WHITHAM THEORY FOR INTEGRABLE SYSTEMS AND TOPOLOGICAL QUANTUM FIELD THEORIES

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1. INTRODUCTION

During the last two years remarkable connections between the non-perturbative theory of two-dimensional gravity coupled with various matter fields, the theory of topological gravity coupled with topological matter fields, the theory of matrix models and, finally, the theory of integrable soliton equations with special Virasoro constraints have been found [1-11]. The main goal of these few lectures is to present the results of perturbation theory of algebraic-geometrical solutions of integrable equations which clarify some of this connections.

All the "integrable" partial differential equations, which are considered in the framework of the "soliton" theory, are equivalent to compatibility conditions of auxiliary linear problems. The general algebraic-geometrical construction of their exact periodic and quasi-periodic solutions was proposed in [12,13]. The cornerstone of this construction is a concept of the Baker-Akhiezer functions - the functions which are uniquely defined with the help of their analytical properties on auxiliary Riemann surfaces. The corresponding analytical properties are so specific that it follows from them and only from them that the Baker-Akhiezer functions are common eigenfunctions of an overdetermined system of linear equations. Consequently, coefficients of these equations are solutions of non-linear equations. The analytical properties of the Baker-Akhiezer functions are the generalization of properties of the Bloch solutions of the finite-gap Shturm-Liuville operators, which were found in a serious of papers by Novikov, Dubrovin, Matveev and Its (see the review of them in [14,15]; the reviews of further development of the finite-gap theory can be found in [16-20]).

Roughly speaking, the algebraic-geometrical construction gives a map from a set of algebraic-geometrical data to a space of solutions of integrable non-linear equations.

$$\{algebraic-geometrical\ data\} \longmapsto \{solutions\ of\ NLPDE\}$$

In a generic case the space of algebraic-geometrical data is a bundle over the moduli space $M_{g,N}$ of smooth algebraic curves Γ_g of genus g with N punctures $P_1, \ldots, P_N$. A bundle over $M_{g,N}$ corresponds to additional data which are: systems of local coordinates $k_\alpha^{-1}(Q)$, $k_\alpha^{-1}(P_\alpha) = 0$, in the neighborhoods of punctures and a set of g points $\gamma_1 \ldots, \gamma_g$ on Γ_g in the general position or equivalently a point on the Jacobian

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$J(\Gamma)$ of the corresponding curve Γ. We shall denote this complete set of data by $\tilde{M}_{g,N}$ and intermediate bundle will be denoted by

$$\hat{M}_{g,N} = \{\Gamma_g, P_\alpha, k_\alpha^{-1}\}$$

It is to be mentioned that $\tilde{M}_{g,N}$ are "universal" data. For the given non-linear integrable equation the corresponding set of data have to be specified. For example, the solutions of the Kadomtsev-Petviashvili (KP) hierarchy can be obtained from the set of data $\tilde{M}_{g,N}$ for $N = 1$.

The algebraic-geometrical construction of [12,13] is by definition a sort of an "inverse transform" : from "spectral" (algebraic-geometrical) data to "solutions". A "direct" transform : from "solutions" to "spectral" data is an unavoidable step in a construction of the perturbation theory of integrable equations. Such theory has to contain answers to the following questions: "how many solutions can be obtained with the help of the algebraic-geometrical construction ?" and "what is a structure of the union of all algebraic-geometrical solutions corresponding to different genera ?". The last question is the question about the solutions of linearized equations on the background of given algebraic-geometrical solution of initial non-linear PDE because they form a tangent space to the manifold of all solutions.

In the next two chapters a brief review of the corresponding results using as an example the KP equation is given. In the fourth chapter the, so-called, Whitham equations are introduced. They are the equations on the moduli space $\hat{M}_{g,N}$ and are "averaging" integrable equations. It is remarkable that they are also "integrable". We shall begin the presentation of the construction of their solutions with the simplest case : the construction of the Whitham equations on $\hat{M}_{0,N}$ which are quasi-classical limit (dispersionless analogue) of soliton equations.

As it was found in [21], the particular solution of dispersionless Lax equations coincides with the perturbed superpotential of topological Landau-Ginzburg models [22]. It turns out, that dispersionless analogue of τ-function for the corresponding solution gives the solution of truncated Virasoro constraints. In the last chapter the relations between general Whitham equations and $1/N$-expansion for loop-equations for matrix models are discussed.

2. BAKER-AKHIEZER FUNCTIONS AND ALGEBRAIC-GEOMETRICAL SOLUTIONS OF THE KP EQUATION

Let Γ be a non-singular algebraic curve of genus g with N punctures P_α and fixed local parameters $k_\alpha^{-1}(Q)$ in the neighborhoods of punctures. For any set of the points $\gamma_1, \ldots, \gamma_g$ in the general position there exists a unique up to constant factor, $c(t_{\alpha,i})$, a function $\psi(t, Q)$, $t = (t_{\alpha,i})$,$\alpha = 1, \ldots, N$; $i = 1, \ldots,$ such that:

(i) the function ψ (as a function of the variable Q which is a point of Γ) is meromorphic everywhere except for the points P_α ant it has at most simple poles at the points $\gamma_1, \ldots, \gamma_g$ (if all of them are distinct).

(ii) at the neighborhood of the point P_α the function ψ has the form

$$\psi(t, Q) = \exp(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i)(\sum_{s=0}^{\infty} \xi_{s,\alpha}(t) k_\alpha^{-s}), k_\alpha = k_\alpha(Q). \tag{2.1}$$

this is the most general definition of a scalar *multipoint* and *multivariable* Clebsh-Gordan-Baker-Akhiezer function (or simply the Baker-Akhiezer function). It depends on the variables $t = \{t_{1,i}, \ldots, t_{n,i}\}$ as on external parameters.

From the uniqueness of the Baker-Akhiezer function it follows that for each pair (α, n) there exists a unique operator $L_{\alpha,n}$ of the form

$$L_{\alpha,n} = \partial_{\alpha,1}^{n} + \sum_{j=1}^{n-1} u_j^{(\alpha,n)}(t)\partial_{\alpha,1}^{j} \tag{2.2}$$

(where $\partial_{\alpha,i} = \partial/\partial t_{\alpha,i}$) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = 0. \tag{2.3}$$

The idea of the proof of theorems of this type which was proposed in [12,13] is universal.

For any formal serious of the form (2.1) their exists a unique operator $L_{\alpha,n}$ of the form (2.2) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = O(k^{-1})\exp(\sum_{i=1}^{\infty} t_{\alpha,i}k_{\alpha}^{i}). \tag{2.4}$$

The coefficients of $L_{\alpha,n}$ are differential polynomials with respect to $\xi_{s,\alpha}$. They can be found after substitution of the serious (2.1) into (2.4).

It turns out that if the series (2.1) is not formal but is an expansion of the Baker-Akhiezer function in the neighborhood of P_α then the congruence (2.4) becomes an equality. Indeed, let us consider the function ψ_1

$$\psi_1 = (\partial_{\alpha,n} - L_{\alpha,n})\psi(t,Q). \tag{2.5}$$

It has the same analytical properties as ψ except only one. The expansion of this function in a neighborhood of P_α starts from $O(k^{-1})$. From uniqueness of the Baker-Akhiezer function it follows that $\psi_1 = 0$ and the equality (2.3) is proved.

Corollary. The operators $L_{\alpha,n}$ satisfy the compatibility conditions

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\alpha,m} - L_{\alpha,m}] = 0. \tag{2.6}$$

Remark. The equations (2.6) are gauge invariant. For any function $g(t)$ operators

$$\tilde{L}_{\alpha,n} = gL_{\alpha,n}g^{-1} + (\partial_{\alpha,n})g^{(-1)} \tag{2.7}$$

have the same form (2.2) and satisfy the same operator equations (2.6). The gauge transformation (2.8) corresponds to the gauge transformation of the Baker-Akhiezer function

$$\psi_1(t,Q) = g(t)\psi(t,Q)$$

Example. One-point Baker-Akhiezer function.

In the one point case the Baker-Akhiezer function has an exponential singularity at a single point P_1 and depends on a single set of variables. Let us choose the normalization of the Baker-Akhiezer function with the help of the condition $\xi_{1,0} = 1$, i.e. an expansion of ψ in the neighborhood of P_1 equals

$$\psi(t_1, t_2, \ldots, Q) = \exp(\sum_{i=1}^{\infty} t_i k^i)(\sum_{s=0}^{\infty} \xi_s(t)k^{-s}). \tag{2.8}$$

In this case operator L_n has the form

$$L_n = \partial_1^n + \sum_{i=o}^{n-2} u_i^{(n)} \partial_1^i. \tag{2.9}$$

For example, for $n = 2, 3$ we have

$$L_2 = \partial_1^2 - u(t), L_3 = \partial_1^3 - \frac{3}{2} u \partial_1 + w.$$

Let us define the variables $x = t_1, y = \sigma^{-1} t_2, t = t_3$. Then from (2.6) it follows (for $n = 2$,$m = 3$) that $u(x, y, t, t_4, \ldots)$ satisfies the KP equation

$$\frac{3}{4} \sigma^2 u_{yy} + (u_t - \frac{3}{2} u u_x + \frac{1}{4} u_{xxx})_x = 0. \tag{2.10}$$

Remark. It should be emphasized that algebraic-geometrical construction is not a sort of abstract "existence" and "uniqueness" theorems. It provides the exact formulae for solutions in terms of the Riemann theta-functions. For example, the algebraic-geometrical solutions of the KP equation have the form

$$u(t_1, t_2, \ldots) = 2 \partial_1^2 \ln \theta (\sum_{i=1}^{\infty} U_i t_i + \Phi) + const \tag{2.11}$$

where $\theta(z_1, \ldots, z_g) = \theta(z_1, \ldots, z_g | B(\Gamma)$ is the Riemann theta-function which defined with the help of matrix $B(\Gamma)$ of b-periods of normalized holomorphic differentials on Γ. The vectors $J_i = \{U_{1,k}, \ldots, U_{g,i}\}$ are vectors of b-periods of normalized Abelian differentials with the only pole at P_1 (see details in [13]).

The equations (2.6) for $n = 2, m > 3$ describe evolutions of $u(x, y, t, t_4, \ldots)$ with respect to "higher times" or equivalently the whole KP hierarchy. Here it is necessary to make a few comments. In the original form the equations (2.6) are a set of non-linear equations on the coefficients $u_i^{(n)}$ and do not have the form of evolution equations. It can be shown (see [23]) that they are equivalent to evolution system in the form which was proposed by Kyoto group [24]. We shall show that for algebraic-geometrical solutions.

For any formal series of the form (2.8) there exists a unique pseudo-differential operator $\mathcal{L}$

$$\mathcal{L} = \partial_1 + \sum_{i=1}^{\infty} u_i(t_1, \ldots) \partial_1^{-i}. \tag{2.12}$$

such that

$$\mathcal{L} \psi(t, k) = k \psi(t, k). \tag{2.13}$$

Then the operators L_n which are uniquely defined from the congruence (2.4) are equal to

$$L_n = [\mathcal{L}^n]_+, \tag{2.14}$$

where $[...]_+$ denotes a differential part of a pseudo-differential operator. From (2.3) it follows that if $\psi(t, k)$ is an expansion of the Baker-Akhiezer function then

$$(\partial_n - [\mathcal{L}^n]_+) \psi(t, Q) = 0. \tag{2.15}$$

The comparitability conditions of (2.13) and (2.15) imply the evolution equations

$$\partial_n \mathcal{L} = [[\mathcal{L}^n]_+, \mathcal{L}] \tag{2.16}$$

on the coefficients $u_i(t_1, \ldots)$ of $\mathcal{L}$. The equations (2.16) are the Sato form for the KP hierarchy.

At the end of this chapter we shall make a few comments about multi-point case. For each α the equation (2.6) up to gauge transformation are equivalent to the KP hierarchy corresponding to each set of variables $\{t_{\alpha,i}\}$. "Which is the interaction between two different "KP hierarchies"?

As it was found in [14,25] for the two-point case a full set of equations can be represented in the following form

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\beta,n} - L_{\beta,n}] = D^{\alpha,\beta}_{N,m} H^{\alpha,\beta} , \tag{2.17}$$

where $H^{\alpha,\beta}$ is a two-dimensional Schrödinger operator in a magnetic field

$$H^{\alpha,\beta} = \frac{\partial^2}{\partial_{\alpha,1}\partial_{\beta,1}} + v_1^{\alpha,\beta}\partial_{\alpha,1} + v_2^{\alpha,\beta}\partial_{\alpha,2} + u^{\alpha,\beta} \tag{2.18}$$

and operators $D^{\alpha,\beta}_{N,m}$ are differential operators in variables $t_{\alpha,1}, t_{\beta,1}.\alpha, \beta$ are differential operators in variables $t_{\alpha,1}, t_{\beta,1}$.

The sense of (2.17) is as following. For the given operator $H^{\alpha,\beta}$ any differential operator D in the variables $t_{\alpha,1}, t_{\beta,1}$ why a number of can be uniquely represented in the form

$$D = D_1 H^{\alpha,\beta} + D_2 + D_3$$

where D_2 is a differential operator with respect to the variables $t_{\alpha,1}$ only and D_3 is a differential operator with respect to the variable $t_{\beta,1}$ only. The equation (2.17) implies that the second and the third term in the corresponding representation for the left hand side of (2.17) are equal to zero. This implies $n + m - 1$ equations on $n+m$ unknown functions (the coefficients of operators $L_{\alpha,n}$ and $L_{\beta,m}$). The equations (2.17) are gauge invariant. That is why a number of equations equals a number of unknown functions.

3. PERIODIC PROBLEM FOR THE KP EQUATION

The KP hierarchy in the form (2.16) is a set of commuting evolution equations on a set of the coefficients $u_i(x), x = t_1$, of a pseudo-differential operator $\mathcal{L}$ (2.12). The original form of the KP equation (2.10) is a nonlinear equation on a single function $u(x, y, t)$ and, though it has not a purely evolution form, the Caushy problem can be formulated for it The Caushy data is a function of two variables $v(x, y) = u(x, y, t = 0)$. In the same sense all equations (2.6) for $N = 1, n = 2, m > 3$ can be formally considered as flows on a space of functions depending on the two variables x, y. These flows can be well-defined only on some subspaces of the space of functions on two variables. That's why the exact form of an equivalence between two forms of the KP hierarchy is a delicate subject. It is deeply connected with the problem of an invertibility of the algebraic-geometrical transform

$$\{\tilde{M}_{g,1}\} \longmapsto \{solutions\ of\ the\ KP\ hierarchy\} \tag{3.1}$$

The algebraic-geometrical solutions of the KP hierarchy are quasi-periodic functions of all variables, as it follows from the exact formula (2.11).

Let us consider periodic (in x and y) solutions. The corresponding subset of algebraic-geometrical data can be specified in the following way. For the given algebraic curve Γ with fixed point P_1 and local parameter in its neighborhood the differentials dp and dE can be defined as meromorphic differentials on Γ with the only poles at P_1 of the form

$$dp = idk(1 + O(k^{-2}), dE = i\sigma^{-1}dk^2(1 + O(k^{-3}) \tag{3.2}$$

and as differentials which have *real* periods along any cycle γ on Γ

$$\oint dp = U_\gamma, \mathrm{Im} U_\gamma = 0,$$

$$\oint dE = V_\gamma, \mathrm{Im} V_\gamma = 0. \tag{3.3}$$

If all the periods have the form

$$U_\gamma = \frac{2\pi n_\gamma}{a_1}, V_\gamma = \frac{2\pi m_\gamma}{a_2},$$

where n_γ and m_γ are integral numbers, then the corresponding solutions of the KP equation have periods a_1 and a_2 with respect to the variables x and y . The set of all "periodic" curves is dense in the whole moduli space for $a_1, a_2 \to \infty$.

Now we shall try to invert the map (3.1) in the case of periodic solutions. The solutions $\psi(x, y, w_1, w_2)$ of the non-stationary Shrodinger equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\psi(x, y, w_1, w_2) = 0 \tag{3.4}$$

with a periodic potential $u(x, y)$ are called Bloch solutions, if they are eigenfunctions of the monodromy operators, i.e.

$$\psi(x + a_1, y, w_1, w_2) = w_1\psi(x, y, w_1, w_2)$$
$$\psi(x, y + a_2, w_1, w_2) = w_2\psi(x, y, w_1, w_2). \tag{3.5}$$

The set of pairs $Q = (w_1, w_2)$, for which their exists such a solution is called the Floque set and will be denoted by Γ. The multivalued functions $p(Q)$ and $E(Q)$ such that

$$w_1 = \exp(ipa_1), w_2 = \exp(iEa_2) \tag{3.6}$$

are called quasi-momentum and quasi-energy, respectively.

For the free operator with zero potential $u_0 = 0$, the Floque set is parametrized by the points of the complex k-plane

$$w_1^0 = \exp(ika_1), w_2^0 = \exp(-\sigma^{-1}k^2a_2) \tag{3.7}$$

and the Bloch solutions have the form

$$\psi_0(x, y, k) = \exp(ikx - \sigma^{-1}k^2y). \tag{3.8}$$

It turns out ([18]) that if *Re* $\sigma \neq 0$ then the Floque set of the operator (3.4) with the smooth potential $u(x, y)$ is isomorphic to the Riemann surface Γ (which has infinite genus in a generic case). The corresponding Riemann surfaces have such a specific structure that the theory of Abelian differentials, theta-functions and so on, can be constructed for them as well as for the finite genus case. (See detailed description of Γ in [18].) If Γ has a finite genus then u is an algebraic-geometrical potential. The last statement means that for periodic algebraic-geometrical solutions of the KP equation the corresponding curve in their algebraic-geometrical data coincides with the curve of the Bloch solutions. At the same time the Baker-Akhiezer function is the Bloch function. In [18] it was proved also that the algebraic geometrical solutions are dense in a space of all periodic (in x, y) solutions of the KP equation for $Re\ \sigma \neq 0$.

4. THE PERTURBATION THEORY OF THE ALGEBRAIC-GEOMETRICAL SOLUTIONS. WHITHAM EQUATIONS

The non-linear WKB (or Whitham) method can be applied to any non-linear equation which has a set of exact solutions of the form

$$u_0(x,y,t) = u_0(Ux + Vy + Wt + \Phi | I_1, \dots, I_N), \tag{4.1}$$

where $u_0(z_1, \dots | I_k)$ is a periodic function of the variables z_i depending on parameters (I_k). The vectors U, V, W are also functions of the same parameters : $U = U(I), V = V(I), W = W(I)$.

In the framework of the non-linear WBK-method (for $(1+1)$-systems see [26-28,31]) the asymptotic solutions of the form

$$u(x,y,t) = u_0(\varepsilon^{-1} S(X,Y,T) | I(X,Y,T)) + \varepsilon u_1 + \dots \tag{4.2}$$

are constructed for the perturbed or non-perturbed initial equation. Here $X = \varepsilon x, Y = \varepsilon y, T = \varepsilon t$ are "slow" variables. If the vector $S(X,Y,T)$ is defined from the relations

$$\begin{gathered} \partial_X S = U(I(X,Y,T)) = U(X,Y,T), \\ \partial_Y S = V(X,Y,T), \quad \partial_T S = W(X,Y,T), \end{gathered} \tag{4.3}$$

then the main term u_0 in the expansion (4.2) satisfies the initial equation up to the first order in ε. After that all the other terms of the series (4.2) are defined from the non-homogeneous linear equations. They can be easily solved if a full set of solutions for a homogeneous linear equation are known.

The KP equation linearized on the background u_0 has the form

$$\frac{3}{4}\sigma^2 v_{yy} + (v_t - \frac{3}{2}u_0 v_x - \frac{3}{2}u)_{0x} v + \frac{1}{4}v_{xxx})_x = 0. \tag{4.4}$$

The adjoined linear equation is

$$\frac{3}{4}\sigma^2 \Phi_{yy} - (\Phi_t + \frac{3}{2}u_0 \Phi_x + \frac{1}{4}\Phi_{xxx})_x = 0. \tag{4.5}$$

In the operator form the linearized KP hierarchy is linear equations on the pseudo-differential operator

$$\delta\mathcal{L} = \sum_{i=1}^{\infty} \delta u_i \partial_1^{-i}$$

and has the form

$$\partial_n \delta\mathcal{L} = [[\mathcal{L}_0^n]_+, \delta\mathcal{L}] + [\delta[\mathcal{L}^n]_+, \mathcal{L}_0].$$

Solutions of the linearized equations can be found from the following obvious observation. If a set of solutions of non-linear equation depending on some parameters is known, the derivatives (with respect to these parameters) are solutions of the linearized equation. In other words solutions of the linearized equation form a tangent space for a space of solutions of the non-linear equation.

In the case of algebraic-geometrical solutions there are deformations of algebraic-geometrical data which preserve genus of curves and there are deformations in "transversal" directions which increase genus of curves. The simplest type of the last deformations as follows. One has to take a pair of points Γ and "glue" between them a "thin" handle. For the periodic solutions u_0 of the KP equation (with $Re\ \sigma \neq 0$) it was shown [31] that such deformations together with deformations along $\tilde{M}_{g,1}$ give a full set of periodic solutions of the linearized equation (4.4) and the exact formulae were obtained for them in terms of the Riemann theta-functions . We shall not consider here a complete perturbation theory even for the KP equation. We restrict ourselves only to a part of this theory which contains a new features with respect to the usual perturbation theory.

The most essential part of it is the, so-called, Whitham equations, which define a "slow" dependence of parameters $I(X, Y, T)$ and, hence, define the main term of the series (4.2).

The asymptotic solutions of the form (4.2) can be constructed with an arbitrary dependence of the parameters I_k on slow variables. In this case the expansion (4.2) will be valid on a scale of order 1. The right hand side of the non-homogeneous linear equation for u_1 contains the first derivatives of the parameters I_k. Therefore, the choice of the dependence of I_k on slow variables can be used for the cancellation of the "secular" term in u_1. The necessary conditions of such cancellation are "ortogonality" conditions of the corresponding right hand side of the equation for u_1 to the cotangent bundle to $\hat{M}_{g,1}$. The cotangent bundle to a space of solutions of a non-linear equation is the space of solutions of an adjoint to the linearized equation. In [31] it was proved that the cotangent bundle for $\hat{M}_{g,1}$ corresponds to the product of the Baker-Akhiezer function and its "dual" $\psi^+(t, Q)$.

The dual Baker-Akhiezer function (in one point case) is a function which is meromorphic on Γ outside P_1 and has at most simple poles at the points $\gamma_1^+, \ldots, \gamma_g^+$. (If $\gamma_1, \ldots, \gamma_g$ are poles of the Baker-Akhiezer function then γ_s^+ are defined as a set of g points such that $\gamma_1, \ldots, \gamma_g$ and $\gamma_1^+, \ldots, \gamma_g^+$ are zeros of a differential $d\Omega$ on Γ with the only pole of order 2 at P_1.)

In the neighborhood of P_1 the function $\psi^+(t, Q)$ has the form

$$\psi^+(t_1, t_2, \ldots, Q) = \exp(\sum_{i=1}^{\infty} -t_i k^i)(\sum_{s=0}^{\infty} \xi_s^+(t) k^{-s}). \tag{4.6}$$

It turns out that if ψ satisfies the equations (2.13) and (2.15) then ψ^+ is a solution of the equations

$$\psi^+ \mathcal{L} = k\psi^+(t, k), \tag{4.7}$$

$$-\partial_n \psi^+ = \psi^+ [\mathcal{L}^n]_+, \tag{4.8}$$

where the left action of differential operators is defined as usual

$$\psi^+(w\partial_1^i) = (-1)^i \partial_1^i (\psi^+ w). \tag{4.9}$$

From (2.13,2.15) and (4.7,4.8) it follows that if $\delta\mathcal{L}$ is a solution of the linearized equations then

$$\partial_n < \psi^+ \delta\mathcal{L}\psi > = < \psi^+(-[\mathcal{L}_0^n]_+ \delta\mathcal{L} + [\mathcal{L}_0^n]_+, \delta\mathcal{L}] + [\delta\mathcal{L}^n]_+, \mathcal{L}] + \delta\mathcal{L}[\mathcal{L}_0^n]_+)\psi > = 0. \tag{4.10}$$

(Here $< \ldots >$ denotes the mean value with respect to $x = t_1$). The equality (4.10) proves that $\psi^+(t, Q)\psi(t, Q)$ is a solution of adjoint equation.

From this statement it follows that if F is a right hand side of the equation for u_1 then the necessary conditions for uniform boundness of u_1 are the equalities

$$< \psi^+(t,Q)F(t)\psi(t,Q) > = 0, \tag{4.11}$$

which have to be fulfilled for all $Q \in \Gamma$.

In [31] it was shown that (4.11) can be represented in the following form. Let us consider the differentials dp, dE on Γ, which were defined by the conditions (3.2,3.3) and let us consider also a meromorphic differential $d\Omega$ with the only pole at P_1 of the form

$$d\Omega = idk^3(1 + O(k^{-4}), \tag{4.12}$$

which has *real* periods on Γ. The integrals $p(Q)$, $E(Q)$, $\Omega(Q)$ of these differentials are multivalued functions on the bundle M_g^* over $\hat{M}_{g,1}$

$$M_g^* = (\Gamma, P_1, k^{-1}, Q \in \Gamma).$$

If I_k are local coordinates on $\hat{M}_{g,1}$ and (λ, I_k) are local coordinates on M_g^*, then for any dependence of I_k on the variables X, Y, T the integrals p, E, Ω become functions of the variables X, Y, T: $p = p(\lambda, X, Y, T)$, $E = E(\lambda, X, Y, T)$, $\Omega = \Omega(\lambda, X, Y, T)$.

Theorem [31]. The necessary conditions for the existence of the asymptotic solutions of the equation

$$\frac{3}{4}\sigma^2 u_{yy} + (u_t - \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x + \varepsilon K[u] = 0. \tag{4.13}$$

which has the form (4.2) with uniformly bounded first-order term are equivalent to the equation

$$\frac{\partial p}{\partial \lambda}(\frac{\partial E}{\partial T} - \frac{\partial \Omega}{\partial Y}) - \frac{\partial E}{\partial \lambda}(\frac{\partial p}{\partial T} - \frac{\partial \Omega}{\partial X}) + \frac{\partial \Omega}{\partial \lambda}(\frac{\partial p}{\partial Y} - \frac{\partial E}{\partial X}) = \frac{< \psi^+ K \psi > }{< \psi^+ \psi}\frac{\partial p}{\partial \lambda}. \tag{4.14}$$

Here $K[u]$ is an arbitrary differential polynomial.

The parameters of the algebraic-geometrical solutions of the whole KP hierarchy are points of the infinite dimensional manifold $\hat{M}_{g,1}$. If only a finite number of flows is considered then only a finite-dimensional part of $\hat{M}_{g,1}$ is essential.

Two local parameters k_1 and k would be called m-equivalent, if $k_1 = k + O(k^{-m})$. The class of m-equivalency of the local parameter is denoted by $[k^{-1}]_m$. From the definition of the Baker-Akhiezer function it follows that if it considered as a function of only m first "times" (i.e. $t_i = 0, i > m$) then ψ and the corresponding solutions of first equations (2.16) depend only on $[k^{-1}]_m$. The algebraic-geometrical solutions of the KP equation depend on $[k^{-1}]_3$. Hence, the number of there parameters equals $N = 3g + 2, I = (I_1, \ldots, I_N)$.

The equality (4.14) has to be fulfilled for any $Q \in \Gamma$, but they are not all independent. It turns out that they are equivalent to $3g + 2$ independent equations (i.e. the equations (4.14) are well-defined).

First of all, let us make an important remark. The equations (4.14) are invariant with respect to any change of local coordinate on Γ, $\lambda_1 = \lambda_1(\lambda, I)$. Hence, any particular choice of λ can be used. Let us choose a function $\lambda(Q)$ on Γ with the only pole of order $r = g + 3$ at P_1 such that

$$\lambda^{1/r}(Q) = k(Q) + O(k^{-3}(Q)) \tag{4.15}$$

(in a generic case such function exists and is unique). The function λ defines a local coordinate in the neighborhood of any point of Γ except points q_s where differential $d\lambda$ equals zero: $d\lambda(q_s) = 0$. The number of such points equals $N = 3g + 2$. The requirement that there are no poles in the left hand side of (4.14) implies (4.14).

Important remark. The Whitham equations are real equations because normalization conditions for differentials $dp, dE, d\Omega$ are real. Below we shall consider all times $\sqrt{-1}t_i$ as real variables (but the case where $a_i t_i$ are real for given complex numbers a_i can be considered in the same way).

Let us introduce meromorphic differentials $d\Omega_i$ with the only pole at P_1 of the form

$$d\Omega_i = d(k^i + O(k^{-1}))$$

and such that all their periods on Γ are *real.* (In our previous notations: $dp = d\Omega_1, dE = d\Omega_2, d\Omega = d\Omega_3$). Then the Whitham equations (in the case $K = 0$) can be written for any pair of times $i, j > 1$

$$\frac{\partial p}{\partial \lambda}(\partial_i \Omega_j - \partial_j \Omega_i) - \frac{\partial \Omega_j}{\partial \lambda}(\partial_i p - \partial_x \Omega_i) + \frac{\partial \Omega_i}{\partial \lambda}(\partial_j p - \partial_x \Omega_j) \; = \; 0. \tag{4.16}$$

(we preserve here the same notation t_i for "slow" variables εt_i).

Let us consider now n-th reduction of the KP hierarchy which is the hierarchy of Lax equation. The Lax equations are equations on the coefficients of differential operator

$$L = \partial_1^n + u_{n-2}\partial_1^{n-2} + \ldots + u_0, \tag{4.17}$$

which have the form

$$\partial_i L = [[L^{i/n}]_+, L], i = 1, 2, ..., \tag{4.18}$$

They are the particular case of (2.16) corresponding to pseudo -differential operator $\mathcal{L}$ such that

$$L = \mathcal{L}^n. \tag{4.19}$$

The subset $M_g(n)$ of algebraic geometrical data $\hat{M}_{g,1}$ which give the solutions of (4.18) is the following set of data: { curve Γ with puncture P_1 is such that there exists a function $E(Q)$ on Γ with the only pole at P_1 of order n; the local parameter in the neighborhood of P_1 is $E^{-1/n}$}. The corresponding solutions of (2.16) do not depend on (t_n, t_{2n}, ...) and are solutions of (4.18).

The Whitham equations (4.16) for the choice $\lambda(Q) = E(Q)$ take the form

$$\partial_i p(t, E) \; = \; \partial_x \Omega_i(t, E). \tag{4.20}$$

(for the KdV equation such form of the Whitham equations were obtained for the first time in [27]).

In [31] the construction of exact solutions for the Whitham equations was proposed. Only some particular cases of this construction would be considered below in detail. In the next chapter we shall start with simplest zero genus case.

5. DISPERSIONLESS LAX EQUATIONS

The algebraic-geometrical solutions of the KP hierarchy corresponding to zero genus case are simple constants. Nevertheless, the Whitham equations even in this case are not trivial.

The zero genus curve Γ_0 with puncture can be identified with usual complex p-plane where infinity corresponds to P_1. There are no moduli in this part of data, but there is still infinite number of parameters corresponding to a choice of local parameter K^{-1}

$$K(p) = p + \frac{u}{2}p^{-1} + \ldots . \tag{5.1}$$

Hence, the Whitham equations in this case are the equations on the local parameter $K^{-1}(p)$. They have the form (4.16) where by definition

$$\Omega_i(p) \;=\; [K^i(p)]_+, \tag{5.2}$$

here $[...]_+$ denotes a non-negative part of Laurent series.

Example. From (5.2) we have that

$$\Omega_2 = k^2 + u, \quad \Omega_3 = k^3 + \frac{3}{2}uk + w$$

and equation (2.16) is equivalent to the dispersionless KP equation (which is also called the Khokhlov-Zabolotskaya equation).

$$\frac{3}{4}u_{yy} + (u_t - \frac{3}{2}uu_x) = 0. \tag{5.3}$$

The Whitham equations for genus zero case describe quasi-classical limit of the corresponding initial equations.(We would like to mention here two papers which were devoted to the dispersionless Lax and KP equations. In [29] the construction of integrals for dispersionless Lax equations was proposed. In [30] the construction of particular solution of (5.3) was presented. The dispersionless KdV equation is

$$u_t = \frac{3}{2}uu_x \tag{5.4}$$

and it is well known that all solutions of it can be obtained in the implicit form from the equation

$$x + tu = f(u), \tag{5.5}$$

where $f(u)$ is an arbitrary function. In that sense the dispersionless KdV equation is even "more integrable" than the KdV equation. The construction of the Whitham equations which was proposed in [31] is a deep generalization of (5.5).

Let us consider the dispersionless Lax equations. The corresponding hierarchy describes solutions of (4.18) which are slow functions of all the variables. They can be written as a system of evolution equation on the coefficients of a polynomial $E(p)$

$$E(p) = p^n + u_{n-2}p^{n-2} + \ldots + u_0. \tag{5.6}$$

The analogues of (4.18) are the equations which are a particular case of the Whitham equations

$$\partial_i E = \frac{d\Omega_i}{dp}\partial_x E - \frac{dE}{dp}\partial_x \Omega_i, \tag{5.7}$$

here $\Omega_i(p)$ are the polynomials (5.2), where $K(p)$ is such that $K^n = E(p)$.

Let $p(E)$ be the inverse (multi-valued) function for (5.6)

$$p(E) = K + O(K^{-1}), K^n = E. \tag{5.8}$$

Then $\Omega_i(p) = \Omega_i(p(E)) = \Omega_i(E)$ can be also considered as multi-valued function of the variable E (or K)

$$\Omega_i(E) = K^i + \sum_{j=1}^{\infty} \chi_{i,j} K^{-j}. \tag{5.9}$$

The equations (5.7) are equivalent to the equations

$$\partial_i p(E) = \partial_x \Omega_i(E). \tag{5.10}$$

which is genus zero particular case of representation (4.10)

In this chapter we consider mainly only one special case of the construction [31], which provides solutions of the Whitham equations, corresponding to the loop-equations and topological minimal models. For the dispersionless Lax equations the corresponding construction looks as follows. Let us define the formal series

$$S_+(p) = \sum_{i=1}^{\infty} t_i \Omega_i(p) = \sum_{i=1}^{\infty} t_i K^i + O(K^{-1}) \tag{5.11}$$

(if only a finite number of t_i is not equal to zero, then $S_+(p)$ is a polynomial). The coefficients of S_+ are linear functions of t_i and polynomials on u_i. We introduce the dependence of u_j on the variables t_i with the help of the following relation: the ratio

$$B(p) = \frac{dS_+}{dE} = \sum_{i=0}^{\infty} b_i p^i \tag{5.12}$$

should contain only positive powers of the variable p. (If $t_i = 0$, for $i > N$, then it means that $B(p)$ *should* be a polynomial with respect to p.) The relation (5.12) defines $u_i(t_1, t_2, ...)$ as implicit function.

The defining relations (5.12) can be represented in another form. Let q_s be zeros of the polynomials

$$\frac{dE}{dp}(q_s) \ = \ 0. \tag{5.13}$$

then (5.12) are equivalent to the equalities

$$\frac{dS_+}{dp}(q_s) = 0. \tag{5.14}$$

Remark. From (5.14) it follows that u_i do not depend on the variables $t_n, t_{2n}, t_{3n}, \ldots$, because $\Omega_{pn} = E^p$, $p = 1, 2, 3, \ldots$.

Let us prove that if (5.12) or (5.14) is fulfilled, then

$$\partial_i S_+(E) = \Omega_i(E). \tag{5.15}$$

Consider the function $\partial_i S_+(E)$. From (5.11) it follows that

$$\partial_i S_+(E) = K^i + O(K^{-1}) = \Omega_i(E) + O(K^{-1}). \tag{5.16}$$

Hence, it is enough to prove that $\partial_i S_+(E)$ is a polynomial in p, because Ω_i is the only polynomial in p such that

$$\Omega_i(p) = K^i + O(K^{-1}). \tag{5.17}$$

The function $\partial_i S_+(E)$ is holomorphic everywhere except maybe at q_s. In the neighborhood of q_s the local coordinate is

$$(E - E_s)^{1/2} + \ldots \tag{5.18}$$

and the derivative $\partial_i S_+(E)$ might be singular at the points q_s. But the defining relations (5.14) imply that $\alpha_s = 0$. Therefore, $\partial_i S_+(E)$ is regular everywhere except at the infinity and is a polynomial. The equations (5.15) are proved.

The compatibility conditions of (5.15) imply (5.10) (because $p = \Omega_1, x = t_1$). Hence, (5.12) defines in the implicit form a solution of the dispersionless Lax equations.

Let us define a function

$$F(t_1, t_2, ...) = -\frac{1}{2} \operatorname{res}(SdS_+), \quad S = \sum_{i=1}^{\infty} t_i K^i. \tag{5.19}$$

From the following formulae it follows that the function

$$\tau_K(t_1, t_2, ...) = \exp F(t_1, t_2, ...) \tag{5.20}$$

is an analogue of the τ-function for usual Lax equations.

Remark. The functions F, τ do not depend on $t_n, t_{2n}, ...$.

In [21] it was proved, that

$$\partial_i F = -\operatorname{res}(K^i dS_+). \tag{5.21}$$

Hence, F is a homogeneous function of the variables t_i:

$$\sum_{i=1}^{\infty} t_i \partial_i F = 2F. \tag{5.22}$$

The function F contains all the information about Ω_i. For example, the coefficients $\chi_{i,j}$ of the expansions (5.9) equal:

$$\partial_i \partial_j F = -i\chi_{j,i} = -j\chi_{i,j}. \tag{5.23}$$

The function F satisfies the truncated version of the Virasoro constraints which were obtained in[10,11] for the partition function of two-dimensional gravity models. The dispersionless analogue of these constraints have the form ([21]):

$$\sum_{i=n+1}^{\infty} it_i \partial_{i-n} F + 12 \sum_j Sd\delta_m S_+). \tag{5.29}$$

For $m = -n, 0, n, 2n, \ldots$ from the defining relations (5.14) it follows that the variation

$$\delta_m S_+ = n \frac{dS_+}{dE} E^{\frac{m+n}{n}} \tag{5.30}$$

is an entire function of the variable p. Therefore,

$$\delta_m S_+ = \sum_{i=1}^{\infty} t_i (i[K^{i+m}]_+ - \sum_{j=1}^{m} j\chi_{ij}[K^{m-j}]_+), \tag{5.31}$$

where χ_{ij} are the coefficients of the expansion (5.9).

The substitution of (5.31)into (5.29) gives for $m = 0, n, 2n, \ldots$

$$0 = \sum_{i=1}^{\infty} [it_i \partial_{i+m} F + \frac{1}{2} \sum_{j=1}^{m-1} (t_i \partial_i \partial_j F)(\partial_{m-j} F)]. \tag{5.32}$$

From (5.22) it follows that

$$\sum_{i=1}^{\infty}(t_i\partial_i\partial_j F) = \partial_j(\sum_{i=1}^{\infty} t_i\partial_i F) - \partial_j F = \partial_j F \tag{5.33}$$

and finally we obtain the equations (5.25, 5.26). For $m = -n$ we have

$$0 = -2 \sum_{i=-m+1}^{\infty} it_i\partial_{i+m}F + \mathrm{res}(\sum_{i=1}^{-m} it_i K^{i+m} dS_+). \tag{5.34}$$

The second term in (5.34) equals

$$\mathrm{res}(\sum_{i=1}^{-m} it_i K^{i+m} dS) = -\sum_{i+j=-m} ijt_it_j, \tag{5.35}$$

which proves (5.24).

Remark. A few comments on more general solutions of dispersionless Lax equations. The proof of the statement that the defining relations (5.14) implies that the corresponding polynomial $E(p)$ satisfies dispersionless Lax equations can be repeated without any changes if we introduce instead of S_+ a new function

$$\hat{S}_+ = S_+ + S_0,$$

where S_0 is piecewise holomorphic function with a constant jumps on some contours (see [31]).

6. THE TOPOLOGICAL MINIMAL MODELS

Topological minimal models were introduced in [32] and were considered in [33]. They are a twisted version of the discrete series of $N = 2$ superconformal Landau-Ginzburg (LG) models. A large class of the $N = 2$ superconformal LG models has been studied in [34-36]. It was shown, that a finite number of states are topological, which means that their operator products have no singularities. These states form a closed ring $\mathcal{R}$, which is called a primary chiral ring. It can be expressed in terms of the superpotential $W(p_i)$ of the corresponding model

$$\mathcal{R} = \frac{C[p_i]}{dW = 0}, \quad dW = \frac{\partial W}{\partial p_i} dp_i. \tag{6.1}$$

In topological models these primary states are the only local physical excitations.

In [22], it was shown that correlation functions of primary chiral fields and integrals of their second des can be expressed in terms of perturbed superpotentials $W(p_i, t_0, t_1, ...)$. For A_{n-1} model the unperturbed superpotential has the form:

$$W_0 = \frac{p^n}{n}. \tag{6.2}$$

(we use here the same normalization as in [22]). The equations which define its perturbation are as follows ([22]):

For any polynomial

$$W(p) = \frac{1}{n}(p^n + u_{n-2}p^{n-2} + ... + u_0) \tag{6.3}$$

polynomials Φ_i, $i = 0,1,2,...,n-2$,

$$\Phi_i = p^i + O(p^{i-1}) \tag{6.4}$$

are defined with the help of the orthogonality conditions

$$\langle\Phi_i\Phi_j\rangle = \mathrm{res}(\frac{\Phi_i\Phi_j}{\partial_p W}) = \delta_{i,n-2-j}. \tag{6.5}$$

The equations which define the dependence of $W(p,t_0,t_1,...,t_{n-2})$ on the "coupling constants" t_i have the form:

$$\partial_i W = \frac{\partial W}{\partial t_i} = -\Phi_i. \tag{6.6}$$

The solution of the equations (6.6) with the initial conditions

$$W(p,0,0,...) = W_0(p) \tag{6.7}$$

were found in [22].

The relations of these results to the dispersionless Lax equations are given by the following theorem ([21]).

Theorem. *Let $E(p,t_1,t_2,...t_{n-1},t_{n+1},...)$ be the solution of the dispersionless Lax hierarchy which was constructed above. Then the superpotential of the perturbed A_{n-1} topological minimal model is equal to*

$$W(p,t_0,...t_{n-2}) = \frac{1}{n}E(p,t_0,\frac{t_1}{2},...,\frac{t_{n-2}}{n-1},\frac{1}{n+1},0,0,...). \tag{6.8}$$

The partition function of this model is equal to

$$F = F(t_0,\frac{t_1}{2},...,\frac{t_{n-2}}{n-1},\frac{1}{n+1},0,0,...), \tag{6.9}$$

where $F(t_1,t_2,....)$ is given by the formula (5.21).

By definition, the perturbed A_{n-1} model depends on $n-1$ variables. As it follows from the above-formulated theorem, after the natural extension as a function of an infinite number of "times" the partition function of this model satisfies the constraints (5.24-5.26). Now we are going to consider the relations of these constraints with the loop equations for matrix models.

7. LOOP EQUATIONS

The loop equations have been originally proposed for Yang-Mills theories. (The review of the latest works on the applications of the loop equations to matrix models and $2d$ quantum gravity can be found in [37].)

The hermitean matrix model is defined by the partition function

$$Z_N = \int DM \exp(-\mathrm{Tr}(V(M)), \tag{7.1}$$

where M is $N \times N$ hermitean matrix,

$$V(K) = \sum_{i=0}^{\infty} \tilde{t}_i K^i. \tag{7.2}$$

The Wilson loop corollator is by definition

$$\mathcal{W}(K) = \langle \mathrm{tr} \frac{1}{K-M} \rangle = -\sum_{i=0}^{\infty} K^{-i-1} \partial_i \log Z_N. \tag{7.3}$$

The loop equations are derived from the invariance of the integral (7.1) with respect to the infinitesimal shift of M and have the form

$$[\sum_{i=1}^{\infty} i\tilde{t}_i K^{i-1}) \mathcal{W}(K)]_- = \mathcal{W}^2(K) + \frac{\delta}{\delta V} \mathcal{W}(K). \tag{7.4}$$

$[\ldots]_-$ denotes the negative part of Laurent series.The equation (7.4) has to be supplemented with the condition

$$\mathcal{W}(K) = \frac{N}{K} + O(K^{-2}). \tag{7.5}$$

The leading term of $1/N^{-2}$ expansion of a solution of (7.4) should be the solution of the truncated equation

$$(\sum_{i=1}^{\infty} i\tilde{t}_i K^{i-1}) \mathcal{W}_0(K) = \mathcal{W}_0^2(K). \tag{7.6}$$

Below we consider only the "even" case $\tilde{t}_{2i+1} = 0$.

Let us consider the solution of the dispersionless KdV equation (the $n = 2$ case of the dispersionless Lax equations) which was constructed in the second section. If $F(t_1, t_3, ..., t_{2i+1}, ...)$ is defined by (5.19), the constraints (5.24-5.26) are equivalent to the equation

$$[(\sum_{i=1}^{\infty} (2i+1) t_{2i+1} K^{2i-1})(t_1 K^{-1} + \sum_{j=1}^{\infty} (\partial_{2j-1} F) K^{-2j-1})]_- +$$

$$\frac{1}{2}(t_1 K^{-1} + \sum_{j=1}^{\infty} \partial_{2j-1} F K^{-2j-1}) = 0. \tag{7.7}$$

From (5.21), it follows that

$$\mathcal{W}_0 = -\frac{d}{dE}(t_1 K + S_-(K)), \tag{7.8}$$

where

$$S_- = S - S_+. \tag{7.9}$$

Therefore, if we define

$$\tilde{t}_{2i} = \frac{2i+1}{2i} t_{2i+1}, \quad N = -\frac{1}{2} t_1, \tag{7.10}$$

then the solution of the dispersionless KdV hierarchy gives the solution of (7.6) with the condition (7.5) with the help of the formula (7.8). This solution coincides with, so-called,"one-cut" solution [37]. At the end of this chapter we shall show that the Whitham equations (or more exactly, their special solutions) on the moduli space of hyperelliptic curves give "multi-cut" solutions of the equation (7.6).

Let us consider the Whitham equations for the genus g algebraic-geometrical solutions of the Lax equations (4.18). They are equations on the moduli space $M_g(n)$.

In full analogy with the genus zero case their particular solutions can be obtained with the help of the following defining relations

$$\frac{dS_+}{dE} \text{ is regular function on } \Gamma \text{ outside } P_1, \tag{7.11}$$

where dS_+ is defined with the help of formula (5.11) (the only difference with dispersionless case is the difference in the definition of the differentials $d\Omega_i$.

The relations (7.11) are equivalent to the equations

$$dS_+(q_s) = 0, \tag{7.12}$$

where q_s are zeros of the differential $dE(q_S) = 0$.

Example. $n = 2$ In this case the space $M_g(2)$ is the space of sets of distinct points $E_1, ..., E_{2g+1}$. The corresponding curves are hyperelliptic curves which are defined by the equation

$$y^2 = \prod_{i=1}^{2g+1} (E - E_i) = R(E). \tag{7.13}$$

The differentials $d\Omega_i$ has a'priori the form:

$$d\Omega_{2i+1} = \frac{Q_i(E)}{\sqrt{R(E)}} dE = \frac{2i+1}{2} \frac{E^{g+i} + ...}{\sqrt{R}} dE. \tag{7.14}$$

The coefficients of the polynomial $Q_i(E)$ are uniquely defined from the normalizing conditions for $d\Omega_i$, which are equivalent to a system of linear equations. They become the functions of E_i and can be expressed through complete hyperelliptic integrals. Therefore, the polynomial Q_i is also a function of the variables E_i, i.e.

$$Q_i(E) = Q_i(E|E_1, ...E_{2g+1}). \tag{7.15}$$

The defining relations (7.12) are equivalent to a set of non-linear transcendent (but not differential) equations

$$\sum_{i=1}^{\infty} t_i Q_i(E_m|E_1, E_2, ..., E_{2g+1}) = 0, m = 1, 2, ..., 2g + 1. \tag{7.16}$$

They define E_m as functions of t_i, which are the solution of the Whitham equations on $M_g(n)$.

Let $F(t_1, ...)$ be the function given by the same formula (5.19) where S_+ corresponds to the solution of the Whitham equations on $M_g(n)$. Then all the relations (5.21-5.23) and the constraints (5.24-5.26) would be fulfilled as well. Hence, for $n = 2$, the formulae (7.8,7.9) after redefinition of "times" (7.10) provides "multi-cut solutions of the equation (7.6).

Important remark. These solutions are not analytical functions of t_i, because the normalizing conditions for Ω_i are real equations. Locally, analytical solutions of (7.6) can be obtained if we consider the Whitham-type equations on the Teichmüller space, which covers $M_g(n)$. Corresponding equations have the same form (4.20), but now normalizing conditions which define Ω_i, should be chosen in the form:

$$\oint_{a_i} d\Omega_i = 0, \tag{7.17}$$

where a_i, b_i is a canonical basis of cycles on Γ. After that all the previous statements will be fulfilled.

REFERENCES

1. E. Brézin, V. Kazakov, *Phys. Lett.* (1990),**B236**, 144.

2. M. Douglas, S. Shenker, *Nucl. Phys.* (1990) **B335**, 635.

3. D. J. Gross, A. Migdal, *Phys. Rev.* (1990) **64**, 127.

4. D. J. Gross, A Migdal, *Nucl. Phys.* (1990) **B340**, 333.

5. T. Banks, M. Douglas, N. Seiberg, S. Shenker, *Phys. Lett.* (1990) **238B**, 279.

6. M. Douglas , *Phys. Lett.* **238B** (1 B.Dubrovin, I. Krichever, S. Novikov, *Soviet Doklady* **229**, No 1 (1976), 15.

7. E. Witten, *Nucl. Phys.* **B340** (1990), 176.

8. R. Dijkgraaf, E. Witten, *Nucl. Phys.* **B342** (1990), 486.

9. J. Distler, *Nucl. Phys.* **B342** (1990), 523.

10. M. Fukuma, H. Kawai, Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity, preprint Tokyo University UT-562, May 1990.

11. M. Fukuma, H. Kawai, Infinite dimensional Grassmanian structure of two-dimensional quantum gravity, preprint Tokyo University UT-572, November 1990.

12. I. Krichever, *Soviet Doklady* **227:2** (1976), 291.

13. I. Krichiver, *Funk. Anal. i Pril.* **11**, No 13 (1977), 15.

14. B. Dobrovin, V. Matveev, S. Novikov, *Uspekhi Mat. Nauk.]* **31**, No 1 (1976), 55–136.

15. V. Zakharov, S. Manakov, S. Novikov, L. Pitaevski, Soliton theory, Moscow, Nauka, 1980.

16. B. Dubrovin, *Uspekhi Mat. Nauk* **36**, No 2 (1981),11-80.

17. I.Krichever, S. Novikov, *Uspekhi Mat. Nauk*, **35**, No 6 (1980).

18. I. Krichever, *Uspekhi Mat. Nauk*, **44**, No 2 (1989), 121.

19. B. Dubrovin, I .Krichever, S. Novikov, Integrable systems, *VINITY AN USSR*, 1985.

20. I.Krichever, *Uspekhi Mat. Nauk* **32**, No 6 (1977), 180.

21. I. Krichever, Topological minimal models and dispersionless Lax equations, preprint ISI Turin (to appear in *Comm. Math. Phys.* (1991).

22. E. Verlinder, H. Verlinder, A solution of two-dimensional topological quantum gravity, preprint IASSNS-HEP-90/40, PUPT-1176 (1990).

23. M. Sato,Y. Sato Soliton equations as dynamical systems in an infinite dimensional Grassmann manifolds, in *Nonlinear Partial Differential equations in Applied Sciences* (North-Holland, Amsterdam, 1982).

24. E.Data, M. Kashivara, M. Jimbo, T. Miva Transformation groups for soliton equations, in *Nonlinear Integrable systems - Classical Theory and Quantum Theory* (World Scientific, Singapore, 1983).

25. B.Dubrovin, I. Krichever, *S. Novikov, Soviet Doklady* **229**, No 1 (1976), 15.

26. A. Gurevich, L. Pitaevskii, *JETP* **65**, No 3 (1973), 590.

27. H. Flashka, M. Forest, L.McLaughlin, *Comm. Pure and Appl. Math.* **33** (6).

28. S. Yu. Dobrokhotov, V. P. Maslov, *Soviet Scientific Reviews, Math. Phys. Rev.* OPA Amsterdam 3 (1982), 221-280.

29. V. E. Zakhavor, *Funk. Anal. i Pril.* **14** (1980), 89.

30. Y. Kodama, J. Gibbons, *Phys. Lett.* 135A (1989), 171.

31. I. Krichever, *Funk. Anal. i Pril.* **22(3)** (1988), 37–52.

32. T Eguchi, S.-K. Yang, $N = 2$ superconformal models as topological field theoreies, preprint of Tokyo University UT-564 (1990).

33. K.Li, Topological gravity with minimal matter , Caltech-preprint CALT-68-1662.

34. E. Martinec, *Phys. Lett.* **217B** (1989), 431.

35. C. Vafa, N. Warner, *Phys. Lett.* **218B** (1989), 51.

36. W. Lerche, C. Vafa, N.P. Warner, *Nucl. Phys.* **B324** (1989), 427.

37. Yu. Makeeno, Loop equations in matrix models and in 2D quantum gravity, Submitted to *Mod. Phys. Lett.* A.