The Periodic Problems for Two-Dimensional Integrable Systems

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1. Introduction

Since the middle of the seventies algebraic geometry has become a very powerful tool in various problems of mathematical and theoretical physics. In the theory of integrable equations the algebraic geometrical methods provide a construction of the periodic and quasi-periodic solutions which can be written exactly in terms of theta functions of the auxiliary Riemann surfaces.

All the integrable equations which are considered in the soliton theory can be represented as compatibility conditions of the auxiliary linear problems. One of the most general types of such representations has the form:

$$[\partial_y - L, \partial_t - A] = 0, \tag{1.1}$$

where L, A are differential operators of the form

$$L = \sum_{i=0}^{n} u_i(x,y,t)\partial_x^i, \quad A = \sum_{i=0}^{m} v_i(x,y,t)\partial_x^i \tag{1.2}$$

with scalar or matrix coefficients.

The most important example of these equations is the Kadomtsev-Petviashvilii (KP) equation

$$\frac{3}{4}\sigma^2 u_{yy} + \left(u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\right)_x = 0 \tag{1.3}$$

which is equivalent to (1.1), where

$$L = \sigma(-\partial_x^2 + u(x,y,t)), \quad A = \partial_x^3 - \frac{3}{2}u\partial_x - w(x,y,t). \tag{1.4}$$

The algebraic geometrical construction of the solutions of integrable equations is based on the concept of the Baker-Akhiezer functions which are definded by their very specific analytical properties on the auxiliary Riemann surfaces. For example, the Baker-Akhiezer functions in the case of the KP equation are defined for each smooth algebraic curve Γ (Riemann surface of finite genus g) with the fixed point P_0 on it, and the local parameter $k^{-1}(P)$ in the neighbourhood of this point, $k^{-1}(P_0) = 0$. For any set of generic points γ_j, $j = 1, \ldots, g$, there exists a unique function $\Psi(x,y,t,P)$, $P \in \Gamma$, such that:

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1^0. It is meromorphic on Γ outside the point P_0 and has no more than simple poles at the points γ_j (if they are distinct);

2^0. The function Ψ has the form:

$$\Psi(x,y,t,P) = \left(1 + \sum_{s=1}^{\infty} \xi_s^i(x,y,t)k^{-1}\right) \exp(ikx + \sigma^{-1}k^2 y + ik^3 t) \tag{1.5}$$

$k = k(P)$, near the point P_0.

For any formal series of the form (1.5) there exist unique operators L and A of the form (1.4) such that the following relations

$$\begin{aligned} (\partial_y - L)\Psi &= O(k^{-1}) \exp(ikx + \sigma^{-1}k^2 y + ik^3 t) \\ (\partial_t - A)\Psi &= O(k^{-1}) \exp(ikx + \sigma^{-1}k^2 y + ik^3 t) \end{aligned} \tag{1.6}$$

are valid. From (1.6) it follows that the coefficient $u(x,y,t)$ of these operators is equal to

$$u(x,y,t) = 2i\xi_{1,x}(x,y,t). \tag{1.7}$$

The left hand sides of (1.6) define the functions which have the same analytical properties outside P_0 as Ψ, and have the form (1.6) near this point. From the uniqueness of the Baker-Akhiezer function Ψ, it follows that they are equal to zero. Hence,

$$(\partial_y - L)\Psi = 0, \quad (\partial_t - A)\Psi = 0 \tag{1.8}$$

and $u(x,y,t)$, which is given by (1.7) is a solution of the KP equation.

The Baker-Akhiezer function $\Psi(x,y,t,P)$ can be exactly written in terms of the Abelian differentials and Riemann theta-function. From the corresponding formulae it follows that the above constructed solutions of the KP equation have the form

$$u(x,y,t) = 2\partial_x^2 \ln \theta(Ux + Vy + Wt + \Phi/\tau) + \text{const}. \tag{1.9}$$

Here, $\theta(z_1, \ldots, z_g)$ is the Riemann theta-function which is defined by the matrix τ_{ij} of the b-periods of the normalized holomorphic differentials on Γ. The vectors $2\pi iU$, $2\pi iV$, $2\pi iW$ are the vectors of b-periods of the normalized Abelian differentials of the second kind with the only poles at P_0 of orders 2, 3, 4, respectively. The vector Φ corresponds to the set of the points γ_j and can be considered in (1.9) as an arbitrary vector.

The construction was proposed in [1, 2] and was developed in different ways for various types of integrable equations (see, for example, the reviews [3, 4, 5, 6]. The analytical properties of the Baker-Akhiezer functions are the natural generalization of the analytical properties of Bloch functions of the ordinary periodic finite-gap differential operators which were obtained in the remarkable works by Novikov, Dubrovin, Matveev, Its in which the algebraic geometrical solutions of the KdV equation, sine-Gordon equation and some other Lax-type equations were constructed.

In this report we shall present a brief review of the latest results obtained in the theory of periodic problems for the two-dimensional integrable systems. First of all, why is it algebraic geometry? What is the meaning of the algebraic geometrical solutions for the general periodic (in x and y) initial value problem for such equations? For the one-dimensional evolution integrable equations, the algebraic geometrical solutions are dense in the space of all periodic solutions (though this statement has not been proved rigorously for all such equations). In the case of the two-dimensional integrable equations the situation is much more complicated.

There are two real forms of the KP-1 ($\sigma^2 = -1$) and KP-2 ($\sigma^2 = 1$). It turns out, that the periodic problems for these equations differ dramatically from each other.

The formal non-integrability of the periodic problem for the KP-1 equation was proved in [7]. The proof of the integrability of such problem for the KP-2 equation was obtained by the author [8] and is based on the spectral theory of the operator

$$M = \sigma \partial_y - \partial_x^2 + u(x, y) \tag{1.10}$$

with the periodic potential.

The second problem which will be considered in this talk is the perturbation theory for two-dimensional integrable equations. We shall concentrate our attention on the so-called Whithem equation which is in our case a system of equations on bundles over the Teichmüller spaces. Finally, we shall demonstrate how the Whithem theory and other aspects of the perturbation theory of integrable equations will be married to each other in attemps to solve the Heisenberg relations

$$[L_n, A_m] = 1, \tag{1.11}$$

for the ordinary differential linear operators

$$L_n = \sum_{i=0}^{n} u_i(x) \partial_x^i, \quad A_m = \sum_{i=0}^{m} v_i(x) \partial_x^i, \quad u_n = v_m = 1. \tag{1.12}$$

The latter are the most popular subject in the field of string theory.

2. The Spectral Theory of Two-Dimensional Periodic Linear Differential Operators

The solutions $\Phi(x, y, w_1, w_2)$ of the nonstationary Schrödinger equation

$$\left(\sigma \partial_y - \partial_x^2 + u(x, y)\right) \Phi(x, y, w_1, w_2) = 0 \tag{2.1}$$

with the periodic potential are called Bloch solutions, if they are eigenfunctions of the monodromy operators, i.e.

$$\begin{aligned} \Psi(x + a_1, y, w_1, w_2) &= w_1 \Psi(x, y, w_1, w_2) \\ \Psi(x, y + a_2, w_1, w_2) &= w_2 \Psi(x, y, w_1, w_2). \end{aligned} \tag{2.2}$$

The set of pairs $Q = (w_1, w_2)$, for which there exists such a solution is called the Floque set, and will be denoted by Γ. The multivalued functions $p(Q)$, $E(Q)$ such that

$$w_1 = \exp(ipa_1), \quad w_2 = \exp(iEa_2)$$

are called quasi-momentum and quasi-energy, respectively.

For the "free" operator with zero potential $u_0 = 0$, the Floque set is parametrized by the points of the complex k-plane

$$w_1^0 = \exp(ika_1), \quad w_2^0 = \exp(-\sigma^{-1}k^2 a_2) \tag{2.3}$$

and the Bloch solutions have the form

$$\Psi_0(x, y, k) = \exp(ikx - \sigma^{-1}k^2 y). \tag{2.4}$$

It turns out that if Re $\sigma \neq 0$, then the Floque set of the operator (2.1) with the smooth potential $u(x, y)$ is isomorphic to the Riemann surface Γ (which has in a generic case infinite genus). The corresponding Riemann surface has such a specific structure that the theory of abelian differentials, theta-functions and so on can be constructed for it as well as for the finite genus case.

The source of the difference between the two cases Re $\sigma = 0$ and Re $\sigma \neq 0$ is the difference between the structure of the "resonant" points for the free operators. The resonant points are the points on the complex k-plane which are the pre-images of the self-intersection points of the imbedding $C \to C^2$, which is defined by (2.3). The points k and k' are resonant, if

$$w_i^0(k) = w_i^0(k'), \quad i = 1, 2. \tag{2.5}$$

From (2.3) it follows that such points are parametrized be integers $(N > 0, M)$ and have the form:

$$k = k_{N,M}, \quad k' = k_{-N,-M}, \tag{2.6}$$

where

$$k_{N,M} = \frac{\pi N}{a_1} + i\sigma \frac{M a_1}{N a_2}. \tag{2.7}$$

In case Re $\sigma \neq 0$, the resonant points tend to infinity and, hence, have no limiting points outside infinity. In case Re $\sigma = 0$, the resonant points are dense on the real axis which makes it impossible (at least by means of our methods) to construct the global Riemann surface of the Bloch functions.

For the real smooth potential u the Floque set can be described in the following form. Let us call the set of pairs of the complex numbers $\pi = \{p_{s,1}, p_{s,2}\}$ (where s belongs to any finite or infinite subset of integer pairs $(N > 0, M)$) "adimissible", if

$$\text{Re } p_{s,i} = \frac{\pi N}{a_1}, \quad |p_{s,i} - k_s| = O\left(\frac{1}{|k_s|}\right), \quad i = 1, 2$$

and the intervals $[p_{s,1}, p_{s,2}]$ do not intersect each other. Let us define the Riemann surface $\Gamma(\pi)$ for any admissible set π. It is obtained from the complex k-plane by cutting it along the intervals $[p_{s,1}, p_{s,2}]$ and $[-\overline{p_{s,1}}, -\overline{p_{s,2}}]$ and by sewing after that the left side of the first cut with the right side of the second cut and vise versa.

Theorem 1. *For any real periodic potentials $u(x, y)$, which can be analytically extended in some neighbourhood of the real values x, y, the Bloch solutions of the Equation* (2.1) *with $\sigma = 1$ are parametrized by the points Q of the Riemann surface $\Gamma(\pi)$ corresponding to some admissible set π. The function $\Psi(x, y, Q)$ which is normalized by the condition $\Psi(0, 0, Q) = 1$, is meromorphic on Γ and has a simple pole γ_s on each cycle a_s which corresponds to the cut $[p_{s,1}, p_{s,2}]$. If the admissible set π contains only a finite number of pairs, then $\Gamma(\pi)$ has finite genus and is compactified by only one point $P_0(k = \infty)$, in the neighbourhood of which the Bloch function Ψ has the form* (1.5).

The potentials u for which $\Gamma(\pi)$ has finite genus, are called finite-gap potentials and as it follows from the last statement of the theorem that they coincide with the algebraic geometrical potentials.

Theorem 2. *Any smooth periodic potential u of the Equation* (2.1) *(with* $\mathrm{Re}\,\sigma \neq 0$*), which can be analytically extended in the complex neighbourhood of the real x, y, can be approximated uniformly with any number of the derivatives by means of the finite-gap (algebraic geometrical) potentials.*

The Floque set is the "integral" of the KP equation. From the previous theorems we have:

Theorem 3. *For any smooth periodic function $v(x, y)$ there exists a unique solution of the* KP-2 *equation $u(x, y, t)$, such that $u(x, y, 0) = v(x, y)$. This solution is regular for all t and quasi-periodic in t. Any smooth periodic solutions of the* KP-2 *equation can be approximated by means of the finite-gap solutions.*

3. The Perturbation Theory of the Finite-Gap Solutions. Whithem Equations

The non-linear WKB (or Whithem) method can be applied to any non-linear equation which has the set of the exact solutions of the form

$$u_0(x, y, t) = u_0(Ux + Vy + Wt + \Phi | I_1, \ldots, I_N), \tag{3.1}$$

where $u_0(z_1, \ldots, z_g | I_k)$ is a periodic function of the variable z_i depending on the parameters I_k. The vectors U, V, W are also functions of the same parameters: $U = U(I)$, $V = V(I)$, $W = W(I)$.

In the framework of the non-linear WKB-method the asymptotic solutions of the form

$$u(x, y, t) = u_0(\varepsilon^{-1} S(X, Y, T) | I_k) + \varepsilon u_1 + \ldots \tag{3.2}$$

are constructed for the perturbed or non-perturbed initial equation. Here $X = \varepsilon x$, $Y = \varepsilon y$, $T = \varepsilon t$ are the "slow variables". If the vector $S(X, Y, T)$ is defined from the relations

$$\begin{aligned} \partial_X S &= U(I(X, Y, T) = U(X, Y, T) \\ \partial_Z S &= V(X, Y, T), \ \partial_T S = W(X, Y, T) \end{aligned} \tag{3.3}$$

the main term u_0 in the expansion (3.2) satisfies the initial equation up to the first order in ε. After that all the other terms of the series (3.2) are defined from the non-homogeneous linear equations. The construction of such asymptotic solutions even for integrable equations is very important, because when using the slow modulation of the parameters of their exact solutions one can sometimes solve the integrable equation with "non-integrable boundary conditions".

For the KdV equation and for some other Lax-type equations, the Whithem method was developed and applied to various problems in [9, 10, 11]. For the two-dimensional integrable systems the Whithem method was proposed in [12]. We shall present here only a part of the corresponding results.

The asymptotic solutions of the form (3.2) can be constructed with an arbitrary dependence of the parameters I_k on slow variables. In this case the expansion

(3.2) is valid on the scales of order 1. The right hand side of the non-homogeneous linear equation which defines the first order term u_1 contains the first derivatives of the parameters I_k. Therefore, the choice of the dependence of I_k on slow variables can be used for the cancellation of the "secular" term in u_1. The corresponding equations on I_k are usually called the Whithem equations.

Let us consider again the KP equation as an example of the two-dimensional integrable systems. Its finite-gap solutions have the form (3.1). The set of their parameters are the system of local coordinates of the manifold M_g which has dimension $N = 3g + 1$.

$$M_g((\Gamma, P_0[k^{-1}]_2) \ . \tag{3.4}$$

(Two local parameters are m-equivalent if $k_1 = k + O(k^{-m})$; the corresponding equivalence class of the local parameter is denoted by $[k^{-1}]_m$.)

Let us consider the second kind differentials on Γ with the only poles at the point P_0 of the form

$$dp = dk(1 + O(k^{-2}), \quad dE = i\sigma^{-1}dk^2(1 + O(k^{-3}), \quad d\Omega = dk^3(1 + O(k^{-4}) \tag{3.5}$$

which have the real periods for any cycle on Γ. Their integrals $p(Q)$, $E(Q)$, $\Omega(Q)$ are multivalued functions on the manifold M_g^* which is a bundle over M_g

$$M_g^* = \left(\Gamma, P_0, [k^{-1}]_2, Q \in \Gamma\right) \ . \tag{3.6}$$

If $(\lambda, I_1, \ldots, I_{3g+1})$ is a system of local coordinates on M_g^* and I_k are functions of the variables X, Y, T then $p = p(\lambda, X, Y, T)$, $E = E(\lambda, X, Y, T)$, $\Omega = \Omega(\lambda, X, Y, T)$ become functions of these variables.

Theorem 4. *The necessary conditions for the existence of the asymptotic solutions of the equation*

$$\frac{3}{4}\sigma^2 u_{yy} + (u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x + \varepsilon K[u] = 0 \tag{3.7}$$

which has the form (3.2) *with uniformly bounded first-order term are equivalent to the equation*

$$\frac{\partial p}{\partial \lambda}\left(\frac{\partial E}{\partial T} - \frac{\partial \Omega}{\partial T}\right) - \frac{\partial E}{\partial \lambda}\left(\frac{\partial p}{\partial T} - \frac{\partial \Omega}{\partial X}\right) + \frac{\partial \Omega}{\partial \lambda}\left(\frac{\partial p}{\partial Y} - \frac{\partial E}{\partial X}\right) = \frac{\langle \Psi K \Psi_+ \rangle_x}{\langle \Psi \Psi_+ \rangle_x}\frac{\partial p}{\partial \lambda} . \tag{3.8}$$

Here $K[u]$ is an arbitrary differential polynomial; Ψ, Ψ_+ are the corresponding Baker-Akhiezer function and its dual, respectively.

Remark. It turns out that there are only $3g + 1$ independent equations among the Equation (3.8) which should be fulfilled for any point Q of the curve Γ.

For the KdV equation and $K = 0$ the Equation (3.8) have the form

$$\partial_T p = \partial_X \Omega \tag{3.9}$$

which was obtained in [11]. The construction of the exact solutions of the Equation (3.8) with $K = 0$ was proposed in the work [12]. We shall present the particular case of this scheme in the next section where the Heisenberg relations would be considered.

4. The Heisenberg Relations for the Ordinary Linear Differential Operators

Great progress has been made recently in non-perturbative two-dimensional gravity coupled to various matter fields. It was shown that the dependence of physical quantities (such as specific heat) on scaled coefficients of the models is described by the KP-hierarchy on the space of the ordinary linear differential operators L_n, A_m such that the relations (1.11) are fulfilled. For pure two-dimensional gravity $n = 2$, $m = 3$ the Equation (1.11) is equivalent to the Painlevé 1 equation

$$\frac{1}{4}u_{xxx} - \frac{3}{2}uu_x = 1\,. \tag{4.1}$$

The Equation (1.12) has a simple scaling transformation

$$u_1(x) = \varepsilon^{(i-n)\beta}\widetilde{u_i}\left(\varepsilon^{-\beta}x\right)\,, \quad v_i = \varepsilon^{(i-m)\beta}\widetilde{v_i}\left(\varepsilon^{-\beta}x\right) \tag{4.2}$$

$\beta = (n+m)^{-1}$. For the operators $\widetilde{L}_n, \widetilde{A}_m$ with the coefficients $\widetilde{u}_i, \widetilde{v}_i$ we have

$$[\widetilde{L}_n, \widetilde{A}_m] = \varepsilon\,. \tag{4.3}$$

The formal asymptotic solutions of the equation (4.3) can be constructed using any commuting operators $[L_{n,0}, A_{m,0}] = 0$

$$\widetilde{L}_n = L_{n,0} + \varepsilon L_{n,1} + \dots\,, \; \widetilde{A}_m = A_{m,0} + \varepsilon A_{m,1} + \dots\,. \tag{4.4}$$

Unfortunately, these asymptotic solutions are well-defined only in the interval $x \sim 1$. For our purposes it is necessary to have the solutions for $x \sim \varepsilon^{-1/(n+m)}$. It can be done in framework of the Whithem theory.

The commuting operators of co-prime orders $(n, m) = 1$ are parametrized by the coefficients of the polynomial

$$w^n + E^m + \sum_{in+jm \leq nm-2} \alpha_{ij} w^i E^j = 0 \tag{4.5}$$

and by the points of the Jacobian of the corresponding algebraic curve [13]. In [1, 2] the exact formulae for the coefficients of the generic commuting operators in terms of the Riemann theta-function were found. For example,

$$u_{n-2} = -n\partial_x^2 \ln\,\theta(Ux + \Phi/\tau) + \text{const}\,. \tag{4.6}$$

Here the matrix τ of b-periods of Γ depends on the values α_{ij}. The vector U is also a function of the variables α_{ij}. The phase vector Φ is arbitrary. All the other coefficients have the same structure

$$u_i = u_{i,0}(Ux + \Phi/\tau)\,, \quad v_i = v_{i,0}(Ux + \Phi/\tau)\,. \tag{4.7}$$

Let me consider the operators $L_n^{\#}, A_m^{\#}$ with the coefficients

$$u_i^{\#} = u_{i,0}\left(\frac{1}{\varepsilon}S(X)/\tau(X)\right)\,, \quad v_i^{\#} = v_{i,0}\left(\frac{1}{\varepsilon}S(X)/\tau(X)\right)\,. \tag{4.8}$$

If the vector $S(X)$ is defined by the relation $\partial_X S = U(\alpha_{ij}(X))$ then the operators $L_n^{\#}, A_m^{\#}$ commute up to the order ε. As was shown in [12] in more general situation the requirement that the first order terms in the expansion (4.4) should be uniformly bounded for all x leads to the equations on the variables α_{ij}. They are particular cases of (3.8) and have the form

$$\frac{\partial w(E,X)}{\partial X} = \frac{\partial p(E,X)}{\partial E}. \tag{4.9}$$

It turns out that they are integrable and we present the construction of their solutions below. Our conjecture (which is partly proved now for $n = 2$, $m = 3$) is that all the other terms of the asymptotic solutions (4.4) are also bounded and the series (4.4) are convergent. If this is true, it is possible to make the inverse rescaling and find the limit for $\varepsilon \to 0$. To begin with we shall give the final answer for the KdV equation with the "string" boundary conditions (1.12) $n = 2$, $m = 2k + 1$.

Let us consider an arbitrary hyperelliptic curve Γ

$$y^2 = \prod_{i=1}^{2k+1} (E - E_i) = E^{2k+1} + \prod_{i=1}^{2k} c_i E^i = R(E). \tag{4.10}$$

As is well-known, this curve defines the solutions of the KdV equation which have the form (1.9) (with $V = 0$).

For any given set of the parameters: the complex constants $c_{k,0}, c_{k+1,0}, \ldots, c_{2k,0}$, the real constants h_i, h_i', $i = 1, \ldots, k$, we shall consider the hyperelliptic curve which is defined by the polynomial R with the coefficients

$$c_i = c_{i,0}\,, \; i = k+2, \ldots, 2k\,; \quad c_k = x + c_{k,0}\,; \quad c_{k+1} = t + c_{k+1,0} \tag{4.11}$$

and such that

$$\operatorname{Im} \int_{E_{2i}}^{E_{2i+1}} \sqrt{R}\, dE = h_i\,, \quad \operatorname{Im} \int_{E_1}^{E_{2i}} \sqrt{R}\, dE = h_i', \qquad i = 1, \ldots, k\,. \tag{4.12}$$

The Equation (4.12) are the set of $2k$ real equations which define k unknown complex coefficients c_i, $i = 0, \ldots, k - 1$ of the polynomial $R(E)$. They become functions of the variable x, t. The τ matrix of the corresponding curve becomes a (known) function of the variables x, t. Let us define the vector

$$S_i(x,t) = \frac{1}{\pi} \left(\int_{E_1}^{E_{2i}} \sqrt{R}\, dE - \sum_{j=1}^{k} \tau_{ij} \int_{E_{2j}}^{E_{2j+1}} \sqrt{R}\, dE \right). \tag{4.13}$$

The Main Conjecture. *The functions*

$$u(x,t) = -2\partial_x^2 \ln \theta(S(x,t) + \Phi/\tau(x,t)) - 2r_1(x,t) \tag{4.14}$$

are the exact solutions of the KdV equation with the "boundary conditions" (1.12).

Here $r_1(x,t)$ is the coefficient of the differential

$$d\Omega_1 = \frac{E^k + \sum_{i=0}^{k-1} r_i E^i}{2\sqrt{R}}\, dE\,, \quad \int_{E_{2i}}^{E_{2i+1}} d\Omega_1 = 0\,.$$

Thus the Equations (4.12) are the only transcendental equations in the definition of $u(x, t)$.

Let us consider now the general Heisenberg relations. Any equation of the form (4.5) has the formal solution

$$w = k^m + \sum_{i=-m+2}^{\infty} a_i k^{-i}\,, \quad k^n = E\,. \tag{4.15}$$

This means that the affine curve (4.5) is compactified by a single point P_0. Let us fix a first few coefficients of the expansion (4.15) and denote them by

$$a_{n-j} = \frac{j}{n} t_j\,, \quad j = 1, \ldots, m+n-2\,. \tag{4.16}$$

They uniquely define the following coefficients of (4.5)

$$\alpha_{ij}, im + jn \geq (m-1)(n-1) = 2g\,. \tag{4.17}$$

For any given real numbers h_i, h_i', $i = 1, \ldots, g$, all the other coefficients of the polynomial (4.5) can be defined (at least locally) as functions of the parameters t_j with the help of the relations

$$\operatorname{Im} \int_{a_i} w\, dE = h_i\,, \quad \operatorname{Im} \int_{b_i} w\, dE = h_i'\,. \tag{4.18}$$

They give $2g$ real equations on g complex variables $\alpha_{ij}, in + jm < (n-1)(m-1)$. Therefore, the curve Γ and the algebraic function $w(E)$ become functions of the variables t_j.

Theorem 5. *The function $w(E, t_1, \ldots)$ satisfies the Whithem equations* (4.9) *if $t_1 = x$.*

Let us define the differentials $d\Omega_j$, $j = 1, \ldots, m+n-2$, whose only poles at infinity have the form

$$d\Omega_j = dk^j \left(1 + O(k^{-j-1})\right) \tag{4.19}$$

and such that

$$\operatorname{Im} \int_\gamma d\Omega_j = 0\,, \quad \gamma \in H_1(\Gamma)\,. \tag{4.20}$$

Corollary. *If the relations* (4.18) *are fulfilled, then*

$$\frac{\partial p}{\partial t_j} = \frac{\partial \Omega_j}{\partial x} \quad \frac{\partial \Omega_j}{\partial t_j} = \frac{\partial \Omega_j}{\partial t_i}\,.$$

Remark. It can be shown that the conjecture which was proposed recently in [14] leads to one particular solution of the Painlevé 1 which belongs to our set of solutions.

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