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Algebraic–Geometrical Methods in the Theory of Integrable Equations and Their Perturbations

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1. Introduction

The year of 1995 is not merely the centenary of the Korteweg–de Vries equation which we celebrate at this conference. It is also the year of the 'majority' of the finite-gap or algebraic–geometrical theory of integration of nonlinear equations – one of the most important components of the branch of modern mathematical physics, which is called the theory of integrable systems or the soliton theory. The main goal of this paper is to present the key points of the finite-gap theory and some of its applications. Part of its applications is directly related to the KdV equation, while a part of them lies beyond the framework of not only this particular equation but the theory of soliton equations in general. Corresponding examples refer to the string theory and topological field theory models. Not aspiring to be exhaustive, they manifest versatility of the methods, the origin of which would be forever related to the magic words: *Korteweg–de Vries Equation.*

Same as in the rapidly decreasing case, primerely the program of constructing periodic solutions of KdV equation was completely based on the spectral theory of the Sturm–Liouville operator – one of the so-called Lax operators for the KdV equation.

But unlike the rapidly decreasing case in which the efficiency of direct and inverse spectral transform was sufficient both for constructing multi-soliton solutions and for solving the Cauchy problem in general, in the periodic case the level of efficiency of the corresponding spectral problems was far from sufficient.

The creation of the effective spectral theory of finite-gap Sturm–Liouville operators proposed in the cycle of papers by S. P. Novikov, B. A. Dubrovin, V. B. Matveev and A. R. Its (see their reviews in [1, 2]; part of the corresponding results was obtained a bit later in [3, 4]), made it possible to construct not only a broad class of periodic and quasi-periodic solutions of the KdV equation. It also brought about the reinterpretation of the approach to the spectral theory of ordinary periodic linear differential operators, as whole.

The statement that the Bloch solutions of such operators considered for the arbitrary complex values of spectral parameter E, form a single-valued function on a Riemann surface, quite obvious nowadays, had remained outside the framework of the classical Flouque theory. It turned out that analytical properties of the Bloch functions on this surface are crucial for the solution of the inverse problem of reconstructing the coefficients of operators. In the case when this surface has a finite genus, the solution of the inverse problem is based on the methods of classical algebraic geometry and the theory of theta functions.

The significance of the algebraic–geometrical approach was completely revealed in [5, 6], where the general algebraic–geometrical construction of periodic solutions of two-dimensional soliton equations of the Kadomtsev–Petviashvili type was proposed. This construction is based on the concept of the Baker–Akhiezer function $\psi(x,y,t,Q)$, which is uniquely determined by their analytical properties on the auxiliary Riemann surface Γ, $Q \in \Gamma$. The corresponding analytical properties naturally generalize the analytical properties of the Bloch functions of ordinary linear periodic differential operators. Their peculiarity is that for any function possessing these properties there always exist differential operators L and A of the form

$$L = \sum_{i=0}^{n} u_i(x,y,t)\partial_x^i, \qquad A = \sum_{j=0}^{n} v_j(x,y,t)\partial_x^j, \quad \partial_x = \frac{\partial}{\partial x}. \tag{1.1}$$

such that the Baker–Akhiezer function is the common solution of the linear equations

$$(\partial_y - L)\psi(x,y,t,Q) = 0, \qquad (\partial_t - A)\psi(x,y,t,Q) = 0. \tag{1.2}$$

The compatibility conditions of the overdetermined system of the linear problems (1.2) imply the operator equation

$$[\partial_y - L, \partial_t - A] = 0 \longleftrightarrow \partial_t L - \partial_y A + [L, A] = 0 \tag{1.3}$$

that is equivalent to the system of nonlinear partial differential equations on the coefficients of the operators L and A.

For example, if L and A have the form

$$L = \sigma^{-1}(\partial_x^2 - u(x,y,t)), \qquad A = \partial_x^3 - \tfrac{3}{2}\,u\partial_x + w(x,y,t), \tag{1.4}$$

then (1.3) is equivalent to the KP equation

$$\tfrac{3}{4}\,\sigma^2 u_{yy} = \left(u_t - \tfrac{3}{2}\,uu_x + \tfrac{1}{4}\,u_{xxx}\right)_x. \tag{1.5}$$

If $u = u(x,t)$ is independent of the second variable y and $w = \frac{3}{4}u_x(x,t)$, then (1.3) is equivalent to the KdV equation

$$u_t - \tfrac{3}{2}\,uu_x + \tfrac{1}{4}\,u_{xxx} = 0. \tag{1.6}$$

We present the general algebraic–geometrical construction of exact solutions of the soliton equations in Section 2. It should be specially emphasized that this construction is purely local and is a sort of inverse transform: from a set of algebraic–geometrical data to solutions of the integrable nonlinear partial differential equations

$$\{\text{algebraic–geometrical data}\} \longmapsto \{\text{solutions of NLPDE}\} \tag{1.7}$$

In a generic case, the space of algebraic–geometrical data is a union for all g of the spaces

$$\widetilde{M}_{g,N} = \{\Gamma_g, P_\alpha, k_\alpha^{-1}(Q), \gamma_1, \dots, \gamma_g\}, \quad \alpha = 1, \dots, N, \tag{1.8}$$

where Γ_g is an algebraic curve of genus g with fixed local coordinates $k_\alpha^{-1}(Q)$, $k_\alpha^{-1}(P_\alpha) = 0$, in neighborhoods of N punctures P_α, and $\gamma_1, \dots, \gamma_g$ are points of Γ_g in a general position. (It is to be mentioned that $\widetilde{M}_{g,N}$ are 'universal' data. For the given nonlinear integrable equation, the corresponding subset of data has to be specified.)

A posteriory, it can be shown that these solutions can be expressed in terms of the corresponding Riemann theta functions and are quasi-periodic functions of all variables. Within this approach, it is absolutely impossible to give an answer to the basic questions: "How many algebraic–geometrical solutions are there? And what is their role in the solution of the periodic Cauchy problem for two-dimensional equations of the KP type?"

The answer to the corresponding question in lower dimensions is as follows. For finite-dimensional $(0 + 1)$-systems, a typical Lax representation has the form

$$\partial_t U(t, \lambda) = [U(t, \lambda),\ V(t, \lambda)], \tag{1.9}$$

where $U(t, \lambda)$ and $V(t, \lambda)$ are matrix functions that are rational (or sometimes elliptic) functions of the spectral parameter λ. In that case, *all* the general solutions are algebraic–geometrical and can be represented in terms of Riemann theta functions.

For special one-dimensional evolution equations of the KdV type ($(1 + 1)$-systems), the existence of direct and inverse spectral transform allow one to prove (though it is not always the rigorous mathematical statement) that algebraic–geometrical solutions are dense in the space of all periodic (in x) solutions.

It turns out that the situation for two-dimensional integrable equations is much more complicated. For one of the real forms of the KP equation that is called the KP-2 equation and which corresponds to $\sigma = 1$ in (1.5), the algebraic–geometrical solutions are dense in the space of all periodic (in x and y) solutions [7]. It seems, that the same statement for the KP-1 equation ($\sigma = i$) is wrong. One of the most important problems in the theory of two-dimensional integrable systems which is still unsolved is 'in what sense' the KP-1 equation that has the operator representation (1.3) and for which a wide class of periodic solution has been constructed, is an '*nonintegrable*' system.

The proof of the integrability of the periodic problem for the KP-2 equation is based on the spectral Floque theory of the parabolic operator

$$M = \partial_y - \partial_x^2 + u(x,y), \tag{1.10}$$

with periodic potential $u(x+l_1,y) = u(x,y+l_2) = u(x,y)$. This theory is presented in Section 3. It is a natural generalization of the spectral theory of the periodic Sturm–Liouville operator. We would like to mention that despite its application to the theory of nonlinear equations and related topics, the structure of the Riemann surface of Bloch solutions of the corresponding linear equation that was found in [7] has been used as a starting point for the abstract definition of the Riemann surfaces of the infinite genus [9].

In the last section, we present the algebraic–geometrical perturbation theory of soliton equations and its application to the topological models of quantum field theory.

2. The Baker–Akhiezer Functions. General Scheme

Let Γ be a nonsingular algebraic curve of genus g with N punctures P_α and fixed local parameters $k_\alpha^{-1}(Q)$ in neighborhoods of the punctures. For any set of the points $\gamma_1,\dots,\gamma_g$ in a general position, there exists a unique (up to constant factor $c(t_{\alpha,i})$) function $\psi(t,Q)$, $t=(t_{\alpha,i})$, $\alpha=1,\dots,N$; $i=1,\dots,$ such that:

(i) the function ψ (as a function of the variable Q which is a point of Γ) is meromorphic everywhere except for the points P_α and has at most simple poles at the points $\gamma_1,\dots,\gamma_g$ (if all of them are distinct);

(ii) at the neighborhood of the point P_α the function ψ has the form

$$\psi(t,Q) = \exp\left(\sum_{i=1}^{\infty} t_{\alpha,i}k_\alpha^i\right)\left(\sum_{s=0}^{\infty}\xi_{s,\alpha}(t)k_\alpha^{-s}\right), \quad k_\alpha = k_\alpha(Q). \tag{2.1}$$

This is the most general definition of a scalar *multi-puncture* and *multi-variable* Baker–Akhiezer function. It depends on the variables $t=\{t_{1,i},\dots,t_{N,i}\}$ as on external parameters.

From the uniqueness of the Baker–Akhiezer function, it follows that for each pair (α,n) there exists a unique operator $L_{\alpha,n}$ of the form

$$L_{\alpha,n} = \partial_{\alpha,1}^n + \sum_{j=1}^{n-1} u_j^{(\alpha,n)}(t)\partial_{\alpha,1}^j \tag{2.2}$$

(where $\partial_{\alpha,i} = \partial/\partial t_{\alpha,i}$) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = 0. \tag{2.3}$$

The idea of the proof of theorems of this type which was proposed in [5, 6] is universal.

For any formal series of the form (2.1), their exists a unique operator $L_{\alpha,n}$ of the form (2.2) such that

$$(\partial_{\alpha,i} - L_{\alpha,n})\psi(t,Q) = O(k^{-1}) \exp\left(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\right). \tag{2.4}$$

The coefficients of $L_{\alpha,n}$ are differential polynomials with respect to $\xi_{s,\alpha}$. They can be found after substitution of the series (2.1) into (2.4).

It turns out that if the series (2.1) is not formal but is an expansion of the Baker–Akhiezer function in the neighborhood of P_α, the congruence (2.4) becomes an equality. Indeed, let us consider the function

$$\psi_1 = (\partial_{\alpha,n} - L_{\alpha,n})\psi(t,Q). \tag{2.5}$$

It has the same analytical properties as ψ except for one. The expansion of this function in the neighborhood of P_α starts from $O(k^{-1})$. From the uniqueness of the Baker–Akhiezer function, it follows that $\psi_1 = 0$ and the equality (2.3) is proved.

COROLLARY 2.1. *The operators $L_{\alpha,n}$ satisfy the compatibility conditions*

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\alpha,m} - L_{\alpha,m}] = 0. \tag{2.6}$$

Remark. Equations (2.6) are gauge invariant. For any function $g(t)$ operators

$$\widetilde{L}_{\alpha,n} = gL_{\alpha,n}g^{-1} + (\partial_{\alpha,n}g)g^{-1} \tag{2.7}$$

have the same form (2.2) and satisfy the same operator equations (2.6). The gauge transformation (2.7) corresponds to the gauge transformation of the Baker–Akhiezer function

$$\psi_1(t,Q) = g(t)\psi(t,Q). \tag{2.8}$$

EXAMPLE (one-puncture Baker–Akhiezer function). In the one-puncture case, the Baker–Akhiezer function has an exponential singularity at a single point P_1 and depends on a single set of variables. Let us choose the normalization of the Baker–Akhiezer function with the help of the condition $\xi_{1,0} = 1$, i.e. an expansion of ψ in the neighborhood of P_1 equals

$$\psi(t_1, t_2, \ldots, Q) = \exp\left(\sum_{i=1}^{\infty} t_i k^i\right)\left(1 + \sum_{s=1}^{\infty} \xi_s(t) k^{-s}\right). \tag{2.9}$$

In this case, operator L_n has the form

$$L_n = \partial_1^n + \sum_{i=o}^{n-2} u_i^{(n)} \partial_1^i. \tag{2.10}$$

For example, for $n = 2, 3$ after redefinition $x = t_1$ we have $L_2 = \sigma L,\ L_3 = A$, where L and A are differential operators (1.4) and

$$u(x, t_2, \ldots) = 2\partial_x \xi_1(x, t_2, \ldots), \tag{2.11}$$

Therefore, if we define $y = \sigma^{-1} t_2$, $t = t_3$, then $u(x, y, t, t_4, \ldots)$ satisfies the KP Equation (1.5).

It should be emphasized that a algebraic–geometrical construction is not a sort of abstract 'existence' and 'uniqueness' theorems. It provides the exact formulae for solutions in terms of the Riemann theta functions. They are the corollary of the exact formula for the Baker–Akhiezer function

$$\psi(t, Q) = \exp\left(\sum_{i,\alpha} t_{i,\alpha} \Omega_{i,\alpha}(P)\right) \frac{\theta(A(P) + \sum_{i,\alpha} U_{i,\alpha} t_{i,\alpha} + Z)}{\theta(A(P) + Z)}, \tag{2.12}$$

Here:

$\theta(z) = \theta(z \mid B)$ is the Riemann theta function – the entire function of g complex variables $z = (z_1, \ldots, z_g)$ that is defined by the matrix B of b-periods of normalized holomorphic differentials $\mathrm{d}\omega_i$, $i = 1, \ldots, g$, on Γ:

$$\theta(z_1, \ldots, z_g) = \sum_{m \in Z^g} \mathrm{e}^{2\pi i(m,z) + \pi i(Bm,m)}; \tag{2.13}$$

$\Omega_{i,\alpha}(P)$ is an Abelian integral

$$\Omega_{i,\alpha}(P) = \int^P \mathrm{d}\Omega_{i,\alpha}, \tag{2.14}$$

corresponding to the unique normalized

$$\oint_{a_k} \mathrm{d}\Omega_{i,\alpha} = 0, \tag{2.15}$$

meromorphic differential on Γ with the only pole of the form

$$\mathrm{d}\Omega_{i,\alpha} = \mathrm{d}k_\alpha^i (1 + \mathrm{O}(k_\alpha^{-i-1})) \tag{2.16}$$

at the puncture P_α;

$2\pi i U_{j,\alpha}$ is the vector of b-periods of the differential $\mathrm{d}\Omega_{j,\alpha}$

$$U_{j,\alpha}^k = \frac{1}{2\pi i} \oint_{b_k} \mathrm{d}\Omega_{j,\alpha}; \tag{2.17}$$

$A(P)$ is a vector with coordinates

$$A(P) = \int^{P} \mathrm{d}\omega_k \tag{2.18}$$

(formula (2.18) is called the Abel transformation);

Z is an arbitrary vector (it corresponds to the divisor of poles of the Baker–Akhiezer function).

It follows from (2.11) that in order to get the solution of the KP equation, it is enough to take the derivative of the first coefficient of the expansion at the puncture of the ratio of theta functions in the formula (2.12). The final formula for the algebraic–geometrical solutions of the KP hierarchy has the form

$$u(t_1, t_2, \ldots) = 2\partial_1^2 \ln \theta\left(\sum_{i=1}^{\infty} U_i t_i + Z\right) + \mathrm{const}, \tag{2.19}$$

(see details in [6]).

Formula (2.19) that was derived in [6], has lead to one of the most important pure mathematical applications of the theory of nonlinear integrable systems. This is the solution of the famous Riemann–Shottky problem.

According to the Torrelli theorem, the matrix of b-periods of normalized holomorphic differentials uniquely defines the corresponding algebraic curve. The Riemann–Shottky problem is to describe symmetrical matrices with the positive imaginary part that are the matrices of b-periods of normalized holomorphic differentials on algebraic curves. Novikov conjected that the function

$$u(x, y, t) = 2\partial_1^2 \ln \theta(Ux + Vy + Wt + Z \mid B) \tag{2.20}$$

is a solution of the KP equation iff the matrix B that defines the theta function is the matrix of b-periods of normalized holomorphic differentials on an algebraic curve and U, V, W are vectors of the b-periods of corresponding normalized meromorphic differentials with the only pole at a point of this curve. This conjecture was proved in [10].

Equations (2.6) for $n = 2$, $m > 3$ describe evolutions of $u(x, y, t, t_4, \ldots)$ with respect to 'higher times' or, equivalently, the whole KP hierarchy. Here it is necessary to make a few comments. In the original form, Equations (2.6) are a set of nonlinear equations on the coefficients $u_i^{(n)}$ and do not have the form of evolution equations. It can be shown (see [11]) that they are equivalent to the evolution system in the form which was proposed by the Kyoto group [12]. We show this equivalence for algebraic–geometrical solutions.

For any formal series of the form (2.9), there exists a unique pseudo-differential operator $\mathcal{L}$

$$\mathcal{L} = \partial_1 + \sum_{i=1}^{\infty} u_i(t_1, \ldots)\partial_1^{-i}, \tag{2.21}$$

such that

$$\mathcal{L}\psi(t,k) = k\psi(t,k). \tag{2.22}$$

Then the operators L_n which are uniquely defined from the congruence (2.4) are equal to

$$L_n = [\mathcal{L}^n]_+, \tag{2.23}$$

where $[\ldots]_+$ stands for the differential part of the pseudo-differential operator. From (2.3), it follows that if $\psi(t,k)$ is an expansion of the Baker–Akhiezer function, then

$$(\partial_n - [\mathcal{L}^n]_+)\psi(t,Q) = 0. \tag{2.24}$$

The compatibility conditions of (2.22) and (2.24) imply the evolution equations

$$\partial_n \mathcal{L} = [[\mathcal{L}^n]_+, \mathcal{L}] \tag{2.25}$$

on the coefficients $u_i(t_1,\ldots)$ of $\mathcal{L}$. The Equations (2.25) are the Sato form for the KP hierarchy.

At the end of this section, we shall make a few comments about the multi-puncture case. For each α Equations (2.6), up to the gauge transformation, are equivalent to the KP hierarchy corresponding to each set of variables $\{t_{\alpha,i}\}$. What is the interaction between two different KP hierarchies?

As was found in [13], for the two-puncture case, a full set of equations can be represented in the following form

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\beta,n} - L_{\beta,n}] = \mathrm{D}^{\alpha,\beta}_{N,m} H^{\alpha,\beta}, \tag{2.26}$$

where $H^{\alpha,\beta}$ is the two-dimensional Schrödinger operator in a magnetic field

$$H^{\alpha,\beta} = \frac{\partial^2}{\partial_{\alpha,1}\partial_{\beta,1}} + v_1^{\alpha,\beta}\partial_{\alpha,1} + v_2^{\alpha,\beta}\partial_{\alpha,2} + u^{\alpha,\beta} \tag{2.27}$$

and operators $\mathrm{D}^{\alpha,\beta}_{N,m}$ are differential operators in the variables $t_{\alpha,1}, t_{\beta,1}$.

The sense of (2.26) is as follows. For the given operator $H^{\alpha,\beta}$, any differential operator D in the variables $t_{\alpha,1}$, $t_{\beta,1}$ can be uniquely represented in the form

$$\mathrm{D} = \mathrm{D}_1 H^{\alpha,\beta} + \mathrm{D}_2 + \mathrm{D}_3, \tag{2.28}$$

where D_2 is a differential operator with respect to the variable $t_{\alpha,1}$ only and D_3 is a differential operator with respect to the variable $t_{\beta,1}$ only. Equations (2.26) imply that the second and third terms in the corresponding representation for the left-hand side of (2.26) are equal to zero. This implies $n+m-1$ equations on $n+m$ unknown functions (the coefficients of operators $L_{\alpha,n}$ and $L_{\beta,m}$). Equations

(2.26) are gauge invariant. That's why the number of equations equals a number of unknown functions. Therefore, the operator equations (2.26) are equivalent to the well-defined system of nonlinear partial differential equations.

3. Spectral Theory of Two-Dimensional Periodic Operators

As was mentioned in the introduction, nowadays the origin of the Riemann surfaces in the spectral theory of periodic ordinary linear differential operators looks more or less obvious.

Indeed, let L be an ordinary differential operator

$$L = \sum_{i=0} u_i(x)\partial_x^i \tag{3.1}$$

with periodic coefficients $u_i(x+l) = u_i(x)$ (that are scalar or matrix functions). Then the monodromy operator

$$\widehat{T}\colon\ y(x) \longmapsto y(x+l) \tag{3.2}$$

induces the finite-dimensional linear operator

$$\widehat{T}(E)\colon\ \mathcal{L}(E) \longmapsto \mathcal{L}(E) \tag{3.3}$$

on the space $\mathcal{L}(E)$ of solutions of the ordinary differential equation

$$Ly(x) = Ey(x), \tag{3.4}$$

where E is a complex spectral parameter.

The characteristic equation

$$R(w, E) = \det(w - \widehat{T}(E)) \tag{3.5}$$

defines the Riemann surface Γ of the Bloch solutions of Equation (3.4), i.e. the solutions $\psi(x, Q)$ of (3.4) that are eigenfunctions of the monodromy operator

$$\psi(x+l, Q) = w\psi(x, Q), \tag{3.6}$$

Here Q is a pair $Q = (w, E)$ such that (3.5) is fulfilled, i.e. a point of the corresponding Riemann surface $Q \in \Gamma$.

Equation (3.5) represents the Riemann surface of the Bloch solutions as an N-sheet covering of the complex plane of the spectral parameter E. In [7] another representation of the Riemann surface of the Bloch solutions was proposed.

Let us consider as an example a nonstationary Schrödinger operator (1.10). The solutions $\psi(x, y, w_1, w_2)$ of the nonstationary Schrödinger equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\psi(x, y, w_1, w_2) = 0 \tag{3.7}$$

with a periodic potential $u(x, y) = u(x + a_1, y) = u(x, y + a_2)$ are called the Bloch solutions, if they are eigenfunctions of the monodromy operators, i.e.

$$\psi(x + a_1, y, w_1, w_2) = w_1\psi(x, y, w_1, w_2), \tag{3.8}$$

$$\psi(x, y+a_2, w_1, w_2) = w_2\psi(x, y, w_1, w_2). \tag{3.9}$$

The Bloch functions will always be assumed to be normalized so that $\psi(0, 0, w_1, w_2) = 1$. The set of pairs $Q = (w_1, w_2)$, for which there exists such a solution is called the Floque set and will be denoted by Γ. The multivalued functions $p(Q)$ and $E(Q)$ such that

$$w_1 = \mathrm{e}^{ipa_1}, \qquad w_2 = \mathrm{e}^{iEa_2},$$

are called quasi-momentum and quasi-energy, respectively.

The gauge transformation $\psi \to \mathrm{e}^{h(y)}\psi$, where $\partial_y h(y)$ is a periodic function, transfers the solutions of (3.7) into solutions of the same equation but with another potential $\widetilde{u} = u - \sigma\partial_y h$. Consequently, the spectral sets corresponding to the potentials u and $\widetilde{u}$ are isomorphic. Therefore, in what follows we restrict ourselves to the case of periodic potentials such that

$$\int_0^{a_1} u(x, y)\,\mathrm{d}x = 0. \tag{3.10}$$

To begin with, let us consider as a basic example the 'free' operator

$$M_0 = \sigma\partial_y - \partial_x^2 \tag{3.11}$$

with zero potential $u(x, y) = 0$. The Floque set of this operator is parametrized by the points of the complex plane of the variable k

$$w_1^0 = \mathrm{e}^{ika_1}, \qquad w_2^0 = \mathrm{e}^{-\sigma^{-1}k^2a_2} \tag{3.12}$$

and the Bloch solutions have the form

$$\psi(x, y, k) = \mathrm{e}^{ikx-\sigma^{-1}k^2y}. \tag{3.13}$$

The functions

$$\psi^+(x, y, k) = \mathrm{e}^{-ikx+\sigma^{-1}k^2y} \tag{3.14}$$

are Bloch solutions of the formal ajoint operator

$$(\sigma\partial_y + \partial_x^2)\psi^+ = 0. \tag{3.15}$$

Formulae (3.12) define the map

$$k \in C \longmapsto (w_1^0, w_2^0) \in C^2. \tag{3.16}$$

Its image is the Floque set for the free operator M_0. It is the Riemann surface with self-intersections that correspond to the pairs $k \neq k'$ such that

$$w_i^0(k) = w_i^0(k'), \quad i = 1, 2. \tag{3.17}$$

From (3.12) it follows that

$$k - k' = \frac{2\pi N}{a_1}, \tag{3.18}$$

$$k^2 - (k')^2 = \frac{\sigma 2\pi i M}{a_2}, \tag{3.19}$$

where N and M are integers. Hence, all the resonant points have the form

$$k = k_{N,M} = \frac{\pi N}{a_1} - \frac{\sigma i M a_1}{N a_2}, \quad N \neq 0, \ k' = k_{-N,-M}. \tag{3.20}$$

The basic idea of the construction of the Riemann surface of Bloch solutions of Equation (3.7) that was proposed in [7] is to consider (3.7) as a perturbation of the free operator (3.7), assuming that the potential $u(x, y)$ is formally small.

For any $k_0 \neq k_{N,M}$ it is easy to construct a formal Bloch solution $\widetilde{\psi}$ of the Equation (3.7) as a formal series

$$\widetilde{\psi} = \sum_{s=0}^{\infty} \widetilde{\phi}_s(x, y, k_0), \quad \widetilde{\phi}_0(x, y, k_0) = \psi(x, y, k_0) = \psi_0. \tag{3.21}$$

This series describes a 'perturbation' of the Bloch solution ψ_0 of the nonperturbed equation.

LEMMA 3.1. *If $k_0 \neq k_{N,M}$, then there exists a unique formal series*

$$F(y, k_0) = \sum_{s=1}^{\infty} F_s(y, k_0) \tag{3.22}$$

such that the equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\Psi(x, y, k_0) = F(y, k_0)\Psi(x, y, k_0) \tag{3.23}$$

has a formal solution of the form

$$\Psi(x, y, k_0) = \sum_{s=0}^{\infty} \phi_s(x, y, , k_0), \quad \phi_0 = \psi_0, \tag{3.24}$$

satisfying the conditions

$$\langle \psi_0^+ \Psi \rangle_x = \langle \psi_0^+ \psi_0 \rangle_x, \qquad \psi_0^+ = \psi^+(x, y, k_0), \tag{3.25}$$

(here and below $\langle f(x) \rangle_x$ stands for the mean value in x of the corresponding periodic function f)

$$\Psi(x + a_1, y, k_0) = w_{10}\Psi(x, y, k_0), \quad w_{10} = w_1^0(k_0), \tag{3.26}$$

$$\Psi(x, y + a_2, k_0) = w_{20}\Psi(x, y, k_0), \quad w_{20} = w_2^0(k_0). \tag{3.27}$$

The corresponding solution is unique and is given by the recursion formulae (3.28)–(3.32).

$$\phi_s = \sum_{n \neq 0} c_n^s(y, k_0)\psi_n(x, y), \quad s > 1, \tag{3.28}$$

$$\psi_n = \psi_n(x, y) = \psi(x, y, k_n),$$

$$\psi_n^+ = \psi^+(x, y, k_n), \qquad k_n = k_0 + \frac{2\pi n}{a_1}, \tag{3.29}$$

$$\begin{aligned} c_n^s(y, k_0) &= \sigma^{-1}\frac{w_{2n}}{w_{20} - w_{2n}} \times \\ &\times \int_y^{y+a_2} \left(\sum_{i=1}^{s-1} F_i c_n^{s-i} - \frac{\langle \psi_n^+ u \phi_{s-1} \rangle_x}{\langle \psi_n^+ \psi_n \rangle_x} \right) \mathrm{d}y', \end{aligned} \tag{3.30}$$

$$w_{2n} = w_2^0(k_n), \tag{3.31}$$

$$F_s(y, k_0) = \frac{\langle \psi_0^+ u \phi_{s-1} \rangle_x}{\langle \psi_0^+ \psi_0 \rangle_x}. \tag{3.32}$$

From (3.23), (3.26), (3.27), it follows that the formula

$$\widetilde{\psi}(x, y, k_0) = \frac{\Psi(x, y, k_0)}{\Psi(0, 0, k_0)} \exp\left\{ -\sigma^{-1} \int_0^y F(y', k_0)\, \mathrm{d}y' \right\} \tag{3.33}$$

defines the formal Bloch solution of Equation (3.7):

$$\widetilde{\psi}(x + a_1, y, k_0) = w_{10}\psi(x, y, k_0), \tag{3.34}$$

$$\widetilde{\psi}(x, y + a_2, k_0) = \widetilde{w}_{20}\psi(x, y, k_0), \tag{3.35}$$

where the corresponding Bloch multiplier is equal to

$$\widetilde{w}_{20} = w_{20} \exp\left\{ -\sigma^{-1} \int_0^{a_2} F(y', k_0)\, \mathrm{d}y' \right\}. \tag{3.36}$$

For sufficiently small $u(x, y)$, it is not too hard to show that the above constructed series of the perturbation theory converge outside some neighborhoods of the resonant points (3.20) and determine there a function $\widetilde{\psi}(x, y, k_0)$ which is analytical in k_0. This is true for any σ. The principle distinction between the cases $\mathrm{Re}\,\sigma = 0$ and $\mathrm{Re}\,\sigma \neq 0$ is revealed under an attempt to extend $\widetilde{\psi}$ to a 'resonant' domain. In the case $\mathrm{Re}\,\sigma = 0$, the resonant points are dense on the real axis. In the case $\mathrm{Re}\,\sigma \neq 0$, there is only a finite number of the resonant points (3.20) in any finite domain of the complex plane. The discreteness of the resonant points in the last case is crucial for the extension of $\widetilde{\psi}$ to a 'resonant' domain (and for the proof of the approximation theorem).

In the stationary case, when u does not depend on y, the preceding formulae turn into the usual formulae of the perturbation theory of eigenfunctions corresponding to simple eigenvalues. The condition

$$w_{2n} \neq w_{20} \longleftrightarrow k_0 \neq k_{NM} \tag{3.37}$$

is a simple analog of an eigenvalue of an operator. In cases when it is violated, it is necessary to proceed along the same lines as in the perturbation theory of multiple eigenvalues.

As the set of indices corresponding to the resonances, we can take an arbitrary set of integers $I \in Z$ such that

$$w_{2\alpha} \neq w_{2n}, \quad \alpha \in I,\ n \notin I. \tag{3.38}$$

LEMMA 3.2. *There are unique formal series*

$$F^\alpha_\beta(y, w_1) = \sum_{s=1}^{\infty} F^\alpha_{\beta,s}(y, w_1) \tag{3.39}$$

such that the equations

$$(\sigma\partial_y - \partial_x^2 + u)\Psi^\alpha(x, y, w_1) = \sum_\beta F^\alpha_\beta(y, w_1)\Psi^\beta(x, y, w_1) \tag{3.40}$$

have unique formal Bloch solutions of the form

$$\Psi^\alpha(x, y, w_1) = \sum_{s=0}^{\infty} \phi^\alpha_s(x, y, w_1), \quad \phi^\alpha_0 = \psi_\alpha = \psi(x, y, k_\alpha). \tag{3.41}$$

$$\Psi^\alpha(x + a_1, y, w_1) = w_1\Psi^\alpha(x, y, w_1), \tag{3.42}$$

$$\Psi^\alpha(x, y + a_2, w_1) = w_{2\alpha}\Psi^\alpha(x, y, w_1). \tag{3.43}$$

such that

$$\langle \psi^+_\beta \Psi^\alpha \rangle_x = \delta_{\alpha,\beta}\langle \psi^+_\alpha \psi_\alpha \rangle_x. \tag{3.44}$$

The corresponding formulae for F^α_β and Ψ^α are the matrix generalization of formulae (3.28), (3.29) (see details in [7]).

Let us define the matrix $T = T^\alpha_\beta(y, w_1)$ by the equation

$$\sigma\partial_t T + TF = 0, \quad T(0, w_1) = 1. \tag{3.45}$$

The functions

$$\widehat{\Psi}^\alpha(x,y,w_1) = \sum_\beta T^\alpha_\beta(y,w_1)\Psi^\beta(x,y,w_1) \tag{3.46}$$

are solutions of (3.7). Under the translation by the period in x, they are multiplied by w_1, while under the translation by the period in y, they are transformed as follows

$$\widehat{\Psi}^\alpha(x,y+a_2,w_1) = \sum_\beta \widehat{T}^\alpha_\beta(w_1)w_{2\beta}\widehat{\Psi}^\beta(x,y,w_1), \tag{3.47}$$

where

$$\widehat{T}^\alpha_\beta(w_1) = T^\alpha_\beta(a_2,w_1). \tag{3.48}$$

It is natural to call a finite set of the formal solutions $\widehat{\Psi}^\alpha$ *quasi-Bloch*, since it remains invariant under the translation by the periods in x and y.

The characteristic equation

$$R(w_1,\widetilde{w}_2) = \det\left(\widetilde{w}_2\delta_{\alpha,\beta} - \widehat{T}^\alpha_\beta(w_1)w_{2,\beta}\right) = 0 \tag{3.49}$$

is an analog of the 'secular equation' in the ordinary perturbation theory of multiple eigenvalues.

Let $h_\alpha(w_1,\widetilde{w}_2)$ be an eigenvector of the matrix $\widehat{T}^\alpha_\beta(w_1)w_{2,\beta}$ normalized so that

$$\sum_\alpha h_\alpha(\widetilde{Q})\widehat{\Psi}^\alpha(0,0,w_1) = 1, \quad \widetilde{Q} = (w_1,\widetilde{w}_2), \tag{3.50}$$

then

$$\widetilde{\psi}(x,y,\widetilde{Q}) = \sum_\alpha h_\alpha(\widetilde{Q})\widehat{\Psi}^\alpha(x,y,w_1) \tag{3.51}$$

is the formal Bloch solution of (3.7) with multipliers w_1 and $\widetilde{w}_2$, normalized in the standard way. The last statement means that the Bloch solutions are defined (locally) on the Riemann surface (3.49).

3.1. Structure of the 'global' Riemann surface of Bloch solutions

To begin with, we shall give here an explanation of the structure of the 'global' Riemann surface of Bloch solutions in the case of small u. Let us consider some neighborhoods $R_{N,M}$ and $R_{-N,-M}$ of the resonant pair of the points $k_{N,M}$ and $k_{-N,-M}$, respectively. The function $w_1(k)$ (3.12) identifies them with some neighborhood $\widehat{R}_{N,M}$ of the point $w_1(k_{N,M}) = w_1(k_{-N,-M})$ on the complex plane of the variable w_1. The series (3.28)–(3.32) of the nonresonant perturbation theory diverge in $R_{N,M}$ and $R_{-N,-M}$, but it turns out that the series of the Lemma 3.2 converge in $\widehat{R}_{N,M}$ and in this domain define quasi-Bloch solutions of (3.7) which are analytical in w_1. The characteristic Equation (3.49) in this case has the form

$$\widetilde{w}_2^2 - f_1(w_1)\widetilde{w}_2 + f_2(w_1) = 0 \tag{3.52}$$

and defines two-sheet covering $\widetilde{R}_{N,M}$ over $\widehat{R}_{N,M}$ on which the Bloch solutions of (3.7) are defined. The boundary of $\widetilde{R}_{N,M}$ can be naturally identified with the boundaries of $R_{N,M}$ and $R_{-N,-M}$. Hence, the structure (local) of the Riemann surface Γ of the Bloch functions looks as follows. Let us cut out $R_{N,M}$ and $R_{-N,-M}$ from the complex plane and, instead of them, glue a corresponding piece of the Riemann surface $\widetilde{R}_{N,M}$. From the topological point of view, this surgery is a glueing of a 'handle' between two resonant points.

The remarkable thing is that the perturbation approach works even when $u(x,y)$ is not small. Of course, in that case, the estimations of the perturbation theory series are much more complicated. In [7], it was proved that if the potential $u(x,y)$ can be analytically extended into a domain

$$|\mathrm{Im}\, x| < \tau_1, \qquad |\mathrm{Im}\, y| < \tau_2, \tag{3.53}$$

for some τ_1, τ_2, then the perturbation series for the nonresonant case converge outside some central finite domain R_0 and outside $R_{N,M}$ for $k_{N,M} \notin R_0$. Outside R_0, we again have to perform a surgery of the previous type ('glue' handles between $k_{N,M}$ and $k_{-N,-M}$ for $k_{N,M} \notin R_0$). In the central domain R_0, we have to glue some finite genus piece of the corresponding Riemann surface $\widetilde{R}_0$ instead of disc R_0. As a result, we obtain the global Riemann surface Γ of the Bloch solutions of Equation (3.7) with Re $\sigma \neq 0$.

THEOREM 3.3. *If the potential $u(x,y)$ of Equation (3.20) can be analytically extended into the domain (3.53), then the Riemann surface Γ of the Bloch solutions of this equation is a result of the above-defined glueing of the three types of 'pieces':*

1°. *A complex plane of the variable k without small neighborhoods of the finite or infinite set of points $k_{N,M}, k_{-N,-M}$ and without some central domain $|k| > K_0$;*

2°. *A set of 'handles' $\widehat{R}_{|N,M|}$ that are defined by equations of the form (3.52) as the two-sheets covering of the small neighborhoods of the pairs $k_{N,M}$, $k_{-N,-M}$;*

3°. *A Riemann surface $\widetilde{R}_0$ (with the boundary) of a finite genus g_0.*

The Bloch solutions of (3.7) $\psi(x,y,Q)$, $Q \in \Gamma$, that are normalized by the condition $\psi(0,0,Q) = 1$ are meromorphic on Γ. Their poles do not depend on x,y. It has one simple pole in each of the domains $\widehat{R}_{|N,M|}$. In the domain $\widehat{R}_0$, it has g_0 poles, where g_0 in general positions (when $\widehat{R}_0$ is smooth) equals the *genus of* $\widehat{R}_0$). *Outside these domains, the function* ψ *is holomorphic and has no zeros.*

If there is a finite number of handles that are glued then the corresponding curve is compactified by one point and the corresponding Bloch function is the Baker–Akhiezer function on the compactified Riemann surface.

In the case of real and smooth $u(x,y)$ for $\sigma = 1$ the final form of the Floque set can be represented in the following form [7]. Let us fix some finite or infinite subset S of integer pairs $(N > 0,\ M)$. The set of pairs of complex numbers $\pi = \{p_{s,1}, p_{s,2}\}$, where $s \in S$ would be called 'admissible' if

$$\operatorname{Re} p_{s,i} = \frac{\pi N}{a_1}, \qquad |p_{s,i} - k_s| = o(|k_s|^{-1}), \quad i = 1,2, \tag{3.54}$$

and the intervals $[p_{s,1}, p_{s,2}]$ do not intersect. (Here k_s are resonant points (3.20), $s = (N,M)$.)

Let us define the Riemann surface $\Gamma(\pi)$ for any admissible set π. It is obtained from the complex plane of the variable k by cutting it along the intervals $[p_{s,1}, p_{s,2}]$ and $[-\bar{p}_{s,1}, -\bar{p}_{s,2}]$ and by sewing after that the left side of the first cut with the right side of the second cut and vice versa. After this surgery, for any cut $[p_{s,1}, p_{s,2}]$ there corresponds a nontrivial cycle a_s on $\Gamma(\pi)$.

THEOREM 3.4. *For any real periodic potential* $u(x,y)$ *which can be analytically extended into some neighborhood of the real values* x, y, *the Bloch solutions of Equation* (3.7) *with* $\sigma = 1$ *are parametrized by points* Q *of the Riemann surface* $\Gamma(\pi)$ *corresponding to some admissible set* π. *The function* $\psi(x,y,Q)$ *which is normalized by the condition* $\psi(0,0,Q) = 1$ *is meromorphic on* Γ *and has a simple pole* γ_s *on each cycle* a_s. *If the admissible set* π *contains only a finite number of pairs, then* $\Gamma(\pi)$ *has finite genus and is compactified by only one point* P_1 $(k = \infty)$, *in the neighborhood of which the Bloch function* ψ *has the form* (2.9).

The potentials u for which $\Gamma(\pi)$ has finite genus are called finite-gap potentials and as it follows from the last statement of the theorem, they coincide with the algebraic–geometrical potentials. The following theorem states that the finite-gap potentials are dense in the space of all periodic smooth functions in two variables [7].

THEOREM 3.5. *Each smooth periodic potential* $u(x,y)$ *of Equation* (3.7) *with* $\operatorname{Re}\sigma \neq 0$ *analytically extendable to a neighbourhood of real* x, y *can be approximated by finite-gap potentials uniformly with any number of derivatives.*

4. Algebraic–Geometrical Perturbation Theory of Integrable Systems

The nonlinear WKB (or Whitham) method for the construction of the asymptotic solutions may be applied for any nonlinear system that has exact quasi-periodic solutions of the form

$$u(x, y, t) = u_0(Ux + Vy + Wt \mid I), \tag{4.1}$$

where $u_0(z_1, \ldots, z_g \mid I)$ is a periodic function in variables z_i depending on a set of parameters $I = (I_1, \ldots, I_N)$; $U = U(I)$, $V = V(I)$, $W = W(I)$ are vectors depending on the same set of data.

Let us consider the asymptotical solutions of the same equation of the form

$$\begin{aligned} u(x, y, t) &= u_0(\varepsilon^{-1}S(X, Y, T) \mid I(X, Y, T)) + \\ &\quad + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t), \end{aligned} \tag{4.2}$$

where $X = \varepsilon x,\ Y = \varepsilon y,\ T = \varepsilon t$ are 'slow variables'.

Remark. Parameters I are integrals of the corresponding nonlinear partial differential equation. As usual, in the perturbation theory the 'integrals' of the initial equation become functions of the slow variables. We would like to emphasize that for partial differential equations one of the possible types of perturbation (that does not exist in the finite-dimensional case of the classical Hamiltonian integrable systems) is not the perturbation of the equation by itself, but the perturbation of a class of initial or boundary conditions. The construction of the asymptotic solutions of the form (4.2) corresponds to the perturbation when instead of a class of periodic or quasi-periodic functions one considers functions with slowly modulated periods.

If the vector $S(X, Y, T)$ satisfies the relations

$$\partial_X S = U(I(X, Y, T)) = U(X, Y, T),$$

$$\partial_Y S = V(X, Y, T), \qquad \partial_T S = W(X, Y, T), \tag{4.3}$$

then the main term u_0 in the expansion (4.2) satisfies the initial equation up to the first order in ε. After that, all the other terms of (4.2) are defined from the nonhomogeneous linear equations. They can be easily solved if a full set of solutions for a homogeneous linear equation are known.

The asymptotic solutions of the form (4.2) can be constructed with an arbitrary dependence of the parameters I_k on the slow variables. In this case, the expansion (4.2) will be valid on a scale of order 1. The right-hand side of the nonhomogeneous linear equation for u_1 contains the first derivatives of the parameters I_k. Therefore, the choice of the dependence of I_k on slow variables can be used for the cancellation of the 'secular' term in u_1.

Let us show briefly the way to derive necessary conditions for the existence of the asymptotic solutions of the form (4.2) with uniformly bounded first-order term (u_1) for an equation that has a zero-curvature representation of the form (1.3).

Let L_0, A_0 be the linear operators corresponding to an exact algebraic–geometrical solution of the corresponding equation. We consider the asymptotic solutions of (1.3)

$$L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots, \qquad A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{4.4}$$

assuming that the algebraic–geometrical data that define L_0, A_0 are functions of the slow variables. The corresponding dependence of L_0, A_0 defines the right-hand side $F(L_0, A_0)$ of the nonhomogeneous equation

$$\partial_t L_1 - \partial_y A_1 + [L_1, A_0] + [L_0, A_1] = F(L_0, A_0). \tag{4.5}$$

for the first-order terms of the asymptotic solution (4.4). (The left-hand side of (4.5) is a linearization of (1.3) on the background of the exact solution L_0, A_0.)

Let ψ be a solution of the auxiliary linear problems

$$(\partial_y - L_0)\psi = 0, \qquad (\partial_t - A_0)\psi = 0, \tag{4.6}$$

and ψ^+ be a solution of the adjoint equations

$$\psi^+(\partial_y - L_0) = 0, \qquad \psi^+(\partial_t - A_0) = 0, \tag{4.7}$$

then (4.5), (4.6) imply that

$$\partial_t(\psi^+ L_1 \psi) - \partial_y(\psi^+ A_1 \psi) - (\psi^+ F \psi) = \partial_x(\psi^+ \mathrm{D} \psi), \tag{4.8}$$

where D is a differential operator with coefficients that are differential polynomials in the coefficients of the operators L_0, L_1, A_0, A_1.

The equality (4.8) implies that if the product $\psi^+\psi$ is quasi-periodic in the variables x, y, t, then

$$\langle(\psi^+ F \psi)\rangle_{x,y,t} = 0, \tag{4.9}$$

where $\langle\cdot\rangle_{x,y,t}$ stands for the mean value in the variables x, y, t of the corresponding function.

In [8] it was found that (4.9) and the compatibility conditions of (4.3) are equivalent to a well-defined system of partial differential equations on the moduli space of curves with punctures and fixed local coordinates in the neighborhoods of these punctures.

4.1. Whitham Equations

To begin with, we shall give an algebraic form of the 'universal' Whitham hierarchy (see [14]). Let $\Omega_A(k, T)$ be a set of holomorphic functions of the variable k (which is defined in some complex domain $\mathcal{D}$), depending on a finite or infinite number of variables t_A, $T = \{t_A\}$. (We keep the same notation t_A

for slow variables εt_A because we are not going to consider 'fast' variables in this section any more.) Let us introduce a one-form

$$\omega = \sum_A \Omega_A(k,T)\,\mathrm{d}t_A, \tag{4.10}$$

onto the space with coordinates (k, t_A). Its full external derivative equals

$$\delta\omega = \sum_A \delta\Omega_A(k,T) \wedge \mathrm{d}t_A, \tag{4.11}$$

where

$$\delta\Omega_A = \partial_k\Omega_A\,\mathrm{d}k + \sum_B \partial_B\Omega_A\,\mathrm{d}t_B, \quad \partial_k = \partial/\partial k,\ \partial_A = \partial/\partial t_A. \tag{4.12}$$

The equation

$$\delta\omega \wedge \delta\omega = 0 \tag{4.13}$$

we shall call, by definition, *the Whitham hierarchy*.

The 'algebraic' form (4.13) of the Whitham equations is equivalent to a set of partial differential equations that have to be fulfilled for any triple A, B, C

$$\sum_{\{A,B,C\}} \varepsilon^{\{A,B,C\}}\partial_A\Omega_B\partial_k\Omega_C = 0 \tag{4.14}$$

(summation in (4.14) is taken over all permutations of indices A, B, C and $\varepsilon^{\{A,B,C\}}$ is a sign of permutation).

Equations (4.13) are invariant with respect to an invertable change of variable

$$k = k(p,T), \qquad \partial_p k \neq 0. \tag{4.15}$$

Let us fix an index A_0 and denote the corresponding function by

$$p(k,T) = \Omega_{A_0}(k,T). \tag{4.16}$$

At the same time, we introduce a special notation for the corresponding 'time'

$$t_{A_0} = x. \tag{4.17}$$

After that, all Ω_A can be considered as functions of the new variable p, $\Omega_A = \Omega_A(p,T)$. Equations (4.14) for A, B, $C = A_0$ after this change of variable k have the form

$$\partial_A\Omega_B - \partial_B\Omega_A + \{\Omega_A, \Omega_B\} = 0, \tag{4.18}$$

where $\{f,g\}$ stands for the usual Poisson bracket on the space of functions of the two variables x, p

$$\{f,g\} = \partial_x f\partial_p g - \partial_x g\partial_p f. \tag{4.19}$$

The Whitham equations were obtained in [8] in the form (4.14). In [7], it was found that they can be represented in the algebraic form (4.13). (We would like to mention here the papers [15, 16] where it was shown that the algebraic form of the Whitham equations leads directly to the semiclassical limit of 'string' equations.)

The Whitham equations in the form (4.14) are equations on the set of functions $\Omega_A(p,T)$ and give but a certain 'shape' that has to be filled with a real content. It is necessary to show that they do define correct systems of equations on the moduli spaces $\widehat{M}_{g,N}$ of smooth algebraic curves Γ_g of genus g with local coordinates $k_\alpha^{-1}(P)$ in neighborhoods of N punctures P_α ($k_\alpha^{-1}(P_\alpha)=0$)

$$\widehat{M}_{g,N}=\{\Gamma_g,\ P_\alpha,\ k_\alpha^{-1}(P),\ \alpha=1,\ldots,N\}. \tag{4.20}$$

Let us consider as the first example the zero-genus case ($g=0$). In this case, a point of the 'phase space' $\widehat{M}_{g=0,N}$ is a set of points p_α, $\alpha=1,\ldots,N$, and a set of formal local coordinates $k_\alpha^{-1}(p)$:

$$k_\alpha(p)=\sum_{s=-1}^{\infty} v_{\alpha,s}(p-p_\alpha)^s \tag{4.21}$$

('formal local coordinate' means that the r.h.s of (4.21) is considered as a formal series without any assumption of its convergency). Hence, $\widehat{M}_{0,N}$ is a set of sequences

$$\widehat{M}_{0,N}=\{p_\alpha,\ v_{\alpha,s},\ \alpha=1,\ldots,N,\ s=-1,0,1,2,\ldots\}. \tag{4.22}$$

The Whitham equations define a dependence of points of $\widehat{M}_{0,N}$ with respect to the variables t_A, where the set of indices $\mathcal{A}$ is as follows

$$\begin{aligned}\mathcal{A}\ =\ &\{A=(\alpha,i),\ \alpha=1,\ldots,N,\ i=1,2,\ldots\\ &\text{and for } i=0,\ \alpha\neq 1\}.\end{aligned} \tag{4.23}$$

As it was explained above, we can fix one of the points p_α with the help of an appropriate change of the variable p. Let us choose $p_1=\infty$.

Let us introduce meromorphic functions $\Omega_{\alpha,i}(p)$ for $i>0$ with the help of the following conditions:

$\Omega_{\alpha,i>0}(p)$ has a pole only at p_α and coincides with the singular part of an expansion of $k_\alpha^i(p)$ near this point, i.e.

$$\Omega_{\alpha,i}(p)=\sum_{s=1}^{i} w_{\alpha,i,s}(p-p_\alpha)^{-s}=k_\alpha^i(p)+O(1),$$

$$\Omega_{\alpha,i}(\infty)=0,\quad \alpha\neq 1. \tag{4.24}$$

$$\Omega_{1,i}(p) = \sum_{s=1}^{i} w_{1,i,s} p^s = k_1^i(p) + O(k_1^{-1}). \tag{4.25}$$

These polynomials can be written in the form of the Cauchy integrals

$$\Omega_{\alpha,i}(p,T) = \frac{1}{2\pi i} \oint_{C_\alpha} \frac{k_\alpha^i(z_\alpha, T)\, \mathrm{d}z_\alpha}{p - z_\alpha}. \tag{4.26}$$

Here C_α is a small cycle around the point p_α.

The functions $\Omega_{\alpha,i=0}(p)$, $\alpha \neq 1$, are equal to

$$\Omega_{\alpha,0}(p) = -\ln(p - p_\alpha). \tag{4.27}$$

Remark. The asymmetry of the definitions of $\Omega_{\alpha,i}$ reflects our intention to choose the index $A_0 = (1,1)$ as a 'marked' index.

The coefficients of $\Omega_{\alpha,i>0}(p)$ are polynomial functions of $v_{\alpha,s}$. Therefore, the Whitham equations (4.12) (or (4.18)) can be rewritten as equations on $\widehat{M}_{0,N}$. But still it has to be shown that they can be considered as a correctly defined system.

THEOREM 4.1. *The zero-curvature form* (4.18) *of the Whitham hierarchy in the zero-genus case is equivalent to the Sato-form that is a compatible system of evolution equations*

$$\partial_A k_\alpha = \{k_\alpha, \Omega_A\}. \tag{4.28}$$

Let us demonstrate a few examples.

EXAMPLE 1 (Khokhlov–Zabolotskaya hierarchy). The Khokhlov–Zabolotskaya hierarchy is the particular $N = 1$ case of our considerations. Any local coordinate $K^{-1}(p)$ near the infinity ($p_1 = \infty$)

$$K(p) = p + \sum_{s=1}^{\infty} v_s p^{-s} \tag{4.29}$$

defines a set of polynomials:

$$\Omega_i(p) = [K^i(p)]_+, \tag{4.30}$$

here $[\ldots]_+$ denotes a nonnegative part of Laurent series. For example,

$$\Omega_2 = k^2 + u, \quad \Omega_3 = k^3 + \tfrac{3}{2} uk + w, \tag{4.31}$$

where $u = 2v_1$, $w = 3v_2$. If we denote $t_2 = y$, $t_3 = t$, then Equation (4.18) for $A = 2$, $B = 3$ gives

$$w_x = \tfrac{3}{4} u_y, \qquad w_y = u_t - \tfrac{3}{2} u u_x, \tag{4.32}$$

from which the dispersionless KP (dKP) equation (Khokhlov–Zabolotskaya equation) is derived:

$$\tfrac{3}{4}\, u_{yy} + \left(u_t - \tfrac{3}{2}\, u u_x\right)_x = 0. \tag{4.33}$$

The Khokhlov–Zabolotskaya equation is a partial differential equation and though it has no pure evolution form, one can expect that its solutions are to be uniquely defined by their Cauchy data $u(x, y, t = 0)$, that is a function of the two variables x, y. Up to now, it is not clear if this two-dimensional equation can be considered as the third equivalent form of the Whitham hierarchy (we recall that solutions of the hierarchy (4.28) formally depend on an infinite number of functions of one variable).

EXAMPLE 2 (longwave limit of 2D Toda lattice). The hierarchy of the longwave limit of a two-dimensional Toda equation is the particular $N = 2$ case of our considerations. There are two local parameters. One of them is near the infinity $p_1 = \infty$ and one is near a point $p_2 = a$. They depend on two sets of variables $t_{\alpha,s}$, $\alpha = 1, 2$, $s = 1, 2, \ldots$, and also on the variable t_0. We shall present here only the basic two-dimensional equation of this hierarchy (an analogue of the Khokhlov–Zabolotskaya equation).

Consider three variables $t = t_0$, $x = t_{1,1}$, $y = t_{2,1}$. The corresponding functions are

$$\Omega_0 = \ln(p - a), \qquad \Omega_{1,1} = p, \qquad \Omega_{2,1} = \frac{v}{p - a}. \tag{4.34}$$

Their substitution into the zero-curvature Equation (4.18) gives

$$v_x = a_t v, \qquad v_t + a_y = 0, \qquad w_t = 0. \tag{4.35}$$

From (4.35), it follows that

$$\partial^2_{xy}\phi + \partial^2_t \mathrm{e}^{\phi} = 0, \tag{4.36}$$

where $\phi = \ln v$. This is the longwave limit of the 2D Toda lattice equation

$$\partial^2_{xy}\varphi_n = \mathrm{e}^{\varphi_{n-1} - \varphi_n} - \mathrm{e}^{\varphi_n - \varphi_{n+1}} \tag{4.37}$$

corresponding to the solutions that are slow functions of the discrete variable n, which is replaced by the continuous variable t. Equation (4.36) has arisen independently in general relativity, the theory of wave phenomena in shallow water, long radio-relay lines, and so on. A bibliography can be found in [17] where a representation of solutions of (4.36) in terms of convergent series was proposed.

EXAMPLE 3 (N-layer solutions of the Benny equation). This example corresponds to a general $N+1$ points case, but we consider only one zero-curvature equation. Let us choose three functions

$$\Omega_1 = p, \qquad \Omega_2 = p + \sum_{i=1}^{N} \frac{v_i}{p - p_i}, \qquad \Omega_3 = p^2 + u, \tag{4.38}$$

which are coupled with the variables x, y, t, respectively. In our standard notations, they are

$$x = t_{1,1}, \qquad y = \sum_{\alpha=1}^{N+1} t_{\alpha,1}, \qquad t = t_{1,2}. \tag{4.39}$$

The zero-curvature Equation (4.18) gives the system

$$p_{it} - (p_i^2)_x + u_x = 0, \qquad v_{it} = 2(v_i p_i)_x,$$

$$u_y - u_x + 2\sum_i v_{ix} = 0. \tag{4.40}$$

Solutions of this system that do not depend on y are N-layer solutions of the Benny equation. As it was noticed in [18], the corresponding system

$$p_{it} - (p_i^2)_x + u_x = 0, \qquad v_{it} = 2(v_i p_i)_x, \qquad u = 2\sum_i v_i \tag{4.41}$$

is a classical limit of the vector nonlinear Schrödinger equation

$$i\psi_{it} = \psi_{i,xx} + u\psi_i, \qquad u = \sum_i |\psi_i|^2. \tag{4.42}$$

(Using this observation in [18], the integrals of (4.41) were found.)

In the case of the arbitrary genus curves, the *basic* Whitham hierarchy is generated by Equations (4.13) or (4.18), where the set of indices $\mathcal{A}$ is the same as in genus zero case. The corresponding functions for $A = (\alpha, i > 0)$ are defined by (2.14)–(2.16). The functions $\Omega_{\alpha,0}$ are integrals of the normalized Abelian differentials $\mathrm{d}\Omega_{\alpha,0}$, $\alpha \neq 1$, of the third kind with simple poles at the points P_1 and P_α with residues 1 and -1, respectively,

$$\mathrm{d}\Omega_{\alpha,0} = \mathrm{d}k_\alpha (k_\alpha^{-1} + O(k_\alpha^{-1})),$$

$$\mathrm{d}\Omega_{\alpha,0} = -\mathrm{d}k_1 (k_1^{-1} + O(k_1^{-1})); \tag{4.43}$$

Remark. It should be mentioned here that the corresponding system of equations is defined on the covering $\widehat{M}^*_{g,N}$ of the moduli space $\widehat{M}_{g,N}$ that corresponds to the choice of canonical basis of cycles on the curve. In order to have the system of equations that are defined on $\widehat{M}_{g,N}$ but not on its covering, it is necessary to change the normalization conditions (2.15) by the conditions that do not depend on the choice of basic cycles. The corresponding conditions are

$$\operatorname{Im} \oint_c \mathrm{d}\Omega_A = 0, \quad c \in H_1(\Gamma_g, Z). \tag{4.44}$$

This normalization conditions were obtained for the Whitham hierarchy in [8]. Below, for the simplicity, we consider the Whitham hierarchy on $\widehat{M}^*_{g,N}$, only.

4.2. Algebraic Orbits and Exact Solutions of the Whitham Equations

In [7] a construction of exact solutions of the Whitham equations corresponding to 'algebraic orbits' was proposed. (This construction is a generalization and effectivization of the scheme [19] where 'generalized godograph' transformation was proposed for hydrodynamical-type diagonalizable Hamiltonian systems; see [20–22].)

Let $\Omega_A(k, T)$ be a solution of the general zero-curvature Equation (4.18)

$$\partial_A \Omega_B - \partial_B \Omega_A + \{\Omega_A, \Omega_B\} = 0. \tag{4.45}$$

They are the compatibility conditions for the equations

$$\partial_A E = \{E, \Omega_A\}, \tag{4.46}$$

Therefore, an arbitrary function $E(p, x)$ defines (at least locally) the corresponding solution $E(p, T)$ of (4.46), $E(p, x) = E(p,\ t_{A_0} = x,\ t_A = 0, A \neq A_0)$. In the domain where $\partial_p E(p, T) \neq 0$, we can use a variable E as a new coordinate, $p = p(E, t)$. The derivatives for fixed E and p are interrelated with the help of the following formula

$$\partial_A F(p, t) = \partial_A f(E, t) + \frac{\mathrm{d}f}{\mathrm{d}E} \partial_A E(p, t). \tag{4.47}$$

Therefore, in the new coordinate, Equations (4.45) are equivalent to the equations

$$\partial_A \Omega_B(E, T) = \partial_B \Omega_A(E, T). \tag{4.48}$$

Hence, there exists a potential $S(E, T)$ such that

$$\Omega_A(E, T) = \partial_A S(E, T). \tag{4.49}$$

Due to this potential, the one-form ω (4.10) can be represented as

$$\omega = \delta S(E, T) - Q(E, T)\, \mathrm{d}E, \tag{4.50}$$

where

$$Q(E,T) = \frac{\partial S(E,T)}{\partial E}. \tag{4.51}$$

Hence,

$$\delta\omega = \delta E \wedge \delta Q. \tag{4.52}$$

Formula (4.52) implies that the functions E and Q as functions of two variables p, x satisfy the classical string equation

$$\{Q, E\} = 1. \tag{4.53}$$

They show that

$$\partial_A Q = \{Q, \Omega_A\}, \tag{4.54}$$

also.

A set of the pairs of functions $Q(p,x)$, $E(p,x)$ satisfying the string equation is a group with respect to the composition, i.e. if $Q(p,x)$, $E(p,x)$ and $Q_1(p,x)$, $E_1(p,x)$ are solutions of (4.53) then the functions

$$\widetilde{Q}(p,x) = Q_1(Q(p,x),\ E(p,x)); \quad \widetilde{E}(p,x) = E_1(Q(p,x), E(p,x)) \tag{4.55}$$

are a solution of (4.53), as well. The Lie algebra of this group is the algebra $\mathrm{SDiff}(T^2)$ of two-dimensional vector-fields preserving an area. The action of this algebra on the potential, τ-function (and so on) within the framework of the longwave limit of a 2D Toda lattice was considered in [15, 16].

By definition, the 'algebraic orbits' of the Whitham hierarchy are solutions such that there exists a 'global' solution of Equation (4.46). In the case of genus-zero Whitham equations, 'global' means that $E(p,T)$ *is a meromorphic solution of Equations* (4.46) *such that*

$$\{E(p,T), k_\alpha(p,T)\} = 0. \tag{4.56}$$

The last equality implies that there exist functions $f_\alpha(E)$ of one variable such that

$$k_\alpha(p,T) = f_\alpha(E(p,T)). \tag{4.57}$$

In order not to be lost in a too general setting right at the beginning, let us start with an example.

EXAMPLE (Lax reductions, $N = 1$). Consider solutions of the dKP hierarchy such that some power of local parameter (4.29) is a polynomial in p, i.e.

$$E(p,T) = p^n + u_{n-2}p^{n-2} + \cdots + u_0 = K^n(p,T). \tag{4.58}$$

The relation (4.58) implies that only a few first coefficients of the local parameter are independent. All of them are polynomials with respect to the coefficients u_i of the polynomial $E(p,T)$. The corresponding solutions of dKP hierarchy can be described in terms of dispersionless Lax equations

$$\partial_i E(p,T) = \{E(p,T), \Omega_i(p,T)\}, \tag{4.59}$$

where

$$\Omega_i(p,T) = \left[E^{i/n}(p,T)\right]_+ \tag{4.60}$$

(as before, $[\ldots]_+$ denotes a nonnegative part of corresponding Laurent series). These solutions of KP hierarchy can be also characterized by the property that they do not depend on the variables $t_n, t_{2n}, t_{3n}, \ldots$.

The construction of the exact solutions of the dispersionless Lax equations can be presented in the following form. For each polynomial $E(p)$ of the form (4.58) and each formal series

$$Q(p) = \sum_{j=1}^{\infty} b_j p^j, \tag{4.61}$$

the formula

$$t_i = \frac{1}{i}\operatorname{res}_\infty \left(K^{-i}(p) Q(p)\, \mathrm{d}E(p)\right) \tag{4.62}$$

defines the variables

$$t_k = t_k(u_i, b_j), \quad i = 0, \ldots, n-2,\ j = 0, \ldots . \tag{4.63}$$

as functions of the coefficients of E, Q. Consider the inverse functions

$$u_i = u_i(t_1, \ldots), \qquad b_j = b_j(t_1, \ldots). \tag{4.64}$$

Remark. In order to be more precise, let us consider a case when Q is a polynomial, i.e. $b_j = 0$, $j > m$. From (4.62), it follows that $t_k = 0$, $k > n+m-1$. Therefore, we have $n+m-1$ 'times' t_k, $k = 1, \ldots, n+m-1$, that are linear functions of b_j, $j = 1, \ldots, m$, and polynomials in u_i, $i = 0, \ldots, n-2$. So, locally the inverse functions (4.64) are well defined.

THEOREM 4.2. *The functions $u_i(T)$ are solutions of dispersionless Lax Equation (4.59). Any other solutions of (4.59) are obtained from this particular one with the help of translations, i.e. $\tilde{u}(t_i) = u(t_i - t_i^0)$.*

The construction of solutions of the dispersionless Lax equations that was presented above manifest the general form of the construction of exact solutions of universal Whitham hierarchy on all spaces $\widehat{M}^*_{g,N}$.

The key elements of this construction are as follows:

1. First of all there is the 'big phase space', that is the moduli space

$$\widetilde{\mathcal{N}}_g(n_\alpha) = \{\Gamma_g,\ \mathrm{d}Q,\ \mathrm{d}E,\ a_i, b_i \in H_1(\Gamma_g, Z)\} \tag{4.65}$$

of curves with a fixed canonical basis of cycles, with a fixed normalized meromorphic differential $\mathrm{d}E$ having poles of orders $n_\alpha + 1$ at points P_α and with a fixed normalized differential $\mathrm{d}Q(P)$ that is holomorphic outside the punctures.

2. The second step is the definition of a set of functions on this moduli space. They are given by the formulae

$$\begin{aligned} t_{\alpha,i} &= \frac{1}{i}\,\mathrm{res}_{P_\alpha}\left(k_\alpha^{-i}(p)Q(p)\,\mathrm{d}E(p)\right), \quad i > 0,\ k_\alpha^{n_\alpha}(p) = E(p); \\ t_{\alpha,0} &= \mathrm{res}_{P_\alpha}\left(Q(p)\,\mathrm{d}E(p)\right) \end{aligned} \tag{4.66}$$

and the formulae

$$t_{h,i} = \oint_{a_i} \mathrm{d}S, \quad i = 1, \ldots, g, \qquad \mathrm{d}S = Q\,\mathrm{d}E, \tag{4.67}$$

$$t_{Q,i} = -\oint_{b_i} \mathrm{d}E, \qquad t_{E,i} = \oint_{b_i} \mathrm{d}Q, \quad i = 1, \ldots, g. \tag{4.68}$$

Let us introduce the differentials that are 'coupled' to the new type of the variables t_A. Namely, to the variables $t_{h,k}$, $t_{Q,k}$, $t_{E,k}$.

1. $\mathrm{d}\Omega_{h,k}$ is normalized

$$\oint_{a_i} \mathrm{d}\Omega_{h,k} = \delta_{ik} \tag{4.69}$$

holomorphic differentials.

2. The differentials $\mathrm{d}\Omega_{E,i}$, $\mathrm{d}\Omega_{Q,i}$ are holomorphic on the curve Γ_g everywhere except for the a-cycles where they have 'jumps'. Their boundary values on a_j cycle satisfy the relations

$$\mathrm{d}\Omega_{E,i}^+ - \mathrm{d}\Omega_{E,i}^- = \delta_{ij}\,\mathrm{d}E, \tag{4.70}$$

$$\mathrm{d}\Omega_{Q,i}^+ - \mathrm{d}\Omega_{Q,i}^- = \delta_{ij}\,\mathrm{d}Q, \tag{4.71}$$

These differentials are uniquely normalized by the conditions

$$\oint_{a_i} \mathrm{d}\Omega_{E,j} = \oint_{a_i} \mathrm{d}\Omega_{Q,j} = 0. \tag{4.72}$$

Let

$$\mathcal{D}_g(n_\alpha) \subset \widetilde{\mathcal{N}}_g(n_\alpha) \tag{4.73}$$

be an open subset of the big phase space such that the corresponding differentials $\mathrm{d}E$, $\mathrm{d}Q$ have no common zeros on Γ_g.

THEOREM 4.3. *The map*

$$\mathcal{D}_g(n_\alpha) \longmapsto (T = \{t_A\}) \tag{4.74}$$

that is defined by formulae (4.66)–(4.68) *is nondegenerate, i.e. the set of functions* $\{t_A\}$ *defines a system of coordinates on* $\mathcal{D}_g(n_\alpha)$. *The corresponding dependence of the curve* $\Gamma_g(T)$ *and the differential* $\mathrm{d}E(T)$ *with respect to these coordinates define a solution of the universal Whitham hierarchy on the moduli space* $\widehat{M}^*_{g,N}$.

In other words, the Whitham hierarchy can be considered as a way to define the special system of coordinates on the moduli space of curves with punctures and with jets of local coordinates in neighborhoods of the punctures. The n_α-jets of the local coordinates are the equivalence classes of the coordinates. Two local coordinates $k_{1,\alpha}(P)^{-1}$ and $k_\alpha(P)$ are equivalent iff

$$k_{1,\alpha}(P) = k_\alpha(P) + \mathrm{O}\big(k_\alpha(P)^{-n_\alpha - 1}\big). \tag{4.75}$$

The formula

$$k_\alpha^{n_\alpha}(P) = E(P), \tag{4.76}$$

where $E(P)$ is the integral of the differential $\mathrm{d}E$ defines locally a map from the moduli space of $\{\Gamma_g, \mathrm{d}E\}$ of curves with fixed normalized meromorphic differentials of the second kind with poles of orders $n_\alpha + 1$ at points P_α to the moduli space $\widehat{M}^*_{g,N}$. This maps is a local isomorphism with the moduli space $M_{g,N}(n_\alpha)$ of curves with punctures and with fixed n_α jets of local coordinates.

4.3. Topological Minimal Models

In [23], it was notice that the calculations of [25] of the perturbed primary rings of the so-called topological minimal A_n-models can be identified with the construction of the solutions of the first n equations of the dispersionless Lax hierarchy (4.59). This fact made it possible to include the corresponding deformations of the primary rings into a hierarchy of an infinite number of commuting flows. The calculations in [25] of partition function for perturbed A_n models gave an impulse for the introduction a τ-function for dispersionless Lax equation. The corresponding results were generalized in [14, 24, 26, 27].

Two- and three-points correlation functions

$$\langle \phi_\alpha \phi_\beta \rangle = \eta_{\alpha\beta}, \qquad c_{\alpha\beta\gamma} = \langle \phi_\alpha \phi_\beta \phi_\gamma \rangle \tag{4.77}$$

of any topological field theory with N primary fields $\phi_1, \ldots, \phi_N$ define associative algebra

$$\phi_\alpha \phi_\beta = c_{\alpha\beta}^{\gamma} \phi_\gamma, \qquad c_{\alpha\beta}^{\gamma} = c_{\alpha\beta\mu}\eta^{\gamma\mu}, \qquad \eta_{\alpha\mu}\eta^{\mu\beta} = \delta_\alpha^\beta, \tag{4.78}$$

with a unit ϕ_1

$$\eta_{\alpha\beta} = c_{1\alpha\beta}. \tag{4.79}$$

It turns out that there exists N parametric deformations of the theory such that 'metric' $\eta_{\alpha\beta}$ is a constant and three-point correlators are given by the derivatives of free energy $F(t)$ of the deformed theory

$$c_{\alpha\beta\gamma}(t) = \partial_{\alpha\beta\gamma}F(t), \qquad \eta_{\alpha\beta} = \partial_{1\alpha\beta}F(t) = \text{const.} \tag{4.80}$$

Such deformations are called *potential Fröbenius deformations*. (We recall that the commutative associative algebra with the bilinear form $\langle\cdot,\cdot\rangle$ that is compatible with multiplication, i.e. $\langle ab, c\rangle = \langle a, bc\rangle$ is called the Fröbenius algebra.)

The associativity conditions of algebra (4.78) with structural constants (4.80) are equivalent to a system of partial differential equations on F (WDVV equations). In [28], it was noticed that these equations are equivalent to the equations

$$[\partial_\alpha - \lambda C_\alpha(t), \partial_\beta - \lambda C_\beta(t)] = 0, \tag{4.81}$$

where $C_\alpha(t)$ is a matrix with entries $c^\gamma_{\alpha\beta}$. In other words, the structure constants of the algebra of deformed topological theories define a flat connection on the space of deformed theories with a spectral parameter λ. The coupling constant are flat coordinates for the space of deformed theory.

The topological minimal A_n models are particular examples of such models. Topological states for twisted $N = 2$ superconformal Landau–Ginsburg models are defined by the superpotential E. In case of the A_{n-1} model, the deformed potential has the form

$$E(p) = p^n + u_{n-2}p^{n-2} + \cdots + u_0. \tag{4.82}$$

The coefficients u_i can be considered as coordinates on the space of the deformed topological minimal models. Each polynomial E defines a factor ring

$$\mathcal{R}_E = \frac{C[p]}{E' = 0}, \qquad E'(p) = \frac{\partial E}{\partial p}. \tag{4.83}$$

The formula

$$\langle f, g\rangle = \sum_{q_s} \text{res}_{q_s}\left(\frac{f(p)g(p)}{E_p}\,dp\right), \quad f(p),\ g(p) \subset \mathcal{R}_E, \tag{4.84}$$

defines a nondegenerate bilinear form on $\mathcal{R}_E$, i.e. supplies $\mathcal{R}_E$ with the structure of the Fröbenius algebra.

In [25], it was shown that the functions $t_i(u_0, \ldots, u_{n-2})$, $i = 1, \ldots, n-1$, that are defined by formula (4.62) for $Q(p) = p$ are flat coordinates for the bilinear form (4.84) and define the potential Fröbenius deformation of the algebra (4.83).

THEOREM 4.4 ([23]). *Let $E(p, t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots)$ be the solution of the dispersionless Lax hierarchy which was constructed above. Then the superpotential of the perturbed A_{n-1} topological minimal model is equal to*

$$W(p, t_1, \ldots, t_{n-1}) = E(p, t_1, \ldots, t_{n-1}, 1, 0, 0, \ldots). \tag{4.85}$$

The free energy of this model is equal to

$$F = F(t_1, t_2, \ldots, t_{n-1}, 1, 0, 0, \ldots), \tag{4.86}$$

where $F(t_1, t_2, \ldots.)$ is given by the formula

$$F(T) = \operatorname{res}_{\infty}\left(\sum_{i=1} t_i K^i \, \mathrm{d}S\right), \quad \mathrm{d}S(p) = Q(p)\, \mathrm{d}E(p). \tag{4.87}$$

The function $\tau(T) = \exp(F(T))$ is the τ-function of the dispersionless Lax hierarchy. The general definition of such a function for algebraic orbits of the universal Whitham hierarchy for all genera is given by the formula

$$\ln \tau = \int_{\Gamma_g} \bar{\mathrm{d}}S \wedge \mathrm{d}S, \quad \mathrm{d}S = Q\, \mathrm{d}E. \tag{4.88}$$

Important remark. The integral (4.88) is not equal to zero, because $S(p, T)$ is holomorphic on Γ except for the punctures P_α and some contours, where it has 'jumps'. Therefore, the integral over Γ_g equals to a sum of the residues at P_α and the contour integrals of the corresponding one-forms.

The τ-function is a function of the variables t_A, $\tau = \tau(T)$. As it was shown in [14] it contains full information about the corresponding solutions $\Omega_A(p, T)$ of the Whitham equations. For $g > 0$ in τ a geometry of the moduli spaces is encoded.

At the end of this section we present the results of [14] where it was shown that the open domain $\mathcal{N}_g(n_\alpha)$ of the 'small' phase space

$$\mathcal{N}_g(n_\alpha) = \{\Gamma_g,\ \mathrm{d}E,\ a_i, b_i \in H_1(\Gamma_g, Z)\} \tag{4.89}$$

that is the moduli space of marked smooth genus g algebraic curves with fixed normalized Abelian differential $\mathrm{d}E$ which has poles of orders $n_\alpha + 1$ at some points P_α, is a Fröbenius foliation, i.e. a smooth foliation with leaves such that the fibers of the tangent bundle has a structure of the Fröbenius algebra and they form the potential deformation of these algebras.

The leaves $\mathcal{N}_g^V(n_\alpha)$ of the foliation are defined by the periods of $\mathrm{d}E$

$$\oint_{b_k} \mathrm{d}E = V_k, \tag{4.90}$$

and are parametrized by constants $V = (V_k)$. Let us us denote the normalized differential with the only pole at the point P_1 of the form (4.91) by $\mathrm{d}p$;

$$\mathrm{d}p = \mathrm{d}(E^{1/n_1} + \mathrm{O}(1)). \tag{4.91}$$

From Theorem 4.3, it follows that the functions

$$t_{\alpha,i} = \frac{1}{i}\operatorname{res}_{P_\alpha}\left(k_\alpha^{-i}(p)p\,\mathrm{d}E(p)\right), \qquad \begin{array}{l}\alpha = 1,\ i = 1,\dots,n_1 - 1,\\ \alpha > 1,\ i = 1,\dots,n_\alpha,\end{array} \tag{4.92}$$

$$t_{\alpha,0} = \operatorname{res}_\alpha(p\,\mathrm{d}E(p)); \tag{4.93}$$

$$t_{h,k} = \oint_{a_k} p\,\mathrm{d}E, \quad k = 1,\dots,g; \tag{4.94}$$

$$t_{E,k} = \oint_{b_k} \mathrm{d}p, \quad k = 1,\dots,g; \tag{4.95}$$

define a system of the coordinates on the leaves $\mathcal{N}_g^V(n_\alpha)$ of the foliation $\mathcal{N}_g(n_\alpha)$. Therefore, the vectors $\partial_a = \partial/\partial t_a$ form the basis in the tangent space of $\mathcal{N}_g^V(n_\alpha)$.

THEOREM 4.5. *The formulae*

$$\langle \partial_a, \partial_b \rangle = \sum_{q_s} \operatorname{res}_{q_s}\left(\frac{\mathrm{d}\Omega_a\,\mathrm{d}\Omega_b}{\mathrm{d}E}\right), \quad \mathrm{d}E(q_s) = 0, \tag{4.96}$$

$$C_{abc} = \sum_{q_s} \operatorname{res}_{q_s}\left(\frac{\mathrm{d}\Omega_a\,\mathrm{d}\Omega_b\,\mathrm{d}\Omega_c}{\mathrm{d}p\;\mathrm{d}E}\right), \tag{4.97}$$

(*where the differentials* $\mathrm{d}\Omega_a$ *are defined by* (2.15, 2.16, 4.43, 4.69–4.72)) *define on* $\mathcal{N}_g^V(n_\alpha)$ *the structure of the Fröbenius manifold. The vector* $\Psi(t,\lambda)$ *with coordinates*

$$\Psi_a(t) = \int \mathrm{d}\Omega_a\,\mathrm{e}^{\lambda E} = \partial_a \int \mathrm{d}p\,\mathrm{e}^{\lambda E}, \tag{4.98}$$

are horizontal sections of the flat connection (4.81), *i.e.*

$$(\partial_a + \lambda C_a(t))\Psi(t,\lambda) = 0. \tag{4.99}$$

Within the framework of the theory of topological models (see [28]), this result may be presented in the following way. The linear space spanned by the differentials $\mathrm{d}\Omega_a$ is isomorphic to H^1 cohomology of the operator $\mathrm{D}_1 = \bar\partial + \mathrm{d}E\wedge$ acting in the space of smooth differential forms on the curve Γ_g. The bilinear form (4.96) on the space of such differentials coincides with the natural pairing in the middle dimensional cohomologies of this operator. The periods (4.98) define the class of equivariant cohomologies of the operator

$$\bar\partial + \mathrm{d}E + \lambda\partial \tag{4.100}$$

that are horizontal sections of the flat connection defined with the help of (4.97).

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