of six dimensions in the ten-dimensional superstrings theory [8]. Reminding ourselves that

$$\dim K_3 = 4 , \qquad H^3(K_3) = 0 , \qquad H^4(K_3) \cong \mathbb{Z} ,$$

we consider the chiral field

$$\varphi: \qquad S^4 \to X .$$

Since $H^4(X) \supset H^4(K_3) \simeq \mathbb{Z}$ it contains an indecomposable element corresponding to the volume density Ω_4 of K_3. There are two independent five-forms on X generated by Ω_4 on K_3 and one-forms $\omega_1^{(1)}$, $\omega_1^{(2)}$ on T^2,

$$H^5(X) \cong \mathbb{Z} \oplus \mathbb{Z} . \tag{26}$$

Treating our construction as we did in the previous example, we introduce the basic representation $\varphi_0 : S_0^4 \to X$; then

$$\Gamma = \sum_{j=1,2} \lambda_j \int_{N^5} \tilde{\Omega}_4 \tilde{\omega}_1^{(j)} \, d\Sigma .$$

Here N^5 is a manifold with the boundary $S^4U - S_0^4$ such that there exists an extension of the mapping φ and φ_0 on the whole manifold. As it follows from (26), the constants λ_1 and λ_2 are quantized independently.

Example 6. Finally let us consider two-dimensional models, with fields taking values in Kähler manifolds of complex dimension three, in which the first Chern class vanishes. These models also appear in string theories [8b]. For all the types of models considered in [8b] $H^2(X) \simeq \mathbb{Z}$. Hence in 2d models it is necessary to use the last construction. Because the third Betti numbers differ from zero, WZL can exist; for instance, in the theory with four generations of fermions $b_3 = 12$ [8b]. Thus we have twelve WZ terms with quantizing constants. Perhaps this is an indication of the finiteness of the model.

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