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THE AVERAGING PROCEDURE FOR THE SOLITON-LIKE SOLUTIONS OF INTEGRABLE SYSTEMS

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Since the middle of the seventies algebraic geometry has become a very powerful tool in various problems of mathematical and theoretical physics. In the theory of integrable equations algebraic geometrical methods provide a construction of the periodic and quasi-periodic solutions which can be written exactly in terms of the theta-functions of the auxiliary Riemann surfaces. This construction works for the two-dimensional case the same as for one-dimensional integrable equations such as the KdV and sine-gordon equation.

All integrable equations which are considered in soliton theory can be represented as compatibility conditions of the auxiliary linear problems. One of the most general types of such representations has the form

$$[\partial_y - L, \partial_t - A] = 0 \tag{0.1}$$

where L,A are the differetial operators of the form

$$L = \sum_{i=o}^{n} u_i(x,y,t)\partial_x^i \ , \ A = \sum_{i=0}^{m} v_i(x,y,t)\partial_x^i \tag{0.2}$$

with scalar or matrix coefficients.

The most important example of such equations is the Kadomtsev-Petviashvilii (KP) equation

$$\frac{3}{4}u_{yy}=(u_t-\frac{3}{2}uu_x+u_{xxx})_x$$

The algebraic-geometrical solutions of this equation have the form [1]

$$u(x,y,t)= u_0(Ux+Vy+Wt+\Phi|I_1,...,I_N) \qquad (0.3)$$

where $u_0(z_1,...z_g \mid I_j)$ is the periodic function of the variable z_i, depending on the set of the parameters I_j. the vectors U,V,W are determined by the same set of data

$$U=U(I),\ V=V(I),\ W=W(I).$$

the vector Φ is arbitrary.

The set of parameters $I_1,...,I_N$ for KP-equation is

$$M_g=(\Gamma,\ P_0,[k^{-1}]_3) \qquad (0.4)$$

- the algebraic curve Γ of the genus g with the fixed point P_0 on it and the equivalence class $[k^{-1}]_3$ of the local parameter $k^{-1}(P)$ in the neighbourhood of the fixed point P_0. (Two local parameters k' and k are m-equivalent if $k'=k+O(k^{-m})$; the corresponding equivalent class is denoted by $[k^{-1}]_m$). The number of such parameters equals $N=3g+1$.

The integrable equations and the set of their exact solutions are the starting point for many problems "around them". Usually the next step is the perturbation theory.

For example, it is well-known that for the description of the analogous of shock-waves for the KdV-equation it is not enough to have the finite-gap solutions like (0.3)

$$u(x,t)=u_0(Ux+Wt+\Phi|\ E_1,...,E_{2n+1}) \qquad (0.5)$$

Here U,W, u_0 are determined by hyperelliptic curve Γ:

$$y^2=\prod_{i=1}^{2n+1}(E-E_i).$$

It is necessary to extend the set of the solutions and this can be done in the framework of so-called nonlinear WKB (or Whitham) -method. The general ideas of this method are not specially connected with the KdV-equation or some other integrable equation.

Roughly speaking, if some equation has the set of the exact solutions of the form (0.5), the asymptotic solutions of this (or perturbative) equation can be constructed in the form:

$$\tilde{u}(x,t)=u_0(\varepsilon^{-1}S(X,T) \mid E_i(X,T))+\varepsilon u_1+\varepsilon^2 u_2+\ldots \tag{0.6}$$

Here $X=\varepsilon x, T=\varepsilon t$ are the "slow-variables". The main term of this series satisfies the equation up to the order ε if the vector $S(X,T)$ is defined from the relations

$$\partial_X S=U(E(X,T))=U(X,T) \quad ,\partial_T S=W(X,T) \tag{0.7}$$

It turns out that the first order term in (0.6) has the same structure as the leading term if the parameters E_i depend on slow variables X,T in a way which is described by some special equations, generally called Whithem equations .

For the Lax-type equations the latter can be obtained from arguments which are the generalization of the averaging procedure used for finite-dimensional hamiltonian systems. It should be mentioned that in the case of a multiphase solutions we don't lose too much if we confine ourselves to the analysis of the Lax-type equations only. One-periodic solutions exist for many nonlinear equations. But the integrable equations are the only ones for which there exist wide sets of multiphase solutions.

The Lax-type equations have an infinite number of local integrals, the densities of which are differential polynomials on the unknown functions u_i.The Whithem equations for the finite-gap solutions of the KdV-equation were proposed in [3] and have the form:

$$\partial_T I_n = \partial_x \bar{Q}_n, \tag{0.8}$$

where

$$I_n = \int P_n(u,u',...)dx \ , \ \bar{Q}_n = \int Q_n(u,u',...)dx \tag{0.9}$$

and $P_n(u,u',...)$, $Q_n(u,u',...)$ are local densities of integrals and currents, respectively. The equations (0.9) are the averaged relations of

$$\partial_t P_n(u,u'...) = \partial_x Q_n(u,u',...) \tag{0.10}$$

In [4] the same equations were obtained using the different approach. It was shown that they are necessary for the existence of the first order term in (0.6) which has the form similar to (0.5).

There are two main purposes of this paper. First of all, we shall present here the general averaging procedure for integrable systems and demonstrate that it can be applied even to the more general situation, than that in which it was deduced in the previous work of the author [5].

As a new example we shall consider Benjamin-Ono equation

$$u_t + 2uu_x + \text{P.V.}\int_{-\infty}^{\infty} \frac{u_{yy}}{y-x} dy = 0, \tag{0.11}$$

which formally doesn't belong to the Lax-type equations. This equation is a non-local analogue of the KdV-eqiation. It can be represented as the compatibility conditions of another type of the auxiliary system of lenear problems. The direct and inverse scattering problems for the corresponding linear equation solve the Caushy problem in the case of rapidly decreasing initial data. In the framework of this approach the exact solutions can be constructed and they are the rational "multisoliton" solutions of the Benjamin-Ono equation (see, for example, [6,7]).

The algebraic geometrical construction of the quasi-periodic solutions of the Benjamin-Ono equation and their averaging procedure are presented in the second and the third paragraphs of this paper (see [8]).

The corresponding solutions are not absolutely new. But they give us the possibility to demonstrate that the ideas of the algebraic-geometrical (or finite-gap) scheme can be applied to the construction of a soliton-like solution even more effectively than in the generic case. Of course, all these solutions are the degeneral case of the solutions corresponding to the smooth auxiliary curves of the higher genus $g>0$. But their "algebraic-geometrical" construction can be presented in a closed form without using the results of algebraic geometry.

The representation of the main ideas of the algebraic-geometrical methods in the theory of intgrable systems without using algebraic geometry is the second goal of this paper.

The last section contains the results of the averaging procedure for the intermediate long-wave equations (ILW):

$$u_t+2uu_x+\frac{1}{2\delta}\mathrm{P.V.}\int_{-\infty}^{\infty}\mathrm{cth}(\frac{\pi}{2\delta}(y-x))u_{yy}dy. \tag{0.12}$$

When $\delta\to 0$ or ∞ the equation (0.12) transforms into KdV or Benjamin-Ono equations respectively.

1. The general scheme

Let's consider the general Lax-type equations

$$\partial_t L=[A,L] \tag{1.1}$$

They are the equations on the coefficients of the operator L, because the coefficients of the operator A (as it follows from (1.1)) are the functions of the coefficients of L and their derivatives. For example, in the KdV- case

$$L=-\partial_x^2+u(x,t)\ ,\quad A=\partial_x^3-\frac{3}{2}u\partial_x-\frac{3}{4}u_x \tag{1.2}$$

For these equations or their perturbations

$$\partial_t L=[A,L]+\varepsilon K$$

(where K is the differential operator of the order less than the order of L and its coefficients depend on L, i.e symbolically it can be written as $K=K(L)$) the formal asymptotic solutions

$$L=L_0+\varepsilon L_1+\ldots \quad , \quad A=A_0+\varepsilon A_1+\ldots \tag{1.3}$$

can be easily constructed if the full set of solutions of the linearized equation

$$\delta L_t=[A_0,\delta L] + [\delta A,L_0] \tag{1.4}$$

is known.

For the periodic finite-gap solutions L_0, A_0 of (1.1), such a set has been obtained exactly [5,9].

As it has been already explained, the parameters I_j of the exact finite-gap solutions L_0, A_0 are the functions of slow variables X,T. Consequently, the first term in (1.3) has to satisfy the following non-homogeneous linear equation

$$L_{1t}-[A_0,L_1] - [A_1,L_0] = K+F(L_0) \tag{1.5}$$

where the coefficients of the operator F can be represented in terms of the coefficients of L_0 and A_0 and in terms of their derivatives in respect to slow variables

$$F = \partial_T L - \{L , A \} \tag{1.6}$$

$$\begin{aligned}\{L, A \} = &\sum_{i=0}^{n} u_i \sum_{k=o}^{i} kC_i^k \partial_x^{k-1} (\hat{\partial}_X v_j)\, \partial_x^{i+j-k} - \\ &- \sum_{j=0}^{m} v_j \sum_{k=o}^{j} kC_j^k \partial_x^{k-1} (\hat{\partial}_X u_i)\, \partial_x^{i+j-k}\end{aligned} \tag{1.7}$$

(Here L and A have the form (0.2).)

Let's consider the solutions ψ and ψ^+ of the auxiliary linear problems:

$$L_0\psi=E\psi \ , (\partial_t-A_0)\psi=0 \qquad (1.8)$$

(E-spectral parameter) and conjugate system

$$\psi^+L_0=E\psi^+ \ , \ \ \partial_t\psi^++\psi^+A_0=0 \qquad (1.9)$$

(The right action of a differential operator on any row-vector function $f^+$is defined in the common way

$$f^+(w\partial_x^i)=(-\partial_x)^i(f^+w)$$

The theorem ([5]) The uniformly bounded solutions L_1 of the equation (1.5) exist only if for any pair of the solutions ψ and ψ^+ of the equation (1.8,1.9) such that $\psi^+(x,t)\psi(x,t)$ is qasiperiodic in x the following relations are valid

$$<\psi^+F\psi>_x = <\psi^+K\psi>_x \qquad (1.10)$$

(Here and below $<\cdot>_x$ means

$$<f>_x=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)dx \ .)$$

From the definition of the right action of the differential operator it follows that for any f^+ and g one has

$$(f^+D)g = f^+(Dg)+\partial_x(f^+(\tilde{D}g))$$

The coefficients of the differential operator $\tilde{D}$ are the differential polinomials on the coefficients of the operator D. Then, using (1.8,1.9)

$$\partial_t(\psi^+ L_1 \psi) = (\psi^+((-A_o L_1 + L_{1t} + L_1 A_o)\psi) + \partial_x(\psi^+(-A_o L_1 \psi))$$

and after that

$$\partial_t(\psi^+ L_1 \psi) - \partial_x(\psi^+ A_o L_1 \psi) = \psi^+(F + K)\psi$$

this equality proves (1.10).

The relations (1.10) and the compartibility conditions for (0.7) are a complete set of the Whithem equations For the two-dimentional intgrable systems (and for Lax-type equations which are their partiqular case) they were obtained in an exact form in [5], where the construction of their solutions was also proposed. We shall not describe this in detail here, because we are going to demonstrate how this scheme works for the Benjamin-Ono equation.

2. The multiphase solutions of Benjamin-Ono equation

The Benjamin-Ono equation is equivalent to the compatibility conditions of the system of linear equations

$$\begin{aligned} &(i\partial_t + \partial_x^2 + U_{j,x}(x,t))\psi_j = 0, \quad j=1,2, \\ &i\partial_x \psi_1 + u\psi_1 = \lambda\psi_2, \end{aligned} \tag{2.1}$$

where $U_1(x,t)$ and $U_2(x,t)$ can be analytically extended into the upper and lower complex halfplanes of the variable, respectively.

Indeed, from (2.1) it follows that

$$iu = U_1 - U_2 + ic(t) \tag{2.2}$$

$$u_t + 2uu_x + (U_{1,xx} + U_{2,xx}) = 0 \tag{2.3}$$

The analytical continuations of U_j, as it follows from (2.2), can be represented with the help of Caushy integral. From the Plemiel-Sakhotsky formulae it follows that

$$U_1=\frac{i(u-c)}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u-c}{x-y}dy\ , \tag{2.4}$$

$$U_2=-\frac{i(u-c)}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u-c}{x-y}dy\ .$$

Therefore, (2.3) transforms into (0.7) after the substitution of (2.4).

For any set of numbers a_i, b_i, c_i, $i=1,...,n$, let's define the matrix $M=(M_{j,m})$

$$M_{j,m}=c_m\exp(\ i(a_m-b_m)x-i(a_m^2-b_m^2)t)\bullet\delta_{j,m}-\frac{1}{b_j-a_m} \tag{2.5}$$

Theorem 2.1. Let C, a_m, b_m be real,numbers

$$C<a_1<b_1<a_2<b_2<\ ...<a_n<b_n \tag{2.6}$$

and let

$$|c_i|^2=-\frac{(b_i-C)\prod\limits_{j\neq i}(a_i-a_j)(b_i-b_j)}{(a_i-C)\prod\limits_{j=1}^{n}(b_i-a_j)(a_i-b_j)} \tag{2.7}$$

Then the formula

$$u(x,t)=C+\sum_{m=1}^{n}(a_m-b_m)-2\mathrm{Im}(\ \partial_x\ln\det M(x,t)) \tag{2.8}$$

defines the real non-singular quasi-periodic solutions of the Benjamin-Ono equation.

The remark The solutions (2.8) have the form

$$u=u_0(Kx+Vt+\Phi \mid a_i,b_i,C) \tag{2.9}$$

where the n-periodic function u_0 and the vectors K,V are defined by a set of data (a_i, b_i, C) and the components of the phase vector Φ are equal to

$$\phi_i=\arg c_i \ . \tag{2.10}$$

The proof. Let's consider the the function $\psi_1(x,t,k)$ of the form

$$\psi_1=(1+\sum_{m=1}^{n}\frac{r_m(x,t)}{k-a_m})\exp(ikx-ik^2t) \tag{2.11}$$

which satisfies the relations

$$c_m \operatorname{res}_{k=a_m}\psi_1=\psi_1(x,t,b_m) \tag{2.12}$$

The linear relations (2.12) are equivalent to the system of linear equations for unknown functions $r_m(x,t)$

$$\sum_{m=1}^{n}M_{j,m}(x,t)r_m(x,t)=1 \ . \tag{2.13}$$

Lemma 2.1. The matrix M is non-degenerate for x, such that Im x ≥ 0.

The proof. Let's suppose that $M(x_0,t_0)$ is degenerate for some real numbers x_0,t_0. It means that there exists the function ψ_0 of the form

$$\psi_0(k)=\sum_{m=1}^{n}\frac{r_m^0}{k-a_m}\exp(ikx_0-ik^2t_0) \tag{2.14}$$

which satisfies the relations (2.12). Let's consider the differential

$$d\Omega=\psi_0(k)\overline{\psi}_0(\overline{k})dk\prod_{i=1}^{n}\frac{k-a_i}{k-b_i} \ . \tag{2.15}$$

This differential is meromorphic in respect to the variable k and has a zero residue at the infinity

$$\mathrm{res}_\infty d\Omega = 0.$$

At the same time from (2.12) and (2.6,2.7), it follows that

$$\mathrm{res}_{k=a_m}d\Omega+\mathrm{res}_{k=b_m}d\Omega=|R_m|^2\frac{\prod_{i\neq m}(a_m-a_i)}{\prod_{i=1}^{m}(a_m-b_i)}\left(1-\frac{b_m-C}{a_m-C}\right)>0\ , \tag{2.16}$$

where

$$R_m=r_m^0\exp(\, ia_mx_0-ia_m^2t_0) \ .$$

Hence, the sum of all the residues of $d\Omega$ is positive which is impossible. This contradiction proves invertibility of the matrix M(x,t) for the real x,t .

Let's consider the function

$$U_1=i\sum_{m=1}^{n}(a_m-b_m)-\partial_x\ln\det M(x,t) \ . \tag{2.17}$$

From the definition of M it follows that

$$U_1(x,t)=O(\,e^{-\alpha\mathrm{Im}\,x}\,) \quad , \quad \alpha=\min_m\,(a_m-b_m) \ . \tag{2.18}$$

If for all m, the differences (a_m-b_m) have the form

$$a_m-b_m=\frac{2\pi}{T}s_m \quad , \quad s_m \text{ are integers} \ , \tag{2.19}$$

the matrix M(x,t) is a periodic function of the variable x . The number of zeros of the function
det M(x,t) in the domain $\operatorname{Im} x>0$, $0\le \operatorname{Re} x <T$ equals

$$N=\frac{i}{2\pi}\int_0^T U_1(x,t)\,dx \quad .$$

This number does not change if we continiously change the parameters (a_i, b_i) keeping the relations (2.19) . When $|a_i - a_j| \to \infty$, it is easily seen that N =0 . Therefore, the statement of lemma is proved for the dense subset of data corresponding to the periodic matrix M . The function U_1 analytically depends on the set of data. That's why it should be holomorphic for x, Im x> 0, in a generic case, as well. The lemma is proved.

It is well-known that the function ψ_1 (x,t,k) satisfies the equation

$$(i\partial_t+\partial_x^2 - 2U_{1,x}(x,t))\psi_1(x,t,k) = 0 \quad , \tag{2.20}$$

where $U_1 = i\Sigma\, r_m(x,t)$ is the same as in (2.17). (See, for example [10]).
In addition to the ordinary statements, in our case $U_1(x,t)$ is holomorphic for x, $\operatorname{Im} x \ge 0$. Moreover, as it follows from (2.5) , we have the estimation (2.18) and

$$\psi_1=\exp(ikx - ik^2t)\,(1+O(e^{-\alpha \operatorname{Im} x}) \tag{2.21}$$

Let's consider now the function $\psi_2(x,t,k)$ of the form

$$\psi_2= (1+\sum_{m=1}^{n}\frac{\tilde r_m(x,t)}{k-b_m})\exp(ikx - ik^2t)\ , \tag{2.22}$$

which satisfies the relations

$$\tilde c_j \operatorname{res}_{k=b_j}\psi_2 = \psi_2(x,t,a_j) \tag{2.23}$$

where $\tilde{c}_j$ are some constants.

This function is the solution of the following equation

$$(i\partial_t+\partial_x^2 - 2U_{2,x}(x,t))\psi_2(x,t,k) = 0, \qquad (2.24)$$

$$U_2= - i\sum_{m=1}^{n}(a_m- b_m) - \partial_x \ln \det \tilde{M}\,(x,t)\,, \qquad (2.25)$$

where the matrix $\tilde{M}$ is equal to

$$\tilde{M}_{mj}= \tilde{c}_m\delta_{mj}\exp(- i(a_m- b_m)x+ i\,(a_m^2 - b_m^2)t\,) +\frac{1}{b_j- a_m} \qquad (2.26)$$

The functions ψ_2 and U_2 are holomorphic in respect to the variable x in the lower halfplane, Im x ≤ 0

The proof of these statements and (2.27, 2.28) is absolutely similar to the previous ones

$$U_2(x,t) = O(e^{\alpha\, \mathrm{Im}\, x}) \qquad (2.27)$$

$$\psi_2(x,t,k)=(1+O(e^{\alpha\, \mathrm{Im}\, x}))\exp\,(ikx - ik^2t\,) \qquad (2.28)$$

Let's introduce the function

$$\lambda(k)= (C - k)\,\frac{\prod_m (k - b_m)}{\prod_m (k - a_m)}\,. \qquad (2.29)$$

Lemma 2.2. If the constants c_m and $\tilde{c}_m$ satisfy the relations

$$\tilde{c}_m^{-1} = c_m \frac{b_m - C}{a_m - C} \frac{\prod_{j\neq m} (a_m - a_j)(b_m - b_j)}{\prod_{j=1}^{n} (b_m - a_j)(b_j - a_m)} \tag{2.30}$$

the following equality is valid

$$i\partial_x \psi_1 + u(x,t)\psi_1 - \lambda(k)\psi_2 = 0 \tag{2.31}$$

where

$$u = \sum_{j=1}^{n} (r_j - \tilde{r}_j + b_j - a_j) + C = C - \sum_{j=1}^{n} (a_j - b_j) + i(U_2 - U_1) \tag{2.32}$$

The proof. From the definition of $\psi_j, \lambda(k)$ and from the relations (2.30) it follows that
$\lambda(k)\psi_2(x,t,k)$ satisfies the relations (2.12). Let's define the function $\Psi(x,t,k)$ which equal to the left-hand side of (2.31).It has the form

$$\Psi = \left(\sum_{j=1}^{n} \frac{R_j(x,t)}{k - a_j}\right) \exp (ikx - ik^2 t) \tag{2.33}$$

and as it should satisfy the relations (2.12), the functions R_j should be the solutions of the linear equations

$$\sum_{j=1}^{n} M_{mj} R_j = 0$$

The matrix M is invertible. Therefore, $R_j = 0$ and the equality (2.31) is proved.

To complete the proof of the theorem it is enough to prove that the restrictions of the parameters which were enlisted in the statement of the theorem are sufficient for the reality of u(x,t).

lemma 2.3. If a_m, b_m are real and

$$\tilde{c}_m = -\bar{c}_m, \tag{2.34}$$

then

$$U_1(x,t) = \overline{U}_2(\bar{x},t) \ . \tag{2.35}$$

The proof. Consider the functions

$$\psi_1^+ = \overline{\psi_2(\bar{x},t,\bar{k})} \ , \psi_2^+ = \overline{\psi_1(\bar{x},t,\bar{k})} \tag{2.36}$$

The function

$$\psi_1(x,t,k)\psi_1^+(x,t,k)$$

is the rational function of the variable k and has the poles at the points a_m, b_m. From (2.12) and (2.23) it follows directly that

$$\mathrm{res}_{k=a_m}\psi_1\psi_1^+ + \mathrm{res}_{k=b_m}\psi_1\psi_1^+ = 0$$

Hence, the residue of this function at the infinity is equal to zero.

$$0 = \mathrm{res}_\infty\psi_1\psi_1^+ = \sum_{i=1}^{n}(r_i+\bar{r}_i) = i\,(\overline{U}_2(\bar{x},t) - U_1(x,t)) \tag{2.37}$$

The theorem is proved.

The dual Baker-Akhiezer functions satisfy the equations

$$\begin{aligned} &(-i\partial_t+\partial_x^2 - 2U_{j,x})\psi_j^+ = 0 \ , \ j=1,2, \\ &-\partial_x\psi_2^+ + u\psi_2^+ = \lambda\psi_1^+ \ . \end{aligned} \tag{2.38}$$

3. Whithem equations

The solutions of the Benjamin-Ono equation, which were constructed above, have the form (2.10). Therefore, according to the general scheme, they can be used for the construction of asymptotic solutions of the form

$$u = u_0(\varepsilon^{-1}S(X,T) \mid a_i(X,T), b_i(X,T), C(X,T)) + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \quad (3.1)$$

The vector S is defined from the relations

$$\partial_X S = K(a_i,b_i,C) \quad , \partial_T = V(a_i,b_i,C) \ . \quad (3.2)$$

The righthand sides in (3.2) depend on X,T through the dependence of a_i, b_i, C on these slow variables.

The Benjamin-Ono equation has not the Lax-pair representation. Nevertherless, the general scheme which was proposed in the First paragraph works the same as for Lax-type equations. The first term $w = w_1$ in the series (3.1) is defined from the linear equation

$$w_t + 2u_0 w_x + 2u_{0x} w + (W_{1,xx} + W_{2,xx}) = F[u_0] \quad (3.3)$$

where the functions W_1, W_2 have the analytical continuations in the upper and lower complex halfplanes of the variable x, respectively. The function w is equal to

$$iw = W_1 - W_2 \ . \quad (3.4)$$

The rightgand side of (3.3) equals

$$F[u_0] = \partial_T u_0 + 2u_0 \partial_X u_0 + 2\partial_X(U_{1,x} + U_{2,x}) \ . \quad (3.5)$$

Lemma 3.1. The uniformly bounded solutions of the equation (3.3) exist only if the following equality

$$<\psi_1 \, F[u_0] \, \psi_2^+ > = 0 \quad (3.6)$$

is valid.

The proof. From the equations (2.1,2.38) it follows, that

$$\begin{aligned}
&\psi_1(w_t+2u_0w_x+2u_{ox}w+W_{1,xx}+W_{2,xx})\psi_2^+ = \\
&= \partial_t(\psi_1 w\psi_2^+) - i\partial_x(w (\psi_{1x}\psi_2^+ - \psi_1\psi_{2x}^+)) + \\
&+\partial_x(\psi_1(W_{1,x}+W_{2,x})\psi_2^+ + 2iW_{1,x}\psi_1\psi_1^+ - 2iW_{2,x}\psi_2\psi_2^+
\end{aligned} \tag{3.7}$$

The average values of all the terms in the righthand side of the equality (3.7) except the two last terms are equal to zero because they are the derivatives of the quasi-periodic functions. The average values of the lst two terms equal zero because the contour of the integration can be shifted into the upper and lower halfplanes, respectively, where the integrant are exponentially small. Hence, the average value of the whole righthand side of (3.7) is equal to zero. The lemma is proved.

Theorem 3.1. The relations (3.6) and the compatibility conditions of the equations (3.2) are equivalent to

$$\partial_T a_i = - \partial_X a_i^2 \ , \partial_T b_i = - \partial_X b_i^2 \ , \partial_T C = - \partial_X C^2 \tag{3.8}$$

The proof. Let's consider the variation of the parameters a_i (τ), b_i (τ), C (τ) . The functions ψ_j (x,t,k $|\tau$), u(x,t $|$ τ) become the functions of the parameter τ . Let's introduce the "short derivative" $\hat{\partial}_\tau u$ of the function u which is equal to the derivative of the formulae (2.10) assuming that the vectors K and V are constants. By this definition

$$\hat{\partial}_\tau u = \partial_\tau u - \sum_i (x\partial_\tau K_i+t\partial_\tau V_i) \frac{\partial u}{\partial \phi_i} . \tag{3.9}$$

Lemma 3.2. The following relations

$$<\psi_1 \hat{\partial}_\tau u \psi_2^+> = \partial_\tau \lambda - i\, \partial_\tau K <\psi_1 \psi_1^+>, \tag{3.10}$$

$$<\psi_1 \frac{\partial u}{\partial \phi_i} \psi_2^+> = 0 \tag{3.11}$$

are fulfilled.

The proof. Let functions $\psi_j = \psi_j(x,t,k\,|\tau_1)$ and $\psi_j^+ = \psi_j^+(x,t,k\,|\,\tau_2)$ correspond to the different values of parameter τ. Then

$$i\partial_x (\psi_1 \psi_2^+) + \psi_1 (u(x,t\,|\tau_1) - u(x,t\,|\tau_2))\psi_2^+ = (\lambda(k|\tau_1) - \lambda(k|\tau_2))\psi_2\psi_2^+ \ . \tag{3.12}$$

Consider the derivative of (3.12) in respect to τ_1 and take $\tau_1 = \tau_2$ after that we shall obtain

$$i\, \partial_\tau K\, (\psi_1 \psi_2^+) + \psi_1 \hat{\partial}_\tau u\, \psi_2^+ = \partial_\tau \lambda\, (\psi_2 \psi_2^+) + \mathbf{Q} \tag{3.13}$$

where the term $\mathbf{Q}$ has the form

$$\mathbf{Q} = \sum_s (\alpha_s x + \beta_s t)\, \partial_x \tilde{w}_s\, (Kx+Vt+\Phi) \tag{3.14}$$

α_s, β_s are constants, the functions $\tilde{w}_s = \tilde{w}_s(z_1,\ldots,z_n)$ are periodic functions of the variable z_i. Let's define the subtorus $T_0(\Phi) \subset T^n$ as the closure of the set points $Kx+Vt+\Phi$ for any vector Φ.

Consider the average value of (2.11) in respect to $\Phi \in T_0(\Phi_0)$. The average value of $\mathbf{Q}$ equals zero, as it follows from (3.14). Hence, (3.10) is fulfilled, because

$$< \psi_1 \psi_1^+ > = <\psi_2 \psi_2^+> = 1. \tag{3.15}$$

(The latter equalities can be obtained, when using the shift of the contour of the integration into the complex plane.)

Lemma 3.3. The relations

$$2 < \psi_1 \,(\, \partial_\tau(\, U_{1,x} + U_{2,x} \,) + u\, \partial_\tau u\,)\, \psi_2^+ > \; = 2K\, \partial_\tau \lambda + i \partial_\tau V < \psi_1 \psi_2^+ > \quad (3.16)$$

are valid.

The proof. From (2.1) and (2.38) it follows that

$$i\partial_t\,(\psi_1\psi_2^+) + \partial_x(\,\psi_{1x}\psi_2^+ - \psi_1\psi_{2x}^+\,) =$$
$$2\,(\delta U_{1,x}\,\psi_1\psi_2^+) + 2i\,(u_x\psi_1\psi_2^+) \quad (3.17)$$

where $\delta U_1 = U_1(x,t\,|\tau_1) - U_1(x,t\,|\tau_2)$. We also have

$$u\,(\psi_1\psi_2^+)_x + \delta u\,(\psi_1\psi_{2x}^+) = \lambda\,(\psi_2\psi_{2x}^+ + \psi_{1x}\psi_1^+) + \delta\lambda\,(\psi_2\psi_{2x}^+) \;. \quad (3.18)$$

From (3.17) (with the help of (3.18) and the equality

$$i\psi_{2x}^+ = -\, u\psi_2^+ + \lambda\psi_1^+\,) \quad (3.19)$$

it can be obtained :

$$2\,(\delta U_{1,x} + 2u\,\delta u\,)\psi_1\psi_2^+ - 2\,\delta u\,\lambda\,\psi_1\psi_1^+ - 2i\,\delta\lambda\,(\psi_2\psi_{2x}^+) =$$
$$= i\partial_t\,(\psi_1\psi_2^+) - i\lambda\,(\psi_2\psi_2^+ + \psi_1\psi_1^+)_x - \delta\lambda\,(\psi_2\psi_2^+)_x - 2i\lambda\,(\psi_2\psi_{2x}^+ + \psi_{1x}\psi_1^+)\;. \quad (3.20)$$

Taking the derivative of (3.20) and considering its average value, we shall obtain

$$2 <\psi_1\,(\hat{\partial}_\tau U_{1,x} + u\,\hat{\partial}_\tau u\,)\,\psi_2^+ > - 2\lambda < \hat{\partial}_\tau u\,\psi_1\psi_1^+ > - 2i\,\partial_\tau\lambda < \psi_2\psi_{2x}^+ > =$$
$$i\,\frac{\partial W}{\partial \tau} <\psi_1\psi_2^+> \quad (3.21)$$

(The average value of all the terms, except for the first term, in the lefthand side of (3.20) equals zero. It can be shown, using the shift of the contours of integration into the complex plane.)

Let's denote the value $C - \sum_{m=1}^{n} (a_m - b_m)$ by $A = A(\tau)$ then

$$- \lambda < \hat{\partial}_\tau u \, \psi_1 \psi_1^+ > = i\lambda <\partial_\tau (U_1 - U_2 - A) \, \psi_1 \psi_1^+ > =$$
$$=i <- \partial_\tau U_2 \, \psi_1 \psi_1^+ >\lambda - i A_\tau \lambda = i\lambda <\partial_\tau U_2 (\psi_2 \psi_2^+ - \psi_1 \psi_1^+) > - iA_\tau \lambda =$$
$$= - < \partial_\tau U_2 (\psi_1 \psi_2^+)_x > - i A_\tau \lambda = < \partial_\tau U_{2,x} \psi_1 \psi_2^+ > - i A_\tau \lambda \tag{3.22}$$

(In (3.22) the equalities

$$< \partial_\tau U_1 \psi_1 \psi_1^+ > = < \partial_\tau U_2 \psi_2 \psi_2^+ > = 0$$

were used.)

The eqyality (3.21) is transformed with the help of (3.22) into the equality (3.16). finally, from (3.10) and (3.16) we have that

$$< \psi_1 \, F[u_0] \, \psi_2^+ > = 2k \, \partial_X \lambda - i \, (\partial_T K - \partial_X V) <\psi_1 \psi_2^+> + 2\lambda \partial_X A \tag{3.23}$$

Hence, as the consequence of (3.6) and $\partial_T K = \partial_X V$ (which follows from 3.2), we obtain

$$\partial_T \ln \lambda + 2k \, \partial_X \ln \lambda + 2 \, A_X = 0 \; . \tag{3.24}$$

This equality is equivalent to the statement of the theorem.

The remark. The equations which were obtained for the parameters a_i, b_i, C are integrable. They coincide with the Hopf equation

$$I_t = - (I^2)_x \tag{3.25}$$

It is well-known that the solutions of this equation are given in the following form :

$$I = f(x - 2It)$$

where $f(\xi)$ is the fixed function of one variable, and is the Caushy data for the equation (3.25) : $f(x) = I(x,0)$.

4. The ILW equation

The ILW equations (0.8) are the compatibility conditions of the same system of linear equations as in case of the Benjamin-Ono equation, but with different analytical properties of the coefficients. If $U_1(x,t)$ and $U_2(x,t)$ are boundary values of the function $U(x,t)$ which is holomorphic inside the strip $|\mathrm{Im}\, x| < \delta$

$$U_1(x,t) = U(x+i\delta), \quad U_2(x,t) = U(x - i\delta)$$

the compatibility conditions for the system (2.1) are equivalent to the equation (0.8). [11,12]

The main purpose of this paragraph is the construction of the finite-gap solutions of the ILW equations. It is based on the algebraic geometrical construction of the integrable potentials of the non-stationary Shrödinger operator (see [13]).

Let Γ be a smooth algebraic curve of the genus g with the fixed point P_0 on it and a local parameter $k^{-1}(P)$ in the neighbourhood of this point. For any set of the g points $\gamma_1,...\gamma_g$ in general position there exists the unique function $\psi(x,t,P)$ with the following analytical properties:

1^0. outside the point P_0 the function $\psi(x,t,P)$ is meromorphic and has simple poles at the points γ_j;

2^0. in the neighbourhood of the point P_0 it has the form :

$$\psi(x,t,P) = \exp(ikx - ik^2t)\left(1 + \sum_{s=1}^{\infty} \xi_s(x,t)\, k^{-s}\right), \quad k = k(P). \tag{4.1}$$

This function is the most important example of the, so-called, Baker-Akhiezer functions (see the general definition in [1]).

As it was proved in [1] , the function ψ is the solution of the equation

$$(i\,\partial_t+\partial_x^2 - 2U_x(x,t))\psi(x,t,k) = 0 \tag{4.2}$$

where

$$U = i\,\xi_1\,(x,t) \tag{4.3}$$

The Baker-Akhiezer function can be represented in terms of the Riemann theta-function (see [1]). Using these formulae we shall obtain

$$U(x,t) = i\,\partial_x \ln\theta\,(Kx+Vt+\Phi) \tag{4.4}$$

where the theta-function

$$\theta(z_1,...z_g) = \sum_{m\in Z^g} \exp\,(2\pi i\,(m,z) + \pi i\,(Bm,m))$$

is defined with the help of the matrix B which is a matrix of b-periods of the normalized holomorphic differential on Γ. The vectors K and V are b-periods of the normalized differentials $d\Omega_1$ and $d\Omega_2$ with the only singularity at the point P_0 of the form

$$d\Omega_1 = dk\,(1+O(k^{-2})),\ d\Omega_2 = dk^2\,(1+O(k^{-3})) \tag{4.5}$$

The vector Φ in (4.4) corresponds to the set γ_j and can be considered as an arbitrary vector.

The integrable potentials U_x depends on the set of data (Γ , P_0, $[k^{-1}]_2$), where $[k^{-1}]_2$ is the equivalence class of local parameter, $k' \approx k$ if $k' = k + O(k^{-2})$.

Now we are going to select the subset of the data which give the solutions of the ILW-equation. It should be emphasized that the corresponding subclass of the curves looks like the δ-deformation of the hyperelliptic curves.

Consider the curve Γ with a fixed point P_0, such that there exists the function $\lambda(P)$ on it, which is holomorphic outside P_0 and has the form:

$$\lambda(P) = - k\, e^{2\delta k} (1+\alpha_1 k^{-1}+\alpha_2 k^{-2}+\dots) \tag{4.6}$$

in the neighbourhood of the point P_0. We shall call such curves pseudo-hyperelliptic.

Consider the functions

$$\begin{aligned} \psi_1(x,t,P) &= \psi(x+i\delta,t,P) \\ \psi_2(x,t,P) &= \psi(x - i\delta,t,P)\,. \end{aligned} \tag{4.7}$$

Lemma 4.1. For pseudo-hyperelliptic curves the functions ψ_1, ψ_2 satisfy the relation

$$i\,\psi_{1x}+u\psi_1 - \lambda(P)\,\psi_2 = 0 \tag{4.8}$$

where

$$iu = U(x+i\delta) - U(x - i\delta) - i\alpha_1\,. \tag{4.9}$$

The proof is standart in the frame work of the algebraic-geometry methods. Let's denote the lefthand side of (4.8) by $\Psi(x,t,P)$. This function has simple poles outside P_0 and has the form:

$$\begin{aligned} \Psi &= \exp\,(ik(x+i\delta) - ik^2 t\,)\,(\xi_1(x - i\delta) - \xi_1(x+i\delta) + u + \alpha_1 + O(k^{-1}) = \\ &=O(k^{-1})\exp\,(ik(x+i\delta) - ik^2 t\,) \end{aligned} \tag{4.10}$$

From the uniqueness of the Baker-Akhiezer functions it follows that $\Psi = 0$. The lemma is proved.

Let's suppose that there exist the antiholomorphic involution of Γ, τ: $\Gamma\to\Gamma$, which preserves the point P_0, and such that

$$k(\tau(P)) = \overline{k(P)} . \tag{4.11}$$

We also suppose that the fixed cycles $\mathbf{a}_j$, $j = 1,...,m \le g$, separate the domains Γ^+, Γ^- such that

$$\Gamma^+ = \tau(\Gamma^-) \ , \Gamma = \Gamma^+ \cup \Gamma^- .$$

We shall choose the orientation of the cycles $\mathbf{a}_j$ as on the boundary of the complex domain Γ^+.

*Theorem 4.1.*If the set of poles γ_j of the Baker-Akhiezer function ψ and the set of conjugate points $\tau(\gamma_j)$ are zeros of the Third-type differential $d\Omega$ with the only simple poles at the points P_0 and P_1 , where P_1 is the zero of the function $\lambda(P)$, $\lambda(P_1) = 0$, and if the differential $\lambda(P)d\Omega \ge 0$ is non-negative on cycles $\mathbf{a}_j$, then the formulae (4.9) and (4.4) define the real non-singular quasi-periodic solutions of the ILW-eqation.

The proof. As it follows fromn (4.4) , in the generic case the function U(x,t) in respect to the variable x is the meromorphic function with the possible simple poles with the non-negative integer residues.

As it follows from (4.4) and the exact furmula for ψ (see [1]), the function U(x,y) has the poles at the point x_0, t_0 only, if there exists the function $\psi_0(P)$ which is meromorphic outside poin P_0 with simple poles at the points γ_j and which has the form:

$$\psi_0(P) = O(k^{-1}) \exp(ik_0 x - ik^2 t_0) \tag{4.12}$$

near the point P_0.

Let's prove at the beginning that there exist no such functions for x_0, such that Im $x_0 = \delta$.

The differential

$$d\Omega^* = \psi_0(P) \overline{\psi}_0(\tau(P)) \lambda(P) \, d\Omega$$

is a holomorphic differential on Γ and non-negative an all the cycles $\mathbf{a}_j$. Hence,

$$\sum_{i=1}^{m} \int_{a_i} d\Omega^* > 0 . \tag{4.13}$$

But at the same time the lefthand side of (4.13) should be equal to zero, because the union of the cycles a_j is the boundary of the complex domain Γ^+. This contrudiction proves that the function U(x,t) is regular for x, such that $|\operatorname{Im} x| = \delta$.

Now we are going to prove that the function U(x,t) is holomorphic for all x, such that $|\operatorname{Im} x| \leq 0$. In the set of all data there is a dense subset of the data such that the corresponding function U is periodic in x. In this case the namber of the poles U(x,t) in the strip $|\operatorname{Im} x| \leq \delta$ per the period equals

$$N = \int_0^T (U(x+i\delta, t) - U(x - i\delta,t)) \, dx \ . \tag{4.14}$$

From (4.4) it follows, that N = 0 . Therefore, in the periodic case the function U(x,t) is holomorphic for x , such that $|\operatorname{Im} x| \leq \delta$. The function U analytically depends on the parameters. That's why it should be holomorphic in the quasi-periodic case, as well.

Consider the differential

$$\psi (x,t,P)\overline{\psi} (x,t, \tau(P)) \, \lambda(P) \, d\Omega \tag{4.15}$$

For x , Im x = δ , this differential is meromorphic on Γ with the only double pole at the point P_0 . Its residue at this point should be equal to zero. Hence,

$$\xi_1(x + i\delta,t) +\bar{\xi}_1(x - i\delta,t) -\alpha_1 = 0$$

and u(x,t) is real. The theorem is proved.

At the end of this paragraph we shall present the Whithem equation for the algebraic-geometrical solutions of the ILW-equations. They will have the same form as the Withem equations for the KdV-equation (see [3]), if the function $\lambda(P)$ is used instead of the projection E(P) for the hyper-elliptic curves. Let's formulate it exactly.

Consider the differentials **dp** and **dΩ** on Γ , which have the form

$$\mathbf{dp} = dk\ (\ 1+O(k^{-2})\ )\ ,\quad \mathbf{d\Omega} = dk^2\ (1+O(k^{-3}\ )) \tag{4.16}$$

which are normalized by conditions

$$\mathrm{Im} \int_\sigma \mathbf{dp} = 0\ ,\quad \mathrm{Im} \int_\sigma \mathbf{d\Omega} = 0\ ,\quad \sigma\in H_1(\Gamma) \tag{4.17}$$

If the parameters of construction (Γ, P_0, $[\ k^{-1}]_2$) are the functions of variables X,T, than the integrals of **dp** and **dΩ** can be considered locally as the function of

$$\mathbf{p} = \mathbf{p}\ (\lambda,X,T)\quad ,\quad \Omega = \Omega\ (\lambda,X,T) \tag{4.18}$$

Theorem4.2. The Whithem equations for the ILW equation have the form:

$$\frac{\partial}{\partial T}\,\mathbf{p}\ (\lambda,X,T) = \frac{\partial}{\partial X}\,\Omega\ (\lambda,X,T)\ . \tag{4.19}$$

The proof of this theorem can be obtained in the same way as it was done for the Benjamin-Ono equation.

It should be mentioned that the exact solutions of the equations (4.19) can be constructed using the ideas of the previous work of the author [5] .

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