EVOLUTION OF A WHITHAM ZONE IN KORTEWEG–DE VRIES THEORY

V. V. AVILOV AND ACADEMICIAN S. P. NOVIKOV

We consider the analogy of shock waves in KdV theory. It is well known (see [1], pp. 261–263) that the analogy of a shock front, for instance, in a collisionless plasma described by the KdV equation, has an oscillation zone. This zone is in the framework of the averaging method described by a set of Whitham equations [2] for three slowly varying quantities which in a given cycle of problems were first used by Gurevich and Pitaevskii [3]. These authors studied exact self-similar solutions of the Whitham equations on the basis of which they reached conclusions about the asymptotic behavior of the oscillation zone as $t \to +\infty$.

The aim of the present paper is a correct mathematical statement of the problem of the evolution of the oscillation (Whitham) zone for arbitrary boundary conditions in the framework of the theory of sets of first order equations. This allows us, in particular, to state and solve numerically the problem about the realizability of the self-similar regimes found in [3] as asymptotic regimes for a wide class of initial conditions as $t \to -\infty$.

The KdV equation has the form $u_t + uu_x + u_{xxx} = 0$. Averaging it against the background of a set of periodic cnoidal waves

$$u(x,t) = 2as^{-2}dn^2\left[\left(\frac{a}{6s^2}\right)^{1/2}(x - Vt), s\right] + \gamma \tag{1}$$

with slowly varying parameters a, s, and γ leads to Whitham equations of the form

$$r_{\alpha t} = v_\alpha(a,s,\gamma)r_{\alpha x}, \quad \alpha = 1,2,3, \tag{2}$$

where

$$a = r_2 - r_1, \quad s^2 = \frac{r_2 - r_1}{r_3 - r_1}, \quad \gamma = r_2 + r_1 - r_3; \tag{3}$$

the equations for v_α are given in [1], p. 264. Here $v_3 \geqslant v_2 \geqslant v_1$, $r_3 \geqslant r_2 \geqslant r_1$. If $r_2 = r_3$, the solution (1) describes a soliton; if $r_2 = r_1$, (1) is a constant.

According to the physical representations worked out in [3], the oscillation zone in the problems studied extend over the entire allowable range of variation of the parameters r_α; i.e., at any time t it is determined in the region (4), which is not known beforehand,

$$x^-(t) \leqslant x \leqslant x^+(t) \tag{4}$$

where

$$\begin{aligned} r_1 &\to r_2, \quad x \to x^-(t), \\ r_2 &\to r_3, \quad x \to x^+(t). \end{aligned} \tag{5}$$

Date: Submitted April 9, 1986.
Dokl. Akad. Nauk SSSR 294, 325–329 (May 1987). Translated by D. ter Haar.

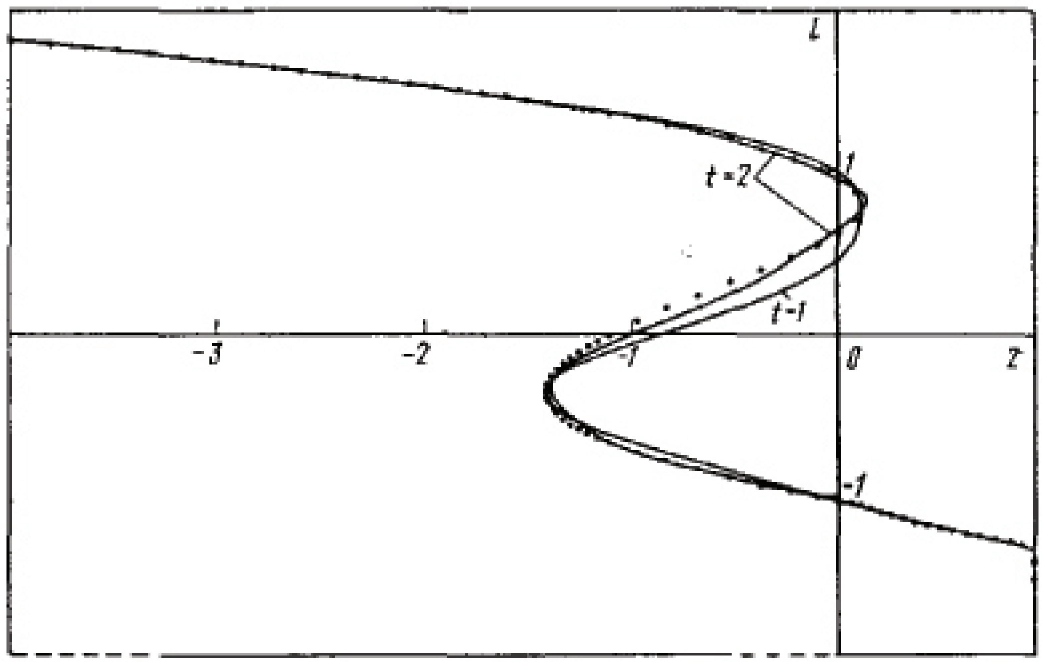

FIGURE 1. Evolution of the single-valued function $l(z,t) = t^{-1/2}r(z,t)$ $(z = xt^{-3/2})$ of problem 2. The initial condition $(t = 1)$ in the oscillation zone corresponds to a perturbation of the self-similar solution; at $t = 2$ this distortion is appreciably diminished. The self-similar solution is indicated by dots.

Outside the range (4) there are no oscillations and the functions $r_\alpha(x,t)$ are not defined; at the boundary of (4) the solution of Eq. (2) must be joined continuously to the solution of the Hopf equation $u_t + uu_x = 0$, obtained from the KdV by dropping the dispersion term, where $u(x,t)$ is determined outside the zone (4):

$$u(x^-(t),t) = r_3(x^-(t),t),$$
$$u(x^+(t),t) = r_1(x^+(t),t). \quad (6)$$

It follows from (5) and (6) that the solution on the whole x axis is described by the continuous function $r(x,t)$ which is three-valued in the oscillation region (4), $r = (r_1, r_2, r_3)$, and single-valued outside that region, $r = u(x,t)$ (see Fig. 1).

Problem. Give the mathematically correct statement of the Cauchy problem for multiple-valued functions $r(x,t)$ which allows us to study the temporal evolution of the oscillation zone (4).

For a more complete and rigorous study the basis of such a statement must follow of course from the exact KdV theory. We have, however, deliberately restricted the discussion to the theory of first-order systems which may have a broader meaning than the KdV theory.

Hydrodynamic systems without dissipation have the form

$$u_t^\alpha = v_\beta^\alpha(u)u_x^\beta, \tag{7}$$

where $u^\alpha(x,t)$ is a vector function, $\alpha = 1, 2, \ldots, k$. The form of (7) is conserved under nonlinear substitution of $u(w)$. If the matrix $v_\beta^\alpha(u)$ is diagonal, the fields u^α are called Riemann invariants. For instance, for Whitham's system (2), $k = 3$, the quantities $u^\alpha = r_\alpha$ are the Riemann invariants of (2). Recently, a theory of Hamiltonian systems of the form (7) and of the Poisson brackets connected with them has been developed [4]. One of the authors of that paper hypothesized that a system of the form (7) is integrate if, first, it is Hamiltonian and, secondly, it has Riemann invariants. In some sense, this hypothesis was proved in [5]. The procedure of [5] allows us to find some exact ("on average finite-zoned") solutions which have as yet not been studied. The theorem following from [5] about the complete integrability of Whitham's system (2) is of a formal, local nature. Its applicability to a particular global class of functions $r_\alpha(x,t)$ has not been studied. Even more questionable is the applicability of this statement to the physically interesting class described above, where the Whitham equation acts only in the finite range (4) and is joined on the boundary to the solution of the trivial Hopf equation; the range (4) changes here with time in a way not known as yet.

The system (2) possesses self-similar solutions of the form (8) with arbitrary exponent γ:

$$r_\alpha(x,t) = t^\gamma l_\alpha(xt^{-1-\gamma}) = t^\gamma l_\alpha(z). \tag{8}$$

An important role below is played by the solution for $\gamma = 1/2$ found in [3] (see [1], pp. 280–284). Let $z = xt^{-3/2}$. Outside the zone (4) we have $u(x,t) = t^{1/2}\theta(z)$, where $z = \theta - \theta^3$. At the edges of (4), where $x = x^\pm(t)$, all $l_\alpha^\pm$ can be expressed in terms of $z^\pm$ from the conditions of continuity and constancy of the zone (4) in the self-similar variable z (see [1], p. 281).

Such a solution exists and is unique, if $l_3 > 0$, $l_1 < 0$, and all l_α are continuous along with their first derivatives in the region $z^- < z < z^+$. At the point z_0, where $l_2(z_0) = 0$, the second derivatives apparently are no longer continuous. Calculations show that

$$z^- \approx -1.141, \quad z^+ \approx 0.117, \quad z_0 \approx -1.11. \tag{9}$$

We now turn to our problem. The class of multiple-valued continuous functions $r(x,t)$ must satisfy conditions (4)–(6). Moreover, these functions must be smooth class C^1 functions outside the points $x^\pm(t)$. They should be assumed to be smoother outside zone (4). In the vicinity of the points on the curves $x^\pm(t)$, the hypothesis of single-valuedness and of smoothness of the inverse function must be satisfied. This means that for any fixed $t \geqslant 1$ the behavior of the quantity r_α as $x \to x^\pm(t)$ is determined from Eqs. (10)–(13) at the given time t:

$$x'' = (a_+ + b_+(r - r^+))f(1 - s^2) + O(r - r^+)^3, \tag{10}$$
$$x'' = x - x^+ \leqslant 0, \quad f(y) = y^2[\log(16/|y|) + 1/2];$$

$$x' = a_-(r - r^-)^2 + b_-(r - r^-)^3 + o(r - r^-)^3, \quad x' = x - x^- \geqslant 0. \tag{11}$$

Here we have

$$dx^+/dt = v_2^+ = v_3^+, \quad dr^+/dt = -|r_3^+ - r_1^+|^2/(12a_+), \tag{12}$$

$$dx^-/dt = v_1^- = v_2^-, \quad dr^-/dt = -1/(2a_-). \tag{13}$$

If these conditions are satisfied, we call a multiple-valued function admissible at the given time t.

Assertion. The Whitham equation, together with the Hopf equation, determines uniquely the temporal evolution of the admissible multiple-valued functions $r(x,t)$.

Although there is no mathematically rigorous proof of this assertion, the present authors have constructed a numerical realization of this evolution applicable to two functional classes (boundary conditions) corresponding to two physical problems (see [1]).

Problem 1. Decay of an initial discontinuity for the KdV. Here we have

$$r(x,t) \to 1, \quad x \to -\infty,$$
$$r(x,t) \to 0, \quad x \to +\infty.$$

Problem 2. Dispersive analog of a shock front. Here the boundary conditions are

$$r(x,t) - u_0(x,y) \to 0, \quad |x| \to \infty, \quad x = u_0 t - u_0^3.$$

In each problem we assume that the rate at which the limit is reached as $|x| \to \infty$ is sufficiently fast (exponential).

Using (2), we carry out the numerical calculation by the characteristics method outside small regions near the points $x^{\pm}(t)$. In the vicinity of these points the quantity $r(x,t)$ was interpolated by Eqs. (10)–(13), where $a_{\pm}(t)$ and $b_{\pm}(t)$ were determined by joining up with the numerical solution. This enabled us to construct in each step in time the extension of the oscillation zone (4), since the derivatives $\dot{x}^{\pm}(t)$ and $\dot{r}^{\pm}(t)$ are known. Along the line $x^-(t)$ the characteristics for r_1 and r_2 are tangent and along $x^+(t)$ the characteristics for r_2 and r_3 are tangent. These are the singular lines: the characteristic r_1 reaches the line $x^-(t)$, touches it, and then leaves it as the characteristic r_2. In exactly the same way, the characteristic r_3 arrives along $x^+(t)$, touches the line $x^+(t)$, and then leaves as r_2 in the region of the (x,t)-plane inside the curves $[x^+(t), x^-(t)]$. At each step in time around the boundaries there is a transition of one characteristic into another. In the numerical calculation the characteristics themselves are not calculated in small regions around the curves $x^{\pm}(t)$.

As boundary conditions at $t = 1$ for problem 2 we introduced various admissible perturbations of the self-similar Gurevich–Pitaevskii regime (see above).

The conclusions are the following: any admissible initial condition $r(x,t)$, which is sufficiently C^1-close to the self-similar solution of problem 2 (see above), evolves an infinite time without the appearance of any singularities (see Fig. 1) and as $t \to \infty$ the functions $l_\alpha(z,t)$ tend to the self-similar solution (8), where $r_\alpha(x,t) = t^{1/2} l_\alpha(z,t)$. There exists a finite threshold – the degree of remoteness of the initial perturbation from the self-similar solution – after which the evolution may be (and sometimes is) such that there appears the usual hydrodynamic steepening and afterwards the inversion of the front for r_α. We do not know the numerical characteristics of this threshold. In order that the evolution of $r(x,t)$ be extended over an infinite time as $t \to +\infty$, it is necessary (although not sufficient) that each single-valued continuous branch of the function $r(x,t)$ be a monotonic function of x at time t. This statement is correct in all problems considered by us, as the numerical experiment shows. Under the same necessary condition in problem 1, a wide class of admissible (although not all) initial conditions in the evolution process tend to

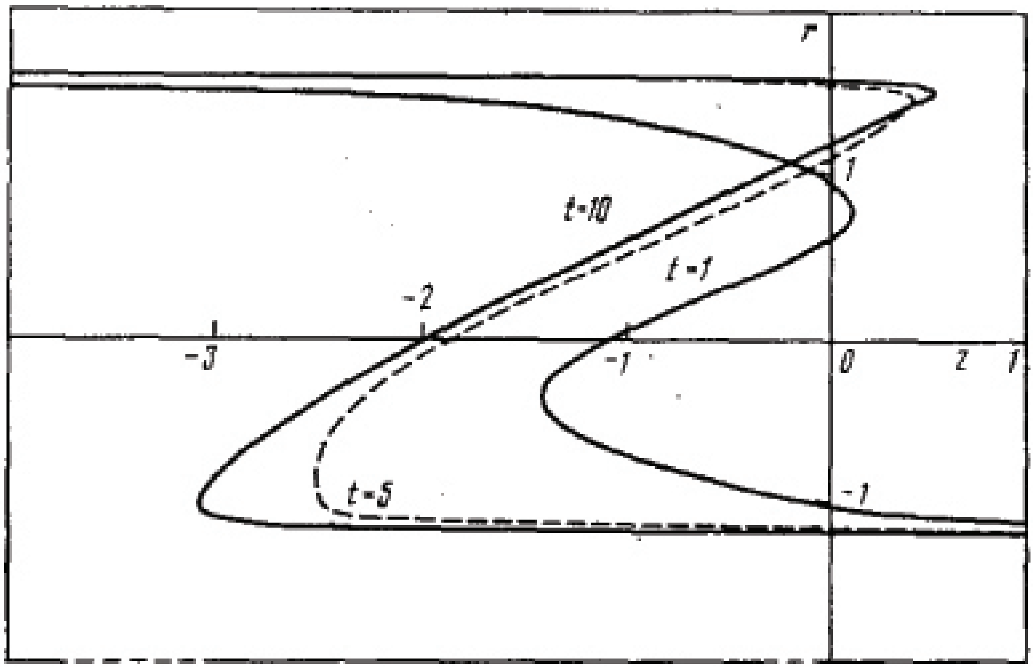

FIGURE 2. Evolution of the single-valued function $r(z,t)$ ($z = xt^{-1}$) of problem 1. The initial condition ($t = 1$) inside the oscillation zone corresponds to the self-similar solution of problem 2 and outside this zone tends to a constant.

a regime which is self-similar with exponent $\gamma = 0$, where $z = xt^{-1}$, $r_1 \to \text{const}$, $r_3 \to \text{const}$, and $v_2 \to z$. This limiting regime is described in [1], pp. 268–270. By itself it is not contained among the admissible functions, but is found to be a limit (see Fig. 2).

From a methodological point of view, it is useful also to consider the case (problem 3) when $x^+ = +\infty$, $x^- = -\infty$, and $r_\alpha(x,t) \to r_\alpha^\pm$ as $x \to \pm\infty$, where $r_1^- = r_2^- < r_3^-$, $r_2^+ = r_3^+ > r_1^+$ (infinite oscillation zone).

Numerical calculations show that in order that the evolution does not for all $t \to +\infty$ lead to the usual hydrodynamic inversion of the front (i.e., in order that $|r_{\alpha x}|$ be finite for all α), it is necessary and sufficient that at the initial time $t = 1$ the condition for a monotonic increase of $r_\alpha(x,t)$ is satisfied:

$$r_{\alpha x} \geqslant 0, \quad -\infty < x < +\infty. \tag{14}$$

If $r_1^+ < r_3^-$ there is a finite range of the self-similar $z = xt^{-1}$, where the solution tends, as $t \to \infty$, to the self-similar solution of problem 1: $r_3 = r_3^-$, $r_1 = r_1^+$, $v_2 = z$ (see Fig. 3). Hence it follows that the condition that there be no singularities in the evolution process can in principle be formulated only in terms of the Riemann invariants r_α. It cannot be expressed in terms of the physical characteristics of the initial condition – such as the average velocity $\bar{u}$, the quantities $u_{\min}$ and $u_{\max}$ (see [1], p. 264), the average momentum density $\bar{p} = \overline{u^2}$, and the average energy $\bar{\varepsilon}$,

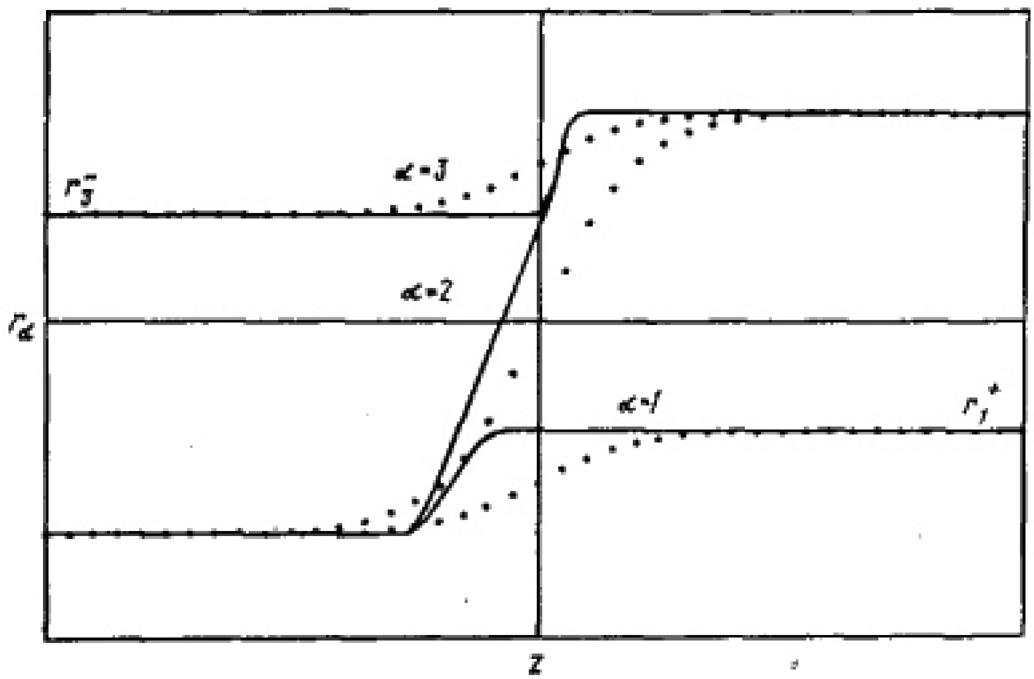

FIGURE 3. Evolution of the infinite oscillation zone of problem 3. The functions $r_\alpha(z,t)$ ($z = xt^{-1}$), $\alpha = 1, 2, 3$, at the initial time $t = 1$ are shown by dots, the full drawn lines correspond to $t = 11$.

whose graphs may appear to be physically meaningful both when (14) is satisfied and when it is violated.

REFERENCES

[1] S. P. Novikov (ed.), Theory of Solitons, Consultants Bureau, New York.
[2] G. B. Whitham, Proc. R. Soc. **A283**, 238 (1965).
[3] A. V. Gurevich and L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **65**, 590 (1973) [Sov. Phys. JETP **38**, 291 (1974)].
[4] B. A. Dubrovin and S. P. Novikov. Dokl. Akad. Nauk SSSR **270**, 781 (1984).
[5] S. P. Tsarev, Dokl. Akad. Nauk SSSR **282**, 280 (1985).

L.D. LANDAU INSTITUTE OF THEORETICAL PHYSICS, ACADEMY OF SCIENCES OF THE USSR, MOSCOW