General Rational Reductions of the KP Hierarchy and Their Symmetries

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The main goal of this paper is to prove the existence of a new general type of reductions for the hierarchy of the Kadomtsev–Petviashvili (KP) equation. The KP equation

$$\tfrac{3}{4}u_{yy} + (u_t - \tfrac{3}{2}uu_x - \tfrac{1}{4}u_{xxx})_x = 0 \tag{1}$$

was the first spatially two-dimensional equation included in the framework of the inverse problem method. As was observed in [1, 2], Eq. (1) possesses the zero-curvature representation

$$[\partial_y - L, \partial_t - A] = 0, \tag{2}$$

where

$$L = \partial_x^2 + u(x,y,t), \qquad A = \partial_x^3 + \tfrac{3}{2}u\partial_x + w(x,y,t). \tag{3}$$

The zero-curvature representation was also used to describe an infinite set of nonlinear equations compatible with the KP equation. These equations form a system of equations for the coefficients $v_{i,n}$ of an infinite set of ordinary linear differential operators

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} v_{i,n}(t_1, t_2, \ldots)\partial_x^i, \tag{4}$$

and are equivalent to the operator equations

$$[\partial_m - L_m, \partial_n - L_n] = 0, \qquad \partial_n = \partial/\partial t_n. \tag{5}$$

In this form the "hierarchy" of the KP equation is an infinite system of equations for infinitely many unknown functions depending on infinitely many variables.

The KP hierarchy was defined in [3] as a system of commuting evolution equations on the space of infinite tuples $u_i(x)$, $i = 1, 2, \ldots$, of functions of one variable. The corresponding equations are equivalent to the Lax equations

$$\partial_n \mathcal{L} = [\mathcal{L}_+^n, \mathcal{L}] \tag{6}$$

for the pseudodifferential operator

$$\mathcal{L} = \partial_x + \sum_{i=1}^{\infty} u_i \partial_x^{-i}. \tag{7}$$

(From now on D_+ stands for the differential part of a pseudodifferential operator D.) The equivalence of these two definitions of KP hierarchy was proved in [4].

The basic type of reductions that has been considered in the theory of KP hierarchy is the reduction to stationary points of one of the flows of the hierarchy (or of a linear combination of such flows). The corresponding invariant submanifolds are characterized by the property that the nth power of the corresponding pseudodifferential operator $\mathcal{L}$ is a differential operator, that is,

$$\mathcal{L}^n = \mathcal{L}_+^n = L = \partial_x^n + \sum_{i=1}^{n-2} w_i(x). \tag{8}$$

The coefficients of $\mathcal{L}$ are differential polynomials in the coefficients w_i of the differential operator L, and w_i parametrize the corresponding invariant subspace of the KP hierarchy.

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Let $\mathcal{K}_{m,n}$ be the manifold of pseudodifferential operators $\mathcal{L}$ such that

$$\mathcal{L}^n = L_1^{-1} L_2 \,, \tag{9}$$

where L_1 and L_2 are coprime differential operators of degrees m and $n+m$, respectively. (Two differential operators are said to be coprime if their kernels do not intersect.) The coefficients of these operators

$$L_1 = \partial_x^k + \sum_{i=1}^{m-1} w_{1,i} \partial_x^i \,, \qquad L_2 = \partial_x^{n+k} + \sum_{j=1}^{n+m-1} w_{2,j} \partial_x^j \tag{10}$$

are parameters that specify the points of $\mathcal{K}_{m,n}$. The normalization of $\mathcal{L}$ such that there is no free term on the right-hand side in (7) is equivalent to the only relation on the coefficients of L_1 and L_2:

$$w_{1,m-1} = w_{2,n+m-1} \,. \tag{11}$$

The coefficients of pseudodifferential operators that belong to $\mathcal{K}_{m,n}$ are differential polynomials in $w_{1,i}$ and $w_{2,j}$.

A priori, the definition of $\mathcal{K}_{m,n}$ depends on the order of factors in the right-hand side in (9). But we are going to show that the corresponding submanifold of pseudodifferential operators is independent of this order and depends only on the degrees of the numerator and denominator of the noncommutative fraction.

Lemma 1. *For any coprime differential operators L_3 and L_4 of degrees m and $n+m$, respectively, there exist unique normalized differential operators L_1 and L_2 of degrees m and $n+m$, respectively, such that*

$$L_1^{-1} L_2 = L_4 L_3^{-1} . \tag{12}$$

(A differential operator is said to be normalized if its leading coefficient is equal to 1.)

Remark. This result is well known. Exact formulas for the coefficients of L_1 and L_2 can be found in [5]. Nevertheless, we are going to present the proof in a form that will be useful later.

Proof. Let $\mathcal{O}_i$, $i = 3, 4$, be the kernel of L_i, i.e.,

$$y(x) \in \mathcal{O}_i : \quad L_i y(x) = 0 \,. \tag{13}$$

The dimension of the linear space $\mathcal{O}_4$ is $n+m$. It follows from the hypothesis of the lemma that the image $L_3(\mathcal{O}_4)$ of this space has the same dimension. Therefore, the equation

$$L_2 y(x) = 0, \qquad y(x) \in L_3(\mathcal{O}_4), \tag{14}$$

uniquely defines a normalized differential operator L_2 of degree $n+m$. The operator L_1 of degree n is defined by the equation

$$L_1 y(x) = 0, \qquad y(x) \in L_4(\mathcal{O}_3) \,. \tag{15}$$

It follows from the definition of L_1 and L_2 that the differential operators $L_2 L_3$ and $L_1 L_4$ of degree $2n+m$ have the same kernel $\mathcal{O}_3 + \mathcal{O}_4$. Therefore, they are equal. The equation

$$L_1 L_4 = L_2 L_3 \tag{16}$$

is equivalent to (12). The lemma is thereby proved.

In the same way one can prove the converse statement: for any coprime differential operators L_1 and L_2 there exist unique normalized differential operators L_3 and L_4 such that Eq. (12) is valid.

Theorem 1. *For any n and m the space $\mathcal{K}_{m,n}$ is invariant with respect to the* KP *hierarchy* (6).

Proof. Equation (6) for $\mathcal{L} \in \mathcal{K}_{m,n}$ is equivalent to the equation

$$\partial_i(L_1^{-1}L_2) = [(L_1^{-1}L_2)_+^{i/n}, L_1^{-1}L_2]. \tag{17}$$

It follows from (17) that

$$(\partial_i L_2)L_3 - (\partial_i L_1)L_4 = L_1(L_1^{-1}L_2)_+^{i/n}L_4 - L_2(L_1^{-1}L_2)_+^{i/n}L_3, \tag{18}$$

where L_3 and L_4 are differential operators such that (12) is satisfied.

To prove the theorem, it suffices to show that for given L_1 and L_2 Eq. (18) uniquely defines operators $\partial_i L_1$ and $\partial_i L_2$ of degrees $m-1$ and $n+m-1$, respectively.

Let $\mathcal{D}$ be the operator defined by the right-hand side in (18). The coefficients of this operator are differential polynomials in the coefficients of L_1 and L_2, that is, $\mathcal{D} = \mathcal{D}(L_1, L_2)$. In the same way as in the proof of the correctness of the KP hierarchy, one can show that this operator is of degree $n+2m-1$. Indeed, any operator commutes with the powers of itself (including fractional powers). Hence, the differential part of the operator $(L_1^{-1}L_2)^{i/n}$ can be replaced in (17) by its integral part

$$(L_1^{-1}L_2)_-^{i/n} = (L_1^{-1}L_2)^{i/n} - (L_1^{-1}L_2)_+^{i/n}. \tag{19}$$

Therefore,

$$\mathcal{D} = L_2(L_1^{-1}L_2)_-^{i/n}L_3 - L_1(L_1^{-1}L_2)_-^{i/n}L_4. \tag{20}$$

The last equation shows that the degree of the differential operator $\mathcal{D}$ is less than or equal to $n+2m-1$.

Lemma 2. *For any differential operator $\mathcal{D}$ of degree $n+2m-1$ and for any coprime differential operators L_3 and L_4 of degrees m and $n+m$, respectively, there exist unique differential operators A_1 and A_2 of degrees $m-1$ and $n+m-1$, respectively, such that*

$$A_2L_3 - A_1L_4 - \mathcal{D} = 0. \tag{21}$$

Proof. A differential operator A_1 of degree $m-1$ is uniquely determined by its action on any m-dimensional linear space. Therefore, it can be specified by the equation

$$A_1y(x) = \mathcal{D}y(x), \qquad y(x) \in L_4(\mathcal{O}_3). \tag{22}$$

A similar equation,

$$A_2y(x) = \mathcal{D}y(x), \qquad y(x) \in L_3(\mathcal{O}_4), \tag{23}$$

specifies the operator A_2. It follows from Eqs. (21) and (22) that the differential operator specified by the left-hand side in (21) is of degree $n+2m-1$ and that the dimension of its kernel is greater than or equal to $n+2m$. Hence, this operator is zero. Lemma 2 is thereby proved.

If L_3, L_4 and $\mathcal{D}$ are given by Eqs. (12) and (20), then the operators A_i are uniquely determined by L_1 and L_2, that is, $A_i = A_i(L_1, L_2)$. Therefore, the equations

$$\partial_i L_1 = A_1(L_1, L_2), \qquad \partial_i L_2 = A_2(L_1, L_2) \tag{24}$$

correctly define the evolution of the operators L_1 and L_2. The theorem is proved.

When discussing this result with the author, T. Shiota proposed an explicit form of the equations that define the evolution of L_1 and L_2.

Theorem 2. *The restriction of the* KP *hierarchy to $\mathcal{K}_{m,n}$ is equivalent to the equations*

$$\partial_i L_1 = L_1(L_1^{-1}L_2)_+^{i/n} - (L_2L_1^{-1})_+^{i/n}L_1, \tag{25}$$

$$\partial_i L_2 = L_2(L_1^{-1}L_2)_+^{i/n} - (L_2L_1^{-1})_+^{i/n}L_2. \tag{26}$$

Proof. It follows from Theorem 1 that (17) correctly defines the evolution of L_1 and L_2. On the other hand, it can readily be verified that Eq. (17) follows from Eqs. (25) and (26). Therefore, to prove the theorem it suffices to verify that the right-hand sides in Eqs. (25) and (26) are differential operators of degrees not greater than $m-1$ and $n+m-1$, respectively. The last statement follows from the fact that, by virtue of the identities

$$L_1^{-1}(L_2L_1^{-1})^{i/n}L_1 = (L_1^{-1}L_2)^{i/n}, \qquad L_2^{-1}(L_2L_1^{-1})^{i/n}L_2 = (L_1^{-1}L_2)^{i/n}, \tag{27}$$

Eqs. (25) and (26) are equivalent to the equations

$$\partial_i L_1 = (L_2L_1^{-1})_-^{i/n}L_1 - L_1(L_1^{-1}L_2)_-^{i/n}, \tag{28}$$

$$\partial_i L_2 = (L_2L_1^{-1})_-^{i/n}L_2 - L_2(L_1^{-1}L_2)_-^{i/n}. \tag{29}$$

The theorem is proved.

Remark. Equations (28) and (29) are similar to the formulas obtained in [6] that describe the evolution of the factors $L = L_1L_2$, where L_i are differential operators or pseudodifferential operators of some special kind.

Example. A particular case of the cited statements is given by the results in the papers [8, 9], where it was proved that the hierarchy of the nonlinear Schrödinger equation can be obtained as a special reduction of the KP hierarchy. In the notation of the present work, the corresponding reduction is the reduction to the space $\mathcal{K}_{1,2}$.

The coefficients of the pseudodifferential operator

$$\mathcal{L} = (\partial_x + v)^{-1}(\partial_x^2 + v\partial_x + w) = \partial_x + \sum_{i=1}^{\infty} u_i\partial_x^{-i} \tag{30}$$

are differential polynomials in the functions v and w and are recursively defined by the equations

$$u_1 = w, \qquad u_{i+1} + u_{ix} + vu_i = 0, \quad i > 1. \tag{31}$$

Let us consider the second flow of the KP hierarchy (6), that is, the equation that defines the dependence of $\mathcal{L}$ on the variable t_2. For $n = 2$, we have $\mathcal{L}_+^2 = \partial_x^2 + 2u_1$. The corresponding equations for the first two coefficients of $\mathcal{L}$ can be reduced to a closed system of two equations for two unknown functions by using Eq. (31) for $i = 1, 2$:

$$\partial_2 u_1 = u_{1xx} + 2u_{2x}, \qquad u_2 = -u_{1x} + vu_1, \tag{32}$$

$$\partial_2 u_2 = u_{2xx} + 2u_{3x} + 2u_1u_{1x} = u_{2xx} - 2(u_{2x} + vu_2)_x + 2u_1u_{1x}. \tag{33}$$

Let us define two new unknown functions r and q by setting

$$u_1 = rq, \quad v = -r_x/r. \tag{34}$$

Equations (32) and (33) are equivalent to the system

$$\partial_2 r = r_{xx} + rqr, \qquad \partial_2 q = -q_{xx} + qrq, \tag{35}$$

which for $r = \pm q^*$ and $t_2 = it$ transforms into the nonlinear Schrödinger equation

$$ir_t - r_x x \mp |r|^2 r = 0. \tag{36}$$

Remark. Equations (35) admit the Lax representation $L_t = [A, L]$, where L is the Dirac operator. It seems quite natural to conjecture that the proposed reductions of the KP hierarchy contain all equations that have a Lax representation with differential operators L with matrix coefficients of an arbitrary dimension.

The restrictions of the KP hierarchy to the spaces $\mathcal{K}_{m,n}$ give a "quantization" of the corresponding algebraic orbits [7] of the dispersionless KP hierarchy. The hierarchy of the dispersionless KP equation is a system of commuting evolution equations

$$\partial_i K = \{K_+^i, K\} = \partial_p(K_+^i)\partial_x K - \partial_x(K_+^i)\partial_p K \tag{37}$$

for the coefficients of the Laurent series

$$K(p) = p + \sum_{i=1}^{\infty} v_i(x, t_1, \ldots) p^{-i}. \tag{38}$$

(In (37), $[\ldots]_+$ stands for the part of the Laurent series containing the nonnegative powers of p.) The reductions of the hierarchy (37) to spaces of Laurent series such that

$$K^n = E(p,t) = p^n + u_{n-2}p^{n-2} + \cdots + u_0 + \sum_{\alpha=1}^{N} \sum_{i=1}^{n_\alpha} v_{i,\alpha}(p - p_\alpha)^{-i} \tag{39}$$

were referred to as algebraic orbits in [7]. We should like to point out that it was shown in [7] that the algebraic orbits of the dispersionless KP hierarchy have additional symmetries, i.e., that the reductions of the hierarchy (37) to a space of Laurent series K satisfying Eq. (39) are compatible with the evolution equations

$$\partial_{i,\alpha} K = \{\Omega_{i,\alpha}, K\} = \partial_p(\Omega_{i,\alpha})\partial_x K - \partial_x(\Omega_{i,\alpha})\partial_p K, \tag{40}$$

where $\Omega_{i,\alpha}$ is a polynomial of degree i in the variable $(p - p_\alpha)^{-1}$ such that

$$\Omega_{i,\alpha} = \sum_{s=1}^{i} w_{s,i,\alpha}(p - p_\alpha)^{-s} = E^{i/n_\alpha} + O(p - p_\alpha) \tag{41}$$

as $p \to p_\alpha$. The following theorem shows that these symmetries can be "quantized."

Theorem 3. *The restriction of the* KP *hierarchy to the invariant spaces $\mathcal{K}_{m,n}$ is compatible with the Lax evolution equations*

$$\mathcal{L}_\tau = [A_{-i}, \mathcal{L}], \tag{42}$$

where

$$A_{-i} = M_1^{-1} M_2, \tag{43}$$

M_1 is a normalized differential operator of degree i, and M_2 is a differential operator of degree $i-1$.

Proof. Equation (42) for $\mathcal{L} = L_1^{-1} L_2$ is equivalent to the equation

$$(\partial_\tau L_2)L_3 - (\partial_\tau L_1)L_4 = L_1 A_{-i} L_4 - L_2 A_{-i} L_3, \tag{44}$$

where L_3 and L_4 are operators for which Eq. (12) is satisfied. The pseudodifferential operator A_{-i} has a negative degree. Therefore, the right-hand side $\mathcal{D}$ in (44) is a pseudodifferential operator of degree less than or equal to $n + 2m - 1$. The final part of the proof of Theorem 1 shows that (44) is a well-defined evolution system if and only if the pseudodifferential operator $\mathcal{D}$ on its right-hand side is a differential operator. Let us show that this condition allows one to express the coefficients of M_1 and M_2 via the coefficients of L_1 and L_2, thus obtaining a closed system of equations. By Lemma 1, for a generic A_{-i} the operator $\mathcal{D}$ can uniquely be represented in the form

$$\mathcal{D} = D_1 D_2^{-1}, \tag{45}$$

where D_1 and D_2 are differential operators and D_2 has the degree $2i$. Therefore, the condition that D_1 is divisible by D_2 is equivalent to a system of $2i$ ordinary differential equations for $2i$ unknown coefficients of the operators M_1 and M_2. The theorem is proved.

Remark 1. It should be pointed out that the above-proved theorem claims the existence of symmetries of the form (42), (43) but does not provide an answer to the question of their complete classification. In particular, the question about how many of them exist for given i and what algebra they form is still open. The commutativity conditions for two symmetries of the form (43) have the form of the zero-curvature equations

$$[\partial/\partial\tau_1 - M_1^{-1}M_2, \partial/\partial\tau_2 - M_3^{-1}M_4] = 0, \tag{46}$$

which were considered in the recent paper [10], with no relation to the symmetries of rational reductions of the KP hierarchy, as a new form of generating integrable equations.

Remark 2. We would like to mention that symmetries of the form (42), (43) exist for the usual Lax reductions of the KP hierarchy as well. For example, for the KdV equation (i.e., for $\mathcal{L}^2 = L = \partial_x^2 + u$) and $i = 1$ the equation

$$L_\tau = u_\tau = [(v\partial_x + w)^{-1}, L] \tag{47}$$

is equivalent to the equations

$$u_\tau w^2 + v(u_\tau w)_x = w_{xx} - vu_x, \qquad v(u_\tau v)_x + 2u_\tau vw = v_{xx} + 2w_x, \tag{48}$$

$$u_\tau v^2 = 2v_x. \tag{49}$$

Let

$$w = \tfrac{1}{2}v_x; \tag{50}$$

then it follows from Eqs. (48) and (49) that

$$u = \tfrac{1}{2}\varphi_{xx} - \tfrac{1}{4}\varphi_x^2, \tag{51}$$

where φ is defined from the relation

$$v = e^\varphi. \tag{52}$$

Equations (51) and (49) imply that

$$\partial_\tau(\tfrac{1}{2}\varphi_{xx} - \tfrac{1}{4}\varphi_x^2) = 2\varphi_x e^{-\varphi} \tag{53}$$

is equivalent to the Lax equation (47) with condition (50).

In the forthcoming publication we are going to consider the cited open questions as well as the problem of constructing algebraic-geometrical solutions to general "rational" reductions of the KP hierarchy and their rational symmetries.

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