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Spectral theory of two-dimensional periodic operators and its applications

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CONTENTS

Introduction

The development of the effective spectral theory of finite-gap Sturm-Liouville operators undertaken in the series of papers by Novikov, Dubrovin, Matveev, and Its (a survey of which is given in [1], [2]; some of those results were obtained slightly later in [3], [4]) has not only enabled us to construct a wide class of periodic and quasi-periodic solutions of the

(1) Also called "finite-zone or "finite-band". (Editor)

Korteweg-de Vries equation. It has led to the revaluation of the whole approach to the development of the spectral theory of arbitrary one-dimensional linear operators with periodic solutions.

The assertion that the Bloch functions of such operators, considered for arbitrary complex values of the spectral parameter E, are values on different sheets of a Riemann surface of a single-valued (on this surface) function, which now looks self-evident, remained beyond the framework of the classical Floquet spectral theory. It has turned out that analytic properties of the Bloch functions on this Riemann surface are crucial for solving the inverse problem of recovering coefficients of the operators from the spectral data. In the case when this Riemann surface has finite genus, the solution of the inverse problem is based on the technique of classical algebraic geometry and the theory of theta-functions. (A generalization of the algebraic geometry language and theta-functions to the case of a hyperelliptic curve of infinite genus, corresponding to the Sturm–Liouville operator with general periodic potential, was obtained in [5].)

The meaning of the algebraic geometry approach was clarified completely in [6], [7] where, for the first time, a general construction for periodic solutions of space two-dimensional equations admitting a commutation relation (equations of Kadomtsev–Petviashvili (KP) type) was suggested. In the framework of this construction the inverse problem for operators of the following form was solved:

$$(1) \qquad \sigma\partial_y - L, \ \partial_t - A, \ \partial_t = \frac{\partial}{\partial t}, \ \partial_y = \frac{\partial}{\partial y},$$

where the coefficients of L and A

$$(2) \qquad L = \sum_{i=0}^{n} u_i(x, y, t)\,\partial_x^i, \ A = \sum_{j=0}^{m} v_j(x, y, t)\,\partial_x^j, \ \partial_x = \frac{\partial}{\partial x}$$

are scalar or matrix-valued functions of their arguments. These coefficients are uniquely determined by the data that characterize analytic properties on an auxiliary algebraic curve Γ (a Riemann surface of finite genus) of a function $\psi(x, y, t, Q)$, $Q \in \Gamma$, called the *Baker–Akhieser–Clebsch–Gordan function*. These analytic properties naturally generalize analytic properties of the Bloch functions of finite-gap one-dimensional periodic operators. Their specific features are such that for any function that possesses them there are always operators L and A of the form (2) such that

$$(3) \qquad (\sigma\partial_y - L)\psi(x, y, t, Q) = 0, \quad (\partial_t - A)\psi(x, y, t, Q) = 0.$$

The non-linear equations on u_i and v_j

$$(4) \qquad [\sigma\partial_y - L, \ \partial_t - A] = 0 \Longleftrightarrow L_t - \sigma A_y + [L, A] = 0,$$

equivalent to the comptability condition for the overdetermined system (3), are just KP type equations.

From the point of view of the problem of constructing solutions of non-linear equations it would be sufficient to solve the inverse problem for finite-gap operators, even without setting the direct spectral problem. (Surveys of different stages of the development of the "finite-gap theory" can be found in [1], [8]-[14]). However, such an approach left completely open the question of the role and the place of the solutions obtained in the periodic problem for the space two-dimensional equations of KP type.

In the one-dimensional case of Lax type equations

$$L_t + [L, A] = 0 \tag{5}$$

the existence of the direct and inverse spectral transforms for operators L with periodic coefficients enables us in principle to prove (though this is not always brought to the level of rigorous mathematical theorems) that the set of finite-gap solutions is dense among all smooth periodic solutions. In the two-dimensional case the situation turns out to be considerably more complicated.

One of the main purposes of this paper is the investigation of this question on the example of the periodic problem for a KP equation

$$\frac{3}{4}\sigma^2 u_{yy} + \partial_x\left(u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\right) = 0, \quad \sigma^2 = \pm 1, \tag{6}$$

which has a representation (4) (found in [14], [15]), where

$$L = \partial_x^2 - u(x, y, t), \quad A = -\partial_x^3 + \frac{3}{2}u\partial_x + w(x, y, t). \tag{7}$$

The answer is different in principle for two versions of this equation: the KP-1 equation ($\sigma^2 = -1$) and the KP-2 equation ($\sigma^2 = 1$).

As shown in [17], the periodic problem for the KP-1 equation is not integrable even formally. It will be shown below that the same problem for the KP-2 equation is integrable and any smooth periodic solution of this equation can be approximated by finite-gap solutions (this was proved locally in the author's papers [18], [19]).

This assertion follows from the spectral theory for the operator

$$M = \sigma\partial_y - \partial_x^2 + u(x, y), \quad \mathrm{Re}\ \sigma \neq 0, \tag{8}$$

with periodic potential $u(x, y)$, to the development of which the first chapter of the paper is devoted.

In an unpublished paper of Taimanov it was proved by methods completely analogous to the methods of [30] that the Bloch functions of the operator M with smooth real periodic potential, defined as solutions of the equation $M\psi = 0$, that are eigenfunctions for the operators of translation by the periods in x and y, can be parametrized (as in the one-dimensional case) by the points of a Riemann surface Γ. The multipliers $w_1(Q)$ and $w_2(Q)$, the eigenfunctions of the translation operators, are holomorphic functions on this surface, $Q \in \Gamma$. This proof is based on a theorem of Keldysh on the resolvents of a family of completely continuous operators holomorphically dependent on parameters. Unfortunately, in the framework of this approach we are unable to obtain detailed information on the structure of Γ, which is necessary for the proof of the main approximation theorem.

The approach to the construction of the Riemann surface of the Bloch functions, we suggest, has a constructive nature and is more effective. In the first section of the paper formal Bloch solutions are constructed with the help of series that are analogous to the series of perturbation theory. In the next section the convergence of these series is proved in different domains that "paste" further into a global Riemann surface. It turns out that outside any neighbourhood of "infinity" this surface has finite genus. Roughly speaking, it is this fact that enables us to approximate an arbitrary potential by finite-gap ones, that is by those potentials for which the corresponding Riemann surfaces have finite genus.

Section 3 of Chapter I is devoted to the spectral theory of finite-gap operators. In addition to the presentation of the scheme of the solution of the inverse spectral problem for such operators, we present in the same section theorems on the completeness of Bloch functions. In Section 5 we prove a theorem on the completeness of products of Bloch functions and their conjugates in the space of square-integrable functions periodic in x and y. This assertion plays a crucial role in the construction of the perturbation theory of finite-gap solutions $u_0(x, y, t)$ of the KP-2 equation. In particular, it enables us to prove that the solution given in §2 of Chapter II of the linearized KP-2 equation

$$\frac{3}{4} v_{yy} + \partial_x\left(v_t - \frac{3}{2} u_0 v_x - \frac{3}{2} u_{0x} v + \frac{1}{4} v_{xxx}\right) = 0 \tag{9}$$

form for each t a basis in the space of square-integrable periodic (in x, y) functions. Knowing this basis, it is easy to write down an asymptotic solution of the form

$$u(x, y, t) = u_0(x, y, t) + \sum_{i=1}^{\infty} \varepsilon^i w_i(x, y, t) \tag{10}$$

both for the KP-2 equation itself and for its perturbations (ε is a small parameter). By analogy with the multiphase non-linear WKB-method (the Whitham method, see [20], [21]) in the space one-dimensional case, even the requirement of uniform boundedness of the first term of the series (10) leads to the fact that the parameters $I_1, \ldots, I_N$ of a finite-gap solution must depend on the "slow" variables $X = \varepsilon x$, $Y = \varepsilon y$, $T = \varepsilon t$. Equations that describe the slow modulation $I_k = I_k(X, Y, T)$ are called *Whitham equations.* For space two-dimensional systems they were obtained for the first time in the paper [22], the results of which will be presented in the last sections of Chapter II. For these equations, which represent a system of partial differential equations on the Teichmüller space, we suggest a construction of precise solutions. In the space one-dimensional case this construction yields an effective statement of the scheme of [23], where a generalization of the "hodograph" method for the solution of "diagonalizable" Hamiltonian systems of hydrodynamic type was suggested. (The theory of Hamiltonian systems of hydrodynamic type was developed in [24], [25].)

As an important special case of an application of these results we present separately in the final section of Chapter II a construction of solutions of the Khokhlov–Zabolotskaya equation, well-known in the theory of non-linear waves,

$$\frac{3}{4}\sigma^2 u_{yy} + \partial_x\left(u_t - \frac{3}{2}uu_x\right) = 0 \tag{11}$$

(a detailed bibliography of papers devoted to this equation can be found in [26]). We note that the equation (11) is a quasi-classical limit of the KP equation.

In the final third chapter we again return to the spectral theory of two-dimensional periodic operators, this time on the example of the two-dimensional Schrödinger operator

$$H_0 = \partial_x^2 + \partial_y^2 + u(x, y). \tag{12}$$

The inverse problem for the two-dimensional Schrödinger operator with magnetic field

$$H = (\partial_x - iA_1(x, y))^2 + (\partial_y - iA_2(x, y))^2 + u(x, y), \tag{13}$$

based on the spectral data corresponding to one energy level $E = E_0$, was posed and considered in [27]. In that paper a class of operators that are "finite-gap on a given energy level" was constructed, which can be distinguished from the point of view of spectral theory by the fact that the Riemann surface of the Bloch functions corresponding to this energy level, being a "complex Fermi-curve", has a finite genus.

In [28], [29] conditions on the algebraic geometry data of the construction of [27] were found that single out smooth real potential ($A_i \equiv 0$) operators $H = H_0$. Novikov has formulated a conjecture that the corresponding potentials form a dense family among all periodic potentials $u(x, y)$.

The main aim of Chapter III is the proof of Novikov's conjecture. Again, as in the proof of the approximation theorem in Chapter I, we shall need detailed information on the structure of the Riemann surface of the Bloch functions of the operator H_0 corresponding to a fixed energy level E_0. (The existence of such a Riemann surface is proved in [30].) From the purely technical formula point of view the construction of formal Bloch solutions of the equation $H_0\psi = E_0\psi$ differs essentially from the construction of Bloch solutions of the equation $M\psi = 0$, where M is an operator of the form (8). However, in the most essential matters of principle the construction of the spectral theory of the operators (8) and (12) proceeds absolutely in parallel. This enables the author to hope that the approach developed in the framework of this paper can be applied to the construction of the spectral theory of arbitrary two-dimensional periodic linear operators.

Before we proceed to the presentation of the main material, we make two digressions. Up to now we have spoken about Riemann surfaces only in connection with the spectral theory of linear periodic differential operators. The points of those surfaces parametrize the Bloch functions, which are defined non-locally, in terms of the operator of translation by the period. We called *finite-gap operators* those operators for which the corresponding Riemann surface has finite genus. However, the initial definition in [6], [7] of "finite-gap solutions" of KP type equations was purely local. (Under such an approach it would be more correct to call such solutions *algebro-geometrical.*) They were singled out by the condition that for the corresponding operators L and A there are operators

$$L_1=\sum_{i=0}^{n_1} \tilde{u}_i(x, y, t)\partial_x^i, \quad L_2=\sum_{i=0}^{m_1} \tilde{v}_i(x, y, t)\partial_x^i, \tag{14}$$

which commute with each other

$$[L_1, L_2] = 0 \tag{15}$$

and commute with the operators (1)

$$[L_i, \sigma\partial_y - L] = 0, \quad [L_i, \partial_t - A] = 0. \tag{16}$$

This definition of "finite-gap" solutions goes back to the pioneering paper by Novikov [31], where he considered restrictions of the KdV equation to stationary solutions of "higher analogues of the KdV equation", that is, to the solutions of the commutation equation of the Sturm–Liouville operator L and an operator A_n of order $2n+1$

$$[L, A_n] = 0. \tag{17}$$

The problem of classification of commuting ordinary linear differential operators with scalar coefficients was posed for the first time and solved partially in the remarkable papers [32], [33] by Burchnall and Chaundy in the early 20's. They proved that for any such operators there is a polynomial in two variables $R(\lambda, \mu)$ such that

$$R(L_1, L_2) = 0 \tag{18}$$

In the case of operators of coprime orders $(n_1, m_1) = 1$, to each point Q of the curve Γ, defined by the equation $R(\lambda, \mu) = 0$, there corresponds a unique (up to a multiplicative constant) common eigenfunction $\psi(x, Q)$ of the operators L_1, L_2 ($y = y_0$, $t = t_0$):

$$L_1\psi(x, Q) = \lambda\psi(x, Q); \quad L_2\psi(x, Q) = \mu\psi(x, Q), \quad Q = (\lambda, \mu). \tag{19}$$

The logarithmic derivative $\psi_x\psi^{-1}$ is a meromorphic function on Γ that has in its affine part g poles $\gamma_1(x), \ldots, \gamma_g(x)$, where g is the genus of Γ. The operators L_1 and L_2 themselves (of coprime order in this case) are uniquely determined by the polynomial R and by fixing g points $\gamma_s(x_0)$ on Γ. Definitive formulae in those papers were not obtained.

The programme of effectivization of the results of [32], [33] was suggested by Baker [34], who noticed the coincidence of the analytic properties of $\psi(x, Q)$ on Γ with those taken at the end of the last century by Clebsch and Gordan as the basis of the definition of an analogue of the "exponential function" on algebraic curves (see [35]). Unfortunately Baker's program was not fulfilled and those papers were undeservedly forgotten for a long time.

In the author's papers [6], [7], where the equations (15) were considered in connection with the problem of constructing solutions of KP type equations, the results of the 20's were considerably effectivized and generalized to the case of operators with matrix coefficients. For the coefficients of commuting scalar operators of coprime orders explicit expressions in terms of the Riemann theta-function were found, which showed that the general solutions of the equation (15) in this case were quasi-periodic functions. This enabled us to connect the local theory of commuting operators with the spectral theory of the Floquet operators with periodic coefficients.

Initially the classification problem was posed in [32], [33] for operators of arbitrary orders, but it was noted that in the case when the orders are not coprime there was not even an approach to its solution. The first progress in this most complicated case was obtained in [36] on the basis of algebraization of the scheme of [6], [7]. The problem of classification of commuting operators in general position was solved completely by the author in [37]. (We note that the principal idea of this solution was suggested in the author's preceding paper [8], but its realization contained essential errors.) It turned out that such operators are uniquely determined by a polynomial R, a matrix divisor of rank r, and a set of $r-1$ arbitrary functions $w_0(x), \ldots, w_{r-2}(x)$. The recovery of the coefficients from these data reduces to the linear Riemann problem. Here r is a divisor of the orders of L_1 and L_2. It is equal to the number of linearly independent solutions of (19).

Let us give a brief description of the principal stages of the proof of the assertion just formulated, in order to present more completely the different mechanisms of the appearance of algebraic geometry constructions. (The reader interested only in the spectral theory of periodic operators can omit this part of the introduction and proceed to the contents of subsequent chapters without particular detriment to understanding the main material.)

Any two operators L_1 and L_2 with scalar coefficients satisfying (15) can be reduced, by a change of the variable x and the conjugation $L_i \to g(x)L_ig^{-1}(x)$, to the form in which $u_{n_1} = 1, u_{n_1-1} = 0, v_{m_1}(x) = v_{m_1} =$ const. This form will be assumed in what follows.

The canonical basis $c_i(x, \lambda;\ x_0)$ in the n_1-dimensional space $\mathcal{L}(\lambda)$ of solutions of the linear equation

(20) $$L_1 y(x) = \lambda y(x)$$

is usually normalized by the conditions

$$\partial_x^j c_i(x,\ \lambda;\ x_0)\,|_{x=x_0} = \delta_{i,j},\ 0 \leqslant i,\quad j \leqslant n_1 - 1.$$

By (15) the operator L_2 induces on $L(\lambda)$ a finite-dimensional linear operator $L_2(\lambda)$ whose matrix entries in the basis c_i are polynomials in λ. Therefore the characteristic polynomial

(21) $$R(\lambda,\ \mu) = \det\,(\mu \cdot 1 - L_2(\lambda))$$

is a polynomial not only in μ but also in λ. It follows from its definition that

(22) $$R(L_1,\ L_2)\ y\ (x) = 0$$

for any solution of (20). Since $R(L_1,\ L_2)$ is an ordinary linear operator, this can be satisfied only when it is zero. Therefore the first of the assertions by Burchnall and Chaundy is proved.

The equation

(23) $$R(\lambda,\ \mu) = 0$$

determines in C^2 the affine part of the curve Γ. To clarify its behaviour at infinity, we consider a formal solution of the equation

(24) $$L_1 \psi\,(x,\ k) = k^{n_1} \psi\,(x,\ k),$$

of the form

(25) $$\psi\,(x,\ k) = e^{k(x-x_0)}\left(\sum_{s=-N}^{\infty} \xi_s\,(x)\,k^{-s}\right).$$

Substituting (25) in (24) and finding successively the ξ_s, we can easily see that there is a unique solution normalized by the condition $N_0 = 0$, $\xi_0 = 1$, $\xi_s(x_0) = 0$, $s > 0$. We denote it by $\psi(x, k;\ x_0)$. Any other solution of the form (25) is uniquely representable in the form

$$\psi(x,\ k) = A(k)\psi(x,\ k;\ x_0).$$

Since the operator L_2 commutes with L_1, it follows that $L_2\psi(x,\ k;\ x_0)$ satisfies (24) and has the form (25). Therefore

(26) $$L_2 \psi\,(x,\ k;\ x_0) = \mu\,(k)\,\psi\,(x,\ k;\ x_0),$$

$$\mu\,(k) = v_{m_1} k^{m_1} + \sum_{s=-m_1+1}^{\infty} \mu_s k^{-s}.$$

We denote by $\widetilde{\mathscr{L}}(k)$ the n_1-dimensional subspace generated by the formal expressions $\psi(x, \varepsilon_j k; x_0)$, $\varepsilon_j^{n_1} = 1$, over the field of Laurent series in the variable k^{-1}. In the initial basis $\psi(x, \varepsilon_j k; x_0)$ the operator L_2 is diagonal. But if we consider in $\widetilde{\mathscr{L}}(k)$ the basis with the canonical normalization conditions, then the matrix entries of this operator in this basis coincide with the matrix entries $L_2^{ij}(\lambda, x_0)$ of the operator L_2 in the basis $c_i(x, \lambda; x_0)$, $\lambda = k^{n_1}$. Therefore

$$R(\lambda, \mu) = \prod_{j=0}^{n_1-1} (\mu - \mu(\varepsilon_j k)). \tag{27}$$

We are ready now to discuss the role of coprimeness of the orders of operators. If $(n_1, m_1) = 1$, it follows from (26) that the equation (22) for large λ, and so for almost all λ, has distinct roots. Furthermore, this means that the curve Γ is irreducible, and it is completed at infinity by a single point P_0 in a neighbourhood of which $k^{-1}(Q) = \lambda^{-1/n_1}$ is a local parameter. In this case, to each point $Q = (\lambda, \mu) \in \Gamma$ there corresponds a unique eigenvector $h(Q, x_0)$ of the matrix $L_2(\lambda, x_0)$ normalized by the condition $h_0 \equiv 1$. Its remaining coordinates $h_i(Q, x_0)$, $i = 1, \ldots, n_1 - 1$, are meromorphic functions on Γ. The function

$$\psi(x, Q; x_0) = \sum_{i=0}^{n_1-1} h_i(Q, x_0) c_i(x, \lambda; x_0), \quad Q = (\lambda, \mu), \tag{28}$$

is a unique solution of (19) under the normalization condition $\psi(x_0, Q; x_0) \equiv 1$.

We consider analytic properties of ψ on Γ. The functions c_i are entire functions of the variable λ. Therefore ψ is meromorphic on Γ outside the point P_0. Moreover, its poles $\gamma_s(x_0)$ coincide with the poles of h_i and so do not depend on x. In a neighbourhood of P_0 it has the form

$$\psi(x, Q; x_0) = e^{k(x-x_0)} \left(1 + \sum_{s=1}^{\infty} \xi_s(x) k^{-s}\right). \tag{29}$$

In the general case the curve Γ is non-singular, and the number of poles of ψ is equal to g, the genus of Γ. The last assertion follows from an examination of the function

$$F(\lambda, x_0) = [\det(\partial_x^i \psi(x, Q_j, x_0))]^2, \tag{30}$$

where the $Q_j = (\lambda, \mu_j) \in \Gamma$ are the inverse images of λ under the natural projection of Γ onto the λ-plane. It has poles of multiplicity 2 at the projections of the poles $\gamma_s(x_0)$ of ψ. Moreover, it has a pole of multiplicity $(n-1)$ at the point $\lambda = \infty$, which follows easily from (29). The zeros of F coincide with the branch points of the covering $\lambda : \Gamma \to C^1$. The equality of the number of zeros and poles of the rational function $F(\lambda, x_0)$ and the formula $2g - 2 = \nu - 2n$, which expresses the genus of an n-sheeted curve in terms of the number ν of branch points, enable us to obtain the desired assertion on the number of poles of ψ.

Thus the common eigenfunction $\psi(x, Q; x_0)$ of the commuting operators L_1 and L_2 is defined on Γ, outside P_0 it has poles $\gamma_1, \ldots, \gamma_g$ not depending on x, and it can be represented in a neighbourhood of P_0 in the form (29). Such functions are called *Clebsch–Gordan–Baker–Akhiezer functions* (more often for brevity they will be called simply *functions of Baker–Akhiezer type*).

The construction of the inverse correspondence, that is, the recovery of the whole commutative ring $\mathcal{A}$ generated by a pair of commuting operators of coprime orders from a non-singular curve Γ with a distinguished point P_0 and a collection of g points in general position, consists of two key stages. The first is the proof of the fact that for any such collection $(\Gamma, P_0, \gamma_1, \ldots, \gamma_g)$ there is a unique corresponding Baker–Akhiezer function. This assertion can easily be obtained with the help of the usual Riemann–Roch theorem. We omit it because we can not only prove the existence and uniqueness of ψ but also obtain explicit expressions for it in terms of the Riemann theta-function. (These expressions in a more general situation will be constructed in §3 of Chapter I.)

The second crucial point is the proof of the fact that for any function $A(Q)$ that has on Γ a pole only at P_0 (the ring of such functions is denoted by $\mathcal{A}(\Gamma, P_0)$), there is a unique operator L_A such that

$$L_A\psi(x, Q; x_0) = A(Q)\psi(x, Q; x_0). \tag{31}$$

The degree of L_A is equal to the order of the pole of $A(Q)$. For the proof of this assertion it is sufficient to prove the existence and uniqueness of the Baker–Akhiezer function. Since it is typical for finite-gap integration, we present it briefly.

For any formal series of the form (29) there is a unique operator L_A such that

$$\begin{gathered}(L_A - A(Q))\,\psi(x, k; x_0) \equiv O(k^{-1})\,e^{k(x-x_0)},\\ A(Q) = a_{-n}k^n + a_{-n+1}k^{n-1} + \ldots\end{gathered} \tag{32}$$

The coefficients of L_A can be found successively if we substitute in (32) the formal series (29) and the expansion of $A(Q)$ in a neighbourhood of P and equate to zero the coefficients of k^s, $s = n, n-1, \ldots, 0$, on the left-hand side. We consider the function $\widetilde{\psi} = L_A\psi(x, Q; x_0) - A(Q)\psi(x, Q; x_0)$, where L_A is the operator just constructed. Since the poles of ψ do not depend on x, it follows that $\widetilde{\psi}$ satisfies all but one of the requirements that define a Baker–Akhiezer function. As follows from (32), the constant term of the pre-exponential factor in its expansion in a neighbourhood of P_0 is equal to zero. It follows from the uniqueness of ψ that $\widetilde{\psi} \equiv 0$, and (31) is proved. It follows that all such operators commute with each other. We emphasize once more that the quasi-periodicity of the coefficients of these operators and the coincidence of the Baker–Akhiezer functions with the Bloch functions are consequences of explicit theta-function formulae for $\psi(x, Q; x_0)$.

From the technical point of view the problem of classifying commuting operators of arbitrary orders is considerably more complicated, but it is close in spirit to the case just treated of operators of coprime orders. In the general case the series $\mu(\varepsilon_j k)$ can take the same values for different ε_j only in the case when $\mu(k) = \widetilde{\mu}(k^r)$. Moreover, as follows from (26), the number r is necessarily a common divisor of the orders n_1 and m_1 of the operators L_1 and L_2. Hence the polynomial R is then equal to

$$R(\lambda, \mu) = \prod_{j=0}^{n'-1} (\mu - \widetilde{\mu}(\widetilde{\varepsilon_j k}))^r = (\widetilde{R}(\lambda, \mu))^r, \tag{33}$$

where $(\widetilde{\varepsilon_j k})^{n'} = \lambda$, $n'r = n_1$.

We keep the notation Γ for the curve given now by the irreducible equation $\widetilde{R}(\lambda, \mu) = 0$. At infinity this curve is completed by a single point in a neighbourhood of which $\lambda^{-1/n'}(Q)$ serves as a local parameter. It follows from (33) that in a neighbourhood of infinity, and so everywhere, to each point Q of Γ there corresponds the r-dimensional space $L_2(\lambda, x_0)$ of eigenvectors with eigenvalue μ, $Q = (\lambda, \mu)$. We choose in this space a basis $h^i(Q, x_0)$, $i = 0, \ldots, r-1$, with the no normalization conditions

$$h_j^i(Q, x_0) = \delta_{ij}, \quad 0 \leqslant i, \quad j \leqslant r-1. \tag{34}$$

All other coordinates h_i^j, $j = r, \ldots, n_1 - 1$, of the vectors h^i are meromorphic functions on Γ. The functions

$$\psi_i(x, Q; x_0) = \sum_j h_j^i(Q, x_0)\, c_j(x, \lambda, x_0) \tag{35}$$

form a basis in the space of solutions of (19), normalized by

$$\partial_x^j \psi_i(x, Q; x_0)|_{x=x_0} = \delta_{ij}, \quad 0 \leqslant i, \quad j \leqslant r-1. \tag{36}$$

The number r is called the rank of the commuting pair L_1 and L_2 (or of the whole commutative ring $\mathcal{A}$ generated by L_1 and L_2).

The vector-valued functions $h^i(Q, x_0)$ determine in the trivial bundle over Γ an algebraic r-dimensional subbundle $\hat{h}(x_0)$. It is the starting point of the investigations of [36]. How can we find the dependence $\hat{h}(x_0)$? For $r = 1$ it was determined by differential equations and its properties played an important role in [1], [2], [37] and other papers. For $r > 1$, as shown in [38], the situation becomes considerably more complicated. "Possible" movements of $\hat{h}$ turn out to be covered by a non-integrable r-distribution on the space of modules of r-dimensional sheaves over Γ with a fixed flag at P_0. The variation of the normalization point x_0 determines a path tangential to this distribution. At this point the investigations of [36], [38] terminate.

Our method consists not in the description of x_0-variations of the sheaf but in finding the eigenfunctions $\psi_i(x, Q; x_0)$, $x_0 = \text{const}$, themselves from their analytic properties. Again, as in the case $r = 1$, the functions ψ_i are meromorphic on Γ outside P_0. By analogy with the calculation of poles of the Baker–Akhiezer function, it can be shown that in general position the ψ_i have poles at rg points $\gamma_s(x_0)$. Moreover the residues of these functions satisfy the relations

$$\alpha_{sj}(x_0)\,\mathrm{res}_{\gamma_s}\psi_i = \alpha_{si}(x_0)\,\mathrm{res}_{\gamma_s}\psi_j, \tag{37}$$

where the constants $\alpha_{si}(x_0)$ do not depend on x (but depend on the normalization point x_0). The collection (γ_s, α_{si}), where α_{si} is an r-dimensional vector defined up to proportionality, that is, $\alpha_s \in CP^{r-1}$, are called *Tyurin parameters*. They characterize ([39]) "matrix divisors" determined by stable r-dimensional sheaves over Γ with a fixed "frame", that is, a set of basic sections.

To determine the behaviour of ψ_i in a neighbourhood of P_0 we consider the matrix $\Psi(x, Q; x_0)$ with matrix entries $\Psi_j^i = \partial_x^i \psi_j(x, Q; x_0)$. Its logarithmic derivative does not depend on the choice of a basis in the space of solutions of (19). Therefore in a neighbourhood of P_0 it can be computed with the help of the series (25) $\psi(x, \varepsilon_j k'; x_0)$, $(\varepsilon_j)^r = 1$, $(k')^r = k$, $j = 0, \ldots, r-1$, where $k^{-1}(Q)$ is a local parameter. We obtain

$$(\partial_x \Psi)\,\Psi^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0, & 0 & \ldots & 0 & 1 \\ k+\tilde{w}_0 & \tilde{w}_1, & \tilde{w}_2 & \ldots & \tilde{w}_{r-2} & 0 \end{pmatrix} + O(k^{-1}). \tag{38}$$

The functions $\widetilde{w}_i(x_0)$ are differential polynomials in the coefficients of the operator L_1.

We define an entire function $\Psi_0(x, k; x_0)$ of the parameter k by requiring that in a neighbourhood of $k = \infty$ it is representable in the form

$$\Psi_0 = \left(\sum_{s=0}^{\infty} \chi_s k^{-s}\right) \Psi(x, k(Q); x_0). \tag{39}$$

The problem of finding Ψ_0 is the Riemann problem of factorizing Ψ on a contour surrounding a small neighbourhood of P_0. It reduces to a system of singular integral equations and has for almost all x a unique solution normalized by the condition $\chi_0 \equiv 1$. It follows from (38) that

$$(\partial_x \Psi_0)\,\Psi_0^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ k+w_0 & w_1 & w_2 & \ldots & w_{r-2} & 0 \end{pmatrix}. \tag{40}$$

For it follows from (38) and (39) that $\Psi_{0x}\Psi_0^{-1}$ has the form (40) in a neighbourhood of $k = \infty$ up to $O(k^{-1})$. Since $\det \Psi_0 = \det \Psi = 1$, this logarithmic derivative is holomorphic outside $k = \infty$. Therefore the equality (40) holds precisely.

Inverting the equality (39), we find that the row-vector ψ with coordinates ψ_i has in a neighbourhood of P_0 the form

$$\psi(x, Q; x_0) = \Big(\sum_{s=0}^{\infty} \xi_s(x, x_0) k^{-s}\Big) \Psi_0(x, k, x_0), \tag{41}$$

where the ξ_s are row-vectors, $\xi_0 = (1, 0, ..., 0)$, and Ψ_0 is determined by (40) and the initial condition $\Psi_0(x_0, k; x_0) = 1$.

A vector-valued function $\psi(x, Q; x_0) = (\psi_0, \ldots, \psi_{r-1})$, meromorphic outside P_0, having rg poles γ_s, satisfying (37), and representable in the form (41) in a neighbourhood of P_0, is called a *vector analogue of Baker–Akhiezer functions* corresponding to the set of data

$$(\Gamma, P_0, \gamma_s, \alpha_s, w_0(x), \ldots, w_{r-2}(x)). \tag{42}$$

Here the $w_i(x)$ are arbitrary functions. (For $r = 1$ we have the usual Baker–Akhiezer functions.)

The inverse problem of recovering commuting operators of rank r can be solved again in two stages. First we can prove that for the data (42) in general position there exists a unique vector-valued function corresponding to them. Its construction reduces to the Riemann problem on Γ of factorizing Ψ_0 on a small contour around P_0. A method of solving matrix Riemann problems on arbitrary algebraic curves was developed in [40], [41].

It follows directly from (40) and (41) that for any function $A(Q)$ there is a unique operator L_A of degree rn, where n is the order of the pole of $A(Q)$, such that

$$(L_A - A(Q))\psi \equiv O(k^{-1})\Psi_0. \tag{43}$$

It follows from the uniqueness of the vector analogue of a Baker–Akhiezer function that each component ψ_i satisfies (31).

The correspondence

$$\hat{L}: A \to L_A$$

determines a homomorphism of the ring $\mathcal{A}(\Gamma, P_0)$ of functions on Γ with a single pole at the distinguished point P_0 to the ring of ordinary differential operators. This homomorphism is determined by a set of data (42) in general position.

Summarizing what we have said above, we arrive at the definitive statement of the classification theorem.

Theorem [37]. *For any commutative ring $\mathcal{A}$ of differential operators there is a curve Γ with a distinguished point P_0 such that $\mathcal{A}(\Gamma, P_0)$ is isomorphic to $\mathcal{A}$. For almost all rings $\mathcal{A}$ the curve Γ is non-singular. Moreover, there is a matrix divisor (γ_s, α_s), $s = 1, ..., rg$, where g is the genus of Γ, and a collection of functions $w_0(x), \ldots, w_{r-2}(x)$ such that the image of the homomorphism $\hat{L}$ determined by them coincides with $\mathcal{A}$ up to the change of variable $x = f(x')$ and conjugation by some function: $\mathcal{A} = \varphi(x)\operatorname{Im}\hat{L}\varphi^{-1}(x)$.* ***The number r is the greatest common divisor of the orders of operators in $\mathcal{A}$.***

In some cases, as shown in [42], we can avoid the necessity of solving the Riemann problem and obtain explicit formulae for the coefficients of commuting operator of rank $r > 1$. In particular, an operator L of order 4 commuting with an operator of order 6 has the form

$$(44)\quad L=(\partial_x^2+u)^2+c_x(\wp(\gamma_2)-\wp(\gamma_1))+ \\ +\partial_x(c_x(\wp(\gamma_2)-\wp(\gamma_1))-\wp(\gamma_2)-\wp(\gamma_1)),$$

$$\gamma_1=c(x)+y,\quad \gamma_2=y-c(x)+c_0,$$

$$8u=(c_{xx}^2-1)c_x^{-2}+8\Phi c_{xx}+4c_x\left(\frac{\partial\Phi}{\partial c}-\Phi\right)-2c_{xxx}c_x^{-1},$$

$$\Phi(c,\ y)=\zeta(-2c)+\zeta(c-y)+\zeta(c+y),$$

where $c(x)$ is an arbitrary function: ζ, $\wp$ are the Weierstrass functions [43].

We omit further details of the theory of commuting operators of rank $r > 1$, since they will not be used in the main part of the paper (in contrast with the construction of rank 1). We mention only the paper [44], where the spectral theory of "finite-gap" periodic operators of rank 2 was constructed, and the papers [10], [42], [45], where a multiparametric generalization of vector analogues of Baker–Akhiezer functions was introduced and with their help solutions of the KP equations were constructed.

To conclude this section, we characterize briefly a construction of solutions of equations that belong to the "KP hierarchy", which was suggested in the series of papers [46] and developed in [47]. This construction was based on a formal generalization of the "local" approach to the axiomatics of Baker–Akhiezer functions of rank 1.

Consider a formal series $\psi(x_1, x_2, x_3, \ldots; k)$ of the form

$$(45)\qquad \psi(\vec{x};\ k)=\exp\left(\sum_{i=1}^{\infty}x_ik^i\right)\left(1+\sum_{s=1}^{\infty}\xi_s(\vec{x})\,k^{-s}\right).$$

For any such formal series there are unique differential operators L_n, $n = 2, 3, \ldots$, in the variable $x = x_1$ (whose coefficients depend on all variables $x_1, x_2, x_3, \ldots$) such that

$$(46)\qquad \left(\frac{\partial}{\partial x_n}-L_n\right)\psi(\vec{x},\ k)=O(k^{-1})\exp\left(\sum_{i=1}^{\infty}x_ik^i\right).$$

The order of L_n is equal to n. Its coefficients (like the coefficients in the construction of commuting operators of rank 1) can be found by successively equating to zero the coefficients of k^s, $s = n, n-1, \ldots, 0$, of the pre-exponential factor on the left-hand side of (46).

In the case when the series (45) is not arbitrary but satisfies the property that its pre-exponential factor converges to a function holomorphic in a neighbourhood of $k = \infty$ and the function itself extends analytically to some algebraic curve of genus g and has g poles there, the relation (46) turns into the precise equality

$$\left(\frac{\partial}{\partial x_n} - L_n\right) \psi(\vec{x}, k) = 0. \tag{47}$$

The conditions of compatibility of the linear equations (47)

$$\left[\frac{\partial}{\partial x_n} - L_n, \frac{\partial}{\partial x_m} - L_m\right] = 0 \tag{48}$$

are just the so-called "KP hierarchy".

It turns out that (47) follows from (46) not only when (45) is an expansion of a multi-parametric Baker–Akhiezer function but also in a more general situation. The corresponding series in the construction of [46], [47] were uniquely determined by the points W of the universal Grassmann manifold. Unfortunately, in the framework of this approach solutions that are interesting from the physical point of view with controllable global analytic properties were not found, except for "finite-gap solutions of rank 1" (which are quasi-periodic functions) and their various degenerations (multi-soliton, rational, and others).

We note that the solutions of the KP equation constructed in [48], [49] are also a special case of general solutions of [46], [47]. It should be emphasized that their construction, which uses tensor fields of Baker–Akhiezer type, enables us to prove that they are "asymptotically finite-gap".

The question of constructing an analogue of the construction [46], [47], in the case of vector-valued Baker–Akhiezer functions that arise in the theory of commuting operators of rank $r > 1$, is still open.

The proof of Novikov's conjecture in the Schottky problem is an important mathematical application of the theory of commuting operators of rank 1 and of the theory of the KP equation. In the author's paper [7] the formula

$$u(x, y, t) = 2\partial_x^2 \log \theta(Ux + Vy + Wt + \zeta | B) \tag{49}$$

was obtained for finite-gap solutions of the KP equation. Here $\theta(z_1, \ldots, z_g | B)$ is the Riemann theta-function constructed from the matrix B of b-periods of holomorphic differentials on an algebraic curve Γ. The vectors U, V, W are determined by the distinguished point P_0. The vector ζ is arbitrary.

The Riemann–Schottky problem consists in describing symmetric matrices B with positive definite imaginary part that are the matrices of b-periods of algebraic curves. Novikov's conjecture was that the function $u(x, y, t)$ given by (49) satisfies the KP equation if and only if B is the matrix of b-periods of some curve Γ. Thus all the necessary relations on B can be obtained by substituting (49) in (6). This conjecture was already partially proved in [50], where the corresponding equations on B were derived and it was proved that they determine an algebraic variety, one of the connected components of which coincides with the variety of the matrices of b-periods.

Novikov's conjecture was completely proved in [51]. The crucial point in [51] is the proof of the fact that if $u(x, y, t)$ of the form (49) satisfies the KP equation, then there are vectors U^s, $s > 3$, such that the function

$$u(x_1, \ldots, x_n, \ldots) = 2\partial_x^2 \log \theta\left(\sum_{s=1}^{\infty} U^s x_s | B\right) \tag{50}$$

determines solutions of the whole KP hierarchy ($x = x_1$, $y = x_2$, $t = x_3$). Since among the vectors U^s there cannot be more than g linearly independent ones, it follows that among the linear combinations of the operators L_n there are two commuting operators of coprime orders and so by [7] B is the matrix of b-periods corresponding to these commuting operators of the curve

Chapter I

THE SPECTRAL THEORY OF THE NON-STATIONARY SCHRÖDINGER OPERATOR

§1. The perturbation theory for formal Bloch solutions

By *Bloch solutions* $\psi(x, y, w_1, w_2)$ of the non-stationary Schrödinger equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\psi = 0 \tag{1.1}$$

with periodic potential $u(x, y) = u(x + l_1, y) = u(x, y + l_2)$ we mean solutions that are eigenfunctions of operators of translation by the periods in x and y, that is,

$$\begin{aligned} \psi(x + l_1, y, w_1, w_2) &= w_1\psi(x, y, w_1, w_2); \\ \psi(x, y + l_2, w_1, w_2) &= w_2\psi(x, y, w_1, w_2). \end{aligned} \tag{1.2}$$

The Bloch functions will always be assumed to be normalized so that $\psi(0, 0, w_1, w_2) = 1$. The set of pairs $Q = (w_1, w_2)$ for which there are Bloch solutions will be denoted by Γ and will be called the *spectral Floquet set*. (For brevity the corresponding Bloch functions will be denoted by $\psi(x, y, Q)$, $Q \in \Gamma$.) The multi-valued functions $p(Q)$ and $E(Q)$ on Γ defined by

$$w_1 = e^{ipl_1}, \quad w_2 = e^{iEl_2}, \tag{1.3}$$

are called the *quasi-momentum* and *quasi-energy* respectively. If Γ is a smooth analytic manifold, then the differentials dp and dE are single-valued holomorphic differentials. Their periods with respect to any cycle on Γ are multiples of $2\pi/l_1$ and $2\pi/l_2$ respectively.

Suppose that to each point $Q = (w_1, w_2) \in \Gamma$ there corresponds a Bloch solution $\psi^+(x, y, Q)$ of the equation conjugate to (1.1)

$$(-\sigma\partial_y - \partial_x^2 + u(x, y))\psi^+ = 0 \tag{1.4}$$

such that

$$\psi^+(x+l_1,\ y,\ Q)=w_1^{-1}\psi^+(x,\ y,\ Q),$$
$$\psi^+(x,\ y+l_2,\ Q)=w_2^{-1}\psi^+(x,\ y,\ Q). \tag{1.5}$$

Then the following assertion is true.

***Lemma* 1.1.** *The following equality holds*:

$$\sigma\, dE\, \langle\psi\psi^+\rangle_x = dp\, \langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y. \tag{1.6}$$

(Here and in what follows $\langle\cdot\rangle_x$ and $\langle\cdot\rangle_y$ denote the mean values in x and y respectively.)

The equality (1.6) for the case of finite-gap operators was obtained for the first time in [52]. A generalization of it to the case of operators of arbitrary order with matrix coefficients is contained in [22].

Proof. Let $\widetilde{\psi} = \psi(x, y, \widetilde{Q})$ and $\psi^+ = \psi^+(x, y, Q)$, where Q and $\widetilde{Q}$ are arbitrary points of Γ. It follows from (1.1) and (1.4) that

$$\sigma\partial_y(\widetilde{\psi}\psi^+) = \partial_x(\widetilde{\psi}_x\psi^+ - \widetilde{\psi}\psi_x^+). \tag{1.7}$$

Averaging (1.7) in x and y and making $\widetilde{Q}$ tend to Q, we obtain the desired equality with the help of (1.2) and (1.5).

The gauge transform $\psi \to e^{\alpha(y)}\psi$, where $\partial_y\alpha(y)$ is a periodic function, transfers the solutions of (1.1) into solutions of the same equation but with another potential $\widetilde{u} = u(x, y) - \sigma\partial_y\alpha$. Consequently, the spectral sets corresponding to the potentials u and $\widetilde{u}$ are isomorphic. Therefore in what follows we restrict ourselves to the case of periodic potentials satisfying the condition

$$\langle u(x,\ y)\rangle_x = 0. \tag{1.8}$$

The main purpose of this section is to construct the perturbation theory for formal Bloch solutions of (1.1), which enables us to express these solutions in terms of the basis data $\psi_n(x, y)$ of Bloch solutions of the "unperturbed" equation (1.1) with some potential $u_0(x, y)$. More precisely, we fix a complex number w_1. The sequence of Bloch solutions

$$\psi_n = \psi_n(x,\ y) = \psi(x,\ y,\ Q_n),\quad Q_n = (w_1,\ w_{2n}) \in \Gamma_0, \tag{1.9}$$

of the equation (1.1) with $u = u_0(x, y)$ will be called a *basic sequence* if any continuously differentiable function $f(x)$ such that

$$f(x + l_1) = w_1 f(x), \tag{1.10}$$

can be represented as a convergent series

$$f(x) = \sum_n r_n(y)\,\psi_n(x,\ y). \tag{1.11}$$

An important example. Let $u_0 \equiv 0$. Then for any complex number w_1 the functions

$$\psi_n = \exp(ik_n x - \sigma^{-1} k_n^2 y) \tag{1.12}$$

form a basic sequence, where the k_n are the roots of the equation

$$w_1 = e^{ik_n l_1}, \text{ that is, } k_n = k_0 + \frac{2\pi}{l_1} n. \tag{1.13}$$

Besides the ψ_n we shall need a "dual sequence"

$$\psi_n^+ = \psi^+(x, y, Q_n) \tag{1.14}$$

of Bloch solutions of the formally conjugate equation

$$(\sigma\partial_y + \partial_x^2 - u_0(x, y))\psi_n^+ = 0 \tag{1.15}$$

that satisfy the orthogonality conditions

$$\langle \psi_n \psi_m^+ \rangle_x = \langle \psi_n \psi_n^+ \rangle \delta_{n,m}. \tag{1.16}$$

Having at our disposal the sequences ψ_n and ψ_n^+, we can easily construct in the "resonance-free case", that is when

$$w_{20} \neq w_{2n}, \quad n \neq 0, \tag{1.17}$$

a Bloch solution $\widetilde{\psi}(x, y, Q_0)$ of (1.1) as a formal series

$$\widetilde{\psi}(x, y, Q_0) = \sum_{s=0}^{\infty} \widetilde{\varphi}_s(x, y, Q_0), \quad \widetilde{\varphi}_0 = \psi_0. \tag{1.18}$$

This series describes a "perturbation" of the Bloch solution ψ_0 of the non-perturbed equation. (Here and in what follows series of the type (1.18) are taken in powers of the small formal parameter δu.)

***Lemma* 1.2.** *If* (1.17) *is satisfied, then there is a unique formal series*

$$F(y, Q_0) = \sum_{s=1}^{\infty} F_s(y, Q_0) \tag{1.19}$$

such that the equation

$$(\sigma\partial_y - \partial_x^2 + u_0 + \delta u)\Psi(x, y, Q_0) = F(y, Q_0)\Psi(x, y, Q_0) \tag{1.20}$$

has a formal solution of the form

$$\Psi(x, y, Q_0) = \sum_{s=0}^{\infty} \varphi_s(x, y, Q_0), \quad \varphi_0 = \psi_0 = \psi(x, y, Q_0), \tag{1.21}$$

satisfying the conditions

$$\langle \psi_0^+ \Psi \rangle_x = \langle \psi_0^+ \psi_0 \rangle_x, \tag{1.22}$$

$$\begin{aligned} \Psi(x + l_1, y, Q_0) &= w_1 \Psi(x, y, Q_0), \\ \Psi(x, y + l_2, Q_0) &= w_{20} \Psi(x, y, Q_0). \end{aligned} \tag{1.23}$$

The corresponding solution is unique. The terms of (1.21) *and the* F_s *are given by the recursion formulae* (1.25)–(1.29).

We note that it follows from the uniqueness of F and from (1.23) that the function $F(y, Q_0)$ is periodic in y.

Proof. The equation (1.20) is equivalent to the system of equations

$$(\sigma\partial_y - \partial_x^2 + u_0)\varphi_s = \sum_{i=1}^{s} F_i\varphi_{s-i} - \delta u\varphi_{s-1}. \tag{1.24}$$

Since ψ_n is a basic sequence, the desired functions φ_s can be represented in the form

$$\varphi_s = \sum_n c_n^s(y, Q_0)\psi_n(x, y), \quad c_n^0 = \delta_{n,0}. \tag{1.25}$$

The requirement (1.22) is equivalent to the fact that

$$c_0^s = 0, \quad s \geqslant 1. \tag{1.26}$$

Substituting (1.25) in (1.24) and equating the coefficients in ψ_n, $n \neq 0$, in the expansions in ψ_n on the left-hand and right-hand sides of this equality, we obtain

$$\sigma\partial_y c_n^s = \sum_{i=1}^{s-1} F_i c_n^{s-i} - \frac{\langle\psi_n^+\delta u\varphi_{s-1}\rangle_x}{\langle\psi_n\psi_n^+\rangle_x}, \quad n \neq 0. \tag{1.27}$$

This equation together with the condition $w_{2n}c_n^s(y + l_2) = w_{20}c_n^s(y)$, equivalent to (1.23), uniquely determines the c_n^s (and so the φ_s):

$$c_n^s(y, Q_0) = \sigma^{-1}\frac{w_{2n}}{w_{20} - w_{2n}}\int_y^{y+l_2}\left(\sum_{i=1}^{s-1} F_i c_n^{s-i} - \frac{\langle\psi_n^+\delta u\varphi_{s-1}\rangle_x}{\langle\psi_n^+\psi_n\rangle_x}\right)dy'. \tag{1.28}$$

It follows from (1.26) that the coefficient of ψ_0 in the expansion of the right-hand side of (1.24) is equal to zero. Therefore

$$F_s(y, Q_0) = \frac{\langle\psi_0^+\delta u\varphi_{s-1}\rangle_x}{\langle\psi_0^+\psi_0\rangle_x}. \tag{1.29}$$

The proof of the lemma is completed.

Corollary. *The formula* (1.30)

$$\tilde\psi(x, y, Q_0) = \exp\left(-\sigma^{-1}\int_0^y F(y', Q_0)\,dy'\right)\frac{\Psi(x, y, Q_0)}{\Psi(0, 0, Q_0)} \tag{1.30}$$

determines a formal Bloch solution of the equation (1.1)

$$\tilde\psi(x + l_1, y, Q_0) = w_1\tilde\psi(x, y, Q_0), \tag{1.31}$$

$$\tilde\psi(x, y + l_2, Q_0) = \tilde w_{20}\tilde\psi(x, y, Q_0), \tag{1.32}$$

where the corresponding multiplier $\tilde w_{20}$ *is equal to*

$$\tilde w_{20} = w_{20}\exp\left(-\sigma^{-1}\int_0^{l_2} F(y', Q_0)\,dy'\right). \tag{1.33}$$

In the stationary case, when u does not depend on y, the preceding formulae turn into the usual formulae of the perturbation theory of eigenfunctions corresponding to the simple eigenvalues. The condition (1.17), as we said above, is an analogue of the condition of simplicity of an eigenvalue of an operator. In those cases when it is violated, it is necessary to proceed along the same lines as in the perturbation theory of multiple eigenvalues.

As the set of indices corresponding to the resonances we can take an arbitrary set of integers $I \subset \mathbf{Z}$ such that

$$w_{2\alpha} \neq w_{2n}, \quad \alpha \in I, \quad n \notin I \tag{1.34}$$

(up to the end of this section, integral indices belonging to I will be denoted by Greek letters, and all the others by Latin).

Lemma 1.3. *There are unique formal series*

$$F^{\alpha}_{\beta}(y, w_1) = \sum_{s=1}^{\infty} F^{\alpha}_{\beta s}(y, w_1) \tag{1.35}$$

such that the equations

$$(\sigma\partial_y - \partial_x^2 + u_0 + \delta u)\, \Psi^{\alpha}(x, y, w_1) = \sum_{\beta} F^{\alpha}_{\beta}(y, w_1)\, \Psi^{\beta}(x, y, w_1) \tag{1.36}$$

have formal Bloch solutions of the form

$$\Psi^{\alpha} = \sum_{s=0}^{\infty} \varphi^{\alpha}_{s}(x, y, w_1), \quad \varphi^{\alpha}_{0} = \psi_{\alpha} = \psi(x, y, Q_{\alpha}), \tag{1.37}$$

$$\Psi^{\alpha}(x + l_1, y, w_1) = w_1 \Psi^{\alpha}(-x, y, w_1), \tag{1.38}$$

$$\Psi^{\alpha}(x, y + l_2, w_1) = w_{2\alpha} \Psi^{\alpha}(x, y, w_1), \tag{1.39}$$

satisfying the conditions

$$\langle \psi^{+}_{\beta} \Psi^{\alpha} \rangle_x = \delta_{\alpha,\beta} \langle \psi^{+}_{\alpha} \psi_{\alpha} \rangle_x. \tag{1.40}$$

The corresponding solutions Ψ^{α} *are unique and given by* (1.41)–(1.43).

The proof of the lemma is completely analogous to the proof of Lemma 1.2, which is a special case of it. Therefore we only give definitive formulae for the $F^{\alpha}_{\beta s}$ and the coefficients of the series:

$$\varphi^{\alpha}_{s}(x, y, w_1) = \sum_{n \notin I} c^{s,\alpha}_{n}(y, w_1)\, \psi_n(x, y), \quad s \geqslant 1. \tag{1.41}$$

We have

$$c^{s,\alpha}_{n} = \sigma^{-1} \frac{w_{2n}}{w_{2\alpha} - w_{2n}} \int_{y}^{y+l_2} \left(\sum_{\beta} \sum_{i=1}^{s-1} F^{\alpha}_{\beta i} c^{s-i,\beta}_{n} - \frac{\langle \psi^{+}_{n} \delta u \varphi^{\alpha}_{s} \rangle_x}{\langle \psi^{+}_{n} \psi_n \rangle_x} \right) dy', \tag{1.42}$$

$$F^{\alpha}_{\beta s}(y, w_1) = \frac{\langle \psi^{+}_{\beta} \delta u \varphi^{\alpha}_{s-1} \rangle_x}{\langle \psi^{+}_{\beta} \psi_{\beta} \rangle_x}. \tag{1.43}$$

We define the matrix $T^{\alpha}_{\beta}(y, w_1)$ by the equation

$$\sigma T_y + TF = 0, \quad T(0) = 1. \tag{1.44}$$

A formal solution of this equation can be found in the form

$$T(y, w_1) = \sum_{s=0}^{\infty} T_s(y, w_1), \quad T_0 = 1, \tag{1.45}$$

where the T_s, $s \geqslant 1$, are given by the recursion formulae

$$T_s = -\sigma^{-1} \int_0^y \Big(\sum_{i=1}^{s-1} T_i(y', w_1) F_{s-i}(y', w_1) \Big) dy'. \tag{1.46}$$

The functions

$$\hat{\Psi}^{\alpha}(x, y, w_1) = \sum_{\beta} T^{\alpha}_{\beta}(y, w_1) \Psi^{\beta}(x, y, w_1) \tag{1.47}$$

are solutions of (1.1). Under the translation by the period in x they are multiplied by w_1, while under the translation by the period in y they are transformed as follows:

$$\hat{\Psi}^{\alpha}(x, y + l_2, w_1) = \sum_{\beta} \hat{T}^{\alpha}_{\beta}(w_1) w_{2\beta} \hat{\Psi}^{\beta}(x, y, w_1), \quad \hat{T}(w_1) = T(l_2, w_1). \tag{1.48}$$

It is natural to call a finite collection of formal solutions $\hat{\Psi}^{\alpha}$ *quasi-Bloch*, since it remains invariant under the translations by the periods in x and y. The characteristic equation

$$R(w_1, \tilde{w}_2) = \det(\tilde{w}_2 \delta^{\alpha}_{\beta} - \hat{T}^{\alpha}_{\beta}(w_1) w_{2\beta}) = 0 \tag{1.49}$$

is an analogue of the "secular equation" in the ordinary perturbation theory of multiple eigenvalues.

Corollary. *Let $h_{\alpha}(w_1, \tilde{w}_2)$ be an eigenvector of the matrix $\hat{T}^{\alpha}_{\beta}(w_1) w_{2\beta}$, normalized so that*

$$\sum_{\alpha} h_{\alpha}(\tilde{Q}) \hat{\Psi}^{\alpha}(0, 0, w_1) = 1, \quad \tilde{Q} = (w_1, \tilde{w}_2). \tag{1.50}$$

Then

$$\tilde{\psi}(x, y, \tilde{Q}) = \sum_{\alpha} h_{\alpha}(Q) \hat{\Psi}^{\alpha}(x, y, w_1) \tag{1.51}$$

is a formal Bloch solution of (1.1) *with multipliers w_1 and $\tilde{w}_2$, where $\tilde{w}_2$ is a root of the equation* (1.47), *normalized in the standard way.*

By analogy with the above we can construct formal Bloch solutions for the equation (1.4) formally conjugate to (1.1).

***Lemma* 1.4.** *If the conditions* (1.34) *are satisfied, then there are unique formal series*

$$F_{\beta}^{+\alpha}(y, w_1) = \sum_{s=1}^{\infty} F_{\beta s}^{+\alpha}(y, w_1) \tag{1.52}$$

such that the equations

$$(\sigma\partial_y + \partial_x^2 - u_0 - \delta u)\,\Psi^{+\alpha}(x, y, w_1) = \sum_{\beta} F_{\beta}^{+\alpha}(y, w_1)\,\hat{\Psi}^{\beta}(x, y, w_1) \tag{1.53}$$

have formal Bloch solutions of the form

$$\Psi^{+\alpha} = \sum_{s=0}^{\infty} \varphi_s^{+\alpha}(x, y, w_1), \quad \varphi_0^{\alpha} = \psi_{\alpha}^{+} = \psi^{+}(x, y, Q_{\alpha}), \tag{1.54}$$

$$\Psi^{+\alpha}(x + l_1, y, w_1) = w_1^{-1}\Psi^{+\alpha}(x, y, w_1), \tag{1.55}$$

$$\Psi^{+\alpha}(x, y + l_2, w_1) = w_{2\alpha}^{-1}\Psi^{+\alpha}(x, y, w_1), \tag{1.56}$$

satisfying the conditions

$$\langle \Psi^{+\alpha}\psi_{\beta}\rangle_x = \delta_{\beta}^{\alpha}\,\langle \psi_{\alpha}\psi_{\alpha}^{+}\rangle_x. \tag{1.57}$$

The corresponding solutions are unique and given by

$$\varphi_s^{+\alpha} = \sum_{n\in I} c_n^{+s,\alpha}(y, w_1)\,\psi_n^{+}(x, y), \quad s \geqslant 1, \tag{1.58}$$

$$c_n^{+s,\alpha} = \sigma^{-1}\frac{w_{2\alpha}}{w_{2\alpha} - w_{2n}} \int_y^{y+l_2} dy' \left(\sum_{\beta}\sum_{i=1}^{s-1} F_{\beta i}^{+\alpha} c_n^{+s-i,\beta} - \frac{\langle \psi_n \delta u \varphi_{s-1}^{+\alpha}\rangle_x}{\langle \psi_n^{+}\psi_n\rangle_x}\right), \tag{1.59}$$

$$F_{\beta s}^{+\alpha} = \frac{\langle \psi_{\beta}\delta u \varphi_{s-1}^{+\alpha}\rangle_x}{\langle \psi_{\beta}^{+}\psi_{\beta}\rangle_x}. \tag{1.60}$$

We define the matrix $T_{\beta}^{+\alpha}(y, w_1)$ by the equation

$$-\sigma T_y^{+} + T^{+}F^{+} = 0, \quad T^{+}(0, w_1) = 1. \tag{1.61}$$

Then the functions

$$\hat{\Psi}^{+\alpha}(x, y, w_1) = \sum_{\beta} T_{\beta}^{+\alpha}(y, w_1)\,\Psi^{+\beta}(x, y, w_1) \tag{1.62}$$

are solutions of (1.4). Under the translation by the period in y they are transformed as follows:

$$\hat{\Psi}^{+\alpha}(x, y + l_2, w_1) = \sum_{\beta} \hat{T}_{\beta}^{+\alpha}(w_1)\,w_{2\beta}^{-1}\hat{\Psi}^{+\beta}(x, y, w_1), \quad \hat{T}^{+} = T^{+}(l_2, w_1). \tag{1.63}$$

Corollary. *The following equality holds:*

$$\sum_{\gamma}' \hat{T}_{\gamma}^{\alpha}\hat{T}_{\gamma}^{+\beta} = \delta^{\alpha\beta}. \tag{1.64}$$

Since $\hat{\Psi}^{\alpha}$ and $\hat{\Psi}^{+\beta}$ are solutions of formally conjugate equations, the $\langle\hat{\Psi}^{+\beta}\hat{\Psi}^{\alpha}\rangle_x$ do not depend on y. Since $T(0) = T^{+}(0) = 1$, it follows that

$$\langle\hat{\Psi}^{+\beta}\hat{\Psi}^{\alpha}\rangle_x = \delta^{\alpha\beta}\langle\psi_{\alpha}^{+}\psi_{\alpha}\rangle_x. \tag{1.65}$$

Therefore

$$\delta^{\alpha\beta}\langle\psi_{\alpha}^{+}\psi_{\alpha}\rangle_x = \langle\hat{\Psi}^{+\beta}(x, y+l_2, w_1)\,\hat{\Psi}^{\bar{\alpha}}(x, y+l_2, w_1)\rangle_x = \\ = \sum \hat{T}_{\nu}^{+\beta}\hat{T}_{\gamma}^{\alpha}\langle\psi_{\gamma}\psi_{\gamma}^{+}\rangle_x. \tag{1.66}$$

Corollary. *The formal Bloch solutions of* (1.4) *are defined on the surface given by* (1.49) *and have multipliers* w_1^{-1} *and* $\tilde{w}_2^{-1}$.

§2. The structure of the Riemann surface of Bloch functions

In this section we shall consider the formal series of the perturbation theory constructed above by taking for an unperturbed potential $u_0 \equiv 0$. The Bloch solutions of the "unperturbed" equation (1.1) and its conjugate

$$(\sigma\partial_y - \partial_x^2)\,\psi(x, y, k) = 0, \quad (\sigma\partial_y + \partial_x^2)\,\psi^{+}(x, y, k) = 0 \tag{2.1}$$

are parametrized by the points of the complex k-plane and have the form

$$\psi = e^{ikx - \sigma^{-1}k^2 y}, \quad \psi^{+} = e^{-ikx + \sigma^{-1}k^2 y}. \tag{2.2}$$

The corresponding eigenvalues of the operators of translation by l_1 and l_2 in x and y are equal to

$$w_1 = e^{ikl_1}, \quad w_2 = e^{-\sigma^{-1}k^2 l_2}. \tag{2.3}$$

For any complex k_0 the functions $\psi_n = \psi(x, y, k_n)$, where

$$k_n = k_0 + \frac{2\pi n}{l_1}, \tag{2.4}$$

form, as we said above, a basic sequence for the continuously differentiable functions $f(x)$ satisfying (1.5) for $w_{10} = w_1(k_0)$. The dual sequence $\psi_m^{+} = \psi^{+}(x, y, k_m)$ satisfies (1.11)

$$\langle\psi_n\psi_m^{+}\rangle_x = \delta_{nm}. \tag{2.5}$$

Therefore the formulae (1.21), (1.25), (1.28), (1.29), (1.30), in which δu must be replaced by $u(x, y)$, determine a formal Bloch solution of (1.1) if k_0 satisfies the resonance-free condition (1.17), which we are going to consider in more detail.

It follows from (2.3) that for $u_0 \equiv 0$ the resonances can only be simple, that is, the equations

$$w_1(k^{(i)}) = w_1(k^{(j)}), \quad w_2(k^{(i)}) = w_2(k^{(j)}) \tag{2.6}$$

can have at most two roots $k^{(1)}$ and $k^{(2)}$. The corresponding pairs of resonance points have the form

$$k^{(1)}=k_{N,M},\quad k^{(2)}=k_{-N,-M}, \tag{2.7}$$

where

$$k_{N,M}=\frac{\pi N}{l_1}+\frac{Ml_1\sigma}{2Nil_2},\quad \text{where } N\neq 0,\quad M \text{ are integers.} \tag{2.8}$$

So if

$$k_0\neq k_{N,M} \tag{2.9}$$

for any integers $N \neq 0$ and M, then we have a formal Bloch solution of (1.1).

Anticipating what follows, we note that with the help of estimates considerably simpler than those we shall obtain below, we can show that for sufficiently small $u(x, y)$ analytically extendable to some neighbourhood of real x, y, the series of the perturbation theory converge outside some neighbourhood of the resonance points (2.8) and determine there a function $\widetilde{\psi}(x, y, k_0)$ analytic in k_0. This is true for any value of σ. The principal distinction between the cases Re $\sigma = 0$ and Re $\sigma \neq 0$ even for small $u(x, y)$ is revealed under an attempt to extend $\widetilde{\psi}$ to the "resonance" domain. The impossibility of such an extension (at least by the methods developed in the paper) for Re $\sigma = 0$ is connected with the fact that in this case the points k_{NM} are dense on the real axis. It would be very interesting and important to find a language that enables us to describe the situation in a neighbourhood of this continuous resonance set. We shall return briefly to this question.

In the case Re $\sigma \neq 0$ the resonance points k_{NM} have only one accumulation point $k = \infty$. This fact is crucial for all subsequent constructions. Up to the end of this section we restrict ourselves to the case $\sigma = 1$, though all its assertions (in particular Theorem 2.1) proved for complex potentials u are valid for all Re $\sigma \neq 0$. For $\sigma = 1$ it is natural to single out the case of real periodic potentials $u(x, y)$, in which general assertions admit an essential further effectivization.

We denote by R_{NM} the neighbourhoods of the resonance points k_{NM} given by the inequalities (we emphasize once more that in what follows $\sigma = 1$)

$$\left|\operatorname{Re} k-\frac{\pi N}{l_1}\right|<\frac{a_1}{N},\quad \left|\operatorname{Im} k-\frac{M}{2N}\frac{l_1}{l_2}\right|<\frac{a_1}{N}, \tag{2.10}$$

where a_1 is a constant chosen for the time being arbitrarily, so that these neighbourhoods are disjoint, that is $a_1 < \pi/2l_1$, $a_1 < l_1/4l_2$. For each point k_0 not belonging to R_{NM} for any integers $N \neq 0, M$ the following inequalities hold:

$$|1-e^{(k_n^2-k_0^2)l_2}|>h,\quad |1-e^{(k_0^2-k_n^2)l_2}|>h, \tag{2.11}$$

where

$$h=\min(1-e^{-a_2}, \sin a_2),\quad a_2=\frac{2\pi l_2}{l_1}a_1. \tag{2.12}$$

In what follows we shall assume that the periodic function $u(x, y)$ under consideration extends analytically to some neighbourhood of real x, y and is bounded there by some constant U, that is,

$$|u(x, y)| \leqslant U, \quad |\operatorname{Im} x| \leqslant \tau_1, \quad |\operatorname{Im} y| \leqslant \tau_2. \tag{2.13}$$

We fix a constant ε satisfying the following inequalities:

$$\varepsilon < \min(\varepsilon_0, 1), \quad C(\varepsilon) < \frac{1}{2}, \tag{2.14}$$

where ε_0 is a root of the discriminant of the quadratic equation

$$aC^2 + bC + \varepsilon^2 U = 0, \quad a = 2U\frac{l_2}{h}, \quad b = \varepsilon U - 1, \tag{2.15}$$

and $C(\varepsilon)$ in the second of the inequalities (2.14) is the value at ε of the branch of the root of the equation (2.15) which is analytic in a neighbourhood of $\varepsilon = 0$, $C(\varepsilon) = \varepsilon^2 U + O(\varepsilon^3)$ (by the second inequality this branch at ε is well-defined).

Let R_0 be the rectangular domain in the complex plane

$$|\operatorname{Re} k| \leqslant N_1, \quad |\operatorname{Im} k| \leqslant N_2, \quad q_j = e^{-2\pi\tau_j/l_j}, \tag{2.16}$$

where N_1, N_2 are arbitrary fixed numbers such that

$$\frac{l_2}{h} q_1^{\frac{l_1 N_1}{2\pi}} \leqslant \varepsilon^2, \quad \frac{l_1 l_2}{\pi h N_1} \log \frac{4l_1}{\pi} N_1 \leqslant \varepsilon^3, \quad N_2 > \frac{l_2}{h} N_1, \quad \frac{l_2}{h} q_2^{2N_2} \leqslant \frac{l_1}{4\pi N_2}. \tag{2.17}$$

We denote by $\widetilde{R}$ the complement to R_0 and the neighbourhoods $R_{N,M}$ of the resonance points.

***Lemma* 2.1.** *For $k_0 \in \widetilde{R}$ the series of the perturbation theory constructed by Lemma* 1.2 *and its corollary absolutely converge uniformly in $\widetilde{R}$ and determine Bloch solutions $\widetilde{\psi}(x, y, k_0)$ of* (1.1) ($\sigma = 1$) *analytic in the domain $k_0 \in \widetilde{R}$, $|\operatorname{Im} x| \leqslant \tau_1$, $|\operatorname{Im} y| \leqslant \tau_2$ and non-vanishing there.*

Proof. It follows from the translation properties of the $c_n^s(y, k_0)$ defined by (1.28) that

$$c_n^s(x, k_0) = \widetilde{c}_n^s(y, k_0)\, e^{(k_n^2 - k_0^2) y}, \tag{2.18}$$

where the function $\widetilde{c}_n^s(y, k_0)$ is periodic in y. Let us prove by induction that for $k_0 \notin R_{NM}$, $|\operatorname{Im} k_0| > N_1$, the following inequalities hold ($s \geqslant 1$):

$$|\widetilde{c}_n^s(y, k_0)| \leqslant C_s q_1^{|n|} \times \begin{cases} \varepsilon^{s+1} f_n(k_0), & n \neq n_0, \\ \frac{l_2}{h}\varepsilon^{s-1}, & n = n_0. \end{cases} \tag{2.19}$$

Here n_0 is an integer such that $|2\pi n_0/l_1 + 2\operatorname{Re} k_0| < 1/2$. The constants C_s are defined successively by

$$C_1 = 1, \quad C_s = U\left(C_{s-1} + 2\frac{l_2}{h}\sum_{i=2}^{s-1} C_{i-1} C_{s-i}\right), \quad s \geqslant 2. \tag{2.20}$$

The non-negative numbers $f_n(k_0)$ satisfy the condition

$$\sum_{n\neq 0,\, n_0} f_n(k_0) \leqslant 1. \tag{2.21}$$

Suppose that (2.19) is valid for all $s' \leqslant s-1$. Then for the same $s' \geqslant 1$ the following inequalities hold:

$$|\varphi_{s'}\psi_0^+| \leqslant C_{s'}\left(\varepsilon^{s'+1} + \varepsilon^{s'-1}\frac{l_2}{h}\left(q_1 e^{\frac{2\pi}{l_1}|\operatorname{Im} x|}\right)^{|n_0|}\right). \tag{2.22}$$

From this inequality for $\operatorname{Im} x = 0$ and from the fact that $l_2 q_1^{|n_0|}/h \leqslant \varepsilon^2$ by (2.17), it follows that

$$F_1 = 0, \quad |F_{s'}(y, k_0)| \leqslant 2UC_{s'-1}\varepsilon^{s'}, \ s' \geqslant 2. \tag{2.23}$$

The equality $F_1 = 0$ is valid by the normalization conditions (1.3).

It follows from (2.19) and (2.23) that

$$\left|\frac{e^{(k_0^2-k_n^2)y}}{e^{(k_n^2-k_0^2)l_2}-1}\int_y^{y+l_2}\left(\sum_{i=2}^{s-1} F_i c_a^{s-i}\right)dy'\right| \leqslant \tag{2.24}$$

$$\leqslant 2Uq^{|n|}J_{n,0}\left(\sum_{i=2}^{s-1} C_{i-1}C_{s-i}\right) \times \begin{cases}\varepsilon^{s+1}, & n \neq n_0,\\ \varepsilon^{s-1}, & n = n_0,\end{cases}$$

where the constant $J_{n,0}$ is equal to

$$J_{n,0} = \frac{|e^{\operatorname{Re}(k_n^2-k_0^2)l_2}-1|}{|\operatorname{Re}(y_n^2-k_0^2)|\,|e^{(k_n^2-k_0^2)l_2}-1|}. \tag{2.25}$$

To estimate

$$I_{n,s} = \frac{e^{(k_0^2-k_n^2)y}}{e^{(k_n^2-k_0^2)l_2}-1}\int_y^{y+l_2}\langle\psi_n^+ u\varphi_{s-1}\rangle_x\, dy' \tag{2.26}$$

we estimate the Fourier coefficients of the expansion in x of the function $(u\varphi_{s-1}\psi_0^+)$

$$\left|\langle e^{\frac{2\pi i n}{l_1}x} u\varphi_{s-1}\psi_0^+\rangle_x\right| \leqslant U\sum_{k\neq 0} q_1^{|k|}|\tilde{c}_{n-k}^{s-1}| \leqslant \tag{2.27}$$

$$\leqslant UC_{s-1}q_1^{|n|} \times \begin{cases}\varepsilon^{s-2}\left(\frac{l_2}{h}+\varepsilon^2\right) \leqslant \varepsilon^{s-2}\left(\frac{l_2}{h}+1\right), & n \neq n_0,\\ \varepsilon^s, & n = n_0.\end{cases}$$

(The summation in (2.27) is taken over $k \neq 0$, since the zero Fourier coefficient of u is absent by (1.3).) From (2.27) and from the fact that

$$\psi_n\psi_0^+ = \exp\left(\frac{2\pi i n}{l_1}x + (k_0^2 - k_n^2)\,y\right)$$

it follows that

$$|I_{n,1}| \leqslant UJ_{n,0}q_1^{|n|}, \tag{2.28}$$

$$|I_{n,s}| \leqslant UC_{s-1}q_1^{|n|}J_{n,0} \times \begin{cases}\varepsilon^{s-2}(l_2h^{-1}+1), & n \neq n_0,\\ \varepsilon^s, \quad n = n_0, & s \geqslant 2.\end{cases}$$

If $k_0 \notin R_{NM}$, then

$$J_{n,0} \leqslant \min\left(\frac{l_2}{h},\ |\operatorname{Re}(k_n^2-k_0^2)^{-1}|\right). \tag{2.29}$$

Moreover

$$\sum_{n\neq n_{0,0}} J_{n,0} \leqslant \sum_{n\neq n_{0,0}} |\operatorname{Re}(k_n^2-k_0^2)^{-1}| \leqslant \frac{l_1}{\pi} |\operatorname{Re} k_0^{-1}| \log\left(\frac{4l_1}{\pi} |\operatorname{Re} k_0|\right). \tag{2.30}$$

It follows from (2.17) that the constants f_n defined by

$$f_n = \varepsilon^{-3}\left(\frac{l_2}{h}+1\right) J_{n,0}, \quad n\neq n_0, \tag{2.31}$$

satisfy the condition (2.21). Summing up (2.28) and (2.24) and taking (2.31) into account, we obtain the desired inequalities (2.19).

For $|\operatorname{Re} k_0| < N_1$, $|\operatorname{Im} k_0| > N_2$ we prove that for all n (including $n = n_0$) the first of the inequalities of (2.19) holds. Moreover, the constants f_n satisfy the condition (2.21), in which the summation is taken over all n. We note that by the induction hypothesis the left-hand side of (2.24) is estimated for all n in terms of the first row of the right-hand side of this inequality.

We deform the contour of integration in (2.26) in the complex domain so that it joins first the points y, $y' \pm i\tau_2$ ($y' = \operatorname{Re} y$), then $y' \pm i\tau_2$, $y' \pm i\tau_2 + l_2$ and $y' \pm i\tau_2 + l_2$, $y + l_2$ by rectilinear intervals. We denote by $I_{n,s}^j$, $j = 1, 2, 3$, the integrals (2.26) over each of these intervals. Since u and φ_{s-1} are analytic for $|\operatorname{Im} y| < \tau_2$,

$$I_{n,s} = I_{n,s}^1 + I_{n,s}^2 + I_{n,s}^3. \tag{2.32}$$

We have

$$I_{n,s}^1 + I_{n,s}^3 = e^{(k_0^2-k_n^2)y} \int_y^{y+l_2} \langle \psi_n^+ u \varphi_{s-1}\rangle_x \, dy'. \tag{2.33}$$

Taking into account that by the induction hypothesis the left-hand side in (2.27) can be estimated for all n in the case under consideration in terms of $UC_{s-1} q_1^{|n|} \varepsilon^s$, $s \geqslant 2$, we obtain

$$|I_{n,s}^1 + I_{n,s}^3| \leqslant U q_1^{|n|} |\operatorname{Im} |(k_n^2-k_0^2)^{-1}| \times \begin{cases} 1, & s=1, \\ C_{s-1}\varepsilon^s, & s\geqslant 2. \end{cases} \tag{2.34}$$

We have for the second summand

$$|I_{n,s}^2| \leqslant U I_{n,0} e^{-\tau_2 |\operatorname{Im}(k_n^2-k_0^2)|} q_1^{|n|} \times \begin{cases} 1, & s=1, \\ C_{s-1}\varepsilon^s, & s\geqslant 2. \end{cases} \tag{2.35}$$

Thus for $I_{n,s}$ two types of inequalities are valid: the first one follows from (2.34) and (2.35), while the second one is the inequality (2.28) which by the induction hypothesis, changed in the domain specified (the first of the inequalities (2.19) holds for all n), acquires the form

$$|I_{n,s}| \leqslant UC_{s-1} J_{n,0} \varepsilon^s q_1^{|n|}, \quad n\neq 0.$$

We define the quantities f_n by

$$(2.36)\quad f_n=\varepsilon^{-2}J_{n,0},\quad |n|>\frac{2N_1l_1}{\pi},\quad f_n=\varepsilon^{-2}\left|\frac{2\pi n}{l_1}\operatorname{Im}k_0\right|^{-1},\quad |n|\leqslant\frac{2N_1l_1}{\pi}.$$

It follows from (2.17) that they satisfy (2.21). Using (2.34) and (2.35) to estimate $|I_{n,s}|$ for $|n|\leqslant 2N_1l_1/\pi$ and the modified inequality (2.28) for $|n|>2N_1l_1/\pi$, we obtain the desired assertion of the lemma.

It follows from (2.20) that the constants C_s are the coefficients of the expansion at the origin

$$(2.37)\qquad C(\varepsilon)=\sum_{s=1}^{\infty}C_s\varepsilon^{s+1}$$

of the analytic branch of the equation (2.15). Hence for $\varepsilon<\varepsilon_0$ this series converges absolutely. Therefore the series (1.19) and (1.21) determine analytic functions $\Psi(x, y, k_0)$ and $F(y, k_0)$, $k_0\in\widetilde{R}$. By the second inequality in (2.14) and also by (2.22) we have for $|\operatorname{Im} x|\leqslant\tau_1/2$

$$(2.38)\qquad |\Psi(x, y, k_0)|\geqslant 1-2C(\varepsilon)>0.$$

Therefore the Bloch function $\widetilde{\psi}$ defined by (1.30) is analytic for $k_0\in\widetilde{R}$, $|\operatorname{Im} x|\leqslant\tau_1/2$, $|\operatorname{Im} y|\leqslant\tau_2$ and does not vanish. The lemma is proved.

We now construct Bloch solutions in resonance domains. As in Lemma 1.3, let I be a finite set of resonance indices.

***Lemma* 2.2.** *If for all $n\notin I$, $\alpha\in I$ the inequalities*

$$(2.39)\qquad \sum_{|n|\geqslant N}J_{n,\alpha}+\sum_{|n|<N}\left(\left|\frac{4\pi n}{l_1}\operatorname{Im}k_0\right|^{-1}+q_2^{2|\operatorname{Im}k_0||n|}\right)\leqslant\varepsilon^2$$

hold for some integer N, where the $J_{n,\alpha}$ are given by (2.25) *with k_0 replaced by k_α, then the series* (1.35) *and* (1.37) *converge absolutely and determine analytic functions $F^{\alpha}_{\beta}(y, w_1)$ and $\Psi^{\alpha}(x, y, w_1)$ satisfying* (1.36).

The proof of the lemma is completely analogous to the proof of Lemma 2.1. The corresponding estimates for the terms of these series have the form

$$(2.40)\qquad |\tilde{c}_n^{s,\alpha}|\leqslant C_s\varepsilon^{s+1}q_1^{|n-\alpha|}f_{n,\alpha}\tilde{c}_n^{s,\alpha}=c_n^{s,\alpha}e^{(k_\alpha^2-k_n^2)y},$$

$$(2.41)\qquad \sum_{n\notin I}f_{n,\alpha}\leqslant 1,$$

$$(2.42)\qquad |F^{\alpha}_{\beta s}l^{(k_\alpha^2-k_\beta^2)y}|\leqslant\begin{cases}U\Phi_{s-1}\varepsilon^s q_1^{|\alpha-\beta|}, & s\geqslant 2,\\ Uq_1^{|\alpha-\beta|}, & \alpha\neq\beta,\quad s=1.\end{cases}$$

We consider consequences of this assertion. Suppose that $k_0\notin R_0$ but it belongs to one of the neighbourhoods R_{NM} of the resonance points. Then if we take $\{0, -2N\}$ for I, the inequalities (2.39) will be satisfied. Therefore for $w_1\in w_1(R_{NM})$ the analytic functions $\Psi^{\alpha}(x, y, w_1)$ and $F^{\alpha}_{\beta}(y, w_1)$, $w_1\in\widetilde{R}_{(N,M)}$, are defined so that (1.36) holds. The matrix $T(y, w_1)$ defined by (1.44) is also analytic in the domain $\tilde{R}_{|N,M|}$ of the complex plane onto which the function $w_1(k)$ maps $R_{N,M}$ and $R_{-N,-M}$. It follows that the Bloch solutions of (1.1), defined by (1.50) and (1.51) for any point of the two-sheeted covering of $\hat{R}_{|N,M|}$ over $\tilde{R}_{|N,M|}$ given by (2.43), are meromorphic functions on $\hat{R}_{|N,M|}$:

$$\tilde{w}_2^2 - \tilde{w}_2 \,\mathrm{Sp}\,(\hat{T}^\alpha_\beta(w_1)\, w_{2\beta}) + \det(\hat{T}^\alpha_\beta(w_1)\, w_{2\beta}) = 0, \quad w_1 = w_1(k_0),\ k_0 \in R_{N,M},\ \alpha, \beta = 0, -2N. \tag{2.43}$$

The poles of $\tilde{\psi}(x, y, Q)$ coincide with the poles of h^α and so do not depend on $x, y, Q \in \hat{R}_{|N,M|}$.

In what follows we shall assume that the constant ε is chosen so that besides the inequalities (2.14) the following inequality also holds:

$$\varepsilon \leqslant \frac{h}{2(1+h\cdot 2)\, l_2 U}. \tag{2.44}$$

In this case the discriminant of the equation (2.42) can vanish only inside the domain $\hat{R}_{|N,M|}$. This assertion follows from the fact that on the boundary of $R_{N,M}$ and $R_{-N,-M}$ both the assumptions of Lemma 2.1 and of Lemma 2.2 are satisfied. It follows from the construction of the Bloch solutions $\tilde{\psi}(x, y, k_0)$ and $\tilde{\psi}(x, y, k'_0)$, $w_1 = w_1(k_0) = w_1(k'_0)$, that the passage to them corresponds to the diagonalization process of the matrix $\hat{T}^\alpha_\beta(w_1) w_{2\beta}$. Therefore the eigenvalues of this matrix coincide on the boundary with $\tilde{w}_2(k'_0)$, $\tilde{w}_2(k_0)$ defined by (1.33) for the resonance-free domain. Since by (2.23)

$$\left| \int_0^{l_2} F(y, k_0)\, dy \right| \leqslant \varepsilon l_2 U C(\varepsilon),. \tag{2.45}$$

we have

$$\left| \frac{\tilde{w}_2(k'_0)}{\tilde{w}_2(k_0)} - 1 \right| \geqslant \left| \frac{w_2(k'_0)}{w_2(k_0)} - 1 \right| - 2 \left| \frac{w_2(k'_0)}{w_2(k_0)} \right| \varepsilon l_2 U C(\varepsilon) \geqslant \geqslant h - 2\varepsilon l_2 U \Phi(\varepsilon)(1 + 2h) \geqslant h(1 - C(\varepsilon)) > 0. \tag{2.46}$$

Therefore on the boundary of $\tilde{R}_{|N,M|}$ the equation (2.43) has distinct roots and its discriminant can have zeros only inside the domain.

All the facts proved above are valid for any potential satisfying (2.13), in particular for the potentials $u_\tau = \tau u(x, y)$, $0 \leqslant \tau \leqslant 1$. Since under such a deformation the number of zeros of the discriminant inside the domain is preserved, and for $\tau = 0$ it has a zero of multiplicity 2 at the point $w_1^{MN} = w_1(k_{NM})$, we arrive at the conclusion that the discriminant of the equation (2.43) has either two simple zeros or one zero of multiplicity 2.

Definition. A pair of integers ($N > 0$, M) such that $k_{NM} \in \tilde{R}$ will be called *distinguished* if the discriminant of the equation (2.43) has a zero of multiplicity 2.

In this case $\hat{R}_{|N,M|}$ is reducible, that is, it splits into two sheets. Then the Bloch function $\tilde{\psi}(x, y, k_0)$ extends analytically to the domains $R_{N,M}$ and $R_{-N,-M}$, which are split sheets of $\hat{R}_{|N,M|}$. For non-distinguished pairs the two-sheeted surface $\hat{R}_{|N,M|}$ is non-singular.

***Lemma* 2.3.** *The Bloch function $\tilde{\psi}(x, y, Q)$ has one simple pole on $\hat{R}_{|N,M|}$ (for non-distinguished pairs $N > 0$, M).*

Before we proceed to the proof of the lemma we note that in exactly the same way as above we can prove that the series of the perturbation theory for the formally conjugate Bloch function $\tilde{\psi}^+(x, y, k_0)$ converge in the resonance-free domain and determine there an analytic function. It follows from the corollary of Lemma 1.4 that $\tilde{\psi}^+(x, y, Q)$ is defined in the same way as $\tilde{\psi}$ on $\hat{R}_{|N,M|}$, where it is meromorphic and its poles do not depend on x, y.

Proof. We consider an arbitrary periodic variation δu of the potential u. By analogy with the proof of (1.6) (see also [22], [52]) we can obtain

$$i\delta E \langle\langle\tilde{\psi}\tilde{\psi}^+\rangle_x - i\delta p \langle\tilde{\psi}_x\tilde{\psi}^+ - \tilde{\psi}\tilde{\psi}_x^+\rangle_y + \langle\langle\tilde{\psi}\delta u\tilde{\psi}^+\rangle\rangle = 0. \tag{2.47}$$

It follows from this equality that the functions $\langle\tilde{\psi}\tilde{\psi}^+\rangle_x$ and $\langle\tilde{\psi}_x\tilde{\psi}^+ - \tilde{\psi}\tilde{\psi}_x^+\rangle_y$ cannot have coinciding zeros. For otherwise at this point $\langle\langle\tilde{\psi}\delta u\tilde{\psi}^+\rangle\rangle = 0$ (where $\langle\langle\cdot\rangle\rangle$ denotes the mean value in x, y), which cannot be true for all δu. Let us now apply (1.6). By what we have proved above the zeros of $\langle\tilde{\psi}\tilde{\psi}^+\rangle_x$ coincide with the zeros of dp which, in turn, coincide with the zeros of the discriminant of the equation (2.43). Therefore there are two of them. Outside the resonance domain $\langle\tilde{\psi}\tilde{\psi}^+\rangle_x \neq 0$. Therefore in $\hat{R}_{|N,M|}$ the number of zeros is equal to the number of poles, that is, each of the functions $\tilde{\psi}$ and $\tilde{\psi}^+$ has one simple pole in this domain. The lemma is proved.

From the topological point of view "pasting" the two-sheeted covering $\hat{R}_{|N,M|}$, to which the Bloch function $\tilde{\psi}$ extends from the resonance-free domain, instead of two domains $R_{N,M}$ and $R_{-N,-M}$, is the simplest reconstruction corresponding to "adding a handle" between two resonance points $k_{N,M}$ and $k_{-N,-M}$.

We consider the extension $\tilde{\psi}$ inside the central resonance domain R_0 defined by the inequalities (2.16) in which without loss of generality we can assume that $N_1' = l_1N_1/2\pi$ is an integer. The function w_1 (2.3) maps R_0 as a $2N_1'$-sheeted covering of the annulus $\exp(-N_2 l_1) < w_1 < \exp(N_2 l_1)$ in the w_1-plane.

As a set I of resonance indices for w_1 that satisfy the preceding inequalities we choose all indices for which $|\mathrm{Re}\, k_\alpha| < N_1$. Then the conditions of applicability of Lemma 2.2 are satisfied. By analogy with the above, we obtain the result that $\tilde{\psi}(x, y, k_0)$ extends from the resonance-free domain to the Riemann surface $\hat{R}_0$, which is defined over the annulus $\exp(-N_2 l_1) < w_1 < \exp(N_2 l_1)$ by the characteristic equation (1.49) for the $2N_1' \times 2N_1'$ monodromy matrix of quasi-Bloch solutions constructed as perturbations of the solutions $\exp(ik_\alpha x - k_\alpha^2 y)$ of the free equation (2.1). By Lemma 2.2 this matrix $\hat{T}_\beta^\alpha(w_1)w_{2\beta}$ is analytic in w_1 in the domain of its definition. Thus, we arrive at the following lemma.

***Lemma* 2.4.** *The Bloch function $\tilde{\psi}(x, y, k_0)$ extends analytically from the resonance-free domain to $\hat{R}_0$, where it is a meromorphic function whose poles do not depend on x, y. Their number g_0 does not exceed the number of pairs $(N > 0, M)$ such that $k_{NM} \in R_0$. In the general position when $\hat{R}_0$ is non-singular, g_0 is equal to the genus of $\hat{R}_0$.*

Anticipating what follows, we note that for the real potentials $u(x, y)$ the surface $\hat{R}_0$ is always non-singular.

We denote by Γ the Riemann surface obtained from the complex k-plane by "pasting" $\hat{R}_0$ instead of R_0 and "pasting" $\hat{R}_{|N,M|}$ instead of $R_{N,M}$ and $R_{-N,-M}$ (for non-distinguished pairs $N > 0$, M). This surface is smooth everywhere except for finitely many points in $\hat{R}_0$.

Renotation. Up to now Bloch solutions of the equation (1.1) constructed with the help of perturbation theory have been denoted by $\tilde{\psi}$. In what follows for brevity we shall omit the tilde sign, denoting them by $\psi(x, y, Q)$. In a similar way we shall omit the tilde sign over the eigenvalue $w_2(Q)$ of the operator of translation by the period in y.

***Theorem* 2.1.** *The Riemann surface Γ is isomorphic to "the Floquet spectral set" for the operator* (1.1). *The Bloch solutions $\psi(x, y, Q)$ of this equation, normalized by the condition $\psi(0, 0, Q) = 1$, are meromorphic on Γ. The poles of ψ do not depend on x, y. In each domain $\hat{R}_{|N,M|}$ the function ψ has one simple pole. In $\hat{R}_0$ it has g_0 poles, where g_0, in the general position when $\hat{R}_0$ is non-singular, is equal to the genus of $\hat{R}_0$. Outside $\hat{R}_{|N,M|}$ and $\hat{R}_0$ the function ψ is holomorphic and has no zeros.*

Proof. All the assertions of the theorem except for the first one follow from the construction of Γ itself. To each point $Q \in \Gamma$ there correspond eigenvalues $w_1(Q)$ and $w_2(Q)$ of the operators of translation by the periods in x and y. They determine a map of Γ into C^2 with coordinates w_1 and w_2. The fact that it determines an isomorphism between Γ and the "Floquet spectral set" follows from the next lemma.

***Lemma* 2.5.** *For any continuously differentiable function $f(x)$ satisfying* (1.9) (*that is,* $f(x + l_1) = w_0 f(x)$) *the series*

$$S = \sum_n \psi(x, y, Q_n) \frac{\langle f \psi_n^+ \rangle_x}{\langle \psi_n^+ \psi_n \rangle_x} \tag{2.48}$$

converges to $f(x)$. (Here, as before, we denote by $Q_n = Q(w_0)$ the point of Γ such that $w_1(Q_n) = w_{10}$, $\psi_n = \psi(x, y, Q_n)$.)

The proof of the lemma in the special case of finite-gap operators was first suggested in [52]. It extends to the general case practically without changes. From Lemma 1.1 and from the fact that the functions $\langle \psi\psi^+\rangle_x$ and $\langle \psi_x\psi^+ - \psi\psi_x^+\rangle_y$ have no common zeros it follows that the differential

$$d\Omega = \frac{dp}{\langle\psi\psi^+\rangle_x} = \frac{dE}{\langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y} \tag{2.49}$$

is holomorphic on Γ and has zeros at the poles of ψ and ψ^+.

We consider the integral

$$S_N = \frac{1}{l_1}\int\limits_{C_N}\int\limits_0^{l_1} f(x')\frac{\psi(x, y, Q)\psi^+(x', y, Q)}{1 - w_{10}w_1^{-1}(Q)}\, d\Omega\, dx', \tag{2.50}$$

where we take for the contour C_N the boundary of the square $|\mathrm{Re}\, k| \leqslant \pi(2N+1)/2l_1$, $|\mathrm{Im}\, k| \leqslant \pi(2N+1)/2l_1$, where N is a sufficiently large integer. The integrand has poles at the points Q_n, and its residues at these points are equal to the corresponding terms of the series (2.48). On the other hand, using (2.22), we can easily see that S_N is equal to the sum of the first N terms of the usual Fourier series for the function $f(x)$. Making N tend to infinity, we obtain the desired assertion.

Let (w_{10}, w_2') be an arbitrary point of the Floquet spectral set and ψ' the Bloch function corresponding to it. If w_2' does not coincide with any value $w_{2n} = w_2(Q_n)$, then

$$\langle\psi'(x, y, w_{10}, w_2')\psi^+(x, y, Q_n)\rangle_x = 0, \tag{2.51}$$

since the left-hand side does not depend on y and on the other hand under the translation of y by l_2 it is multiplied by $w_2' w_{2n}^{-1}$. It follows from Lemma 2.5 that $\psi' \equiv 0$. The theorem is proved.

We emphasize once more that it is valid for all (including complex) potentials satisfying (2.13). For real $u(x, y)$ it can acquire a more effective form. Before doing this, we give the following definition.

Definition. A potential $u(x, y)$ is called *finite-gap* if all except finitely many pairs $(N > 0, M)$ for it are distinguished, that is, Γ has finite genus.

For finite-gap potentials the surface Γ, corresponding to them, coincides outside some finite domain with a neighbourhood of infinity on the usual complex plane. Therefore it can be compactified by one "infintely distant" point $P_0 = \infty$. In what follows we keep the notation Γ for the corresponding Riemann surface (algebraic curve).

Corollary. *The Bloch solutions $\psi(x, y, Q)$, $Q \in \Gamma$, of the equation* (1.1) *for finite-gap potentials u are defined on the compact Riemann surface Γ. Outside the distinguished point P_0 the function ψ is meromorphic and has g*

poles not depending on x, y, where in the general position when Γ *is non-singular g is equal to the genus of* Γ. *In a neighbourhood of* P_0 *the function* $\psi(x, y, Q)$ *has the form*

$$\psi = e^{ikx-k^2y}\left(1+\sum_{s=1}^{\infty}\xi_s(x, y)k^{-s}\right), \tag{2.52}$$

where $k^{-1} = k^{-1}(Q)$ *is a local parameter in a neighbourhood of* P_0.

All the assertions of the corollary except for the last one follow directly from the definition of finite-gap potentials and Theorem 2.1. To obtain (2.52) we use the fact that in a neighbourhood of infinity $\psi(x, y, k)$ is given by series of the perturbation theory for the resonance-free case. It follows from (2.22) that the function

$$\psi(x, y, k)\, e^{-ikx+k^2y}, \tag{2.53}$$

which is holomorphic in a deleted neighbourhood of P_0, is bounded. Therefore it is holomorphic in this neighbourhood and can be expanded in the series

$$\psi(x, y, k)\, e^{-ikx+k^2y} = \sum_{s=0}^{\infty}\xi_s(x, y)k^{-s}. \tag{2.54}$$

It follows from the normalization (1.8) that $\xi_0 = 1$, and the corollary is proved.

We call a set of pairs of complex numbers $\pi = \{(p_s', p_s'')\}$, where s ranges over a finite or infinite set of pairs of integers ($N > 0, M$), admissible if

$$\operatorname{Re} p_s' = \operatorname{Re} p_s'' = \frac{\pi N}{l_1}, \quad |p_s' - k_s| = o\left(\frac{1}{|k_s|}\right), \tag{2.55}$$
$$|p_s'' - k_s| = o\left(\frac{1}{|k_s|}\right),$$

and the intervals $[p_s', p_s'']$ parallel to the imaginary axis are disjoint.

For each admissible set π we construct a Riemann surface $\Gamma(\pi)$ by making vertical slits between the pairs of points p_s', p_s'' and $-\bar{p}_s'$, $-\bar{p}_s''$ and by pasting together the left bank of the right slit with the right bank of the left slit and vice versa. After such pasting, to each pair of slits (p_s', p_s'') and $(-\bar{p}_s', -\bar{p}_s'')$ there corresponds a cycle non-homologous to zero, which will be denoted by a_s.

***Theorem* 2.2.** *For any real periodic potential* $u(x, y)$ *analytically extendable to a neighbourhood of real x, y, the Bloch solutions of the equation* (1.1) *with* $\sigma = 1$ *are parametrized by the points Q of a Riemann surface* $\Gamma(\pi)$ *for some admissible set* π. *The corresponding function* $\psi(x, y, Q)$ *is meromorphic on* $\Gamma(\pi)$ *and has one simple pole on each of the cycles* a_s.

Proof. For real potentials u the complex conjugation transforms Bloch solutions of (1.1) into Bloch solutions of the same equation. Therefore the correspondence

$$\tau: (w_1, w_2) \to (\overline{w_1}, \overline{w_2}) \tag{2.56}$$

is an anti-holomorphic involution of the Floquet spectral set, which by Theorem 2.1 induces an anti-holomorphic involution of the "spectral curve" Γ. We can verify that such a curve exists directly from the construction of Γ. In particular, it follows from Lemma 1.2 that in the resonance-free domain τ has the form $k_0 \to -\overline{k}_0$ and moreover $\psi(x, y, k_0) = \overline{\psi}(x, y, -\overline{k}_0)$.

We consider the neighbourhoods $R_{N,M}$ of the resonance points lying outside the central domain R_0. The invariance of $\hat{R}_{|N,M|}$ under τ means that two zeros of the discriminant of the equation (2.43) either both lie on the straight line $\operatorname{Re} k = \pi N/l_1$ or they are placed symmetrically outside this line. The latter is impossible, because on the intersection of this line with the boundary of $R_{N,M}$ the signs of the imaginary parts of the eigenvalues of the operator of translation by the period in y are different (this is seen directly from (2.3) for the free equation (2.1) and from (2.46) for the general operator (1.1)). Consequently, inside an interval of the line there is a point at which w_2 is real. Therefore both zeros of the discriminant, which we have denoted by p'_s, p''_s, lie on the line $\operatorname{Re} k = \pi N/l_1$. The slit between them corresponds to the cycle a_s, which is singled out by the conditions that on it both multipliers w_1 and w_2 are real, $s = (N, M)$. This cycle is a "forbidden zone", which appears at the place of the resonance point k_s. Let us prove that the pole of the Bloch function lies on the cycle a_s. On this cycle ψ and ψ^+ are real. Since $\psi(x, y, Q) = \overline{\psi}(x, y, \tau(Q))$, the poles of ψ must be invariant under τ. Since on $\hat{R}_{|N,M|}$ there is only one pole of ψ, it must be a fixed point under τ and so it belongs to the cycle a_s.

For sufficiently small potentials $u(x, y)$, when the central domain R_0 is empty, the theorem is proved. We shall increase $u(x, y)$. The structure of Γ described above is topologically stable and can be destroyed only under the confluence of cycles a_s for different s. (At that moment Γ will have singularities.) The condition of periodicity of u is an obstacle to such a confluence. The condition of periodicity of u in x separates the cycles a_s and $a_{s'}$ if $N \neq N'$. The periodicity of u in y is an obstacle to the confluence of cycles over intervals of one line $\operatorname{Re} k = \pi N/l_1$. If we cut Γ along the cycles a_s and along the line $\operatorname{Re} k = 0$, then in the domain $\operatorname{Re} k > 0$ a single-valued branch of the quasi-energy $E(Q)$ is defined. Since the differential dE is purely imaginary on a_s, the real part of $E(Q)$ extends continuously to a_s and is identically equal there to $\pi M/l_2$, $s = (N, M)$. Thus, the cycles a_s are sepatated by the values of the real parts of the quasi-momentum and the quasi-energy and cannot join. Therefore the desired theorem is valid for all u, not only for small u.

It follows from the construction of Γ that for sufficiently large $|s| = |N| + |M|$ the points p'_s and p''_s are localized in neighbourhoods R_s of the resonance points k_s, which is reflected in (2.55). In the case under consideration of potentials u analytic in a neighbourhood of real x, y, we can show that

$$|p'_s - p''_s| = O(e^{-\alpha|N| - \beta|M|}). \qquad (2.57)$$

This relation is not proved in this paper and is not included in the definition of admissible sets in connection with the following circumstances.

The representation of Γ described in Theorem 2.2 is well known (see [53], [54]) in the spectral theory of the Sturm–Liouville operator with periodic potential $u(x)$. The corresponding curves Γ are hyperelliptic. The collections p'_s, p''_s for $s = (N, 0)$ correspond to them. Moreover, $p'_s = \bar{p}''_s$. For independent parameters uniquely determining u we can take $d_s = \operatorname{Im} p'_s$ and points γ_s, one on each of the cycles. In terms of these parameters the process of approximation of u by finite-gap potentials u_G looks very simple. The potential u_G corresponds to the collection of data in which it is supposed that $d^G_s = d_s$, $|s| \leqslant G$, $d^G_s = 0$, $|s| > G$ ([53]).

Such an approach to the proof of the approximability of an arbitrary periodic potential by finite-gap ones in the non-stationary case is very complicated, because the parameters p'_s and p''_s are not independent (they were dependent in the stationary case too, but their connection was explicit there). As will be seen later, to any finite admissible collections there correspond finite-gap potentials periodic in x and quasi-periodic in y (see §4). The condition of periodicity in y leads to the fact that among the p'_s and p''_s only one half is independent (for example, p'_s or $p'_s - p''_s$). Therefore if we try to construct a process of approximation by finite-gap periodic potentials, it is necessary to "shut" the zones $[p'_s, p''_s]$ for large $|s|$, correcting the remaining ones at the same time. In principle this way is possible, but technically it is rather difficult to realize it. Below we shall give a proof of the approximation theorem based on a different idea, which is also applicable in the case of the spectral theory of operators for which the poles of the Bloch functions do not lie on fixed ovals of the corresponding anti-involution (the spectral theory of operators that are used for the construction of finite-gap solutions of the sine-Gordon equation or a non-linear Schrödinger equation with repulsion and so on apply to a number of these cases). Since in the course of a detailed proof the explicit parametrization of u with the help of admissible collections π will not be used, we do not specify necessary and sufficient conditions that characterize the admissible collections.

§3. The approximation theorem

Suppose that the potential $u_1(x, y)$ of the equation (1.1) with Re $\sigma \neq 0$ is a trigonometric polynomial. Since

$$(3.1)\quad \psi(x, y, k_{NM})\psi^{+}(x, y, k_{-N,-M}) = \exp\left(\frac{2\pi iNx}{l_1}+\frac{2\pi iMy}{l_2}\right)$$

(in this section we adopt the initial definitions and notations of the first section and the beginning of the second, that is, $\psi(x, y, k)$ is a solution of the free equation (2.1) and $\widetilde{\psi}(x, y, Q)$ are solutions of equations of the type (1.1)); this means that for some G

$$(3.2)\quad \langle\langle\psi(x, y, k_{N,M})\psi^{+}(x, y, k_{-N,-M})u_1(x, y)\rangle\rangle = 0, \quad |N| + |M| > G.$$

It follows from the formulae of Lemma 1.2 that under the condition (3.2) the first-order term $\varphi_1(x, y, k_0)$ of the perturbation theory has no poles at the resonance points $k_{N,M}$ for $|N| + |M| > G$ and can be extended to them by continuity. The poles at these points arise in the next order of the perturbation theory. The main idea of the subsequent construction relies on the possibility of constructing a formal series $U(x, y)$ with principal term u_1, the subsequent terms of which are chosen so that the corresponding terms of the series of the perturbation theory have no poles at the k_{NM}, $|N| + |M| > G$.

***Lemma* 3.1.** *Let $u_1(x, y)$ be a periodic function satisfying* (3.2). *Then there is a unique formal series*

$$(3.3)\quad U(x, y) = \sum_{s=1}^{\infty} u_s(x, y),$$

in which for $s \geqslant 2$

$$(3.4)\quad u_s^{NM} = \langle\langle\psi(x, y, k_{NM})\psi^{+}(x, y, k_{-N,-M})u_s(x, y)\rangle\rangle = 0, \quad |N| + |M| \leqslant G,$$

and such that for any $k_0 \neq k_{NM}$, $|N| + |M| \leqslant G$, there is a unique formal series

$$(3.5)\quad F(y, k_0) = \sum_{s=1}^{\infty} F_s(y, k_0),$$

for which the equation

$$(3.6)\quad (\sigma\partial_y - \partial_x^2 + U(x, y))\Psi(x, y, k_0) = F(y, k_0)\Psi(x, y, k_0)$$

has a formal solution of the form

$$(3.7)\quad \Psi(x, y, k_0) = \sum_{s=0}^{\infty} \varphi_s(x, y, k_0), \quad \varphi_0 = \psi(x, y, k_0),$$

satisfying the relations

$$\langle \Psi(x, y, k_0)\, \psi^+(x, y, k_0) \rangle_x = 1, \tag{3.8}$$

$$\begin{aligned} \Psi(x + l_1, y, k_0) &= w_1(k_0)\Psi(x, y, k_0); \\ \Psi(x, y + l_2, k_0) &= w_2(k_0)\Psi(x, y, k_0). \end{aligned} \tag{3.9}$$

Proof. The equation (3.6) is equivalent to the system

$$(\sigma\partial_y - \partial_x^2)\, \varphi_s = \sum_{i=1}^{s} (F_i - u_i)\, \varphi_{s-i}. \tag{3.10}$$

For $k_0 \neq k_{NM}$ the terms of the series (3.5) and (3.7) are given by formulae completely analogous to (1.25)–(1.29):

$$F_s(y, k_0) = \sum_{i=1}^{s} \langle \psi_0^+ u_i \varphi_{s-i} \rangle_x, \qquad \psi_0^+ = \psi^+(x, y, k_0), \tag{3.11}$$

$$\varphi_s = \sum_{n \neq 0} c_n^s(y, k_0)\, \psi_n(x, y), \quad \psi_n = \psi\left(x, y, k_0 + \frac{2\pi n}{l_1}\right), \tag{3.12}$$

$$c_n^s = \sigma^{-1} \frac{w_{2n}}{w_{20} - w_{2n}} \int_y^{y+l_2} \sum_{i=1}^{s} (F_i c_n^{s-i} - \langle \psi_n^+ u_i \varphi_{s-i} \rangle_x)\, dy'. \tag{3.13}$$

Suppose that the terms u_i of (3.3) with numbers $i \leqslant s-1$ are constructed so that the $\varphi_i(x, y, k_0)$ have no poles if $k_0 = k_{NM}$ for $|N| + |M| > G$. Hence, the φ_i can also be defined at these points by continuity. The next term $u_s(x, y)$ of the series (3.3) can be found from

$$\begin{aligned} u_s^{NM} = \int_y^{y+l_2} \sum_{i=1}^{s-1} (F_i c_N^{s-i} - \\ - \langle \psi^+(x, y', k_{-N,-M})\, u_i \varphi_{s-i}(x, y', k_{NM}) \rangle_x)\, dy', \quad |N| + |M| > G. \end{aligned} \tag{3.14}$$

The equalities (3.14) together with the normalization conditions (3.4) and (1.8) determine all the Fourier coefficients of the periodic function $u_s(x, y)$. It follows from (3.14) that $\varphi_s(x, y, k_0)$ has no poles at k_{NM} for $|N| + |M| > G$. The lemma is proved.

***Theorem* 3.1.** *Each smooth periodic potential $u(x, y)$ of the equation* (1.1) *with* Re $\sigma \neq 0$, *analytically extendable to a neighbourhood of real x, y, can be approximated by finite-gap potentials uniformly with any number of derivatives.*

The proof of this assertion will be given only for $\sigma = 1$. It extends to the general case Re $\sigma \neq 0$ practically without changes (as in the proof of Theorem 2.1). For any integer G we denote by $u_0^G(x, y)$ and $u_1^G(x, y)$ periodic functions such that

$$u(x, y) = u_0^G(x, y) + u_1^G(x, y), \quad \langle u_0^G \rangle_x = \langle u_1^G \rangle_x = 0, \tag{3.15}$$

and such that for the u_0^G the conditions (3.4) and for the u_1^G the conditions (3.2) are satisfied. By Lemma 3.1, to the potential u_1^G there corresponds the unique formal series (3.3) $U^G(x, y)$.

***Lemma* 3.2.** *There is a constant G_0 depending on the quantities U, τ_1, τ_2 from* (2.13) *such that for $G > G_0$ the corresponding formal series* (3.3) *converges and determines a smooth finite-gap periodic potential $U^G(x, y)$ of the equation* (1.1).

Proof. If u satisfies (2.13), then for $|\text{Im}\, x| \leqslant \tau_1$, $|\text{Im}\, y| \leqslant \tau_2$

$$(3.16)\qquad |u_0^G(x, y)| \leqslant U_0^G = U\exp(-\sqrt{2}(\pi\tau_1/l_1 + \pi\tau_2/l_2)G)$$

Therefore

$$(3.17)\qquad |u_1^G(x, y)| \leqslant U_1^G = U + U_0^G.$$

As in the proof of Lemma 2.1 we represent the coefficients c_n^s of the series (3.12) in the form (2.14). Then for $k_0 \notin R_0$, $k_0 \notin R_{NM}$ the inequalities (2.19) hold with C_s replaced by $\widetilde{C}_s$, defined recursively by

$$(3.18)\qquad \widetilde{C}_1 = 1, \quad \widetilde{C}_s = \frac{l_2}{h}\sum_{i=2}^{s-1}\Big(\sum_{j=1}^{i} U_j\Phi_{i-j}\Big)\widetilde{C}_{s-i} + \sum_{i=1}^{s} U_i\Phi_{s-i},$$

$$(3.19)\qquad \Phi_s = 2\widetilde{C}_s, \quad s \geqslant 1.$$

The constants U_i in (3.18) bound the terms of (3.3):

$$(3.20)\qquad |u_s(x, y)| \leqslant U_s\varepsilon^{s-1}.$$

To obtain recursion formulae for the U_s, we note that if the inequalities (2.19) are valid for $k_0 \notin R_0$ (R_0 is the central resonance domain) and for $k_0 \in R_{NM}$, then they also remain valid for $k_0 \notin R_0$, $k_0 \in R_{NM}$, because the functions $c_n^s(y, k_0)$ are regular in R_{NM} and so we can apply the maximum modulus principle to them. It follows from this remark that if the inequalities (2.19) are proved up to the order $s-1$, then in the relations (3.14) that determine the Fourier coefficients of u_s with numbers N, M, $|N| + |M| > G$ (the remaining Fourier coefficients vanish by (3.4)) we can apply the inequalities (2.22) to $\varphi_i(x, y, k_{NM})$. We obtain finally

$$(3.21)\qquad U_s = \Big(\frac{l_2}{h}\sum_{i=2}^{s-1}\Big(\sum_{j=1}^{i} U_j\Phi_{i-j}\Big)\widetilde{C}_{s-i} + \sum_{i=1}^{s-1} U_i\Phi_{s-i}\Big) e^{-\sqrt{2}(\pi\tau_1/l_1 + \pi\tau_2/l_2)G}.$$

It follows from (3.18)–(3.21) that for sufficiently large G the series (3.3) converges and determines a smooth periodic function $U^G(x, y)$. (It is sufficient to choose G so that the points k_{NM} with $|N| + |M| > G$ satisfy the condition $|\text{Re}\, k_{NM}| > \widetilde{N}_0/\sqrt{2}$, $|\text{Im}\, k_{NM}| > \widetilde{N}_0/\sqrt{2}$, where $\widetilde{N}_0$ can be found by analogy with N_1, N_2 in Lemma 2.1 from the conditions of convergence of the generating series for $\widetilde{C}_s$, U_s.) At the same time we obtain the result that for k_0, $|k_0| > \widetilde{N}_0$, the series (3.7) converges and determines a Bloch solution of the equation (3.6) which is analytic and does not vanish for any k_0, $|k_0| > \widetilde{N}_0$. Therefore U^G is a finite-gap potential. The lemma is proved.

It follows directly from (3.21) and (3.16) that for $|\operatorname{Im} x| \leqslant \tau_1$, $|\operatorname{Im} y| \leqslant \tau_2$

$$(3.22) \qquad |u(x, y) - U^G(x, y)| \leqslant M \exp(-\sqrt{2}(\pi\tau_1/l_1 + \pi\tau_2/l_2)G),$$

where the constant M depends only on U, τ_1, τ_2. Therefore the sequence $U^G(x, y)$ of finite-gap potentials tends to $u(x, y)$ as $G \to \infty$ uniformly with any number of derivatives. The theorem is proved.

§4. The spectral theory of finite-gap non-stationary Schrödinger operators

The definition of finite-gap potentials given in the second section refers formally only to the potentials of the equation (1.1) with Re $\sigma \neq 0$. However, although when Re $\sigma = 0$ for a general periodic potential $u(x, y)$ the Floquet spectral set globally is not a Riemann surface, we can introduce the notion of finite-gap potentials in this case too. Moreover, the general definition of finite-gap potentials refers not only to periodic but also to quasi-periodic potentials with a finite group of periods. Solutions of the equation (1.1) with such potentials u are called *Bloch solutions* if the logarithmic derivatives $\psi_x\psi^{-1}$, $\psi_y\psi^{-1}$ have the same group of periods as $u(x, y)$. The set of such solutions is exactly the Floquet spectral set. In the case when it is a Riemann surface Γ of finite genus $g < \infty$ the corresponding potential is called *finite-gap*. From the solution of the inverse problem of recovering u from the corresponding algebraic geometry data, which was posed and solved in [6], [7] and is presented below, it follows that this definition is non-empty.

Let Γ be a non-singular algebraic curve of genus g with a distinguished point P_0 and a fixed local parameter $k^{-1}(Q)$ in its neighbourhood, $k^{-1}(P_0) = 0$. For any set $\gamma_1, \ldots, \gamma_g$ in general position there is a unique function $\psi(x, y, Q)$ such that

(4.1) 1° outside P_0 it is meromorphic and has at most simple poles at the points γ_s (if all of them are distinct);

2° in a neighbourhood of P_0 it has the form

$$(4.2) \qquad \psi(x, y, Q) = e^{ikx - \sigma^{-1}k^2y}\left(1 + \sum_{s=1}^{\infty} \xi_s(x, y)k^{-s}\right), \quad k = k(Q).$$

We note that ψ depends only on the equivalence class $[k^{-1}]_2$ of the local parameter. (For any positive integer m we call k^{-1} and k_1^{-1} *m-equivalent local parameters* if $k_1(Q) = k(Q) + O(k^{-m}(Q))$. The equivalence class will be denoted by $[k^{-1}]_m$. (In what follows we shall mean by the local parameter its equivalence class unless otherwise specified.)

We fix on Γ a basis of cycles a_i, b_i with canonical intersection matrix $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$. In a standard way we can define (see [7] or [9]) the basis of normalized holomorphic differentials ω_k, $k = 1, \ldots, g$, the vectors $B_k = (B_{ki})$ of their b-periods, and the corresponding Riemann theta-function, an entire function of g complex variables which under the translations of the arguments by the basis unit vectors e_k in C^g and by the vectors B_k is transformed as follows:

$$\theta(\tau + e_k) = \theta(\tau), \quad \theta(\tau + B_k) = e^{-\pi i B_{kk} - 2\pi i \tau_k}\theta(\tau). \tag{4.3}$$

Let q be an arbitrary point of Γ. The *Abel map* is by definition the correspondence which associates with a point $Q \in \Gamma$ the vector $A(Q)$ with coordinates $A_k(Q) = \int_q^Q \omega_k$. For any collection of g points $\gamma_1, \ldots, \gamma_g$ in general position the function $\theta(A(Q) + Z)$, where

$$Z = K - A(\gamma_1) - \ldots - A(\gamma_g) \tag{4.4}$$

(K being the vector of Riemann constants) has exactly g zeros coinciding with the γ_s (we note that by (4.3) the zeros of a multi-valued function on Γ are well-defined).

We denote by $\Omega^{(s)}$, $s = 1, 2$, the meromorphic differentials on Γ that have the only poles at the point P_0 of the form

$$\Omega^{(1)} = dk(1 + O(k^{-2})), \quad \Omega^{(2)} = i\sigma^{-1}dk^2(1 + O(k^{-3}))$$

and normalized by the condition

$$\int_{a_i} \Omega^{(s)} = 0. \tag{4.5}$$

The vectors of their b-periods will be denoted by

$$2\pi U_k = \oint_{b_k} \Omega^{(1)}, \quad 2\pi V_k = \oint_{b_k} \Omega^{(2)}. \tag{4.6}$$

A function $\psi(x, y, Q)$ of Baker–Akhiezer type determined by its analytic properties (4.1), (4.2) has the form

$$\psi = \exp\left(i \int_q^Q x\Omega^{(1)} + y\Omega^{(2)}\right) \frac{\theta(A(Q) + Ux + Vy + Z)\,\theta(A(P_0) + Z)}{\theta(A(Q) + Z)\,\theta(A(P_0) + Ux + Vy + Z)}. \tag{4.7}$$

The proof of (4.7) consists in a direct verification of the fact that the right-hand side does not change when going round any cycle on Γ (that is, the function ψ on Γ is well-defined) and satisfies all the necessary analytic properties.

***Theorem* 4.1** ([7]). *The function* $\psi(x, y, Q)$ *satisfies* (1.1) *with potential* $u(x, y)$ *equal to*

$$u(x, y) = 2\partial_x^2 \log \theta(A(P_0) + Ux + Vy + Z) - 2c, \tag{4.8}$$

where the constant c *is determined from the expansion*

$$\int_q^Q \Omega^{(1)} = k(Q) + c_0 + ck^{-1}(Q) + O(k^{-2}(Q)). \tag{4.9}$$

Proof. We consider the function

$$\tilde{\psi} = (\sigma\partial_y - \partial_x^2 + u)\psi(x, y, Q), \quad u(x, y) = 2i\xi_{1x}(x, y), \tag{4.10}$$

where ξ_1 is the coefficient in (4.2). It possesses all but one of the analytic properties of ψ. The expansion of its pre-exponential factor in a neighbourhood of P_0 begins with a term of order k^{-1}, while for ψ it begins with 1. It follows from the uniqueness of ψ that $\tilde{\psi} \equiv 0$. To obtain (4.8), it is sufficient to expand the right-hand side of (4.7) in a neighbourhood of P_0 using the following relation (a consequence of the bilinear Riemann relations):

$$A(Q) = A(P_0) + iUk^{-1}(Q) + O(k^{-2}(Q)). \tag{4.11}$$

For a curve in general position the corresponding potentials $u(x, y)$ are quasi-periodic. The conditions that single out the curves which correspond to the periodic potentials can be formulated as follows.

Let dp and dE be meromorphic differentials on Γ having the only singularities at P_0 of the form

$$dp = dk(1 + O(k^{-2})), \quad dE = i\sigma^{-1}dk^2(1 + O(k^{-3})) \tag{4.12}$$

and uniquely normalized so that their periods along all cycles of Γ are real. If for any cycle C on Γ

$$\oint_C dp = \frac{2\pi n_C}{l_1}, \quad \oint_C dE = \frac{2\pi m_C}{l_2}, \text{ where } n_C, m_C \text{ are integers,} \tag{4.13}$$

then the potentials u corresponding to such curves Γ have periods l_1 and l_2 in x and y respectively. The Baker–Akhiezer functions coincide with the Bloch solutions of the equation (1.1). The differentials dp and dE are the differentials of quasi-momentum and quasi-energy, and the corresponding "multipliers" $w_1(Q)$ and $w_2(Q)$ are equal to

$$w_1(Q) = \exp\left(il_1 \int_q^Q dp\right), \quad w_2(Q) = \exp\left(il_2 \int_q^Q dE\right). \tag{4.14}$$

(The conditions (4.13) guarantee that the $w_i(Q)$ do not depend on the path of integration.) The proof of the above assertions follows from the fact that

$$\psi(x + l_1, y, Q) = w_1(Q)\psi(x, y, Q), \tag{4.15}$$

$$\psi(x, y + l_2, Q) = w_2(Q)\psi(x, y, Q), \tag{4.16}$$

since the right-hand and left-hand sides of these equalities have the same analytic properties.

Formally conjugate or dual Baker–Akhiezer functions being solutions of (1.4) are defined in the following way. Let $d\Omega$ be the unique differential meromorphic on Γ with a single pole of the second order at P_0 and having zeros at $\gamma_1, \ldots, \gamma_g$. Besides the γ_s, the differential $d\Omega$ also has g zeros, which will be denoted by $\gamma_1^+, \ldots, \gamma_g^+$. A function $\psi^+(x, y, Q)$ that is meromorphic on Γ outside P_0 and has poles at $\gamma_1^+, \ldots, \gamma_g^+$ will be called a *dual Baker–Akhiezer function*. In a neighbourhood of P it has the form

$$\psi^+(x, y, Q) = e^{-ikx+\sigma^{-1}k^2y}\left(1+\sum_{s=1}^{\infty} \xi_s^+(x, y)\, k^{-s}\right). \tag{4.17}$$

***Lemma* 4.1** ([56]). *For the coefficients ξ_1 and ξ_1^+ of the expansions* (4.2) *and* (4.17) *the following equality holds*:

$$\xi_1(x, y) + \xi_1^+(x, y) = 0. \tag{4.18}$$

Proof. It follows from (4.2), (4.17), and the definition of $\gamma_1^+, \ldots, \gamma_g^+$ that the differential

$$d\widetilde{\Omega}(x, y, Q) = \psi(x, y, Q)\psi^+(x, y, Q)d\Omega(Q) \tag{4.19}$$

is holomorphic outside P_0, where it has a pole of the second order. Therefore the residue of $d\widetilde{\Omega}$ at P_0 is equal to zero. Since it is equal to the left-hand side of (4.18), the lemma is proved.

***Corollary*.** *The dual Baker–Akhiezer function ψ^+ is a solution of the equation* (1.4) *formally conjugate to the equation* (1.1) *which ψ satisfies.*

***Lemma* 4.2.** *If Γ, P_0, $\gamma_1, \ldots, \gamma_g$ are such that the potential u corresponding to them is non-singular, then the differential $d\Omega$ is equal to*

$$d\Omega = \frac{dp}{\langle\psi\psi^+\rangle_x} = \frac{\sigma\, dE}{\langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y}. \tag{4.20}$$

Proof. By complete analogy with the proof of Lemma 2.3 it can be shown that if u is non-singular, then $\langle\psi\psi^+\rangle_x$ and $\langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y$ cannot have common zeros. It follows from (1.6) that the right-hand sides of (4.20) are holomorphic outside P_0 and have zeros at the poles of ψ, ψ^+ and a pole of the second order at P_0. Since these properties uniquely determine Ω, the lemma is proved.

***Theorem* 4.2.** *For a real smooth periodic potential u of the equation* (1.1) *the corresponding curve Γ is isomorphic to the Floquet spectral set.*

The proof of the theorem for an arbitrary σ completely repeats the proof of the first assertion of Theorem 2.1, since the relation (4.20) is sufficient to carry it out.

The potentials u corresponding to an arbitrary set of data $(\Gamma, P_0, k^{-1}, \gamma_s)$ are complex meromorphic functions. The identification of real and non-singular potentials in the cases $\sigma = 1$ and $\sigma = i$ turns out to be different in principle.

The case $\sigma = i$. For u to be real it is necessary that there is an anti-holomorphic involution τ on Γ such that $\tau(P_0) = P_0$. The local parameter k^{-1} must be chosen so that $k(\tau(Q)) = \bar{k}(Q)$. The poles γ_s under the action of τ must be transformed into the dual collection $\tau(\gamma_s) = \gamma_s^+$, that is, the $\gamma_s, \tau(\gamma_s)$ must be zeros of $d\Omega$ with a single pole of the second order at P_0.

If these conditions are satisfied, then by the coincidence of the analytic properties the following functions are equal to each other:

$$\psi^+(x, y, Q) = \bar{\psi}(x, y, \tau(Q)). \tag{4.21}$$

Therefore

$$\xi_1(x, y) = \bar{\xi}_1^+(x, y), \tag{4.22}$$

and by (4.18) $u = 2i\xi_{1x}$ is real.

For a potential u to be smooth it is sufficient that the anti-involution τ is of splitting type, that is, its fixed ovals $a_0, \ldots, a_l$, $l \leq g$, split Γ into two domains $\Gamma^{\pm}$. If $d\Omega$ corresponding to $\gamma_1, \ldots, \gamma_g$ is non-negative on a_s with respect to the orientation given on these ovals as on the boundary of Γ^+, then u has no singularities for real x, y.

The sufficiency of the above conditions for the smoothness of u was first obtained in [12]. Their necessity was proved recently in [57] on the basis of a detailed analysis of the theta-function formula (4.8). We shall give below a brief sketch of another method of the proof.

First of all we note that it is sufficient to prove the necessity of the above conditions for the periodic potentials, because the set of curves with a distinguished point P_0 that correspond to them as $l_1, l_2 \to \infty$ is dense in the set of all finite-gap potentials.

The correspondence

$$(w_1, w_2) \to (\bar{w}_1^{-1}, \bar{w}_2^{-1}) \tag{4.23}$$

for $\sigma = i$ leaves invariant the Floquet spectral set. Since this set is isomorphic to Γ, it follows that (4.23) induces an anti-holomorphic involution $\tau : \Gamma \to \Gamma$. The fixed ovals τ split Γ into two domains Γ^+, where $|w_1| > 1$, and Γ^-, where $|w_1| < 1$. On these ovals dp is positive, and by (4.20) the differential $d\Omega$ is also positive. The assertion is proved.

The fixed ovals $a_0, \ldots, a_l$ of the anti-involution τ are the "spectrum" of the operator (1.1) in the space of square integrable functions on the real line.

***Theorem* 4.3** ([52]). *Suppose that the parameters* $(\Gamma, P_0, k^{-1}, \gamma_s)$ *satisfy the above conditions that guarantee that the corresponding finite-gap potential* $u(x, y)$ *is real and non-singular. Then*

$$(4.24) \qquad \delta(x-x') = \int_{(\bigcup_s a_s)\setminus P_0} \psi(x, y, Q)\,\psi^+(x', y, Q)\,d\Omega.$$

The theorem is proved in [52] in a more general situation with the help of the standard method of contour integration.

We note that for $Q \in a_s$ the functions $\psi(x, y, Q)$ and $\psi^+(x, y, Q)$ are complex conjugate to each other and bounded, since $|w_i(Q)| = 1$.

The case $\sigma = 1$. Finite-gap solutions of the equation (1.1) with $\sigma = 1$ are real and non-singular if and only if their data $(\Gamma, P_0, k^{-1}, \gamma_s)$ satisfy the following conditions: there is an anti-holomorphic involution τ on the curve Γ that has $g+1$ fixed ovals (such curves are called *M-curves*); each fixed oval of τ contains one of the points $P_0, \gamma_1, \ldots, \gamma_g$; the local parameter k^{-1} in a neighbourhood of P_0 must be chosen so that $k(\tau(Q)) = -\bar{k}(Q)$.

Remark. On the fixed ovals $\tau^* dp = -\overline{dp}$, $\tau^* dE = -\overline{dE}$, therefore the condition that the periods of this differentials are real means that the integrals of dp, dE along $a_1, \ldots, a_g$ are equal to zero. Hence in the case $\sigma = 1$ the differentials dp and dE coincide with the differentials $\Omega^{(1)}$ and $\Omega^{(2)}$, where the $\Omega^{(s)}$ are defined at the beginning of the section (see (4.5)).

§5. The completeness theorem for products of Bloch functions

In this section we restrict ourselves to the case of real non-singular finite-gap potentials of the equation (1.1) with $\sigma = 1$. As shown above, they are determined by an M-curve Γ with a distinguished point $P_0 \in a_0$ (where $a_0, \ldots, a_g$ are the fixed ovals of the anti-involution $\tau : \Gamma \to \Gamma$) and by a collection of points $\gamma_s \in a_s$. Moreover, they depend on the equivalence class $[k^{-1}]_2$ of a local parameter such that $k(\tau(Q)) = -\bar{k}(Q)$. The real dimension of the manifold of such data

$$(5.1) \qquad M_g = (\Gamma, P_0, [k^{-1}]_2)$$

is equal to $3g+1$, where g is the genus of Γ. The submanifold M_g^0 of data (5.1), corresponding to the potentials with zero mean value in x, has dimension $3g$ and, as seen from (4.8, 9) and from the fact that (for $\sigma = 1$) $dp = \Omega^{(1)}$, is determined by the condition $p^{-1}(Q) \in [k^{-1}]_2$, where $p(Q)$ is an arbitrary branch of the quasi-momentum.

The main aim of this section is to construct, from products of Bloch functions corresponding to finite-gap periodic operators and their dual functions, an analogue of the Fourier basis in the space of functions periodic in x and y. Before we present these results we shall need detailed information on the structure of "resonance points" on the curves corresponding to such potentials.

Suppose that the set of data $(\Gamma, P_0, [k^{-1}]_2) \in M_g^0$ satisfies the conditions (4.13), which are necessary and sufficient for the periodicity of u. Then the functions $w_i(Q)$, $i = 1, 2$, being the eigenvalues of the operators of translation by the periods in x and y, are defined on Γ. Two points Q and Q' are called *resonance points* if $w_i(Q) = w_i(Q')$.

On each of the domains $\Gamma^{\pm}$ into which the cycles $a_0, \ldots, a_g$ split Γ we can choose a single-valued branch of the integrals

$$p(Q) = \int_q^Q dp, \quad E(Q) = \int_q^Q dE, \quad q \in a_0. \tag{5.2}$$

(For Γ^+ we take the domain on which $\operatorname{Re} p > 0$.)

Lemma 5.1 ([18]). *For any M-curve Γ the map*

$$\Gamma^+ \ni Q \to (\operatorname{Re} p(Q), \operatorname{Re} E(Q)) \tag{5.3}$$

is a real diffeomorphism of Γ^+ onto the right half-space $\mathbf{R}^2$ with g deleted points. The curve Γ and P_0 correspond to the periodic potentials of the equation (1.1) *with $\sigma = 1$ if and only if the coordinates of these points on Γ have the form $(\pi N_s l_1^{-1}, \pi M_s l_2^{-1})$, where $N_s > 0$, M_s are integers. All the pairs of resonance points on Γ are the points $P_{NM}^{\pm}$ such that $P_{NM}^- = \tau(P_{NM}^+)$ and P_{NM}^+ is the inverse image under the map* (5.3) *of the point with coordinates $(\pi N l_1^{-1}, \pi M l_2^{-1})$, where $N > 0$, M are integers, $(N, M) \neq (N_s, M_s)$, $s = 1, \ldots, g$.*

Proof. The differentials dp and dE are purely imaginary on the fixed ovals $a_0, \ldots, a_g$. Therefore the map (5.3) extends continuously to these ovals. Moreover, the cycle a_0 is mapped to the origin, while the cycles a_s are mapped to the points with coordinates

$$\frac{1}{2}\oint_{b_s} dp = \pi U_s, \quad \frac{1}{2}\oint_{b_s} dE = \pi V_s, \quad s = 1, \ldots, g \tag{5.4}$$

(the b_s are cycles of Γ complementing the a_s to a canonical basis).

We consider the level curve of the function $\operatorname{Re} p = r$ on Γ^+. The function $\operatorname{Re} E$ has no extrema on this curve. First we shall prove this for $r \neq \pi U_s$, $s = 1, \ldots, g$. Suppose that $\operatorname{Re} E$ has an extremum at a point Q on the curve $\operatorname{Re} p = r$. Then at this point

$$\frac{dE}{dp}(Q) = \lambda, \tag{5.5}$$

where λ is real.

The differential $dE - \lambda dp$ has $2g - 1$ zeros. It is real on the cycles $a_0, \ldots, a_g$. Its integrals over $a_1, \ldots, a_g$ are equal to zero since, as explained at the end of the previous section, so are the integrals of dE and dp over these cycles. Hence $dE - \lambda dp$ has at least two zeros on each of these cycles.

One more zero belongs to a_0. Consequently, this differential cannot vanish at Q, which contradicts (5.5). In a similar way it can be proved that Re E is monotone on all connected components of the level curve Re $p = \pi U_s$. The first assertion of the lemma is proved, while the second follows from (5.4) and (4.13).

For the proof of the last assertion of the lemma it is sufficient to consider the following map on Γ:

$$(5.6)\qquad Q \to (\operatorname{Im} p(Q),\ \operatorname{Im} E(Q)).$$

By analogy with the above it can be proved that the inverse image of any point of $\mathbf{R}^2$ consists of at most two points of Γ. Since Im p and Im E are even with respect to τ, the two inverse images are conjugate to each other. Under conjugation Re p and Re E change sign. The resonance condition for the two points

$$(5.7)\quad \operatorname{Re} p\,(P^+_{NM}) - \operatorname{Re} p\,(P^-_{NM}) = \frac{2\pi N}{l_1},\quad \operatorname{Re} E\,(P^+_{NM}) - \operatorname{Re} E\,(P^-_{NM}) = \frac{2\pi M}{l_2}$$

implies the assertion of the lemma.

Let $\psi(x, y, Q)$ be the Baker–Akhiezer function constructed by the data (5.1) and the collection of poles γ_s. If the conditions (4.13) of periodicity of the correspondent potential are satisfied, then the products

$$(5.8)\qquad \Phi^{\pm}_{NM}(x, y) = \psi(x, y, P^{\pm}_{NM})\,\psi^+(x, y, P^{\mp}_{NM})$$

are, by the definition of resonance points, periodic functions of x, y. The products $\psi(x, y, Q)\psi^+(x, y, Q)$ are periodic functions too. It follows from the Riemann–Roch theorem that among the latter there are only $g+1$ linearly independent ones. Indeed, for any x, y the function $\psi\psi^+$, as a function of Q, is meromorphic with possible poles at the points γ_s^+, γ_s. By the Riemann–Roch theorem the dimension of the space of such functions is equal to $g+1$. (It follows from this reasoning that the dimension of the space of functions $\psi(x, y, Q)\psi^+(x, y, Q)$ is at most $g+1$. In the proof of Theorem 5.1 it will be shown that it is equal to $g+1$.)

We denote by $\Phi_s^+(x, y)$ the periodic functions

$$(5.9)\quad \Phi_s^+(x, y) = \frac{\psi(x, y, P_{2s})\,\psi^+(x, y, P_{2s})}{\langle\psi(x, y, P_{2s})\,\psi^+(x, y, P_{2s})\rangle_x},\qquad s = 1, \ldots, g,$$

where the P_j, $j = 1, \ldots, 2g$, are the zeros of the differential dp numbered so that P_{2s-1}, P_{2s} lie on the oval a_s.

Let $L_2^0 = L_2^0(T^2)$ be the space of square integrable functions periodic in x, y and with zero mean value in x. We denote the dual space by $(L_2^0)^*$. Let us define elements $\Phi_s^- \in (L_2^0)^*$ which, as will be shown below, together with $\Phi^{\pm}_{NM}$ and Φ_s^+ form an analogue of the Fourier basis in $(L_2^0)^*$.

We define the functions $r_s(x, y)$ by the formula

$$(5.10)\quad r_s(x, y) = \exp\left(i \int_q^{\gamma_s} (x\,dp + y\,dE)\right) \frac{\theta(A(\gamma_s) + Ux + Vy + Z)}{\theta(A(P_0) + Ux + Vy + Z)}\,.$$

which up to the constant factor $\theta(A(P_0)+Z)$ coincides with the coefficient of the singular term in the expansion of $\psi(x, y, Q)$ in the local parameter $\theta(A(Q)+Z)$ in a neighbourhood of its pole. (We recall that $\theta(A(\gamma_s)+Z) = 0$.) Let Q_n^s be points of Γ such that $w_1(\gamma_s) = w_1(Q_n^s)$. We consider the series

$$(5.11)\qquad \Phi_s^-(x, y) = \sum_{n\neq 0} \frac{w_2(Q_n^s)}{w_2(\gamma_s) - w_2(Q_n^s)} \frac{r_s(x, y)\psi^+(x, y, Q_n^s)}{\langle \psi(x, y, Q_n^s)\psi^+(x, y, Q_n^s)\rangle_x}\,.$$

***Lemma* 5.2.** *The series* (5.11) *for all x and $y < l_2$ converges and determines a smooth analytic function $\Phi_s^-(x, y)$ periodic in x. For any continuously differentiable function $v(x, y)$ with periods l_1, l_2 in x, y there is a finite limit*

$$(5.12)\qquad \lim_{l\to l_2} \int_0^l \langle \Phi_s^-(x, y)\, v(x, y)\rangle_x\, dy,$$

which determines the element $\Phi_s^- \in (L_2^0)^*$.

Proof. We have $k(Q_n^s) = 2\pi n/l_1 + p_s$ as $|n| \to \infty$. Hence

$$(5.13)\qquad w_2(Q_n^s) \approx \exp\left(-\frac{4\pi^2 n^2 l_2}{l_1^2} - \frac{4\pi n p_s}{l_1} l_2\right).$$

In a similar way, up to a finite factor

$$(5.14)\quad \psi^+(x, y, Q_n^s) \approx \exp\left(-\frac{2\pi i n x}{l_1} + \frac{4\pi^2 n^2 y}{l_1^2} + \frac{4\pi n p_s y}{l_1}\right).$$

Therefore for $y < l_2$ the terms of (5.11) decay exponentially. The periodicity of Φ_s^- in x follows from the fact that by the definition of Q_n^s all terms of the series are periodic in x. We denote by r_s^0 the periodic function $r_s^0 = r_s(x, y)\exp(-ip_s x - iE_s y)$, where p_s and E_s are the values of the quasi-momentum and the quasi-energy at γ_s. We have

$$(5.15)\quad \frac{r_s(x, y)\psi^+(x, y, Q_n^s)}{\langle \psi(x, y, Q_n^s)\psi^+(x, y, Q_n^s)\rangle_x} =$$

$$= \left(r_s^0 + O\left(\frac{1}{|n|}\right)\right) e^{-\frac{2\pi i n x}{l_1} + \frac{4\pi^2 n^2 y}{l_1^2} + \frac{4\pi n p_s}{l_1} + iE_s y}\,.$$

Therefore the left-hand side of (5.12) is represented by the sum of a series whose terms for $|n| > N_0$ are uniformly bounded by the Fourier coefficients of the periodic function $r_s^0(x, y)v(x, y)$, which implies the last assertion of the lemma.

***Theorem* 5.1.** *The functions $\Phi_s^{\pm}$ and $\Phi_{NM}^{\pm}$ form a minimal basis in $(L_2^0)^*$.*

Proof. To prove the completeness of this set, it is sufficient to show that for any continuously differentiable periodic function $v(x, y)$ it follows from the equalities

$$(5.16)\qquad \text{a) } \langle\langle v\Phi_s^+\rangle\rangle = 0, \quad \text{b) } \int_0^{l_2} \langle \Phi_s^-(x, y)\, v(x, y)\rangle_x\, dy = 0,$$

$$(5.17)\qquad \text{a) } \langle\langle v\Phi_{NM}^{\pm}\rangle\rangle = 0, \quad \text{b) } \langle v\rangle_x = 0,$$

that $v \equiv 0$. (Here and in what follows $\langle\langle\cdot\rangle\rangle$ denotes the mean value in x, y.)

For any point $Q_0 \in \Gamma$ such that $w_1(Q_0) \neq w_1(P_j)$, $Q_0 \neq \gamma_s$, $Q_0 \neq P_{NM}^{\pm}$, we consider the series

$$(5.18)\qquad \varphi(x, y, Q_0) = \sum_{n\neq 0} \psi_n(x, y) \frac{w_{2n}}{w_{2n} - w_{20}} \int_y^{y+l_2} \frac{\langle \psi_n^+ v \psi_0\rangle_x}{\langle \psi_n^+\psi_n\rangle_x}\, dy',$$

where $\psi_n(x, y) = \psi(x, y, Q_n)$, $w_{2n} = w_2(Q_n)$, and Q_n as before is defined from the condition $w_1(Q_n) = w_1(Q_0)$. Asymptotically the terms of this series coincide with the terms of the series $\varphi_1(x, y, Q_0)$ considered in §2. Therefore (5.18) converges and determines an analytic function of the variable Q_0. It follows from (5.17) that it extends analytically to all resonance points $P_{NM}^{\pm}$. Let us show that it can also be extended by continuity to points $Q_0 \neq P_j$ such that $w_1(Q_0) = w_1(P_j)$. We consider $\varphi(x, y, Q_0')$, where Q_0' is close to Q_0. Making Q_0' tend to Q_0, we see that among the terms of the series (5.18) there are two terms tending to infinity. They correspond to the indices n_0, n_0+1 such that the corresponding points Q_{n_0}, Q_{n_0+1} lie in a neighbourhood of P_j. (These terms tend to zero, since $\langle\psi_{n_0}\psi_{n_0}^+\rangle_x$ and $\langle\psi_{n_0+1}\psi_{n_0+1}^+\rangle_x$ tend to zero as $Q_0' \to Q_0$.) The sum of the two terms tends to a finite limit. In fact, the terms of (5.18) coincide for $n \neq 0$ with the residues at the points Q_n of the differential

$$(5.19)\qquad \int_y^{y+l_2} dy' \int_0^{l_1} \frac{\psi(x, y, Q)\, \psi^+(x', y', Q)\, \psi(x', y', Q_0)\, v(x', y')\, dx'}{(w_1(Q_0)\, w_1^{-1}(Q) - 1)\,(w_2(Q_0)\, w_2(Q_0') - 1)}\, d\Omega(Q),$$

which locally depends smoothly on Q_0. Therefore the sum of the two terms of (5.18) that tend to infinity tends to the integral of the differential (5.19) over a small contour surrounding P_j.

Thus if v satisfies (5.17a), then $\varphi(x, y, Q_0)$ is an analytic function on Γ outside the points P_j, γ_s and the distinguished point P_0. At the points γ_s it possibly has simple poles, while at the points P_j, $j = 1, \ldots, 2g$, it can have poles of multiplicity 2. It follows from the equality

$$(5.20)\qquad \frac{w_{2n}}{w_{2n} - w_{20}} \int_y^{y+l_2} \chi_n(y')\, dy' = \frac{w_{2n}}{w_{2n} - w_{20}} \int_0^{l_2} \chi_n(y')\, dy' - \int_0^{y} \chi_n\, dy',$$

$$\chi_n = \frac{\langle \psi_n^+ v \psi_0\rangle_x}{\langle \psi_n^+ \psi_n\rangle_x}$$

that the function

$$\tilde{\varphi}(x, y, Q_0) = \varphi(x, y, Q_0) - \varphi(0, 0, Q_0)\psi(x, y, Q_0) \tag{5.21}$$

has no poles at the points P_j if v satisfies (5.16a). It follows from (5.16b) that $\varphi(0, 0, Q_0)$ has no poles at the points γ_s. Hence $\tilde{\varphi}$ is meromorphic on Γ outside P_0 and possibly has simple poles at the points γ_s. By analogy with (2.19) for $s = 1$ we have

$$\tilde{\varphi}(x, y, Q_0)\psi^+(x, y, Q_0) = O(k^{-1}(Q_0)). \tag{5.22}$$

Therefore φ is a function of Baker–Akhiezer type, but in the expansion (4.2) for $\tilde{\varphi}$ the pre-exponential factor begins with $O(k^{-1})$. From the uniqueness of the Baker–Akhiezer function we conclude that $\tilde{\varphi} = 0$.

By Lemma 2.4 the sequence $\psi_n = \psi(x, y, Q_n)$ is a basic sequence (in the sense of the definition given in § 1). Comparing formulae (1.25), (1.28) with (5.18), we obtain

$$(\partial_y - \partial_x^2 + u_0)\varphi(x, y, Q_0) = -v\psi_0 + \frac{\langle\psi_0^+ v\psi_0\rangle_x}{\langle\psi_0^+\psi_0\rangle_x}\psi_0, \tag{5.23}$$

where u_0 is the finite-gap potential corresponding to the Baker–Akhiezer function $\psi(x, y, Q)$. Since $\tilde{\varphi} = 0$, the left-hand side of (5.23) is equal to zero. We conclude from (5.16a) that $v \equiv 0$. The completeness of the family $\Phi_s^{\pm}$, $\Phi_{NM}^{\pm}$ is proved.

The proof of minimality of this family follows from the following construction of a "dual" basis in L_2^0. We consider an arbitrary variation $u(x, y, \tau)$ of a finite-gap potential $u_0 = u(x, y, 0)$. For any point $Q_0 \neq P_j$, $P_{NM}^{\pm}$ we denote by $Q(\tau)$ the point of the Riemann surface Γ_τ corresponding to the potential $u(x, y, \tau)$, which is determined by the condition $w_1(Q(\tau)) = w_1(Q_0)$. We put

$$\psi_\tau = \psi_\tau(x, y, Q_0) = \partial_\tau\psi(x, y, Q(\tau))\,|_{\tau=0}. \tag{5.24}$$

By definition this function has Bloch behaviour in x with multiplier $w_1(Q_0)$.

***Lemma* 5.3.** *For any variation $u(x, y, \tau)$ the function $\varphi(x, y, Q_0)$ defined by* (5.18), *where*

$$v(x, y) = \partial_\tau u(x, y, \tau)\,|_{\tau=0}, \tag{5.25}$$

is equal to

$$\varphi(x, y, Q_0) = \psi_\tau - \frac{\langle\psi_\tau\psi_0^+\rangle_x}{\langle\psi_0\psi_0^+\rangle_x}\psi_0, \quad \psi_0 = \psi(x, y, Q_0). \tag{5.26}$$

Proof. The right-hand side of (5.26) is a Bloch function with multipliers w_{10}, w_{20} and satisfies the normalization condition $\langle\varphi\psi_0^+\rangle_x = 0$. Differentiating (1.1) with respect to τ, we see that it is a solution of (5.23). As shown in § 1, such a solution is unique and is given by (5.18). The lemma is proved.

We first consider finite-gap variations u that preserve the periods of u_0. Such variations are those that do not change Γ but move the poles γ_s of a Bloch function. We put

$$(5.27)\qquad v_s^-(x, y) = \frac{\partial}{\partial\gamma_s} u(x, y \mid \Gamma, \gamma_1, \ldots, \gamma_g).$$

(These functions are linear combinations of $\partial u/\partial z_i$, where u is given by (4.8) and the z_i are the coordinates of the vector Z.) Moreover there are variations that preserve the γ_s but change Γ. For example, if we take the endpoints p_{2s} of the slits in the model of Γ in §2 for the parameters determining Γ (we recall that for variations of Γ preserving the periods of u_0, among the endpoints of the slits only half of them are independent), then we can define the functions

$$(5.28)\qquad v_s^+(x, y) = \partial/\partial p_{2s} u(x, y \mid p_2, \ldots, p_{2g}, \gamma_1, \ldots, \gamma_g).$$

***Lemma* 5.4.** *The functions $v_s^\pm$ satisfy the following conditions*:

$$(5.29)\qquad \langle\langle v_s^\pm \Phi_{NM}^\pm\rangle\rangle = \langle\langle v_s^\pm \Phi_{NM}^\mp\rangle\rangle = 0,$$

$$(5.30)\qquad \langle\langle v_s^+, \Phi_{s'}^-\rangle\rangle = 0, \quad \langle\langle v_s^- \Phi_{s'}^+\rangle\rangle = 0.$$

$$(5.31)\qquad \langle\langle v_s^+ \Phi_{s'}^+\rangle\rangle = \delta_{ss'}, \quad \langle\langle v_s^- \Phi_{s'}^-\rangle\rangle = a_s\delta_{ss'}, \quad a_s \neq 0.$$

Proof. For both types of variations under consideration, ψ_τ (where τ is either γ_s or p_{2s}) has no poles at the points $P_{NM}^\pm$. This implies (5.29). The function $\partial\psi/\partial\gamma_s$ has a pole at γ_s of multiplicity 2 and simple poles at $\gamma_{s'}$, $s' \neq s$. It is analytic at the remaining points. Comparing these properties with those that follow from (5.18), we obtain the second equalities in (5.30), (5.31). Under variations of p_{2s} the derivatives $\partial\psi/\partial\ p_{2s}$ have poles at the points P_{2s}. Hence we obtain the first equalities in (5.30), (5.31). The lemma is proved.

Its assertions say that the $\Phi_s^\mp$ form a basis in the cotangent bundle to the manifold of periodic finite-gap potentials, corresponding to the curves of genus g. Below we shall show that $\Phi_{NM}^\pm$ are dual to the variations transversal to this manifold, which "open a gap" at the place of the resonance points $P_{NM}^\pm$.

We consider small neighbourhoods $R_{NM}^\pm$ of some pair of points $P_{NM}^\pm$. The function w_1 identifies each of these neighbourhoods with some neighbourhood $\widetilde{R}_{NM}$ of $w_1(P_{NM}^\pm)$. If for $w_1 \in \widetilde{R}_{NM}$ we put $P^\pm(w_1) \in R_{NM}^\pm$, $w_1(P^\pm) = w_1$, then $w_2^\pm(w_1) = w_2(P^\pm)$ are analytic functions in $\widetilde{R}_{NM}$. Let $\hat{R}_{NM}$ be a two-sheeted covering of $\widetilde{R}_{NM}$ given by the equation

$$(5.32)\qquad z^2 - (w_2^+(w_1) + w_2^-(w_1))z + (1-\varepsilon^2)w_2^+(w_1)w_2^-(w_1) = 0.$$

For sufficiently small ε the boundary of $\hat{R}_{NM}$ splits into two circles, each of which can naturally be identified with the boundaries of $R_{NM}^\pm$. This identification enables us to paste the domain $\hat{R}_{NM}$ to the complement of the domains $R^{\pm}_{NM}$ in Γ. As a result we obtain a Riemann surface of genus $g+1$. We denote it by $\Gamma^{\varepsilon}_{NM}$. The involution τ extends naturally to $\Gamma^{\varepsilon}_{NM}$, where it has besides the old fixed cycles $a_0, \ldots, a_g$ a new one $a_{g+1} \in \hat{R}_{NM}$.

We present briefly the necessary information about holomorphic differentials on $\Gamma^{\varepsilon}_{NM}$ [58]. Let $\tilde{\omega}_1, \ldots, \tilde{\omega}_{g+1}$ be a basis of normalized vectors on $\Gamma^{\varepsilon}_{NM}$. If $\omega_1, \ldots, \omega_g$ is a basis of normalized holomorphic differentials on Γ and $\hat{\omega}_{NM}$ is the normalized differential of the third kind on Γ with residues $\pm 1/2\pi i$ at the points $P^{\pm}_{NM}$, then we have outside $R^{\pm}_{NM}$

$$(5.33) \quad |\omega_i \tilde{\omega}_i^{-1} - 1| = O(\varepsilon^2), \quad i = 1, \ldots, g, \quad |\hat{\omega}_{NM} \tilde{\omega}_{g+1}^{-1} - 1| = O(\varepsilon^2).$$

Let $\tilde{B}^{\varepsilon}$ and B be the matrices of periods of the curves $\Gamma^{\varepsilon}_{NM}$ and Γ respectively. Then it follows from (5.33) that

$$(5.34) \qquad \tilde{B}^{\varepsilon}_{ij} = B_{ij} + O(\varepsilon^2), \quad i, j \leqslant g,$$

$$(5.35) \qquad \tilde{B}_{g+1, i} = \oint_{b_i} \hat{\omega}_{NM} = A_i(P^{+}_{NM}) - A_i(P^{-}_{NM}) = A^{i}_{NM}$$

(the second of these equations is a consequence of the Riemann relations). We have for the matrix entry $\tilde{B}_{g+1,g+1}$

$$(5.36) \qquad \tilde{B}_{g+1, g+1} = \frac{1}{\pi i} (\log \varepsilon + r_{NM} + O(\varepsilon^2)).$$

The theta-function $\tilde{\theta} = \theta(z_1, \ldots, z_{g+1})$ constructed from the matrix $\tilde{B}^{\varepsilon}$ is equal to

$$(5.37) \qquad \tilde{\theta} = \theta(z) + \varepsilon e^{r_{NM}} [\theta(z + A_{NM}) e^{2\pi i z_{g+1}} - \\ - \theta(z - A_{NM}) e^{-2\pi i z_{g+1}}] + O(\varepsilon^2),$$

where $z = (z_1, \ldots, z_g)$, $A_{NM} = (A^{i}_{NM})$ and A^{i}_{NM} are defined in (5.35).

We consider the finite-gap potential $\tilde{u}(x, y)$ corresponding to the curve $\Gamma^{\varepsilon}_{NM}$ and the divisor of the poles $\gamma_1, \ldots, \gamma_{g+1}$. It is given by (4.8), in which the theta-function is $\tilde{\theta}$. The vectors of the b-periods of the differentials dp and dE on $\Gamma^{\varepsilon}_{NM}$ are equal to

$$(5.38) \qquad \begin{aligned} &\tilde{U}_i = U_i + O(\varepsilon^2), & &\tilde{V}_i = V_i + O(\varepsilon^2), \quad i = 1, \ldots, g, \\ &\tilde{U}_{g+1} = \frac{2\pi i N}{l_1} + O(\varepsilon^2), & &\tilde{V}_{g+1} = \frac{2\pi i M}{l_2} + O(\varepsilon^2). \end{aligned}$$

From (4.8) and (5.37) we obtain

$$(5.39) \qquad \delta u = \tilde{u} - u = \varepsilon (v^{+}_{NM} e^{2\pi i z_{g+1}} + v^{-}_{NM} e^{-2\pi i z_{g+1}}) + O(\varepsilon^2),$$

where the functions $v^{\pm}_{NM}$ are given by

$$(5.40) \quad v^{\pm}_{NM} = e^{r_{NM}} \cdot 2\partial_x^2 \frac{\theta(Ux + Vy + Z \pm A_{NM})}{\theta(Ux + Vy + Z)} e^{\pm \frac{2\pi i N}{l_1} x \pm \frac{2\pi i M}{l_2} y}.$$

***Lemma* 5.5.** *The functions $v^{\pm}_{NM}$ satisfy the relations*

$$\langle\langle v^{\pm}_{NM}\Phi^{\pm}_{s}\rangle\rangle = 0, \qquad \langle\langle v^{\pm}_{NM}\Phi^{\mp}_{s}\rangle\rangle = 0, \tag{5.41}$$

$$\langle\langle v^{\pm}_{NM}\Phi^{\pm}_{N_1M_1}\rangle\rangle = 0, \quad \langle\langle v^{\pm}_{NM}\Phi^{\mp}_{N_1M_1}\rangle\rangle = \delta_{NN_1}\delta_{MM_1}. \tag{5.42}$$

Proof. Considering the derivative of the Bloch function with respect to ε, we see that the corresponding function ψ_ε has simple poles at the points γ_s and P_j and a pole at the pair of points $P^{\pm}_{NM}$. Comparing its residue with (5.18), we obtain (5.42). The equalities (5.41) follow from the fact that the poles of ψ_ε at the points γ_s and P_{2s} are simple.

The lemmas proved above enable us to conclude that the basis $\Phi^{\pm}_{s}$, $\Phi^{\pm}_{NM}$ in $(L^0_2)^*$ is minimal. At the same time they prove the following theorem.

***Theorem* 5.2.** *The functions $v^{\pm}_{s}$ and $v^{\pm}_{NM}$ defined by* (2.57), (5.28), *and* (5.40) *form a minimal basis in L^0_2.*

Chapter II

THE PERIODIC PROBLEM FOR EQUATIONS OF KADOMTSEV-PETVIASHVILI TYPE

As mentioned in the introduction, equations of KP type are a system of non-linear equations for the coefficients u_i and v_j of operators L, A of the form (2) equivalent to the operator equation (4). (In what follows we shall assume that $u^{\alpha\beta}_n = u^\alpha_n\delta_{\alpha\beta}$, $v^{\alpha\beta}_m = v^\alpha_m\delta_{\alpha\beta}$ are constant diagonal matrices, $v^{\alpha\alpha}_{m-1} = 0$.) This definition needs a refinement. It turns out that if $n \leqslant m$, then the system (4) reduces to a sheaf of systems only on the coefficients of A that are parametrized by constants $h_{\alpha i}$, $\alpha = 1, ..., l$; $i = 0, ..., n$. (See [7] for the details.) In what follows, by equations of KP type we shall mean reduced systems of equations for the coefficients of A.

§1. Necessary information on finite-gap solutions

The initial object in the construction of [7] of finite-gap solutions of (4) is a non-singular algebraic curve Γ of genus g with distinguished points P_α, $\alpha = 1, ..., l$, in the neighbourhoods of which the local parameters $k^{-1}_\alpha(Q)$ are fixed, $k^{-1}_\alpha(P_\alpha) = 0$. We put $R_\alpha(k) = \sum_{i=0}^{n} h_{\alpha i}k^i$ (where the $h_{\alpha i}$ are constants parametrizing the systems of equations of KP type together with constants v^α_m that are diagonal elements of the leading coefficient of A).

For any collection of points $\gamma_1, \ldots, \gamma_{g+l-1}$ in general position there is a unique meromorphic function $\psi_\alpha(x, y, t, Q)$, $Q \in \Gamma$, which

1° *is meromorphic on Γ outside the points P_α and has poles at γ_s (at most simple if the γ_s are distinct)*;

2° *in a neighbourhood of* P_β *is representable in the form*

$$\psi_\alpha = l^{\,i(k_\beta x + R_\beta(k_\beta)y + v_m^\beta k_\beta^m t)}\left(\sum_{s=0}^{\infty} \xi_s^{\alpha\beta}(x, y, t)\, k_\beta^{-s}\right), \tag{1.1}$$

where $\xi_0^{\alpha\beta} = \delta_{\alpha\beta}$, $k_\beta = k_\beta(Q)$.

We denote by $\psi(x, y, t, Q)$ the column vector with coordinates ψ_α. As shown in [7], there are unique operators L and A of the form (2) with $(l \times l)$ matrix coefficients such that

$$(\partial_y - L)\psi(x, y, t, Q) = 0, \quad (\partial_t - A)\psi(x, y, t, Q) = 0. \tag{1.2}$$

Since the equalities (1.2) are satisfied for all Q, it follows that L and A satisfy (4) (with $\sigma = 1$). It follows easily from the uniqueness of ψ_α that they do not change under substitutions of local parameters such that $k'_\beta = k_\beta + O(k_\beta^{-m})$. The local parameters related to each other in this way belong to one equivalence class $[k_\beta^{-1}]_m$.

The complex dimension of the manifold of collections

$$M_g = (\Gamma, P_\alpha, [k_\alpha^{-1}]_m), \quad \Gamma \text{ has genus } g, \tag{1.3}$$

is equal to $N = 3g - 3 + (m+2)l$. We can introduce a complex analytic structure on M_g. Let $I = (I_1, \ldots, I_N)$ be an arbitrary (local) system of coordinates on M_g. The dependence of all magnitudes on I in the subsequent formulae is complex-analytic.

We denote by dp, dE, $d\Omega$ the meromorphic differentials on Γ with poles at the points P_α of the form dk_α, $dR_\alpha(k_\alpha)$, $v_m^\alpha dk_\alpha^m$ respectively, uniquely normalized by the condition that their integrals over all cycles are real. Let a_i, b_j be the canonical basis of cycles on Γ. We define a g-dimensional real vector U with coordinates

$$U_k = \frac{1}{2\pi}\oint_{b_k} dp, \quad U_{k+g} = -\frac{1}{2\pi}\oint_{a_k} dp, \quad k = 1, \ldots, g. \tag{1.4}$$

In a similar way, starting from dE, $d\Omega$ we can define $2g$-dimensional vectors V, W. Cutting Γ along a_i, b_j, we can choose a single-valued branch of the integrals $p(Q)$, $E(Q)$, $\Omega(Q)$. In a neighbourhood of P_α they have the form

$$p = k_\alpha - a_\alpha + O(k_\alpha^{-1}), \quad E = R_\alpha(k_\alpha) - b_\alpha + O(k_\alpha^{-1}), \tag{1.5}$$
$$\Omega = v_m^\alpha k_\alpha^m - c_\alpha + O(k_\alpha),$$

and p, E, Ω can be normalized uniquely by the condition $a_l = b_l = c_l = 0$.

In [22] with the help of explicit theta-function formulae it is proved that the finite-gap solutions constructed above have the following form. If $a = a(I)$, $b = b(I)$, $c = c(I)$ are diagonal matrices $a_\alpha\delta_{\alpha\beta}$, $b_\alpha\delta_{\alpha\beta}$, $c_\alpha\delta_{\alpha\beta}$, then

$$L = g\hat{L}g^{-1}, \quad A = g\hat{A}g^{-1}, \tag{1.6}$$

where $g = \exp(i(ax + by + ct + \Phi))$, $\Phi_{\alpha\beta} = \Phi_\alpha \delta_{\alpha\beta}$, and the coefficients $\hat{u}_i$, $\hat{v}_j$ of the operators $\hat{L}$ and $\hat{A}$ have the form

$$(1.7) \qquad \hat{u}_i = \hat{u}_i(Ux + Vy + Wt + Z \mid I), \quad \hat{v}_j = \hat{v}_j(Ux + Vy + Wt + Z \mid I).$$

Here $\hat{u}_i(z_1, \ldots, z_{2g}|I)$, $\hat{v}_j(z_1, \ldots, z_{2g}|I)$ are functions with unit periods in the variables z_i. The real coordinates of the vector Z and the complex constants Φ_α are determined by the collection $\gamma_1, \ldots, \gamma_{g+l-1}$. In formulae (1.6), (1.7) they can be assumed to be arbitrary.

To avoid burdening the presentation with superfluous technical detail, we refer the reader to [22] for details of the construction of explicit formulae for $\hat{u}_i$, $\hat{v}_j$.

As an example we consider finite-gap solutions of the KP equation [6]. Solutions of this equation are constructed with the help of the Baker–Akhiezer function $\psi(x, y, t, Q)$, which is meromorphic on Γ outside P_0, has poles $\gamma_1, \ldots, \gamma_g$, and in a neighbourhood of P_0 has the form

$$(1.8) \qquad \psi = e^{ikx - \sigma^{-1}k^2 y + ik^3 t}\left(1 + \sum_{s=1}^{\infty} \xi_s(x, y, t) k^{-s}\right), \quad k = k(Q).$$

This function has a form similar to (1.4.7) (here and in what follows, in the triple numbering the first number indicates the number of the chapter)

$$(1.9) \qquad \psi = e^{\int_q^{\Omega} (x\Omega^{(1)} + y\Omega^{(2)} + t\Omega^{(3)})} \frac{\theta(A(Q) + Ux + Vy + Wt + Z)\,\theta(A(P_0) + Z)}{\theta(A(Q) + Z)\,\theta(A(P_0) + Ux + Vy + Wt + Z)},$$

where $\Omega^{(1)}$, $\Omega^{(2)}$ are the same as in Chapter I, and $\Omega^{(3)}$ is a normalized Abelian differential with a pole at P_0 of the form dk^3. The corresponding finite-gap solution $u(x, y, t)$ is given by

$$(1.10) \quad u(x, y, t) = 2\partial_x^2 \log \theta(Ux + Vy + Wt + A(P_0) + Z) + \text{const}.$$

Coming back to finite-gap solutions of the general equations (4), we define following [56] the notion of a dual Baker–Akhiezer function. For any collection $\gamma_1, \ldots, \gamma_{g+l-1}$ in general position there is a unique (up to proportionality) differential $\hat{\omega}$ of the second kind with poles of the second order at the points P_α and vanishing at the points γ_s. The collection of points $\gamma_1^+, \ldots, \gamma_{g+l-1}^+$ that are the remaining zeros of $\hat{\omega}$ is called *dual* to the collection $\gamma_1, \ldots, \gamma_{g+l-1}$.

If $\psi(x, y, t, Q)$ is a vector-valued Baker–Akhiezer function defined above by the set of data (1.3) and the poles γ_s, then the dual vector-valued function $\psi^+(x, y, t, Q)$ is the row vector with coordinates ψ_α^+, which are meromorphic on Γ outside P_α with poles at the γ_s^+ and have in a neighbourhood of P_β the form

$$(1.11) \qquad \psi_\alpha^+ = \exp(i(-k_\beta x - R_\beta(k_\beta) y - v_m^\beta k_\beta^m t)\left(\sum_{s=0}^{\infty} \xi_s^{+\alpha\beta}(x, y, t) k_\beta^{-s}\right).$$

It is proved in [56] that ψ^+ satisfies the equations

$$\psi^+ (\partial_y - L) = 0, \quad \psi^+ (\partial_t - A) = 0, \tag{1.12}$$

where the operators L and A are the same as in (1.2). The proof of [56] is based on the fact that by the definition of ψ and ψ^+ the differentials

$$d\Lambda_{\alpha\beta} = \psi_\alpha (x, y, t, Q) \psi^+_\beta (x', y', t', Q) \hat{\omega} (Q) \tag{1.13}$$

are holomorphic outside $P_1, \ldots, P_l$, therefore

$$\sum_{\gamma=1}^{l} \operatorname{res}_{P_\gamma} d\Lambda_{\alpha\beta} = 0. \tag{1.14}$$

The bilinear relations, introduced in the papers by Sato, Miva, Jimbo, Date (see [46], [47], [68]) for the determination of τ-functions, are a generalization of the relation (1.14).

§2. The perturbation theory for finite-gap solutions of the Kadomtsev-Petviashvili −2 equation

We consider the problem of constructing asymptotic solutions of the equation

$$\frac{3}{3} u_{yy} + \left(u_t - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}\right)_x + \varepsilon K [u] = 0, \tag{2.1}$$

where ε is a small parameter, and $K[u]$ is a differential polynomial. There are several ways of stating this problem. One of them is related to the investigation of the influence of the perturbing term on the solutions of the periodic problem for the KP-2 equation. In this case an asymptotic series is constructed for the solution of the Cauchy problem with the initial data $u(x, y, 0)$ that belong to a neighbourhood of a finite-gap solution of the KP-2 equation. The second way of stating the problem is also meaningful in the case $K \equiv 0$. In this, an asymptotic solution of the KP-2 equation is searched for, the first term of which is equal to

$$u_0 (x, y, t) = 2\partial_x^2 \log \theta (\varepsilon^{-1} S (X, Y, T) \mid I (X, Y, T)) + c (X, Y, T), \tag{2.2}$$

where

$$\tilde{u} (z) = 2\partial_x^2 \log \theta (z \mid I), \quad \partial_x = \sum U_i \partial_{z_i}, \tag{2.3}$$

is a periodic function of $z = (z_1, \ldots, z_g)$ whose parameters (that is, the matrix of periods of holomorphic differentials on Γ) depend on the slow variables $X = \varepsilon x$, $Y = \varepsilon y$, $T = \varepsilon t$. The vector-valued function S is determined by the equations

$$\partial_X S = U(X, Y, T), \quad \partial_Y S = V(X, Y, T), \quad \partial_T S = W(X, Y, T), \tag{2.4}$$

where U, V, W are the vectors of the periods of the differentials dp, dE, $d\Omega$. They depend on X, Y, T by means of the dependence of the main parameters (Γ, P_0, k^{-1}) on these variables.

For space one-dimensional systems, in particular for the KdV equation, the main attention has been given to the second way of stating the problem [21], [22], [59]. Combining the two problems, we shall search for a solution of the equation (2.1) in the form

$$u(x, y, t) = u_0(x, y, t \mid X, Y, T) + \sum_{i=1}^{\infty} \varepsilon^i u_i(x, y, t \mid X, Y, T). \tag{2.5}$$

In the case when u_0 is a periodic function of x, y, to construct the series (2.5) it is sufficient to construct a set of solutions of the linearized equation (2.1)

$$\frac{3}{4} v_{yy} + \left(v_t - \frac{3}{2} u_0 v_x - \frac{3}{2} u_{0x} v + \frac{1}{4} v_{xxx}\right)_x = 0, \tag{2.6}$$

that for all t form a basis in the space of functions periodic in x, y. Moreover, it is necessary to have a dual basis of solutions of the conjugate linear equation

$$\frac{3}{4} \Phi_{yy} + \left(-\Phi_t - \frac{3}{2} u_0 \Phi_x + \frac{1}{4} \Phi_{xxx}\right)_x = 0. \tag{2.7}$$

To construct solutions of the equation (2.6) we use the fact that if there is a family of solutions of a non-linear equation, then the derivatives of these solutions with respect to the parameters are solutions of the linearized equation. Therefore the functions

$$v_s^+(x, y, t) = \frac{\partial}{\partial p_{2s}} u_0(x, y, t), \quad v_s^-(x, y, t) = \frac{\partial}{\partial \gamma_s} u_0(x, y, t), \tag{2.8}$$

where $u_0(x, y, t) = u_0(x, y, t \mid \gamma_s, p_{2s})$ are finite-gap solutions given by (1.10), are solutions of (2.6).

Considering variations of Γ analogous to those used in §5 and which correspond to "adding a handle" between the points Q and $\tau(Q)$, we obtain the following assertion.

***Lemma* 2.1.** *The functions* (2.9) *are solutions of the equation* (2.6)

$$v(x, y, t, Q) = 2e^{r(Q)} \partial_x^2 \left[\frac{\theta(Ux+Vy+Wt+Z+A(Q)-A(\tau(Q)))}{\theta(Ux+Vy+Wt+Z)}\right] \times \tag{2.9}$$
$$\times \exp\{i(p(Q)-p(\tau(Q)))x + (E(Q)-E(\tau(Q)))y + (\Omega(Q)-\Omega(\tau(Q)))t\}.$$

Here $r(Q)$ is a real function defined in the following way. Let $\hat{\omega}_Q$ be the normalized differential of the third kind with residues $\pm 1/2\pi i$ at the points Q, $\tau(Q)$. As $Q \to Q'$ we have

$$\int_{\tau(Q')}^{Q'} \hat{\omega}_Q = \log|p(Q') - p(Q)| + 2r(Q) + O((|p(Q) - p(\tau(Q))|)^{-1}). \tag{2.10}$$

By the definition of resonance points, the functions $v^{\pm}_{NM} = v(x, y, t, P^{\pm}_{NM})$ are periodic solutions of the equation (2.6).

We denote by $\Phi^{\pm}_{s}(x, y, t)$ the functions constructed with the help of $\psi(x, y, t, Q)$ and $\psi^{+}(x, y, t, Q)$ in the same way as the functions $\Phi^{\pm}_{s}(x, y)$ were constructed in the last section of Chapter I. Moreover we define periodic functions $\Phi^{\pm}_{NM}(x, y, t) = \Phi(x, y, t, P^{\pm}_{NM})$, where

$$\Phi(x, y, t, Q) = \frac{\psi(x, y, t, Q)\,\psi^{+}(x, y, t, \tau(Q))}{\langle\psi(x, y, t, Q)\,\psi^{+}(x, y, t, Q)\rangle_x}. \tag{2.11}$$

By complete analogy with the results of §5 of Chapter I we obtain the following theorem.

Theorem 2.1. *The functions* $v^{\pm}_{s}, v^{\pm}_{NM}$ *for any* t *form a basis in* L^{0}_{2}. *Moreover for them and for the* $\Phi^{\pm}_{s}$, $\Phi^{\pm}_{NM}$ *the orthogonality relations* (1.5.27, 28) *and* (1.5.40, 41, 42) *hold.*

Corollary. *The functions* $\Phi^{\pm}_{s}$, $\Phi^{\pm}_{NM}$ *are solutions of* (2.7).

The formulae for $v(x, y, t, Q)$ and $\Phi(x, y, t, Q)$ obtained above enable us to determine easily all the terms of the series (2.5) in the case of a periodic solution u_0. A direct analysis of the resulting expressions shows that the corresponding series can be defined for all finite-gap solutions by approximating the latter on any compact set by finite-gap periodic (in x, y) solutions with periods $l_1, l_2 \to \infty$. Under such an approximation the limit of the subset of the resonance points that gives non-trivial contributions to u_i is the set of points $Q \notin a_s$ such that there are integers $\rho = (r_1, \ldots, r_g)$ for which

$$\operatorname{Re} p(Q) = r_1 U_1 + \ldots + r_g U_g, \quad \operatorname{Re} E(Q) = r_1 V_1 + \ldots + r_g V_g. \tag{2.12}$$

Let R be the subgroup of Z^g formed by those collections of integers for which the right-hand sides in (2.12) are equal to zero, $R = R(U, V)$. For any collection $\rho \in Z^g$ we denote by $\bar{\rho}$ the element of the quotient group Z^g/R. The points described in (2.12) are uniquely determined by the class $\bar{\rho}$ (and are denoted in what follows by $Q_{\bar{\rho}}$), which is not equal to zero or to any of the classes $\bar{\rho}^{\pm}_{s}$, where $\rho^{\pm}_{s}$ is a collection in which $r_i = \pm\delta_{is}$. We denote by $F_i[u_0, \ldots, u_{i-1}]$ the "discrepancy" of order ε^i that is obtained by substituting the corresponding partial sum of the series (2.5) in (2.1).

Theorem 2.2. *The term* $u_i(x, y, t \mid X, Y, T)$ *of the series* (2.5) *is equal to*

$$u_i = \sum_{s=1}^{g} \left(c^{+}_{is}(t)\, v^{+}_{s}(x, y, t) + c^{-}_{is}(t)\, v^{-}_{s}(x, y, t)\right) + \sum_{\bar{\rho}\neq 0,\ \bar{\rho}^{\pm}_{s}} c_i(t, Q_{\bar{\rho}})\, v(x, y, t, Q_{\bar{\rho}}), \quad i \geqslant 1. \tag{2.13}$$

Here

$$(2.14)\qquad \begin{aligned} c_{is}^{\pm}(t) &= \tilde{c}_{is}^{\pm} - \int_0^t \langle\langle \Phi_s^{\mp}\partial_x F_i \rangle\rangle\, dt', \\ c_i(t, Q_\rho^-) &= \tilde{c}_i(Q_\rho^-) - \int_0^t \langle\langle \Phi(x, y, t', Q_\rho^-)\,\partial_x F_i \rangle\rangle\, dt'. \end{aligned}$$

We note that in formulae (2.14) only the dependence of all terms on the "rapid" variables x, y, t is reflected, though all of them are also functions of the slow variables X, Y, T that enter the definition of $v_s^{\pm}$, v, $\Phi_s^{\pm}$, Φ by means of the dependence on these variables of the parameters (Γ, P_0, k^{-1}). Moreover, the integration constants $\tilde{c}_{is}^{\pm}$, $\tilde{c}_i(Q_\rho^-)$ in (2.14) can also be functions of X, Y, T. Equations determining their dependence on X, Y, T can be obtained from the requirement of uniform boundedness in t of the term u_{i+1}.

The most interesting point is the determination of the dependence on X, Y, T of the main parameters (Γ, P_0, k^{-1}) of finite-gap solutions, proceeding from the requirement of uniform boundedness in t of the first correcting term u_1. The next section of the paper is devoted to this question.

§3. Whitham equations for space two-dimensional "integrable systems"

The problem of constructing asymptotic solutions of general space two-dimensional equations (4) and their perturbations is posed in the following way. Let $K(A)$ be a differential operator of order $m-1$ whose coefficients are differential polynomials in the coefficients of A. We search for asymptotic solutions

$$(3.1)\qquad \tilde{A} = A_0 + \varepsilon A_1 + \ldots, \quad \tilde{L} = L_0 + \varepsilon L_1 + \ldots$$

of the equation

$$(3.2)\qquad \partial_t L - \partial_y A + [L, A] - \varepsilon K(A) = 0.$$

In the first section of this chapter we have found the general form of finite-gap solutions of the equations (4). In accordance with the general ideas of the Whitham method (the non-linear WKB method), we shall consider asymptotic solutions (3.1), the leading term of which has the form

$$(3.3)\qquad A_0 = G\hat{A}_0 G^{-1}, \quad L_0 = G\hat{L}_0 G^{-1},$$

where $G = \exp(i\varepsilon^{-1}S_0(X, Y, T) + i\Phi(X, Y, T))$, and the coefficients of the operators $\hat{A}_0$, $\hat{L}_0$ are equal to

$$(3.4)\qquad \hat{v}_i(\varepsilon^{-1}S(X, Y, T) + Z(X, Y, T) \mid I(X, Y, T)),$$

$$(3.5)\qquad \hat{u}_i(\varepsilon^{-1}S(X, Y, T) + Z(X, Y, T) \mid I(X, Y, T)).$$

The vector-valued function S and the diagonal matrix S_0 must satisfy the conditions

$$(3.6)\quad \begin{cases} \partial_X S = U(X, Y, T), & \partial_Y S = V(X, Y, T), \quad \partial_T S = W(X, Y, T), \\ \partial_X S_0 = a(X, Y, T), & \partial_Y S_0 = b(X, Y, T), \quad \partial_T S_0 = c(X, Y, T), \end{cases}$$

where U, V, W are vectors of the periods of the differentials dp, dE, $d\Omega$, defined on the curve Γ, that correspond to the collection of data (1.3), parametrized by $I(X, Y, T)$; the diagonal matrices a, b, c are defined in (1.5).

A complete solution of the problem of constructing the whole series (3.1) requires, as in the example of the KP-2 equation treated above, constructing a basic collection of solutions of the linearized equation (4). It turns out that the equations of the dependence of the magnitudes $I(X, Y, T)$ can be obtained without constructing this basis from the requirement that u_1 is bounded.

We consider the manifold

$$(3.7)\qquad \hat{M}_g = (\Gamma,\ P_\alpha,\ [k_\alpha^{-1}]_m,\ Q \in \Gamma),$$

naturally fibered over M_g. Let $(\lambda, I_1, \ldots, I_N)$ be a local coordinate system on $\hat{M}_g$ such that for fixed I_s the function $\lambda(Q)$ parametrizes some domain of the curve $\Gamma = \Gamma(I)$. Any such system will be called a *local connection of the bundle* $\hat{M}_g \to M_g$, since for any path $I(\tau)$ in M_g and a point $Q_0 \in \Gamma(I(\tau_0))$ we can locally define the lifting of this path in $\hat{M}_g$ by defining a point $Q(\tau) \in \Gamma(I(\tau))$ by the condition $\lambda(Q(\tau)) = \lambda(Q_0)$.

The multivalued functions p, E, Ω defined on each curve are multivalued functions on $\hat{M}_g$, that is, $p = p(\lambda, I)$, $E = E(\lambda, I)$, $\Omega = \Omega(\lambda, I)$. If I depends on X, Y, T, then p, E, Ω are functions of λ, X, Y, T.

Theorem 3.1 [22]. *The following equations are necessary conditions for the existence of an asymptotic solution* (3.1) *with principal term of the form* (3.3)–(3.5) *and bounded terms* L_1, A_1:

$$(3.8)\quad \frac{\partial p}{\partial\lambda}\left(\frac{\partial E}{\partial T} - \frac{\partial\Omega}{\partial Y}\right) - \frac{\partial E}{\partial\lambda}\left(\frac{\partial p}{\partial T} - \frac{\partial\Omega}{\partial X}\right) + \frac{\partial\Omega}{\partial\lambda}\left(\frac{\partial p}{\partial Y} - \frac{\partial E}{\partial X}\right) = \frac{\langle\psi^+ K\psi\rangle_x}{\langle\psi\psi^+\rangle_x}\,\frac{\partial p}{\partial\lambda}.$$

The equations (3.8) can be represented in an invariant form not depending on fixing a local connection λ. If the I's depend on X, Y, T, then the inverse image $I(X, Y, T)$ in $\hat{M}_g$ is a four-dimensional manifold $\mathcal{M}^4 \subset \hat{M}_g$. We consider on $\mathcal{M}^4$ the 1-form $\omega = p\,dX + E\,dY + \Omega\,dT$. Then the equations (3.8) in the case $K \equiv 0$ are equivalent to the requirement that on $\mathcal{M}^4$ the exterior square of the differential $d\omega$ (which is a 4-form) must be equal to zero, that is,

$$(3.9)\qquad d\omega \wedge d\omega = 0.$$

The construction of solutions of the equations (4) given in §1 contains, in particular, a construction of solutions of the Lax equations $L_t = [A, L]$. We consider a submanifold M_g^0 of data (1.3) for which the corresponding differential is exact, that is, $E = E(Q)$ is a single-valued function on Γ. In this case the coefficients of L and A do not depend on y, and (4) turns into the Lax equation. The function $E(Q)$ can be used as a local connection. In this case $p = p(E, X, T)$, $\Omega = \Omega(E, X, T)$ and the equation (3.8) turns into

$$\partial_T p - \partial_X \Omega = -\frac{\langle \psi^+ K \psi \rangle_x}{\langle \psi^+ \psi \rangle_x} \frac{\partial p}{\partial E}. \tag{3.10}$$

For $K \equiv 0$ the equation (3.10) coincides with the equation $\partial_T p - \partial_X \Omega = 0$ first obtained in the special case of the KdV equation in [59] as a consequence of the averaged conservation laws.

§4. The construction of exact solutions of Whitham equations

Let $n_\alpha \geqslant m+1$ be integers such that $\sum n_\alpha = g + l(m+1)$. For any curve Γ of genus g with distinguished points P_α in general position and with local parameters k_α^{-1} fixed in their neighbourhoods there is a unique (up to addition of constants) function $\lambda(Q)$ having poles of multiplicity n_α at P_α, holomorphic outside them and such that in a neighbourhood of P_α

$$\lambda^{1/n_\alpha}(Q) = k_\alpha(Q) + O(k_\alpha^{-m}(Q)).$$

In the case of general position we can assume that the zeros q_i, $i = 0, \ldots, N$, of the differential $d\lambda$ are simple. There are $N+1$ of them, where $N = 3g-3+l(m+2)$. We can define the function $\lambda(Q)$ uniquely (that is, fix the indefinite constant) if we require that $\lambda(q_0) \equiv 0$. In this case we can choose as the local coordinates on M_g the magnitudes $\lambda_i = \lambda(q_i)$, $i = 1, \ldots, N$. The collections $(\lambda(Q), \lambda_1, \ldots, \lambda_N)$ form local coordinate systems on $\hat{M}_g$. The connections on $\hat{M}_g$ given in this way will be called *canonical*.

On an arbitrary curve Γ_0 in general position we fix some piecewise smooth contour $\mathscr{L}_0$ (not necessarily closed or connected). Using the connection $\lambda(Q)$, we can carry over this contour to the curves Γ sufficiently close to Γ_0. In a similar way we can define a differential dh on each such contour $\mathscr{L} \in \Gamma$ if we define a piecewise smooth differential dh on the initial contour $\mathscr{L}_0 \subset \Gamma_0$.

In the standard way with the help of Cauchy integrals it can be proved that there is a unique differential $d\Lambda$ on Γ that is meromorphic outside $\mathscr{L}$ with a unique pole in q_0 and continuously extendable to $\mathscr{L}$. Moreover, its limit values on $\mathscr{L}$ must satisfy the "jump" condition

$$d\Lambda^+(\tau) - d\Lambda^-(\tau) = dh(\tau), \quad \tau \in \mathscr{L}. \tag{4.1}$$

In addition, the periods of $d\Lambda$ over all cycles on Γ must be real.

***Theorem* 4.1** ([22]). *Suppose that $\lambda_i = \lambda(q_i)$ depend on X, Y, T so that for any $i = 1, \ldots, N$ one of the following two conditions is satisfied*:

$$(4.2) \qquad \oint \frac{1}{\sqrt{\lambda-\lambda_i}}(d\Lambda + X\,dp + Y\,dE + T\,d\Omega) = 0 \quad \text{or} \quad \lambda_i = \text{const}.$$

Then $p = p(\lambda, X, Y, T)$, $E = E(\lambda, X, Y, T)$, $\Omega = \Omega(\lambda, X, Y, T)$ satisfy the equations

$$(4.3) \qquad \partial_T p = \partial_X \Omega, \quad \partial_Y p = \partial_X E, \quad \partial_T E = \partial_Y \Omega.$$

The integrals in (4.2) are taken over small contours around the points q_i. If the q_i do not lie on $\mathcal{L}$, then the first of the conditions (4.2) means that the differential in the integrand vanishes at the points q_i.

Proof. We consider the differential $d\hat{S} = d\Lambda + Xdp + YdE + Td\Omega$. Since its jump on $\mathcal{L}$ is constant, it follows that the differential $\partial_X d\hat{S}$ is meromorphic on Γ. The conditions (4.2) guarantee that $\partial_X d\hat{S}$ has no poles at the points q_i. Therefore the differential $\partial_X d\hat{S} - dp$ is holomorphic on Γ (it could have a (simple) pole at q_0 but this is impossible by the theorem on residues). Since by the normalization conditions the periods of this holomorphic differential over any cycle is real, it is equal to zero. In a similar way it can be proved that $dE = \partial_Y d\hat{S}$, $d\Omega = \partial_T d\hat{S}$. The equality (4.3) is a consequence of the equality of the mixed derivatives for $d\hat{S}$. The theorem is proved.

Given X, Y, T, the equations (4.2) are a system of N equations with N unknowns λ_i. Its solutions $\lambda_i(X, Y, T)$ determine special solutions of the Whitham equations for the unperturbed equations (4) ($K \equiv 0$). These solutions depend on dh and on the choice of a canonical connection. The class of these solutions can be enlarged by admitting constant poles of $d\Lambda$ (see [22]). As we see from the proof of the theorem, it remains valid if all violations of the analyticity of $d\Lambda$ do not depend on X, Y, T. Apparently, the most general class of exact solutions can be obtained by defining $d\Lambda$ as a solution of the $\bar{\partial}$-problem with a constant right-hand side. We are planning to return to this question in another publication. Besides generalizations of the definition of $d\Lambda$ we can also enlarge the ways of choosing canonical connections.

Let $\mathfrak{M} \subset M_g$ be a submanifold of M_g (possibly coinciding with it). We say that on the bundle $\hat{\mathfrak{M}} \to \mathfrak{M}$, which is the restriction of $\hat{M}_g$ to $\mathfrak{M}$, an admissible connection is given if on each curve Γ in the collection of data Γ, P_α, k_α^{-1}, determining a point of $\mathfrak{M}$, a function $\lambda(Q)$ is defined such that for any number λ_0 in a neighbourhood of $\lambda(P_\alpha)$ the magnitudes $k_\alpha^i(Q)$, $i = 1, \ldots, m$, where Q is determined from the condition $\lambda(Q) = \lambda_0$, are well-defined functions of λ_0, that is, they do not depend on Γ. We note that the canonical connections are admissible. The points q_i at which $d\lambda = 0$ are singularities of the connection.

***Theorem* 4.1′.** *Suppose that* $(\Gamma, P_\alpha, [k_\alpha^{-1}]_m) \in \mathfrak{M}$ *depend on* X, Y, T *so that at each of the singularities of an admissible connection one of the conditions* (4.2) *is satisfied. Then the corresponding Abelian integrals* p, E, Ω *satisfy the equations* (4.3).

In the special case of the submanifold of data $M_g^0 \subset M_g$ that determine solutions of Lax type equations and the connection on M_g^0 given by the function $E(Q)$, the above theorem leads to the following equations (if all $\lambda_i = E(q_i) \neq \text{const}$):

$$(4.4) \qquad w_i(\lambda_1, \dots, \lambda_{N_1}) + v_i(\lambda_1, \dots, \lambda_{N_1}) T + X = 0, \quad N_1 = \dim M_g^0,$$

where

$$(4.5) \qquad v_i = \frac{d\Omega}{dp}(q_i), \quad w_i = \left(\oint \frac{d\Lambda}{\sqrt{\lambda - \lambda_i}}\right)\left(\frac{dp}{\sqrt{\lambda - \lambda_i}}\right)^{-1}.$$

The equations (4.4) were suggested in [23] as a generalization of the "hodograph" method. It should be noted that in [23] there was no effective construction of the functions w_i. The second formula in (4.5) complements the scheme of [23].

In contrast with the general space two-dimensional case, where our construction gives only special solutions of the corresponding Whitham equations, the equations (4.4) enable us to solve the Cauchy problem for the Whitham equations for the parameters of finite-gap solutions of Lax type equations. The differential dh from the definition of $d\Lambda$ and the contour $\mathcal{L}$ are expressed in terms of the initial values $\lambda_i(X, 0)$.

We give a brief sketch of the construction of dh in the case of the KdV equation (the general case of Lax type equations differs very little in principle from this special case). The real finite-gap solutions of the KdV equation are given by a hyperelliptic curve Γ: $y^2 = R(E) = \prod_i (E - \lambda_i)$, $i = 0, \dots, 2g$, where the λ_i are real, and by a collection of poles γ_s. The differentials dp and $d\Omega$ have the form

$$(4.6) \qquad dp = (E^g + r_1 E^{g-1} + \dots + r_g)\frac{dE}{\sqrt{R(E)}},$$

$$d\Omega = (E^{g+1} - s_1 E^g + \tilde{r}_1 E^{g-1} + \dots + \tilde{r}_g)\frac{dE}{\sqrt{R(E)}},$$

where the constants $r_i, \tilde{r}_i$ are determined by the normalization conditions

$$(4.7) \qquad \int_{E_{2j-1}}^{E_{2j}} dp = 0, \quad \int_{E_{2j-1}}^{E_{2j}} d\Omega = 0, \quad j = 1, \dots, g, \; s_1 = \frac{1}{2}\sum \lambda_i.$$

We consider the differential

$$(4.8) \qquad d\hat{S}(X, E) = \int_0^X dp(X', E)\, dX' + d\hat{S}_0, \quad d\hat{S}_0 = d\hat{S}_0(E).$$

If $d\hat{S}_0 \equiv 0$, then this differential is analytic outside the real axis and has a jump on the inverse image of the real axis, which we denote by $dh(E)$. The existence of this jump is related to the fact that on the real axis we cannot choose a single-valued branch of $dp(X, E)$ for all X. By Theorem 4.1′ the differential dh determines a solution of the Whitham equations

$$\lambda_{iT} + v_i \lambda_{iX} = 0, \tag{4.9}$$

which by the construction of dh has the desired initial value.

In some cases we can, by choosing a constant differential $d\hat{S}_0$ (with jumps and poles), arrange that $d\hat{S}$ is meromorphic. As an example of such a situation we give a construction of "average n-gap" solutions of the equations (4.9).

Let $d\Omega^{(n)}$ be a normalized differential on Γ with the only singularity at infinity of the form $d\Omega^{(n)} = dk^n(1 + O(k^{-n}))$.

Corollary. *The equations* (4.4), *where* $w_i = (d\Omega^{(n)}/dp)(\lambda_i)$ *determine the self-similar solutions* $\lambda_i = t^\gamma \lambda_i(x/t^{1+\gamma})$ *of the Whitham equations* (4.9) *with self-similar exponent* $\gamma = 2/(n-3)$.

In [62] a self-similar solution with exponent $\gamma = 1/2$ for $g = 1$ was found numerically that satisfies the boundary conditions $\lambda_2(z^+) = \lambda_3(z^+)$, $\lambda_1(z^+) = u_+$, $\lambda_1(z^-) = \lambda_2(z^-)$, $\lambda_3(z^-) = u_-$, $z^\pm = u_\pm - u_\pm^3$. This solution describes for $t \gg 0$ the structure of a shock wave appearing after the "overturn moment". In recent work of Potemin it was shown that the average 7-gap solution constructed by the above corollary satisfies the required boundary conditions. The boundaries of the oscillation domain turned out to be $z^+ = \sqrt{10/27}$, $z^- = -\sqrt{2}$ (approximate values of these magnitudes found numerically in [62] were $z^+ \approx 0.117$, $z^- \approx 1.41$; the equality $z^- = -\sqrt{2}$ was mentioned in [64]). An important consequence of this result is the smoothness of the self-similar solution in the whole domain $z^- < z < z^+$, though it followed from the scheme of the numerical solution of [62] that this solution possibly had a weak discontinuity inside the domain.

§5. The quasi-classical limit of two-dimensional integrable equations. The Khokhlov-Zabolotskaya eouation

The simplest solutions of the non-linear equations (4) are "zero-gap" solutions corresponding in our construction to the curves Γ of genus $g = 0$. They have the form (1.6), where $\hat{u}_i$ and $\hat{v}_j$ are constant matrices. It turns out that the Whitham equations even in this case are non-trivial and, as will be seen from what follows, in some cases are of independent physical interest. These equations coincide with the classical limit (4). It follows from the results of §3 that they can be represented in the form

$$\left(\frac{\partial E}{\partial t} - \frac{\partial \Omega}{\partial y}\right)\frac{\partial p}{\partial k} - \left(\frac{\partial p}{\partial t} - \frac{\partial \Omega}{\partial x}\right)\frac{\partial E}{\partial k} + \left(\frac{\partial p}{\partial y} - \frac{\partial E}{\partial x}\right)\frac{\partial \Omega}{\partial k} = 0, \tag{5.1}$$

where $p = p(k, x, y, t)$, $E = E(k, x, y, t)$, $\Omega = \Omega(k, x, y, t)$ are rational functions of the variable k.

Example 1. Let $p = k$, $E = \sigma^{-1}(k^2 - u)$, $\Omega = k^3 - 3uk/2 - w$. In this case the equation (5.1) is equivalent to the system

$$w_x = \sigma \frac{3}{4} u_y, \quad \sigma w_y = u_t + \frac{3}{2} u u_x. \tag{5.2}$$

Eliminating w from (5.2), we obtain the Khokhlov–Zabolotskaya equation.

Example 2. In the case $p = k$

$$E = k + \sum_{i=1}^{N} \frac{\eta_i}{k - v_i}, \quad \Omega = k^2 + u, \tag{5.3}$$

and the equation (5.1) leads to the system

$$\begin{cases} v_{it} - 2v_i v_{ix} - u_x = 0, \quad \eta_{it} - 2(v_i \eta_i)_x = 0, \\ u_x - u_y - \sum_{i=1}^{N} \eta_{ix} = 0. \end{cases} \tag{5.4}$$

The solutions of (5.4), not depending on y, correspond to the quasi-classical limit of the vector non-linear Schrödinger equation which, as noticed for the first time in [60], describes N-fibre solutions of the Benney equation. In [60] the classical limits of the general Lax equations were considered and it was shown that they are compatibility conditions of an algebraic and an ordinary differential equation. This implied a construction of integrals of these equations. However, the question of construction of solutions remained open. The scheme of the solution of the Cauchy problem for the system (5.4), based on a development of the ideas of [60], was suggested in [61]. We note that this scheme can easily be obtained as a special case of our construction of solutions of the equations (5.1), which follows from the result of the preceding section.

As an example we consider the construction of solutions of (5.2). It is given in closed form without tracing the literal correspondences between its elements and those of the construction of the preceding section.

We define the polynomial

$$\lambda(k) = k^4 - 2uk^2 - \frac{4}{3} wk - \lambda_0, \tag{5.5}$$

where the constant $\lambda_0 = \lambda_0(u, v)$ can be chosen so that $\lambda(q_0) = 0$, where q_0, q_1, q_2 are zeros of the differential $d\lambda$. It is convenient for what follows to pass from u, w to the variables q_1, q_2 with the help of the relations

$$u = q_1 q_2 - (q_1 + q_2)^2, \quad w = 3q_1 q_2(q_1 + q_2). \tag{5.6}$$

Then

$$\lambda_0 = (q_1 + q_2)^4 - 2u(q_1 + q_2)^2 + \frac{4}{3} w(q_1 + q_2). \tag{5.7}$$

(We note that the choice of $\lambda(k)$ in the form (5.5) corresponds in the terminology of §4 to fixing a canonical connection.) In this case

$$p=\lambda^{1/4}(k)+O(k^{-1}),\quad E=\lambda^{1/2}(k)+O(k^{-1}),\quad \Omega=\lambda^{3/4}(k)+O(k^{-1}).$$

We take an arbitrary contour $\mathscr{L}$ in the k-plane and a smooth differential $dh(\tau)$ on it. We define a function $\mathscr{F}(k)$ by

$$\mathscr{F}(k)=\int_{\mathscr{L}}\frac{dh(\tau)}{k-\xi(\tau)},\tag{5.8}$$

where $\xi(\tau)$ is one of the roots of the equation $\lambda(\xi)=\tau^4$. This function depends on q_1, q_2 as parameters. We reflect this in writing $\mathscr{F}=\mathscr{F}(k\mid q_1, q_2)$.

In the case under consideration the equations (4.2) have the form

$$0=\frac{\partial\mathscr{F}}{\partial k}(q_i\mid q_1, q_2)+x+\sigma^{-1}(2q_i-u)\,y+\left(3q_i^2-\frac{3}{2}u\right)t,\quad i=1,\ 2.\tag{5.9}$$

The system (5.9) determines implicitly the functions $q_1(x, y, t)$, $q_2(x, y, t)$.

Corollary. *If the functions $q_i(x, y, t)$, $i=1, 2$, are determined by* (5.9), *then $u=u(x, y, t)$ and $w=w(x, y, t)$, given by* (5.6), *satisfy* (5.2).

The equations (5.2) have self-similar solutions

$$t^{\gamma}\,u(x/t^{1+\gamma},\ y/t^{1+\gamma/2}),$$

and in a similar way for w. Similar solutions can be obtained by taking $\mathscr{F}(k\mid q_1, q_2)=\Phi_n$, where Φ_n is a polynomial of k of degree n uniquely determined by the relation

$$\Phi_n(k\mid q_1,\ q_2)=\lambda^{n/4}(k\mid q_1,\ q_2)+O(k^{-1}).$$

The self-similar exponent of the corresponding solutions is equal to $\gamma=2/(n-3)$.

To obtain solutions of (5.4) that do not depend on y, we should proceed in the following way. We define a function $\mathscr{F}$ by (5.8), where $\lambda(k)=E(k)$ and ξ is defined from the relation $\lambda(\xi)=\varphi(\tau)$ ($\varphi(\tau)$ being a parametrization of the contour $\mathscr{L}$). The function $\mathscr{F}(k\mid \eta_i, v_i)$ depends on η_i, v_i as parameters. If the $\eta_i(x, t)$, $v_i(x, t)$ are determined from the system of equations

$$\frac{\partial\mathscr{F}}{\partial k}(\varkappa_j\mid\eta_i,\ v_i)+x+2t\varkappa_j=0,\quad j=1,\ \ldots,\ 2N,\tag{5.10}$$

where the $\varkappa_j$ are roots of the equation

$$dE(\varkappa_j)=0 \Longleftrightarrow 1=\sum_{i=1}^{N}\frac{\eta_i}{(\varkappa_j-v_i)^2},\ \ldots\tag{5.11}$$

then they satisfy (5.4).

CHAPTER III

THE SPECTRAL THEORY OF THE TWO-DIMENSIONAL PERIODIC SCHRÖDINGER OPERATOR FOR ONE ENERGY LEVEL

The main aim of this chapter is to develop the spectral theory of the operator

$$H = -\partial_x^2 - \partial_y^2 + u(x, y) \tag{0.1}$$

with smooth periodic potential u. It follows from the results of [30] that the Floquet spectral set $M^2 \subset C^3$ (defined as the set of triples of complex numbers for which the equation

$$H\psi(x, y, t, Q) = E\psi(x, y, t, Q),\ Q = (E, w_1, w_2), \tag{0.2}$$

has a Bloch solution $\psi(x, y, t, Q)$, $Q \subset M^2$, with "multipliers" w_1, w_2) is an analytic submanifold of C^3. The *complex Fermi-curve* Γ_{E_0} *corresponding to the "energy level* $E = E_0$" is by definition the section of $M^2 \subset C^3$ by the hyperplane $E = E_0$. As in the case of the operator (1.1.1), an explicit construction of Γ_E and a detailed description of the structure of this Riemann surface, following from it, is based on a construction with the help of the perturbation theory of formal Bloch solutions of (0.2).

§1. The perturbation theory for formal Bloch solutions

Let $u_0(x, y)$ be an arbitrary smooth periodic potential. We fix a complex number w_{10}. A collection of solutions $\psi_\nu(x, y)$ of the equation

$$(-\partial_x^2 - \partial_y^2 + u_0(x, y))\psi_\nu = 0 \tag{1.1}$$

will be called *basic* if

$$\psi_\nu(x + l_1, y) = w_{10}\psi_\nu(x, y);\quad \psi_\nu(x, y + l_2) = w_{2\nu}\psi_\nu(x, y), \tag{1.2}$$

and if the following conditions are satisfied:

1° there is a "dual" collection of solutions $\psi_\nu^+(x, y)$ of the same equation such that

$$\psi_\nu^+(x + l_1, y) = w_{10}^{-1}\psi_\nu^+(x, y);\quad \psi_\nu^+(x, y + l_2) = w_{2\nu}^{-1}\psi_\nu^+(x, y), \tag{1.3}$$

$$\langle \psi_{\nu y}\psi_\mu^+ - \psi_\nu\psi_{\mu y}^+\rangle_x = r_\nu\delta_{\nu,\mu},\quad r_\nu \neq 0 \tag{1.4}$$

(since ψ_ν, ψ_μ^+ satisfy (1.1, 2, 3), it follows that r_ν does not depend on y);

2° for any continuously differential function $f(x)$ such that

$$f(x + l_1) = w_{10}f(x), \tag{1.5}$$

the series (1.6) and (1.7) converge and are equal to

$$0 = \sum_\nu \frac{\langle \psi_\nu^+ f\rangle_x}{r_\nu}, \tag{1.6}$$

$$f(x) = \sum_\nu \psi_{\nu y}\frac{\langle \psi_\nu^+ f\rangle_x}{r_\nu} = -\sum_\nu \psi_\nu\frac{\langle \psi_{\nu y}^+ f\rangle_x}{r_\nu}. \tag{1.7}$$

Example. Let $u_0 = 4$. Then for any k the functions

$$\psi(x, y, k) = \exp\left(\left(k+\frac{1}{k}\right)x + i\left(k-\frac{1}{k}\right)y\right) \tag{1.8}$$

are Bloch solutions of (1.1) with multipliers

$$w_1(k) = \exp\left(\left(k+\frac{1}{k}\right)l_1\right), \quad w_2(k) = \exp\left(i\left(k-\frac{1}{k}\right)l_2\right). \tag{1.9}$$

It can be verified directly that for any

$$w_{10} = w_1(k_0) \neq \exp(\pm 2l_1) \tag{1.10}$$

the sequence

$$\psi_\nu(x, y) = \psi(x, y, k_\nu) \tag{1.11}$$

is a basic sequence. Here the k_ν are determined from the equation $w_1(k_\nu) = w_{10}$ and are equal to

$$k_\nu = \frac{\pi i n}{l_1} + \frac{1}{2}\left(k_0 + \frac{1}{k_0}\right) \pm \sqrt{\left(\frac{\pi i n}{l_1} + \frac{1}{2}\left(k_0 + \frac{1}{k_0}\right)\right)^2 - 1} \tag{1.12}$$

(the indices ν numbering k_ν form a pair $(n, \pm)$ that consists of an integer and a sign). The dual collection consists of the functions

$$\psi_\nu^+(x, y) = \psi(x, y, -k_\nu). \tag{1.13}$$

Remark. By the definition itself the collection of basic functions is "overdetermined", and so it is impossible to expand $f(x)$ uniquely in ψ_ν or $\psi_{\nu y}$. However, for any pair of functions $f(x)$, $g(x)$ satisfying (1.5) there are *unique* constants $c_\nu(y)$ such that

$$f(x) = \sum_\nu c_\nu(y)\psi_{\nu y}(x, y), \quad g(x) = \sum_\nu c_\nu(y)\psi_\nu(x, y). \tag{1.14}$$

It follows from (1.14) that these constants are equal to

$$c_\nu(y) = \frac{\langle f\psi_\nu^+ - g\psi_{\nu y}^+\rangle_x}{r_\nu}. \tag{1.15}$$

We denote one of the indices ν by "0" and assume that

$$w_{20} \neq w_{2\nu}, \quad \nu \neq 0. \tag{1.16}$$

Lemma 1.1. *If* (1.16) *is satisfied, then for any continuously differential functions* $\delta u(x, y)$ (*with the same periods as* $u_0(x, y)$) *there are unique formal series*

$$F(y, Q_0) = \sum_{s=1}^{\infty} F_s(y, Q_0), \tag{1.17}$$

$$\Psi(x, y, Q_0) = \sum_{s=0}^{\infty} \varphi_s(x, y, Q_0), \quad \varphi_0 = \psi_0(x, y) \tag{1.18}$$

such that

$$(1.19) \qquad (-\partial_x^2-\partial_y^2+u_0+\delta u)\,\Psi=2F\Psi_y+(F_y+F^2)\,\Psi$$

as well as the following conditions:

$$(1.20) \qquad \Psi(x+l_1,y,Q_0)=w_{10}\Psi(x,y,Q_0), \quad \Psi(x,y+l_2,Q_0)= = w_{20}\Psi(x,y,Q_0),$$

$$(1.21) \qquad \langle\Psi_y\psi_0^+-\Psi\psi_{0y}^+\rangle_x+F\,\langle\Psi\psi_0^+\rangle_x=r_0=\langle\psi_{0y}\psi_0^+-\psi_0\psi_{0y}^+\rangle_x$$

(*for the time being* Q_0 *conditionally denotes the pair* (w_{10}, w_{20})).

Proof. The equation (1.19) is equivalent to the system

$$(1.22) \quad (-\partial_x^2-\partial_y^2+u_0)\,\varphi_s= = -\delta u\varphi_{s-1}+\sum_{i=1}^{s}\Big(2F_i\varphi_{s-i,\,y}+F_{iy}\varphi_{s-i}+\sum_{l=1}^{s-i}F_iF_l\varphi_{s-i-l}\Big).$$

We shall search for a solution of (1.22) as a sum

$$(1.23) \qquad \varphi_s=\sum_\nu c_\nu^s(y)\,\psi_\nu(x,y),$$

by assuming that the c_ν^s are chosen so that

$$(1.24) \qquad \varphi_{sy}=\sum_\nu c_\nu^s(y)\,\psi_{\nu y}(x,y).$$

It follows from the above remark that this can be done in a unique way. It follows from (1.23) and (1.24) that

$$(1.25) \qquad \sum_\nu c_{\nu y}^s\psi_\nu=0.$$

Substituting (1.24) and (1.23) in (1.22), we obtain

$$(1.26) \qquad -\sum_\nu c_{\nu y}^s\psi_{\nu y}=R_s,$$

where R_s is the right-hand side of (1.22). It follows from (1.25) and (1.26) that

$$(1.27) \qquad -c_{\nu y}^s=\langle R_s\psi_\nu^+\rangle_x/r_\nu.$$

These equations together with the condition

$$(1.28) \qquad c_\nu^s(y+l_2)=\frac{w_{20}}{w_{2\nu}}\,c_\nu^s(y)$$

enable us to determine the c_ν^s uniquely for $\nu\neq 0$. The condition (1.21) is sufficient for the existence of a periodic solution of (1.27) for $\nu=0$. This condition uniquely determines F_s. The final formulae have the form

$$(1.29) \qquad F_s=\frac{\langle\psi_0^+\delta u\varphi_{s-1}\rangle_x}{r_0}-\sum_{i=1}^{s-1}\Big(F_ic_0^{s-i}-\sum_{l=1}^{s-i}\frac{F_iF_l\,\langle\varphi_{s-i-l}\psi_0^+\rangle_x}{r_0}\Big);$$

$$(1.30) \qquad c_0^0=1, \quad c_0^s=-\frac{1}{r_0}\sum_{i=1}^{s}F_i\,\langle\varphi_{s-i}\psi_0^+\rangle_x, \quad s\geqslant 1.$$

For $\nu \neq 0$, $c_\nu^0 = 0$, and for $s \geqslant 1$ we have

$$(1.31)\quad c_\nu^s = -r_\nu^{-1}\Big(\sum_{i=1}^{s} F_i\langle\varphi_{s-i}\psi_\nu^+\rangle_x + \frac{w_{2\nu}}{w_{20}-w_{2\nu}}\int_y^{y+l_2} dy'\Big(\frac{\langle\psi_\nu^+\delta u\varphi_{s-1}\rangle_x}{r_\nu} - \sum_{i=1}^{s}\Big(F_i c_\nu^{s-i} - \sum_{l=1}^{s-i}\frac{F_iF_l\langle\varphi_{s-i-l}\psi_\nu^+\rangle_x}{r_\nu}\Big)\Big)\Big).$$

Corollary. *The formula*

$$(1.32)\quad \tilde{\psi}(x, y, Q_0) = \exp\Big(\int_0^y F(y', Q_0)\,dy'\Big)\Psi(x, y, Q_0)\Psi^{-1}(0, 0, Q_0)$$

determines a formal Bloch solution of the equation

$$(1.33)\quad (-\partial_x^2 - \partial_y^2 + u(x, y))\tilde{\psi} = 0, \quad u = u_0 + \delta u,$$

$$(1.34)\quad \begin{aligned}\tilde{\psi}(x+l_1, y, Q_0) &= w_{10}\tilde{\psi}(x, y, Q_0),\\ \tilde{\psi}(x, y+l_2, Q_0) &= \tilde{w}_{20}\tilde{\psi}(x, y, Q_0),\end{aligned}$$

where

$$(1.35)\quad \tilde{w}_{20} = w_{20}\exp\Big(\int_0^{l_2} F(y', Q_0)\,dy'\Big).$$

For the resonance case (that is, when the condition (1.16) is not satisfied) we proceed along the same lines as in Chapter I. We denote by I an arbitrary subset of indices ν such that

$$(1.36)\quad w_{2\alpha} \neq w_{2\nu},\ \alpha \in I,\ \nu \notin I.$$

***Lemma* 1.2.** *There is a unique matrix formal series*

$$(1.37)\quad F(y, w_{10}) = \sum_{s=1}^{\infty} F_s(y, w_{10}), \quad F = (F_\beta^\alpha), \quad \alpha, \beta \in I,$$

such that the equation (1.19) (*where F is now a matrix and Ψ is a vector*) *has a formal solution Ψ, whose components satisfy the conditions*

$$(1.38)\quad \begin{aligned}\Psi^\alpha(x + l_1, y, w_{10}) &= w_{10}\Psi^\alpha(x, y, w_{10}),\\ \Psi^\alpha(x, y + l_2, w_{10}) &= w_{2\alpha}\Psi^\alpha(x, y, w_{10}),\end{aligned}$$

$$(1.39)\quad \langle\Psi_y^\alpha\psi_\beta^+ - \Psi^\alpha\psi_{\beta y}^+\rangle_x + \sum_{\gamma\in I} F_\gamma^\alpha\langle\Psi^\gamma\psi_\beta^+\rangle_x = \delta_{\alpha\beta}r_\alpha.$$

The proof of the lemma is analogous to that of Lemma 1.1. We omit for brevity the corresponding recursion formulae for the coefficients F_s and $c_\nu^{s,\alpha}$, since they are complete matrix analogues of the formulae of the resonance-free case.

We define the matrix $T(y, w_{10})$ from the equation

$$(1.40)\quad \partial_y T + TF = 0,\ T_\beta^\alpha(0, w_{10}) = \delta_\beta^\alpha.$$

Then the components $\hat{\Psi}^\alpha$ of the vector $\hat{\Psi} = T\Psi$ are solutions of (1.33). As in Chapter I it can be proved that the assertion of the corollary to Lemma 1.1.3 is valid, that is, to each point of the surface given by the characteristic equation (1.1.49) there corresponds a unique Bloch solution $\widetilde{\psi}$ of the equation (1.33).

Remark. The assertions of §1 of Chapter I on the construction of the "dual" functions $\widetilde{\psi}^+(x, y, Q_0)$, which are defined on the same surfaces as $\widetilde{\psi}(x, y, Q_0)$, go over completely to the case under consideration.

§2. The structure of complex "Fermi-curves"

Let $u_0 = 4$. Then, as we said earlier, for any $w_{10} \neq e^{\pm 2l_1}$ the equation (1.1) has a basic sequence of Bloch solutions. Therefore the formulae of Lemma 1.1 define formal Bloch solutions $\widetilde{\psi}(x, y, k_0)$ of the equation (0.2) if we put in them $\delta u = u - E - 4$ and if k_0 satisfies the resonance-free condition (1.16). It follows from (1.8) that the resonance pairs of points have the form (k^+_{NM}, k^-_{NM}), $(\widetilde{k}^+_{NM}, \widetilde{k}^-_{NM})$, where

$$k^{\pm}_{NM} = \pm z_{NM}\ (1 \pm \sqrt{1 + |z_{NM}|^{-2}}), \tag{2.1}$$

$$\widetilde{k}^{\pm}_{NM} = \pm z_{NM}\ (1 \mp \sqrt{1 + |z_{NM}|^{-2}}), \tag{2.2}$$

$$z_{NM} = \frac{\pi i N}{2l_1} + \frac{\pi M}{2l_2}, \quad N,\ M \text{ being integers.} \tag{2.3}$$

The set of such points has only two limit points $k = 0$, $k = \infty$.

Further constructions and assertions practically completely repeat their analogues in §2 of Chapter I. Therefore we restrict ourselves to brief statements of them, indicating if necessary those minor changes which should be inserted into the proofs and constructions of §2 of Chapter I.

Fixing h, we can choose neighbourhoods $R^{\pm}_{NM}$ and $\widetilde{R}^{\pm}_{NM}$ of the resonance points (2.1), (2.2) so that for any k_0 not belonging to them the following inequalities hold:

$$|w_{20} w_{2\nu}^{-1} - 1| > h, \quad |w_{20}^{-1} w_{2\nu} - 1| > h. \tag{2.4}$$

We can assume that h is chosen small enough in order that these neighbourhoods be disjoint. Suppose that a periodic function $u(x, y)$ is analytically extendable to a neighbourhood of the real variables x, y (that is, it satisfies the inequalities (1.2.13) for some U, τ_1, τ_2).

***Lemma* 2.1.** *There is a constant N_0 such that for k_0 not belonging to $R^{\pm}_{NM}$ and $\widetilde{R}^{\pm}_{NM}$ and satisfying the condition $|k_0| + |k_0|^{-1} > N_0$ the series of the perturbation theory constructed by Lemma 1.1 (for $u_0 = 4$, $\delta u = u - E - 4$) and its corollary converge uniformly and absolutely and determine a Bloch solution $\widetilde{\psi}(x, y, k_0)$ of the equation (0.2) analytic (in x, y, k_0) and non-vanishing.*

Remark. By complete analogy with the above we can construct series of the perturbation theory for the formally conjugate function $\widetilde{\psi}^+(x, y, k_0)$ which is analytic, like ψ, in the resonance-free domain.

We now consider $k_0 \in R^+_{NM}$ (or $\widetilde{R}^+_{NM}$) and $|k_0| + |k_0|^{-1} > N_0$. As a set of resonance indices we choose $\nu = 0$ and ν_0 such that $k_{\nu_0} \in R^-_{NM}$ (or $\widetilde{R}^-_{NM}$ respectively). Then for $w_{10} \in W_{NM} = w_1(R^{\pm}_{NM})$ (or $w_{10} \in \widetilde{W}_{NM} = w_1(\widetilde{R}^{\pm}_{NM})$) the series of the perturbation theory of Lemma 1.2 determine a two-dimensional quasi-Bloch solution of the equation (0.2). The corresponding monodromy matrix $\hat{T} = T(l_2, w_{10})$ determines a two-sheeted covering of $\hat{R}_{NM}$ or $\hat{\widetilde{R}}_{NM}$ over the domains W_{NM} and $\widetilde{W}_{NM}$. Again we call a pair N, M *distinguished* if the discriminant of the characteristic equation for $\hat{T}$ has a zero of multiplicity two.

***Lemma* 2.2.** *For non-distinguished pairs N, M the Bloch function $\widetilde{\psi}$ extends to $\hat{R}_{NM}(\hat{\widetilde{R}}_{NM})$ and has one simple pole there.*

To repeat the gist of the proof of Lemma 1.2.3, it is sufficient to apply the following assertion instead of Lemma 1.1.1.

***Lemma* 2.3.** *Suppose that $\psi(x, y, Q)$ and $\psi^+(x, y, Q)$ are Bloch solutions of the equation* (0.2), *where Q is a non-singular point of the surface Γ_E; then*

$$dp_x \langle\psi_x\psi^+ - \psi_x^+\psi\rangle_y + dp_y \langle\psi_y\psi^+ - \psi\psi_y^+\rangle_x = 0. \tag{2.5}$$

The functions $\langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y$ and $\langle\psi_y\psi^+ - \psi\psi_y^+\rangle_x$ have no common zeros in the non-singular part of Γ_E.

The equality (2.5) can be proved by analogy with the proof of (1.1.6). The second assertion of the lemma follows from the fact that under the variation δu of the potential u of the operator (0.1) we have

$$i\delta p_x \langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y + i\delta p_y \langle\psi_y\psi^+ - \psi\psi_y^+\rangle_x = \langle\langle\psi^+\delta u\psi\rangle\rangle. \tag{2.6}$$

By analogy with Lemma 1.2.4 we can construct an extension of $\widetilde{\psi}(x, y, k_0)$ inside the "central resonance domain" R_0: $|k_0| + |k_0|^{-1} \leq N_0$, which is replaced by a finite-sheeted covering $\hat{R}_0$ of the domain $W_0 = w_1(R_0)$.

We denote by Γ_E the Riemann surface obtained by "pasting" $\hat{R}_{NM}$ and $\hat{\widetilde{R}}_{NM}$ instead of the deleted neighbourhoods of the non-distinguished resonance points and "pasting" $\hat{R}_0$ instead of R_0.

***Theorem* 2.1.** *The Riemann surface Γ_E is isomorphic to the "spectral Fermi-curve" of the operator* (0.1). *The Bloch solutions $\psi(x, y, Q)$, $Q \in \Gamma_E$, of this equation normalized by the condition $\psi(0, 0, Q) \equiv 1$ are meromorphic on Γ_E. The poles of ψ do not depend on x, y. In each of the domains $\hat{R}_{NM}$, $\hat{\widetilde{R}}_{NM}$ (N, M being a non-distinguished pair) ψ has one simple pole.*

In the domain $\hat{R}_0$ it has g_0 poles, where g_0, in the general position when $\hat{R}_0$ is non-singular, is equal to the genus of $\hat{R}_0$. Outside $\hat{R}_0$, $\hat{R}_{NM}$, $\hat{\tilde{R}}_{NM}$ the function ψ is holomorphic.

All the assertions of the theorem except for the first one follow from the construction of Γ_E itself. To each point Q of Γ_E there correspond the multipliers $w_i(Q)$, $i = 1, 2$. They determine a map of Γ_E to the corresponding "Fermi-curve". The fact that this map is an isomorphism follows from the assertion of the following lemma.

For any complex number w_{10} we denote by $Q_\nu \in \Gamma_E$ the solutions of the equation

$$w_1(Q_\nu) = w_{10}, \tag{2.7}$$

and by $\psi_\nu(x, y)$ the functions $\psi(x, y, Q_\nu)$.

***Lemma* 2.4.** *If the equation* (2.7) *has simple roots, then the collection of functions $\psi_\nu(x, y)$ is basic* (*in the sense of the definition given at the beginning of* § 1).

Proof. It follows from Lemma 2.3 that the differential

$$d\Omega = -dp_x\,(\langle\psi_y\psi^+ - \psi\psi_y^+\rangle_x)^{-1} = dp_y\,(\langle\psi_x\psi^+ - \psi\psi_x^+\rangle_y)^{-1} \tag{2.8}$$

is holomorphic on Γ_E and has zeros at the poles of ψ and ψ^+. The assertion of the lemma follows from the examination of the contour integrals

$$S_{1N} = \int_{C_N} d\Omega \int_0^{l_1} f(x')\,\frac{\psi(x, y, Q)\,\psi^+(x', y, Q)}{1 - w_{10}w_1^{-1}(Q)}\,dx', \tag{2.9}$$

$$S_{2N} = \int_{C_N} d\Omega \int_0^{l_1} f(x')\,\frac{\psi_y(x, y, Q)\,\psi^+(x', y, Q)}{1 - w_{10}w_1^{-1}(Q)}\,dx', \tag{2.10}$$

where C_N is the union of two contours surrounding the points $P_\pm$ that have radii of order N and N^{-1} and do not intersect the resonance domains. These integrals tend to zero and $f(x)$ respectively, as $N \to \infty$. Since the residues of the integrands coincide with the terms of the series (1.6) and (1.7), the lemma is proved.

***Corollary* 1.** *The correspondence*

$$(w_1, w_2) \to (w_1^{-1}, w_2^{-1}) \tag{2.11}$$

determines a holomorphic involution $\sigma : \Gamma_E \to \Gamma_E$ of the Fermi-curves.

Proof. To each point $Q \in \Gamma_E$ there correspond a Bloch solution $\psi(x, y, Q)$ with multipliers $w_1(Q)$, $w_2(Q)$ and the "dual" function $\psi^+(x, y, Q)$ with multipliers $w_1^{-1}(Q)$, $w_2^{-1}(Q)$. Since ψ^+ is a Bloch solution of the same equation (0.2) and the points of Γ_E parametrize all Bloch solutions, it follows that the pair $w_1^{-1}(Q)$, $w_2^{-1}(Q)$ belongs to the Fermi-curve, and the lemma is proved. At the same time we obtain

$$\psi^+(x, y, Q) = \psi(x, y, \sigma(Q)). \tag{2.12}$$

***Corollary* 2.** *If the potential $u(x, y)$ is real, then on the curve Γ_E an anti-holomorphic involution τ is defined that is induced by the correspondence*

$$(w_1, w_2) \to (\bar{w}_1, \bar{w}_2), \tag{2.13}$$

and

$$\bar{\psi}(x, y, Q) = \psi(x, y, \tau(Q)). \tag{2.14}$$

Definition. A potential u is called *finite-gap with respect to the level E_0* if all except finitely many pairs N, M for it are distinguished when constructing Γ_{E_0}, that is, when Γ_{E_0} has finite genus.

By the definition of distinguished pairs, for finite-gap potentials with respect to the level E_0 the surface Γ_{E_0} outside some finite domain $|k_0| + |k_0|^{-1} \leqslant N_1$ coincides with neighbourhoods of the points $k = 0$ and $k = \infty$ on the usual complex plane. Therefore it can be compactified by two "infinitely distant" points $P_\pm$. In what follows we shall keep the notation Γ_{E_0} for the corresponding compact Riemann surface.

***Theorem* 2.2.** *The Bloch solutions of the equation* (2.2) *for $E = E_0$ for potentials u that are finite-gap with respect to E_0 are defined outside two points $P_\pm$ of the compact Riemann surface Γ_{E_0}, on which there is a holomorphic involution σ, $\sigma(P_\pm) = P_\pm$. In a neighbourhood of $P_\pm$ this function $\psi(x, y, Q)$, $Q \in \Gamma_{E_0}$, has the form*

$$\psi(x, y, Q) = \exp((x \pm iy) k_\pm) \left(1 + \sum_{s=1}^{\infty} \xi_s^\pm(x, y) k_\pm^{-s}\right), \tag{2.15}$$

where the $k_\pm^{-1} = k_\pm^{-1}(Q)$ are local parameters in neighbourhoods of $P_\pm$ (moreover $k_\pm(\sigma(Q)) = -k_\pm(Q)$). Outside $P_\pm$ the function ψ is meromorphic and has g poles not depending on x, y, where in the general position when Γ_{E_0} is non-singular g is equal to the genus of Γ_{E_0}. In this case the poles γ_s and $\gamma_s^+ = \sigma(\gamma_s)$ are zeros of a differential $d\Omega$ of the third kind with simple poles at the points $P_\pm$ and holomorphic outside them. If the potential $u(x, y)$ is real, then there is an anti-holomorphic involution τ on Γ_{E_0} commuting with σ and such that $\tau(P_\pm) = P_\mp$, $k_\pm(\tau(Q)) = \bar{k}_\mp(Q)$. Moreover, the set of poles of ψ is invariant with respect to τ.

By complete analogy with Theorem 1.3.1 the following assertion can be proved.

***Theorem* 2.3.** *For any E_0 the smooth periodic potential $u(x, y)$ of the operator* (0.1), *analytically extendable to some neighbourhood of real x, y, can be approximated uniformly with any number of derivatives by potentials $u_G(x, y)$ that are "finite-gap with respect to the level E_0".*

§3. The spectral theory of "finite-gap operators with respect to the level E_0" and two-dimensional periodic Schrödinger operators

An important distinction between the spectral theory of the non-stationary Schrödinger operator (1.1.1) with $\sigma = 1$ and the two-dimensional Schrödinger operator in the case of smooth periodic potentials $u(x, y)$ is that in the first case the corresponding spectral curve Γ is always non-singular, while in the second case the "complex Fermi-curve" Γ_{E_0} can have finitely many singular points. A complete description of possible types of singularities has still not been obtained.

We begin this section with a brief presentation of the inverse problem of the recovery of "finite-gap with respect to the level E_0" potentials $u(x, y)$ in the case of non-singular "Fermi-curves" Γ_{E_0} ([28], [29]).

Let Γ be a non-singular algebraic curve of genus g with two distinguished points $P_\pm$, in neighbourhoods of which the local parameters $k_\pm^{-1}(Q)$ are fixed, $k_\pm^{-1}(P_\pm) = 0$. For any collection of g points $\gamma_1, \ldots, \gamma_g$ in general position there is a unique Baker-Akhiezer function $\psi(x, y, Q)$ meromorphic on Γ outside $P_\pm$, having poles at the points γ_s and asymptotics

$$(3.1) \qquad \psi = e^{k_+ z}\left(1 + \sum_{s=1}^{\infty} \xi_s^+(x, y) k_+^{-s}\right), \quad k_\pm = k_\pm(Q), \quad Q \to P_\pm,$$

$$(3.2) \qquad \psi = e^{k_- \bar{z}} c(x, y)\left(1 + \sum_{s=1}^{\infty} \xi_s^-(x, y) k_-^{-s}\right), \quad z = x + iy, \quad \bar{z} = x - iy.$$

It was proved in [27] that such a function ψ satisfies the equation

$$(3.3) \qquad \widetilde{H}\psi = 0, \quad \widetilde{H} = -\partial^2_{z\bar{z}} + A_{\bar{z}}\partial_{\bar{z}} + u,$$

where

$$(3.4) \qquad A_{\bar{z}}(x, y) = \partial_z \log c(x, y), \quad u(x, y) = \partial_{\bar{z}} \xi_1^+(x, y).$$

For the function ψ and also for $A_{\bar{z}}$ and u explicit theta-function formulae have been obtained.

In [28], [29] sufficient conditions on the data $(\Gamma, P_\pm, k_\pm, \gamma_s)$ were found for the operator $\widetilde{H}$ corresponding to them to be purely potential, that is, $A_{\bar{z}} \equiv 0$. These conditions are the following:

1) there is an involution $\sigma : \Gamma \to \Gamma$ on Γ with two fixed points $P_\pm$;

2) the local parameters $k_\pm^{-1}$ must satisfy the condition $k_\pm(\sigma(Q)) = -k_\pm(Q)$;

3) the points γ_s and $\gamma_s^+ = \sigma(\gamma_s)$ form a divisor of the zeros of a differential $d\Omega$ of the third kind with single simple poles at $P_\pm$.

The sufficiency of these conditions follows from the fact that if they are satisfied, then the differential (3.5) is holomorphic outside $P_\pm$, where it has simple poles.

$$(3.5) \qquad d\widetilde{\Omega} = \psi(x, y, Q)\psi^+(x, y, Q)d\Omega(Q), \quad \psi^+(x, y, Q) = \psi(x, y, \sigma(Q)).$$

The equality to zero of the sum of the residues of this differential leads to the fact that $c^2 \equiv 1$ (since $c(0, 0) = 1$, it follows that $c(x, y) \equiv 1$). The latter is sufficient for the equality $A_{\bar{z}} = 0$.

***Theorem* 3.1.** *The above conditions* (1–3) *on the data of the inverse problem* (Γ, $P_\pm$, $k_\pm$, γ_s) *are necessary for the operator* (0.3) *corresponding to them to be potential* (*that is, to have the form* (0.1)), *and the potential* $u(x, y)$ *to be smooth. If the potential* u *is periodic, then* Γ *is isomorphic to the "complex Fermi-curve"* $\Gamma_{E=0}$.

Proof. In the general case the operator $\widetilde{H}$ corresponding to the data (Γ, $P_\pm$, $k_\pm$, γ_s) is quasi-periodic. The periodicity conditions are formulated in exactly the same way as for the case of finite-gap non-stationary Schrödinger operators. We define the differentials of the quasi-momenta dp_x, dp_y as differentials of the second kind on Γ with single poles at the points $P_\pm$ of the form

$$(3.6) \qquad dp_x = -i\, dk_\pm (1 + O(k_\pm^{-2})), \quad dp_y = \pm\, dk_\pm (1 + O(k_\pm^{-2}))$$

and uniquely normalized by the conditions that their periods over all cycles on Γ are *real*. If these periods are multiples of $2\pi/l_1$ for dp_x and $2\pi/l_2$ for dp_y, then the operator $\widetilde{H}$ has periods l_1, l_2 in x and y respectively. For the periodic potential operators the last assertion of the theorem can be proved in exactly the same way as the first assertion of Theorem 2.1. After this the necessity of the conditions (1–3) for the periodic operators follows from Theorem 2.2. The real matrices of the periods of the differentials dp_x, dp_y are non-degenerate functions of the parameters (Γ, $P_\pm$, $[k_\pm^{-1}]_1$). Therefore the set of periodic operators as $l_1, l_2 \to \infty$ is dense among all finite-gap operators with respect to a fixed level of operators (corresponding to smooth curves). This enables us to complete the proof of the theorem.

In a similar way it can be proved that for $u(x, y)$ to be real it is necessary that there is an anti-involution τ on Γ such that $\tau(P_\pm) = P_\mp$, $k_+(\tau(Q)) = \overline{k}_-(Q)$ and that the divisor of the poles $\gamma_1, \ldots, \gamma_g$ is invariant under τ.

In [29] sufficient conditions on the parameters (Γ, σ, τ, $P_\pm$, $k_\pm$, γ_s) were formulated that guarantee the smoothness of the potential u of the operator (0.1) corresponding to them. Besides the above requirements, it is sufficient that Γ is an M-curve with respect to τ, and among its $g+1$ ovals $a_0, \ldots, a_g$ there are g ovals such that $\sigma(a_i) = a_{g_0+i}$ (here g_0 is the genus of the curve Γ/σ; since σ has two fixed points, it follows that $g = 2g_0$), $i = 1, \ldots, g_0$. If the points γ_s are chosen so that there is one point in each oval a_s, $s = 1, \ldots, g$, then the corresponding potential u will be smooth.

Besides these conditions there is another type of sufficient conditions. If the anti-involution $\tau\sigma$ is an anti-involution of splitting type, and the differential $d\Omega$ is positive on all fixed ovals of $\tau\sigma$ with respect to the orientation given on these ovals as on the boundary of one of the domains into which they split Γ, then the potential u will be smooth.

The two types of sufficient conditions given above are analogous to the conditions that guarantee the smoothness of the finite-gap potentials of the operator (1.1.1) with $\sigma = 1$ and $\sigma = i$ respectively. The proofs of these assertions are also completely analogous.

In a recent paper [64] a series of sufficient conditions was found, among which the ones given above occupy diametrically opposite positions. The method of [64] is based on an analysis of theta-function formulae for $u(x, y)$ and differs in principle from the approach developed here. There is still no reformulation of the whole series of conditions of [64] into the form that we need. As shown in [64], the conditions obtained are not only sufficient but also necessary for the smoothness of the potentials u corresponding to the smooth curves $\Gamma_{E=0}$. These potentials have the form

$$u(x, y) = -2\partial_z\partial_{\bar{z}} \log \theta (U_1 z + U_2 \bar{z} + \zeta_0) + c, \tag{3.7}$$

where the constant c depends on Γ, $P_\pm$ (its explicit form was found in [65]) and the theta-function θ is a Prym theta-function, that is, it is constructed from the matrix of the periods of the holomorphic differentials that are odd with respect to σ. For certain types of degeneration of Γ the Prymian of the curve can remain non-degenerate (in contrast with the Jacobian, which is always degenerate). It is this fact that causes the possibility of the existence of smooth finite-gap quasi-periodic potentials corresponding to singular curves. The most interesting case, which gives the principal state of the corresponding operator H, is considered in [29], [66]. More general examples can be constructed by using the well-known technique of the construction of "multi-soliton against a background of finite-gap potentials" (see [52] for the case of operators of the type (1.1.1)). We omit a detailed description of these examples, since at present we do not know a complete description of admissible types of degeneration. To answer this question, we need a more detailed investigation of the direct spectral problem which has been considered in the preceding section. We turn the reader's attention to the fact that a related question of the description of possible types of degeneration is discussed in the letter of Shiota included at the end of the Russian edition of the book [67].

It is seen from the results of the preceding section that the potentials corresponding to smooth curves, that is, having the form (3.7), are dense among all finite-gap potentials (corresponding to curves with possible singularities), therefore the assertion of the theorem on the density of the finite-gap potentials means that the potentials of the form (3.7) are also dense.

In conclusion we note that the restriction on the length of the paper forces us to give up a discussion of applications of the spectral theory of two-dimensional periodic Schrödinger operators in the theory of non-linear equations. The creation of the perturbation theory for the periodic solutions of the Novikov–Veselov equations and the derivation of the Whitham equations for them (which, by the way, have the same form (2.3.8) after the change $dp = dp_x$, $dE = dp_y$) is completely analogous to the constructions of Chapter II. By analogy with §5 of Chapter I we can prove the completeness in the space of functions periodic in x, y of the products of the Bloch solutions ψ, ψ^+ at the resonance points and the products of $\psi(x, y, Q)$, $\psi^+(x, y, Q)$, as well as a number of other assertions.

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