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Integrability and Seiberg-Witten exact solution

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Abstract

The exact Seiberg-Witten (SW) description of the light sector in the $N = 2$ SUSY $4d$ Yang-Mills theory [N. Seiberg and E. Witten, Nucl. Phys. B 430 (1994) 485 (E); B 446 (1994) 19] is reformulated in terms of integrable systems and appears to be a Gurevich-Pitaevsky (GP) [A. Gurevich and L. Pitaevsky, JETP **65** (1973) 65; see also, S. Novikov, S. Manakov, L. Pitaevsky and V. Zakharov, Theory of solitons] solution to the elliptic Whitham equations. We consider this as an implication that the dynamical mechanism behind the SW solution is related to integrable systems on the moduli space of instantons. We emphasize the role of the Whitham theory as a possible substitute of the renormalization-group approach to the construction of low-energy effective actions.

1. The exact expression for the vacuum-condensate dependence of the effective coupling constant in $d = 4$ $N = 2$ SUSY YM theory [1] provides a new basis for the search of a relevant description of the vacuum structure in non-abelian theories. Especially interesting is the emergence of the characteristic features of the $2d$-integrable structures in essentially $4d$ problem. In this letter we explain that the SW *answer* for $4d$ theory is just the same as the GP solution of the elliptic Whitham equations, which in its turn is a simple analog of the solutions to "string equations" arising in the context of $2d$ (world sheet) string theories and gravity models [8].

A more detailed discussion will be presented elsewhere.

2. We begin with a survey of the relevant statements from the general theory of $4d$ YM fields and from Ref. [1].

The simplest dynamical characteristic of YM theory is the effective coupling constant $g^{-2}(\mu)$ (defined as a coefficient in front of $\int \mathrm{tr}\, F^2_{\mu\nu}$ in the effective action) as a function of the normalization point μ (roughly speaking, the IR cut-off in the integration over fast quantum fluctuations).

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[8] To be exact we discuss throughout this letter the *first* GP solution, which arises as a step decay in the KdV theory, while the *second* one rather corresponds to $2d$ string equations.

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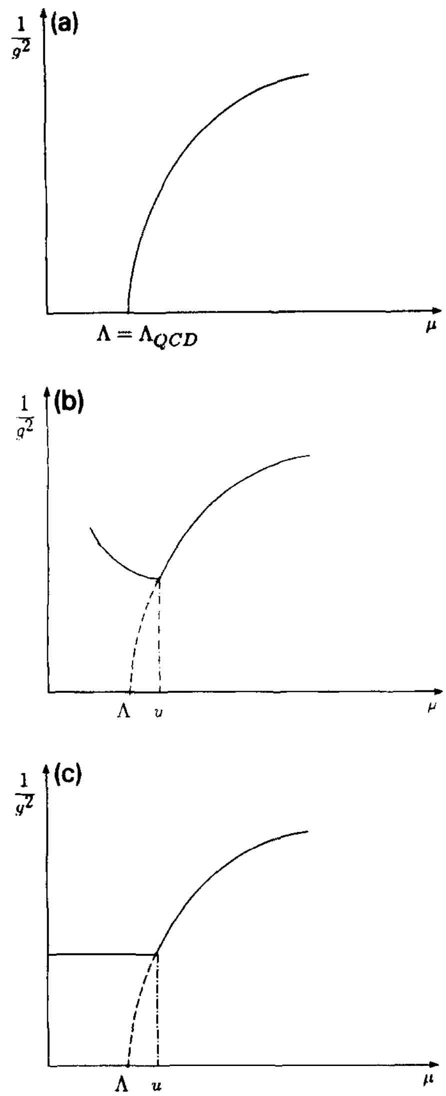

Fig. 1.

Within the perturbation theory for the non-abelian model this function is given by Fig. 1a. If there is a scalar-field condensate [9] u, spontaneously breaking the original gauge symmetry $SU(2)$ to $U(1)$, one gets instead a picture like Fig. 1b. If the original symmetry is larger than $SU(2)$ there is a series of transitions at various points u_a. In the $N = 2$ SUSY YM theory the $U(1)$ β-function is zero, and what one obtains is Fig. 1c. In this model one is actually interested in the function $g^{-2}(\mu|u)$ of *two* variables, since there is a valley in the effective potential and the value of u is a priori arbitrary (u is a dynamical variable). Because of the simple pattern in Fig. 1c, the function

$$g^{-2}(\mu = 0|u) = g^{-2}(\mu = u|0) \tag{1}$$

can be considered as carrying a certain information about the most intriguing quantity $g^{-2}(\mu|u = 0)$. In other words, one can substitute the typical confinement-phase problem (of evaluation of $g^{-2}(\mu|0)$) by the typical Higgs-phase one (of evaluation of $g^{-2}(0|u)$) and the latter one definitely makes sense even beyond the perturbation theory. It is also natural to introduce the full complex coupling constant $\tau = \frac{1}{2\pi}(ig^{-2} + \theta)$, where θ is the coefficient in front of the "topological" term $\int \operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu}$. Within the perturbation theory θ does not depend on μ and u (see, however, [3]).

The definition of $\tau(\mu)$, as well as identifications like Eq. (1), beyond perturbation theory gets ambiguous. However, a qualitative description is well known in the instanton-gas approximation [4]. The new behaviour, as compared to the perturbation theory, is the occurrence of μ-dependence of θ, which results in renormalization of the bare $\theta \neq \pi$ at $\mu = \infty$ to $\theta = 0$ at $\mu = \Lambda$, and deconfinement (occurrence of zero of β-function at $g^{-2} \neq 0$) at $\theta = \pi$. Both effects are described [5,6] by the characteristic renormalization-group flow shown in Fig. 2a. The analytic description is given by the equations

$$\frac{dg^{-2}}{d\log\mu} = b + ce^{-\gamma g^{-2}}\cos\theta,$$
$$\frac{d\theta}{d\log\mu} = ce^{-\gamma g^{-2}}\sin\theta, \tag{2}$$

where $b(g^2) = b_1 + o(g^2)$ and $c(g^2, \mu, u)$ are some positive functions depending on a particular model.

Beyond the instanton-gas approximation one should represent $\tau(\mu)$ as some parameter of the effective theory on the universal moduli space of instantons [10]. In the $N = 2$ SUSY case, where perturbation theory is almost trivial (for example, the perturbative β-function has only one-loop contributions, [8]), one can expect

[9] See [1] for notational details.

[10] In general this theory can have different phases. One of them – believed to be relevant for confinement in QCD – is known in less formal terms as that of the instanton fluid [7].

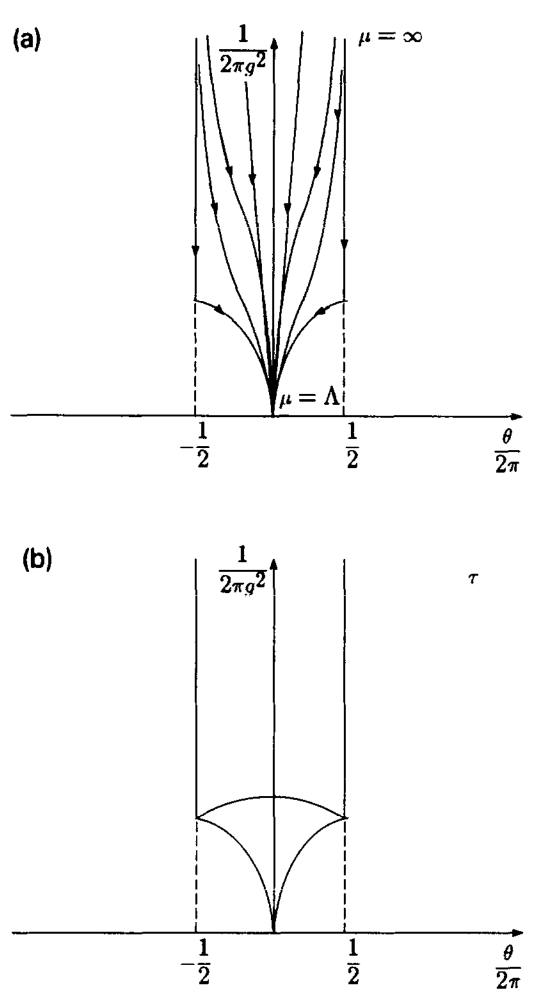

Fig. 2.

that the relevant dynamical system is especially simple. One of the possible ideas is that it somehow possesses integrable properties, peculiar for dynamics on known moduli spaces (see for example [9–11]). The results of [1], as well as their generalizations in [12], look as being consistent with this integrable dynamics.

Namely, in [1] $\tau(\mu = 0|u)$ is identified with the coordinate on the modular half-plane for the one-dimensional complex tori, see Fig. 2b, while u is interpreted as a parameter (one of ramification points) in their elliptic representation

$$y^2 = (z^2 - \Lambda^2)(z - u)\,, \tag{3}$$

i.e.

$$u = \Lambda\left(1 - 2\frac{\theta_2^4(0|\tau)}{}.\theta_3^4(0|\tau)\right) \tag{4}$$

The $SU(N)$ generalizations are described in [12] in terms of moduli of the specific subclass of hyperelliptic surfaces,

$$y^2 = P_N^2(z) - \Lambda^{2N}, \tag{5}$$

where P_N is any polynomial of degree N. These expressions provide an explicit way to avoid the singular point $u = \Lambda$ (where $\tau = 0$, i.e. $g^{-2} = 0$ and $\theta = 0$ – in accordance with qualitative Fig. 2a; note that u was restricted to be real in that picture) by analytic continuation into the complex u-plane. It also introduces one more singularity at another point $u = -\Lambda$ ($\tau = \pm 1$), while the vicinity of the last singular point $u = \infty$ ($\tau = i\infty$) is described by the ordinary perturbation theory (thus, the three "infinitely-remote" points are not identical, and the theory actually lives on the covering of moduli space – again, as suggested by the naive Fig. 2a).

Most impressive, [1] implies that the Riemann surfaces themselves – not just their moduli – have some physical significance. Namely, the spectrum of excitations in the theory is identified as

$$M_{m,n} = |ma + na_D|\,, \tag{6}$$

where

$$a = \oint_A \lambda, \quad a_D = \oint_B \lambda \tag{7}$$

and

$$\lambda = \frac{z-u}{y(z)}dz = \sqrt{u - \Lambda\cos\varphi}\,d\varphi \tag{8}$$

is a particular 1-differential *on the surface* with the double pole and the double zero at the ramification points $z = \infty$ and $z = u$ respectively [11].

[11] In terms of the φ-parametrization of the spectral curve (see (8)), the integrals (7) for $|u| < 1$ are just the action-integrals $\oint p(\varphi)d\varphi$ in the Sine-Gordon model $L_{SG} = \dot{\varphi}^2 - \Lambda\cos\varphi$ over the classically allowed and forbidden domains at a given "energy" u, see Fig. 3 (in this figure, the A-cycle corresponds to the interval I–IV, while the B-cycle is described by the interval III–V; one can certainly choose the cycles in another convention – say, instead of a, one can choose the linear combination $A - B$, which corresponds to the interval II–III). Note that in this parametrization

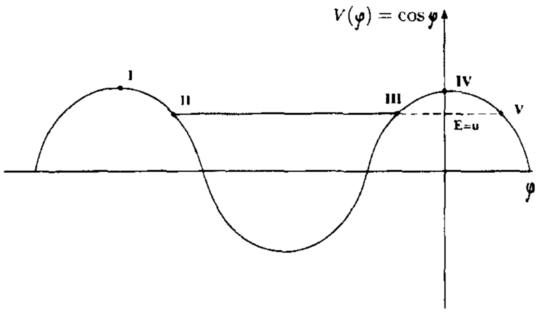

Fig. 3.

This poses the question of *what* is the *reason* for Riemann surfaces to appear in this theory: while τ and u are present in it from the very beginning, the surface is something new and emerges dynamically only in the low-energy effective theory.

3. The answer to this question is, of course, more general than the particular SW example [12].

The effective dynamics in the space of coupling constants, like θ and g^{-2}, substitutes the original dynamics in the ordinary space-time by a set of Ward identities (low-energy theorems), which normally have the form of non-linear differential equations for the effective action (which in this context is often referred to as the generalized τ-function). When these equations belong to (generalization of) the KP/Toda-type hierarchy – as it often happens after an appropriate choice of variables – their solutions (i.e. acceptable shapes of effective actions) are parametrized in terms of some auxiliary "spectral surfaces" also known as "target space" curves (not world-sheet) in the language of string theory.

The family of "vacua" of the original model is thus naturally associated with the family of spectral surfaces, i.e. with their moduli space. It seems that only the moduli space itself has physical meaning, not the spectral surfaces but this is not however quite true. So far we discussed the effective action (KP/Toda-like τ-function) as a function of time-variables (the coupling constants θ, g^{-2} etc). However, if considered as a function of *moduli* (e.g. of scalar condensates u), the *effective* τ-function induces a new (low-energy-sector) dynamics on the space of moduli. This new dynamics implies that the moduli are no longer invariants of motion: instead they are "RG-slow" dynamical variables of the theory [13].

The general approach to the construction of such effective actions is known as the Bogolyubov-Whitham averaging method (see [14,17] for a comprehensive review and references). Though this Whitham dynamics is that of the moduli, its explicit formulation is most simple and natural in terms of connections on spectral surfaces. Thus the low-energy dynamics actually gives a lot of physical significance to the spectral surfaces themselves, and, after all, it is not such a big surprise that the dynamical characteristics – of which (6) is a simplest example – are expressed in terms of them.

In the SW case one could try to be more specific – but in this letter we restrict ourselves to the following simplified scheme.

One begins with considering the u-field-dependent dynamics on the moduli space of instantons. One can further think that some directions in the functional space are most important for the low-energy theory. An obvious candidate for such a variable is $K = \int \epsilon_{ijk}(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k)d^3r$. Effective potentials are periodic in K and associated excitations are always light (unless they *mix* with something else which is also light – as is the case with the η'-meson in QCD). The conjugate variable to K is exactly θ: one of our most significant (along with the g^{-2})

φ is identified with $\varphi + 2\pi$. It also deserves mentioning that for the elliptic solution the surface itself is isomorphic to its Jacobian, thus the periods of differentials play the role of periods of the real motion in potential (8) of the "auxuliary" quantum-mechanical problem.

[12] The answer seems to be similar to that from the $2d$ (string theory) case where the arising (target-space!) spectral curve might be associated with the "scale-parameter" curve. The non-perturbative effects imply that such a surface has a nontrivial topology while the mechanism of arising the higher topologies is not yet clear.

[13] The situation is much similar to the standard renormalization group. Indeed, the RG dynamics is governed by the action of some vector field $d/\log\mu = \sum \beta_i(g)\partial/\partial g_i$ on the space of coupling constants. Analogously, the Whitham dynamics gives an example of *some* vector fields generated by the corresponding "slow"-time flows, the counterpart of the coupling constant space being the moduli space. Certainly, the standard RG approach is unambiguously used only within the perturbative framework, while we deal with the exact solution. Therefore, we consider Whitham as the corresponding generalization of the RG equations beyond the perturbative regime, which would still have the form of the first-order differential equations with respect to the coupling constants (co-ordinates in the moduli space).

"time variables". After "Legendre transform" one can think of the original dynamics of the u-fields as of a RG-like one in the space of coupling constants. Solutions to these "RG"-equations identify the valley vacuum averages with moduli of spectral surfaces. Monodromies on these surfaces are natural variables of the Toda-chain hierarchies, with the length of the chain equal to N for the $SU(N)$ gauge group. Indeed, the simplest type of dynamics for a variable U in the fundamental representation of $SU(N)$ is implied by the Lagrangian

$$\mathcal{L}_U = \mathrm{tr}(U^{-1}\partial U)^2 + \mathrm{tr}\, U, \tag{9}$$

which in the Cartan sector reduces to:

$$\mathcal{L}_{\text{eff}} = \sum_{a=1}^{N} \left((\partial_\mu \phi_a)^2 + e^{\phi_a} \right),$$

$$U \to \mathrm{diag}\left(e^{\phi_a}\right), \quad \sum_{a=1}^{N} \phi_a = 0. \tag{10}$$

Possible higher-derivative corrections to $\mathcal{L}_U$ can be associated with the higher Hamiltonians of the Toda-chain hierarchy.

Now comes the first miracle. According to [13], the finite-gap solutions to the Toda-chain systems are characterized exactly by hyperelliptic surfaces of the peculiar type (5) [14].

The next task is to consider the effective Whitham dynamics. With the "first miracle" in mind – and with the knowledge that all the Toda-chain systems are particular members of the KP/Toda-lattice family – we can just use the well-known Whitham theory of *integrable* hierarchies [14,17] [15] (these are exactly the ones that arised in the recent studies of $2d$ topological theories/gravities [16–19] and describe exact solutions to the string equations [20,21].)

4. Now let us turn to the next observation. If one takes as a characteristic of effective dynamics in the vicinity of the classical solution the SW formulas (6)–(8), one immediately recognizes them as familiar objects from the theory of the Whitham equations. Namely, λ in (7) is exactly the generating 1-differential arising in the first Gurevich-Pitaevsky problem [2] [16]

In formal terms, the Whitham equations can be described as follows. The KP/Toda-type τ-function associated with a given spectral Riemann surface is equal to

$$\mathcal{T}\{t_i\} = e^{t\gamma t}\vartheta\left(\mathbf{\Phi}_0 + \sum_{i=1}^{\infty} t_i \boldsymbol{k}_i\right),$$

$$\boldsymbol{k}_i = \oint_B d\Omega_i(z), \tag{11}$$

[14] We remind that the data (a spectral complex curve, a point on it and a complex coordinate in the vicinity of the point) is always in one-to-one correspondence with the solutions to the KP-hierarchy, the explicit relation being given in terms of the Baker-Ahiezer function (the curve itself can be also described by the evolution-invariant equation $\det(L(z) - y) = 0$, where $L(z)$ is the Lax-operator). Particular reductions of KP correspond to restrictions on the choice of Riemann surfaces. In particular, generic hyperelliptic surfaces correspond to solutions to KdV, while the subclass (5) describes solutions to the Toda-chain hierarchy. The most spectacular in the last relation is that the power of the polynomial P_N in (5) is exactly the length of the chain, i.e. the size of the $SU(N)$ matrices in the fundamental representation.

[15] This context can actually be not so narrow as it seems. As often happens, different original (non-renormalized) models produce the same kind of effective (renormalized) dynamics, and at the end of the day it can happen that integrable systems just label the classes of universality of effective actions. In other words, the concrete type of Whitham dynamics, even if derived from the study of integrable hierarchy, can have much broader significance. Moreover, the Whitham equations are themselves integrable, and – according to the previous remark – it is mostly this integrability that we refer to in the title of this letter.

[16] This problem came from the physics of fluids and concerns the decay of a step (Heavyside) function under the KdV evolution. The exact KdV dynamics,

$$\frac{\partial \tilde{u}}{\partial t_3} = \tilde{u}\frac{\partial \tilde{u}}{\partial t_1} + \frac{\partial^3 \tilde{u}}{\partial t_1^3},$$

drives the initial profile of Fig. 4a into that like Fig. 4b, while the Whitham dynamics describes the smooth enveloping curve, see Fig. 4c. For comparison, Fig. 4d shows the result of the evolution of the same step function according to the naive "quasiclassical" KdV (which is in fact the spherical Whitham equation): the Bateman-Hopf equation,

$$\frac{\partial \bar{u}}{\partial T_3} = \bar{u}\frac{\partial \bar{u}}{\partial T_1}.$$

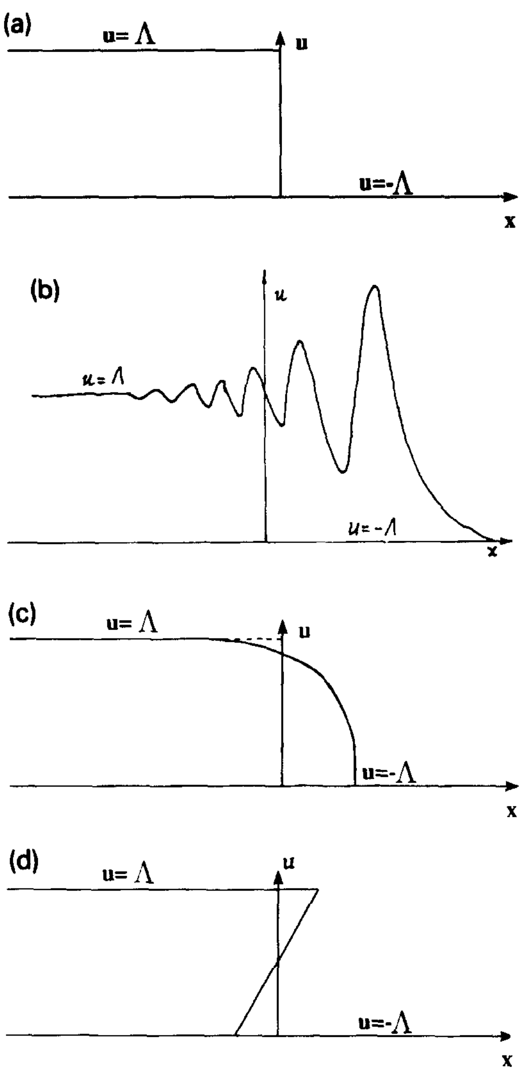

Fig. 4.

where ϑ is a Riemann theta-function and $d\Omega_i(z)$ are meromorphic 1-differentials with poles of the order $i+1$ at a marked point z_0. They are fully specified by the normalization relations

$$\oint_A d\Omega_i = 0 \tag{12}$$

and

$$d\Omega_i(z) = \left(\xi^{-i-1} + o(\xi)\right) d\xi \tag{13}$$

where ξ is the local coordinate in the vicinity of z_0. The moduli $\{u_\alpha\}$ of the spectral surface are invariants of the KP flows

$$\frac{\partial u_\alpha}{\partial t_i} = 0, \tag{14}$$

and label the "vacua" - the (finite-gap) solutions to the KP system. The effective dynamics on the space of these "vacua", generated by the Bogolyubov-Whitham method, arises with respect to some a priori new "slow" Whitham times T_i. The way the moduli depend on T_i is defined by the Whitham equations (induced by the fast KP/Toda-type equations), which for the two-dimensional integrable systems were first derived in [15] in the following form:

$$\frac{\partial d\Omega_i(z)}{\partial T_j} = \frac{\partial d\Omega_j(z)}{\partial T_i}. \tag{15}$$

These equations imply that

$$d\Omega_i(z) = \frac{\partial dS(z)}{\partial T_i} \tag{16}$$

with some "generating" 1-differential $dS(z)$, whose periods can be interpreted as the effective "slow" variables. Note that the self-evident relation (16) was crucially used in constructing the exact solutions to the Whitham equation that was proposed in [15]. The equations for moduli, implied by this system, are of a peculiar linear form:

$$\frac{\partial u_\alpha}{\partial T_i} = v_{ij}^{\alpha\beta}(u)\frac{\partial u_\beta}{\partial T_j} \tag{17}$$

with some (in general complicated) functions $v_{ij}^{\alpha\beta}$, which depend on the *type* of "vacua" under consideration [17].

In the KdV case all the spectral surfaces are hyperelliptic, i takes only odd values $i = 2j+1$, and

$$d\Omega_{2j+1}(z) = \frac{\mathcal{P}_{j+g}(z)}{y(z)}dz, \tag{18}$$

the coefficients of the polynomials $\mathcal{P}_j$ being fixed by normalization conditions (12), (13) (one usually takes $z_0 = \infty$ and the local parameter in the vicinity

[17] These formulas imply a special choice of the basis in the moduli space, taking the co-ordinates (T-variables) coming from commuting KP-flows. The relation $\tau_{ij} = \partial^2 \log \mathcal{T}/\partial T_i \partial T_j$ which defines the period matrix in terms of the $N = 2$ superpotential [1] has also appeared in the theory of topological $2d$-theories, see [18].

of this point is $\xi = z^{-1/2}$). In this case Eqs. (17) can be diagonalized if the coordinates $\{u_\alpha\}$ on the moduli space are taken to be the ramification points:

$$v_{ij}^{\alpha\beta}(u) = \delta^{\alpha\beta} \left.\frac{d\Omega_i(z)}{d\Omega_j(z)}\right|_{z=u_\alpha}. \tag{19}$$

Now an important remark is that after one switches on the Whitham dynamics the periods of the differential dS defined by (16) become the periods of the "modulated" function (11). We will see below that it gives us the SW spectrum.

5. Let us be more specific in the elliptic (GP/SW) case and restrict ourselves to the first two time-variables, $i = 1,3$. The elliptic (one-gap) solution to KdV is

$$\begin{aligned}\tilde{u}(t_1,t_3,\dots|u) &= \frac{\partial^2}{\partial t_1^2}\log\mathcal{T}(t_1,t_3,\dots|u)\\ &= U_0\wp(k_1t_1+k_3t_3+\dots+\Phi_0|\omega,\omega') + \frac{u}{3},\end{aligned} \tag{20}$$

where $\wp(t)$ is the Weierstrass $\wp$-function, and

$$\begin{aligned}dp &\equiv d\Omega_1(z) = \frac{z-\alpha(u)}{y(z)}dz,\\ dQ &\equiv d\Omega_3(z) = \frac{z^2-\frac{1}{2}uz-\beta(u)}{y(z)}dz.\end{aligned} \tag{21}$$

Normalization conditions (12) prescribe that

$$\alpha(u) = \frac{\oint_A \frac{zdz}{y(z)}}{\oint_A \frac{dz}{y(z)}} \quad\text{and}\quad \beta(u) = \frac{\oint_A \frac{(z^2-\frac{1}{2}uz)dz}{y(z)}}{\oint_A \frac{dz}{y(z)}}. \tag{22}$$

The observation, that we referred to at the beginning of Section 3, is that a particular solution $dS(z)$ to Eqs. (16) in the elliptic case is the same as the differential $\lambda(z)$ in (8).

Indeed, as we are going to demonstrate,

$$\begin{aligned}dS(z) &= \left(T_1 + T_3(z+\frac{1}{2}u) + o(T_5) + \dots\right)\\ &\quad\times\frac{z-u}{y(z)}dz\\ &= g(z|T_i,u)\lambda(z),\end{aligned} \tag{23}$$

where $g(z)$ is a calculable function of Whitham times with pole only at $z=\infty$ of the order $\frac{l-1}{2}$, if $T_l \neq 0$ and all the $T_{i>l} = 0$. The reason why $dS(z)$ has this particular form (i.e. possesses double zero at $z=u$) is simple. Normally, the derivative of a meromorphic object over moduli has more poles (since after a change of the complex structure the holomorphic object becomes non-holomorphic), and moduli in the hyperelliptic parametrization are located at the ramification points. In our case there is just one ramification point, u, which is T_i-dependent, and, in order to cancel the pole at $z=u$ in $\partial dS(z)/\partial T_i$ (which does not occur in $d\Omega_i(z)$), one needs to put some power of $(z-u)^{1/2}$ in the numerator of $dS(z)$ – once $y(z)$ appeared in the denominator. Since $(z-u)^{1/2}$ is not a single-valued function on the surface, one needs to take its square.

From (23) one derives:

$$\begin{aligned}\frac{\partial dS(z)}{\partial T_1} &= \left(z-u-(\frac{1}{2}T_1+uT_3)\frac{\partial u}{\partial T_1}\right)\frac{dz}{y(z)},\\ \frac{\partial dS(z)}{\partial T_3} &= \left(z^2-\frac{1}{2}uz-\frac{1}{2}u^2\right.\\ &\quad\left.-(\frac{1}{2}T_1+uT_3)\frac{\partial u}{\partial T_3}\right)\frac{dz}{y(z)},\\ &\dots,\end{aligned} \tag{24}$$

and comparison with explicit expressions (21) implies:

$$\begin{aligned}(\frac{1}{2}T_1+uT_3)\frac{\partial u}{\partial T_1} &= \alpha(u)-u,\\ (\frac{1}{2}T_1+uT_3)\frac{\partial u}{\partial T_3} &= \beta(u)-\frac{1}{2}u^2.\end{aligned} \tag{25}$$

In other words, this construction provides a (GP) solution to the Whitham equation

$$\frac{\partial u}{\partial T_3} = v_{31}(u)\frac{\partial u}{\partial T_1}, \tag{26}$$

with

$$v_{31}(u) = \frac{\beta(u)-\frac{1}{2}u^2}{\alpha(u)-u} = \left.\frac{d\Omega_3(z)}{d\Omega_1(z)}\right|_{z=u}, \tag{27}$$

which can be expressed through elliptic integrals [2].

We see that (7) can be reinterpreted as

$$a = \left.\frac{1}{T_1}\oint_A dS(z)\right|_{T_3,T_5,\dots=0},$$

$$a_D = \frac{1}{T_1} \oint_B dS(z) \Big|_{T_3, T_5, \ldots = 0} \tag{28}$$

being the periods of the "modulated" GP elliptic solution. This implies that in the generic situation (for non-elliptic surfaces and all $T_{2j+1} \neq 0$) the SW formula (6) should be

$$\mathcal{M}_{m,n} \sim \left| m \oint_A dS + n \oint_B dS \right| . \tag{29}$$

Note also that

$$\frac{\partial}{\partial T_i} \oint_A dS = \oint_A d\Omega_i = 0, \tag{30}$$

while

$$\frac{\partial}{\partial T_i} \oint_B dS = \oint_B d\Omega_i = k_i, \tag{31}$$

which are the frequencies in the original KP/Toda-type solution (11). So, the periods of the "modulated" Whitham solution give rise to the mass spectrum in the SW exact solution and its generalizations.

6. All the quantities entering the Whitham equations have the meaning of the averaged characteristics of the bare elliptic solution (3). Note also, that the further speculation of the meaning of the GP solution in the SW context of $4d$ gauge theory is possible. If one relates KP times t_1, t_3 with (the functions of) bare θ and g^{-2} then the main object under consideration – the KdV "potential" $u(t_1, t_3)$ becomes related to the correlator $\langle F\tilde{F}, F\tilde{F}\rangle$. This looks quite hopeful since such correlators contain the information about topological excitations in the gauge theory. Now after the averaging the "slow" times T_1, T_3 in the Whitham system can be identified with the functions of the "renormalized" KP times (coupling constants). Moreover the form of the GP solution suggests its interpretation as a "decay" of the topological excitations in SW theory in the non-perturbative regime $|u| < \Lambda$.

The GP solutions have automodel form and this can be related to the emergence of the holomorphic coupling constant $\tau = \frac{1}{2\pi}(ig^{-2} + \theta)$. The physical implications of the GP solution for the strong-coupling dynamics of the YM theory will be discussed elsewhere.

7. To conclude, we see that the central formula (6) of [1] can be interpreted as (29), i.e. in terms of periods of the central object $dS(z)$ in the theory of the Whitham hierarchy. This observation seems to be important since there exists a general belief that low-energy effective actions are proper objects to be referred to as *generalized* τ-functions. One should add that conceptually the Whitham method is precisely the averaging over fast fluctuations, which is necessary to produce the effective action for slow variables i.e. plays the role of the non-perturbative analog of the renormalization group. We believe that this analogy deserves attention and further studies [18] will put them on a more solid ground.

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[18] Among related studies, one should mention [22–25]. It also deserves noting that – as usual – the study of some $2d$ analogues of the $4d$ YM theory can shed additional light on the problem. The obvious relation is to $2d$ topological theories and string equations. In both cases the Whitham dynamics is known to arise in the description of effective actions (see [26] for a discussion of parallels with the $2d$ physics).

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