Russian Math. Surveys **50**:6 1101–1150

Uspekhi Mat. Nauk **50**:6 3–56 UDC 512

Spin generalization of the Ruijsenaars–Schneider model, the non-Abelian 2*D* Toda chain, and representations of the Sklyanin algebra

I. Krichever and A. Zabrodin

Contents

§ 1. Introduction

In some sense the results presented in this paper are a by-product of our attempt to analyse representations of the Sklyanin algebra, that is, the algebra generated by four elements S_0, S_α, $\alpha = 1, 2, 3$, that satisfy the following homogeneous quadratic relations:

$$[S_0, S_\alpha]_- = iJ_{\beta\gamma}[S_\beta, S_\gamma]_+, \tag{1.1}$$

$$[S_\alpha, S_\beta]_- = i[S_0, S_\gamma]_+, \tag{1.2}$$

where $[A, B]_\pm = AB \pm BA$, and a triple of Greek indices α, β, γ in (1.1), (1.2) stands for any *cyclic* permutation of $(1, 2, 3)$. The structure constants $J_{\alpha\beta}$ of the algebra are given by

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma}, \tag{1.3}$$

The work of I.K. was supported by the International Scientific Foundation under grant MD-8000, and by the Russian Foundation for Fundamental Research under grant 93-011-16087. The work of A.Z. was supported in part by the Russian Foundation for Fundamental Research under grant 93-02-14365, by the International Scientific Foundation under grant MGK300, and by ISTC under grant 015.

where J_α are arbitrary constants. Therefore, (1.1)–(1.3) define a two-parameter family of quadratic algebras. These relations (1.3) were introduced in [1] as the minimal set of conditions under which the operators

$$L(u)=\sum_{a=0}^{3} W_a(u)S_a\otimes\sigma_0 \tag{1.4}$$

satisfy the equation

$$R^{23}(u-v)L^{13}(u)L^{12}(v)=L^{12}(v)L^{13}(u)R^{23}(u-v). \tag{1.5}$$

Here σ_α are Pauli matrices, σ_0 is the unit matrix, the functions $W_a(u)=W_a(u\,|\,\eta,\tau)$, $a=0,\dots,3$, depend on parameters η and τ in a such way that

$$W_a(u)=\frac{\theta_{a+1}(u)}{\theta_{a+1}(\eta/2)}, \tag{1.6}$$

where $\theta_a(x)=\theta_a(x\mid\tau)$ are standard Jacobi theta-functions with characteristics and the modular parameter τ, and the function

$$R(u)=\sum_{a=0}^{3} W_a\Big(u+\frac{1}{2}\eta\Big)\sigma_a\otimes\sigma_a \tag{1.7}$$

is an elliptic solution of the quantum Yang–Baxter equation

$$R^{23}(u-v)R^{13}(u)R^{12}(v)=R^{12}(v)R^{13}(u)R^{23}(u-v), \tag{1.8}$$

corresponding to the so-called 8-vertex model. (Note that the R-matrix of the 6-vertex model is obtained at the limit $\tau\to 0$.)

In relations (1.5), (1.8) we use the following standard notation of the theory of the Yang–Baxter equation. For a module M over the Sklyanin algebra, equation (1.4) defines an operator in the tensor product $M\otimes C^2$. We denote by $L^{13}(u)$ ($L^{12}(u)$) the operator in the tensor product $M\otimes C^2\otimes C^2$ that acts as $L(u)$ on the first and third (second) spaces, and acts as the identity operator on the second (third) one. Similarly, R^{23} acts identically on M and coincides with the operator (1.7) on the second and third spaces.

It is well known (see, for example, the surveys [2]–[4]) that the classification of discrete quantum systems soluble by the quantum inverse scattering method reduces to solving (1.5) for the case when $R(u)$ is a fixed solution of the Yang–Baxter equation (1.8). Generalizations of the Sklyanin algebra, corresponding to more general elliptic solutions of (1.8) obtained in [5], were introduced in [6], [7]. At present only the simplest finite-dimensional representations of the generalized Sklyanin algebras are known. It is of great interest to construct representations of these algebras in terms of difference operators similar to the representations of the original Sklyanin algebra (1.1)–(1.3) described in [8].

As was shown in [8], the operators S_a, $a=0,\dots,3$, admit representations in the form of second-order difference operators acting in the space of meromorphic functions $f(x)$ of one complex variable x. One of the series of such representations has the form

$$(S_a f)(x) = \frac{(i)^{\delta_{a,2}}\theta_{a+1}(\eta/2)}{\theta_1(x)}\big(\theta_{a+1}(x-\ell\eta)f(x+\eta) - \theta_{a+1}(-x-\ell\eta)f(x-\eta)\big). \quad (1.9)$$

By a straightforward but tedious computation one can check that for any τ, η, ℓ the operators (1.9) satisfy the commutation relations (1.1)–(1.3) if the values of the structure constants are given by

$$J_\alpha = \frac{\theta_{\alpha+1}(\eta)\theta_{\alpha+1}(0)}{\theta^2_{\alpha+1}(\eta/2)}. \quad (1.10)$$

Therefore, the quantities τ, η parametrize the structure constants, and ℓ is the parameter of the representation. Note that Sklyanin's original parameter η introduced in [8] corresponds to the parameter $\frac{1}{2}\eta$ in (1.6), (1.7), (1.9) and (1.10).

Let us put $f_n = f(n\eta + x_0)$ and assign to the operators (1.9) the following difference Schrödinger operators with quasiperiodic coefficients:

$$S_a f_n = A^a_n f_{n+1} + B^a_n f_{n-1}. \quad (1.11)$$

The spectrum of a generic operator of this form in the space $l^2(\mathbb{Z})$ of square integrable sequences f_n has a structure of Cantor set type. If η is a rational number, $\eta = p/q$, then the coefficients of the operator (1.11) are q-periodic. In general, q-periodic difference Schrödinger operators have q unstable bands in the spectrum.

In § 5 we show that the operator S_0 defined by (1.9) possesses the following extremely unusual spectral property.

Theorem 1.1. *Given a positive integer 'spin' ℓ and an arbitrary η, the operator S_0 defined by* (1.9) *has 2ℓ unstable bands in the spectrum. Its Bloch functions are parametrized by points of the hyperelliptic curve of genus 2ℓ defined by the equation*

$$y^2 = R(\varepsilon) = \prod_{i=1}^{2\ell+1}(\varepsilon^2 - \varepsilon_i^2). \quad (1.12)$$

The Bloch eigenfunctions $\psi(x, \pm\varepsilon_i)$ of the operator S_0 at the edges of bands span a functional subspace that is invariant for all the operators S_a. The corresponding $(4\ell+2)$-dimensional representation of the Sklyanin algebra is a direct sum of two equivalent $(2\ell+1)$-dimensional representations of the Sklyanin algebra.

Remark. In § 5 we show that there is a unique choice of signs for ε_i such that the Bloch eigenfunctions $\psi(x, \varepsilon_i)$ induce an *irreducible* representation of the Sklyanin algebra. Unfortunately, at present we cannot point out an explicit and constructive procedure for splitting the edges into two parts. We conjecture that if the structure constants (and therefore the parameters ε_i) are real, then the irreducible representation is induced when all the edges of bands are positive, that is, $\varepsilon_i > 0$.

This theorem indicates a connection between representations of the Sklyanin algebra and the theory of finite-gap integration of soliton equations. (The theory of finite-gap difference Schrödinger operators [9]–[12] was developed in the context of solving the Toda chain and difference KdV equations.) Furthermore, the assertion of the theorem implies that S_0 is the proper difference analogue of the classical Lamé operator

$$L = -\frac{d^2}{dx^2} + \ell(\ell+1)\wp(x), \tag{1.13}$$

which is obtained from S_0 as $\eta \to 0$. Finite-gap properties of higher Lamé operators for arbitrary integer values of ℓ are well known (see [13] and the references therein).

In § 6 we propose a relatively simple procedure for deriving the functional realization (1.9) of the Sklyanin algebra by difference operators. This approach partially explains the origin of these operators. The basic tool is the key property of the elementary R-matrix (1.7), which was used by Baxter [14] in solving the eight-vertex model and which he called a 'pair-propagation through a vertex'. A suitable generalization of this property for an arbitrary spin L-operator (1.4) leads to formulae (1.9). This approach needs much less amount of computation than the direct substitution of the operators (1.9) in the commutation relations (1.1), (1.2). Note that this method gives automatically all the three representation series obtained by Sklyanin and an extra one unknown before.

In [15] the remarkable connection between the motion of poles of the elliptic solutions of the KdV equation (which are isospectral deformations of the higher Lamé potentials) and the Calogero–Moser dynamical system was revealed. As was shown in [16], [17], this connection becomes an isomorphism if we consider *the elliptic solutions of the Kadomtsev–Petviashvili (KP) equation*. In [18] the methods of finite-gap integration of the KP equation were applied to integrate the motion equations of the elliptic Calogero–Moser system in terms of Riemann theta-functions. In [19] these results have been extended to spin generalizations of the Calogero–Moser system.

In this paper we extend this theory to construct elliptic solutions of the two-dimensional ($2D$) Toda chain and its non-Abelian analogues. The equations of the $2D$ Toda chain have the form

$$\partial_+\partial_-\varphi_n = e^{\varphi_n-\varphi_{n-1}} - e^{\varphi_{n+1}-\varphi_n}, \qquad \partial_\pm = \frac{\partial}{\partial t_\pm}. \tag{1.14}$$

We consider solutions which are *elliptic with respect to the discrete variable* n, that is, solutions of the form

$$\varphi_n(t_+,t_-) = \varphi(n\eta + x_0, t_+, t_-) \tag{1.15}$$

such that the function

$$c(x,t_+,t_-) = \exp\bigl(\varphi(x,t_+,t_-) - \varphi(x-\eta,t_+,t_-)\bigr) \tag{1.16}$$

is elliptic with respect to the variable x. We show that in this case the function $\exp(\varphi)$ is given by the representation

$$\exp\varphi(x,t_+,t_-) = \prod_{i=1}^{n} \frac{\sigma(x-x_i+\eta)}{\sigma(x-x_i)}, \qquad x_i = x_i(t_+,t_-) \tag{1.17}$$

($\sigma(x|\omega_1, \omega_2)$ is the standard Weierstrass σ-function), and the dynamics of its poles x_i with respect to the time flows t_+, t_- is given by the motion equations for the Ruijsenaars–Schneider system [20]:

$$\ddot{x}_i = \sum_{s \neq i} \dot{x}_i \dot{x}_s \big(V(x_i - x_s) - V(x_s - x_i)\big), \tag{1.18}$$

where

$$V(x) = \zeta(x) - \zeta(x + \eta), \qquad \zeta(x) = \frac{\sigma(x)'}{\sigma(x)}. \tag{1.19}$$

This system is the relativistic analogue of the Calogero–Moser model. Hamiltonians generating the commuting ($t_\pm$)-flows have the form

$$H_\pm = \sum_{j=1}^{n} e^{\pm p_j} \prod_{s \neq j}^{n} \left(\frac{\sigma(x_j - x_s + \eta)\sigma(x_j - x_s - \eta)}{\sigma^2(x_j - x_s)} \right)^{1/2} \tag{1.20}$$

with canonical Poisson brackets $\{p_i, x_k\} = \delta_{ik}$.

The method used in the proof of this assertion also enables us to construct the action-angle variables for the system (1.18) and to integrate the system explicitly in terms of theta-functions. Applied to the non-Abelian analogue of the $2D$ Toda chain, this approach leads to spin generalization of the Ruijsenaars–Schneider model.

This generalized model is a system of N particles on the line with coordinates x_i, and its internal degrees of freedom are described by l-dimensional vectors $a_i = (a_{i,\alpha})$ and covectors $b_i^+ = (b_i^\alpha)$, $\alpha = 1, \dots, l$. The motion equations have the following form:

$$\ddot{x}_i = \sum_{j \neq i} (b_i^+ a_j)(b_j^+ a_i)\big(V(x_i - x_j) - V(x_j - x_i)\big), \tag{1.21}$$

$$\dot{a}_i = \sum_{j \neq i} a_j (b_j^+ a_i) V(x_i - x_j), \tag{1.22}$$

$$\dot{b}_i^+ = -\sum_{j \neq i} b_j^+ (b_i^+ a_j) V(x_j - x_i). \tag{1.23}$$

The potential $V(x)$ is given by (1.19) or by its trigonometric or rational degenerations $V(x) = (\coth x)^{-1} - (\coth(x+\eta))^{-1}$ and $V(x) = x^{-1} - (x - \eta)^{-1}$, respectively. To develop a Hamiltonian formalism for this system needs special consideration; this is beyond the scope of this paper.

Let us count the number of non-trivial degrees of freedom. The original system has $2N + 2Nl$ dynamical variables x_i, $\dot{x}_i$, $a_{i,\alpha}$, b_i^α. The motion equations are invariant under the rescaling

$$a_i \to \lambda_i a_i, \qquad b_i \to \frac{1}{\lambda_i} b_i. \tag{1.24}$$

The corresponding integrals of motion have the form $I_i = \dot{x}_i - (b_i^+ a_i)$; in order to fix their zero value we put

$$\dot{x}_i = (b_i^+ a_i). \tag{1.25}$$

The reduced system is defined by the additional N extra constraints $\sum_\alpha b_i^\alpha = 1$, which destroy the symmetry (1.24). Therefore, the phase space of the reduced system has dimension $2Nl$. Moreover, the system is invariant under the transformations

$$a_i \to W^{-1}a_i, \qquad b_i^+ \to b_i^+ W, \tag{1.26}$$

apart from the symmetry (1.24). Here a matrix $W \in GL(r,\mathbb{R})$ is restricted only by the condition that W preserves the above-mentioned additional conditions on the b_i; in other words, W leaves the vector $v = (1,\dots,1)$ invariant. Taking into account (1.26), we see that the dimension of the completely reduced phase space $\mathcal{M}$ is equal to

$$\dim \mathcal{M} = 2\left[Nl - \frac{l(l-1)}{2}\right]. \tag{1.27}$$

In §§ 2–4 we derive explicit representations for general solutions of the system (1.21)–(1.23) in terms of theta-functions. It should be emphasized that the corresponding formulae are identical to those obtained in [19] for spin generalizations of the Calogero–Moser model. The only change is that the class of auxiliary spectral algebraic curves (in terms of which the theta-functions are constructed) is different. These curves can be described purely in terms of algebraic geometry.

With each smooth algebraic curve Γ of genus N we associate an N-dimensional complex torus $J(\Gamma)$ (Jacobian of the curve). A pair of points $P^\pm \in \Gamma$ defines a vector U in the Jacobian. Let us consider a class of curves that have the following property: there exists a pair of points on the curve such that the complex linear subspace generated by the corresponding vector U is *compact*, that is, it is an *elliptic curve* $\mathcal{E}_0$. This means that there exist two complex numbers $2\omega_\alpha$, $\operatorname{Im}\omega_2/\omega_1 > 0$, such that $2\omega_\alpha U$ belongs to the lattice of periods of holomorphic differentials on Γ. From a purely algebraic-geometrical point of view the problem of describing such curves is transcendental. It turns out that this problem has an explicit solution; moreover, the algebraic equations defining such curves can be written as characteristic equations for the Lax operator of the Ruijsenaars–Schneider system. In the case of general position $\mathcal{E}_0$ intersects the theta-divisor at N points x_i, and if we move $\mathcal{E}_0$ in the direction defined by the vector V^+ (V^-) tangent to $\Gamma \in J(\Gamma)$ at the point P^+ (P^-), then the intersections of $\mathcal{E}_0$ with the theta-divisor move according to the Ruijsenaars–Schneider dynamics. There is an analogous description of spin generalizations of this system. The corresponding curves have two sets of points $P_i^\pm$, $i = 1,\dots,l$, such that in the linear subspace spanned by the vectors corresponding to each pair there exists a vector U with the same property as above.

Note that the geometric interpretation of integrable many-body systems of Calogero–Moser–Sutherland type resides in the representation of these models as reductions of geodesic flows on symmetric spaces [21]. Equivalently, the models can be obtained by means of the Hamiltonian reduction [22] from free dynamics in a larger phase space possessing a rich symmetry. A generalization to infinite-dimensional phase spaces (cotangent bundles to current algebras and groups) was suggested in [23], [24]. The infinite-dimensional gauge symmetry enables us to make a reduction to a system with finitely many degrees of freedom. The Ruijsenaars–Schneider type models and the elliptic Calogero–Moser model are contained in the class of systems described by this procedure.

A further generalization of this approach consists in considering dynamical systems on cotangent bundles to moduli spaces of stable holomorphic vector bundles on Riemann surfaces. Such systems were introduced by Hitchin in the paper [25], where their integrability was proved. An attempt to identify the known many-body integrable systems in terms of the abstract formalism developed by Hitchin was recently made in [26]. To do this, it is necessary to consider vector bundles on algebraic curves with singular points. It turns out that the class of integrable systems corresponding to the Riemann sphere with marked points includes spin generalizations of the Calogero–Moser model and integrable Gaudin magnets [27] as well (see also [28]).

Notwithstanding the fact that Hitchin's approach is of a general nature and offers a clear geometric interpretation, it cannot be used directly to obtain explicit formulae for solutions of the motion equations. Furthermore, in general an explicit form of the motion equations is unknown. We hope that an alternative approach to Hitchin's systems may be based on the approach first suggested in [18] for the elliptic Calogero–Moser system and developed in more detail in this paper. This approach seems to be less invariant but yields more explicit formulae. In our opinion this approach has not yet been used in its full strength. Conjecturally, to each Hitchin system one can assign a linear problem having solutions of a special form (called double-Bloch solutions in this paper), in terms of which one may construct explicit formulae for solutions of the motion equations.

This paper as a whole can be divided into three relatively independent parts. The structure of the first part (§§ 2–4) is very close to that of the paper [19]. Furthermore, to make this paper self-contained and to stress the universal character of the approach suggested in [18], we sometimes use the literal citation of [19]. At the same time we skip some technical details common for both cases and try to stress the specifics of difference equations. In the second part (§ 5) we introduce and study discrete analogues of Lamé operators. Finally, in the third part (§ 6) we use the concept of the vacuum vectors of L-operators to give a simple derivation of difference operators representing the Sklyanin algebra. Actually, we expect a deeper connection between the three main topics of this paper, for which reason we have combined these topics within a single paper; a short discussion on this point is given in § 7.

§ 2. The generating linear problem

The equations of the non-Abelian $2D$ Toda chain have the form

$$\partial_+\big((\partial_- g_n)g_n^{-1}\big) = g_n g_{n-1}^{-1} - g_{n+1}g_n^{-1}. \tag{2.1}$$

These equations are equivalent to the compatibility condition for the overdetermined system of linear problems

$$\partial_+\psi_n(t_+,t_-) = \psi_{n+1}(t_+,t_-) + v_n(t_+,t_-)\psi_n(t_+,t_-), \tag{2.2}$$

$$\partial_-\psi_n(t_+,t_-) = c_n(t_+,t_-)\psi_{n-1}(t_+,t_-), \tag{2.3}$$

where

$$c_n = g_n g_{n-1}^{-1}, \qquad v_n = (\partial_+ g_n)g_n^{-1} \tag{2.4}$$

(g_n is an $(l \times l)$-matrix). As in the case of the Calogero–Moser model and its spin generalizations [18], [19], the statement that the system (1.21)–(1.23) and the pole system defined by elliptic solutions of the non-Abelian $2D$ Toda chain are isomorphic is based on the fact that the auxiliary linear problem with elliptic coefficients has infinitely many *double-Bloch* solutions.

We call a meromorphic vector-valued function $f(x)$ a *double-Bloch function* if it has the following monodromy properties:

$$f(x+2\omega_\alpha) = B_\alpha f(x), \qquad \alpha = 1,2; \tag{2.5}$$

here ω_α are periods of an elliptic curve. The complex numbers B_α are called *Bloch multipliers*. In other words, f is a meromorphic cross-section of a vector bundle over an elliptic curve. Any double-Bloch function can be represented as a linear combination of elementary functions as follows.

We put

$$\Phi(x,z) = \frac{\sigma(z+x+\eta)}{\sigma(z+\eta)\sigma(x)} \left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{x/(2\eta)}. \tag{2.6}$$

Using the addition theorems for the Weierstrass σ-function, it is easy to check that $\Phi(x,z)$ satisfies the difference analogue of the Lamé equation

$$\Phi(x+\eta,z) + c(x)\Phi(x-\eta,z) = E(z)\Phi(x,z), \tag{2.7}$$

where

$$c(x) = \frac{\sigma(x-\eta)\sigma(x+2\eta)}{\sigma(x+\eta)\sigma(x)}. \tag{2.8}$$

Here z plays the role of a spectral parameter, which parametrizes the eigenvalues $E(z)$ of the difference Lamé operator

$$E(z) = \frac{\sigma(2\eta)}{\sigma(\eta)} \frac{\sigma(z)}{(\sigma(z-\eta)\sigma(z+\eta))^{1/2}}. \tag{2.9}$$

The Riemann surface $\widehat{\Gamma}_0$ of the function $E(z)$ is a two-fold covering of the initial elliptic curve Γ_0 with periods $2\omega_\alpha$, $\alpha = 1,2$; its genus is equal to 2.

The function $\Phi(x,z)$ is a doubly-periodic function of z, that is,

$$\Phi(x, z+2\omega_\alpha) = \Phi(x,z), \tag{2.10}$$

If $x/2\eta$ is an integer, then Φ is a well-defined meromorphic function on Γ_0. If $x/2\eta$ is a half-integer, then Φ becomes a single-valued function on $\widehat{\Gamma}_0$. In the general case when x is an arbitrary number one can define a single-valued branch of $\Phi(x,z)$ by cutting the elliptic curve Γ_0 between the points $z = \pm\eta$.

As a function of x, the function $\Phi(x,z)$ is a double-Bloch function, that is,

$$\Phi(x+2\omega_\alpha, z) = T_\alpha(z)\Phi(x,z), \tag{2.11}$$

where the Bloch multipliers are given by

$$T_\alpha(z) = \exp\bigl(2\zeta(\omega_\alpha)(z+\eta)\bigr) \left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right)^{\omega_\alpha/\eta}. \tag{2.12}$$

In the fundamental parallelogram defined by the sides $2\omega_\alpha$ the function $\Phi(x, z)$ has a unique pole at the point $x = 0$:

$$\Phi(x, z) = \frac{1}{x} + A + O(x), \qquad A = \zeta(z + \eta) + \frac{1}{2\eta} \ln \frac{\sigma(z - \eta)}{\sigma(z + \eta)}. \tag{2.13}$$

This implies that if a double-Bloch function $f(x)$ has simple poles at points x_i of the fundamental parallelogram and has Bloch multipliers B_α such that at least one of them is different from 1, then it can be represented in the form

$$f(x) = \sum_{i=1}^{N} s_i \Phi(x - x_i, z) k^{x/\eta}, \tag{2.14}$$

where the variable z and the complex number k are connected by the relation

$$B_\alpha = T_\alpha(z) k^{2\omega_\alpha/\eta}. \tag{2.15}$$

(Note that any pair of Bloch multipliers can be represented in the form (2.15) with an appropriate choice of z and k.)

Indeed, let x_i, $i = 1, \ldots, m$, be poles of $f(x)$ lying in the fundamental domain of the lattice with periods $2\omega_1$, $2\omega_2$. Then there exist vectors s_i such that the function

$$F(x) = f(x) - \sum_{i=1}^{m} s_i \Phi(x - x_i, z) k^{x/\eta}$$

is *holomorphic* in x in the fundamental domain and it is a double-Bloch function with the same Bloch multipliers as f. Moreover, any non-trivial double-Bloch function, having at least one Bloch multiplier different from 1, has at least one pole in the fundamental domain, whence $F = 0$.

The gauge transformation

$$f(x) \mapsto \tilde{f}(x) = f(x) e^{ax}, \tag{2.16}$$

where a is an arbitrary constant, does not change the poles of any functions and transforms a double-Bloch function into another double-Bloch function. If B_α are the Bloch multipliers for f, then the Bloch multipliers for $\tilde{f}$ are equal to

$$\widetilde{B}_1 = B_1 e^{2a\omega_1}, \qquad \widetilde{B}_2 = B_2 e^{2a\omega_2}. \tag{2.17}$$

We say that two pairs of Bloch multipliers are *equivalent* if for some a they are connected by (2.17). Note that for all equivalent pairs of Bloch multipliers the product

$$B_1^{\omega_2} B_2^{-\omega_1} = B \tag{2.18}$$

is a constant depending only on the equivalence class.

Theorem 2.1. *The equations*

$$\partial_t \Psi(x,t) = \Psi(x+\eta,t) + \sum_{i=1}^{N} a_i(t) b_i^+(t) V\big(x - x_i(t)\big) \Psi(x,t), \tag{2.19}$$

$$-\partial_t \Psi^+(x,t) = \Psi^+(x-\eta,t) + \Psi^+(x,t) \sum_{i=1}^{N} a_i(t) b_i^+(t) V\big(x - x_i(t)\big) \tag{2.20}$$

have N pairs of linear independent double-Bloch solutions $\Psi_{(s)}(x,t)$ and $\Psi^+_{(s)}(x,t)$ such that they have simple poles at the points $x_i(t)$ and $(x_i(t) - \eta)$, respectively,

$$\Psi_{(s)}(x + 2\omega_\alpha, t) = B_{\alpha,s} \Psi_{(s)}(x,t), \qquad \Psi^+_{(s)}(x - 2\omega_\alpha, t) = B_{\alpha,s} \Psi_{(s)}(x,t), \tag{2.21}$$

and have equivalent Bloch multipliers (that is, the quantity

$$B_{1,s}^{\omega_2} B_{2,s}^{-\omega_1} = B \tag{2.22}$$

is independent of s) if and only if $x_i(t)$ satisfy equations (1.21), *and the vectors a_i, b_i^+ satisfy the constraints* (1.25) *and the system of equations*

$$\dot{a}_i = \sum_{j \neq i} a_j (b_j^+ a_i) V(x_i - x_j) - \lambda_i a_i, \tag{2.23}$$

$$\dot{b}_i^+ = -\sum_{j \neq i} b_j^+ (b_i^+ a_j) V(x_j - x_i) + \lambda_i b_i^+, \tag{2.24}$$

where $\lambda_i = \lambda_i(t)$ are scalar functions.

Remark. The system (1.21), (2.23), (2.24) is 'gauge equivalent' to the system (1.21)–(1.23). This means that if (x_i, a_i, b_i^+) satisfy equations (1.21), (2.23), (2.24), then x_i and the vector-valued functions

$$\widehat{a}_i = a_i q_i, \quad \widehat{b}_i^+ = b_i q_i^{-1}, \quad q_i = \exp\Big(\int^t \lambda_i(t')\, dt'\Big) \tag{2.25}$$

are solutions of the system (1.21)–(1.23).

Theorem 2.2. *Suppose that equations* (2.19), (2.20) *have N linear independent double-Bloch solutions with Bloch multipliers satisfying* (2.22). *Then there are infinitely many such solutions, all of which can be represented in the form*

$$\Psi = \sum_{i=1}^{N} s_i(t,k,z) \Phi\big(x - x_i(t), z\big) k^{x/\eta}, \tag{2.26}$$

$$\Psi^+ = \sum_{i=1}^{N} s_i^+(t,k,z) \Phi\big(-x + x_i(t) - \eta, z\big) k^{-x/\eta}, \tag{2.27}$$

where s_i is an l-dimensional vector, $s_i = (s_{i,\alpha})$, and s_i^+ is an l-dimensional covector, $s_i^+ = (s_i^\alpha)$. The set of corresponding pairs (z,k) is parametrized by points of the algebraic curve defined by the equation

$$R(k,z) = k^N + \sum_{i=1}^{N} r_i(z) k^{N-i} = 0.$$

Proof of Theorem 2.1. As we have mentioned, $\Psi_{(s)}(x,t)$ (like any double-Bloch function) may be written in the form (2.26) with some values of the parameters z_s, k_s. From (2.22) it follows that the parameters z_s can be chosen as follows:

$$z_s = z, \qquad s = 1, \dots, N.$$

For this value of z we substitute a function $\Psi(x,t,z,k)$ of the form (2.26) in equation (2.19). Since any function with such monodromy properties has at least one pole, it follows that (2.26), (2.23) are satisfied if and only if the right- and left-hand sides of these equalities have the same singular parts at the points $x = x_i$ and $x = x_i - \eta$.

Comparing the coefficients in front of $(x - x_i)^{-2}$ in (2.19), we obtain

$$\dot{x}_i s_i = a_i (b_i^+ s_i), \tag{2.28}$$

whence the vector s_i is proportional to a_i,

$$s_{i,\alpha}(t,k,z) = c_i(t,k,z) a_{i,\alpha}(t), \tag{2.29}$$

and the vectors a_i, b_i satisfy the constraints (1.25). Cancellation of the coefficients in front of $(x - x_i + \eta)^{-1}$ gives the conditions

$$-k s_i + \sum_{j \neq i} a_i b_i^+ s_j \Phi(x_i - x_j - \eta, z) = 0. \tag{2.30}$$

Taking into account (1.25) and (2.29), we can write (2.30) as the matrix equation for the vector $C = (c_i)$,

$$\big(L(t,z) - kI\big) C = 0, \tag{2.31}$$

where I is the unit matrix, and the Lax matrix $L(t,z)$ is defined as follows:

$$L_{ij}(t,z) = (b_i^+ a_j) \Phi(x_i - x_j - \eta, z). \tag{2.32}$$

Finally, cancellation of the poles $(x - x_i)^{-1}$ gives the conditions

$$\dot{s}_i - \Big(\sum_{j \neq i} a_j b_j^+ V(x_i - x_j) + \big(A - \zeta(\eta)\big) a_i b_i^+\Big) s_i - a_i \sum_{j \neq i} (b_i^+ s_j) \Phi(x_i - x_j, z) = 0, \tag{2.33}$$

Taking into account (2.29), we arrive at the motion equations (2.23), where

$$\lambda_i(t) = \frac{\dot{c}_i}{c_i} + \big(\zeta(\eta) - A\big)\dot{x}_i - \sum_{j \neq i} (b_i^+ a_j) \Phi(x_i - x_j, z) \frac{c_j}{c_i}\,. \tag{2.34}$$

We can write (2.31) in the matrix form

$$\big(\partial_t + M(t,z)\big)C = 0, \tag{2.35}$$

where the second operator of the Lax pair is given by

$$M_{ij}(t,z) = \big(-\lambda_i + (\zeta(\eta) - A)\dot{x}_i\big)\delta_{ij} - (1-\delta_{ij})b_i^+ a_j \Phi(x_i - x_j, z). \tag{2.36}$$

Analogously, substituting the vector Ψ^+ of the form (2.27) in (2.20), we obtain the relations

$$s_i^\alpha(t,k,z) = c_i^+(t,k,z)b_i^\alpha(t), \tag{2.37}$$

where the covector $C^+ = (c_i^+)$ satisfies the conditions

$$C^+(L(t,z) - kI) = 0, \tag{2.38}$$

$$-\partial_t C^+ + C^+ M^{(+)}(t,z) = 0, \tag{2.39}$$

the operator L is given by (2.32), and the operator $M^{(+)}$ is obtained from (2.36) by changing $\lambda_i(t)$ to

$$\lambda_i^+(t) = -\frac{\dot{c}_i^+}{c_i^+} + \big(\zeta(\eta) - A\big)\dot{x}_i - \sum_{j\neq i}(b_j^+ a_i)\Phi(x_j - x_i, z)\frac{c_j^+}{c_i^+}. \tag{2.40}$$

Moreover, the covector b_i^+ satisfies the motion equation

$$\dot{b}_i^+ = -\sum_{j\neq i} b_j^+(b_i^+ a_j)V(x_j - x_i) + \lambda_i^+ b_i^+. \tag{2.41}$$

The assumption of the theorem implies that equations (2.31), (2.35) and (2.38), (2.39) have N linear independent solutions corresponding to different values of k. The compatibility conditions for these equations have the form of the Lax equations

$$\dot{L} + [M, L] = 0, \qquad \dot{L} + [M^{(+)}, L] = 0, \tag{2.42}$$

from which it follows that $\lambda_i = \lambda_i^+$.

The function $\Phi(x,z)$ satisfies the following functional relations:

$$\Phi(x-\eta,z)\Phi(y,z) - \Phi(x,z)\Phi(y-\eta,z) = \Phi(x+y-\eta,z)\big(V(-x) - V(-y)\big), \tag{2.43}$$

$$\Phi'(x-\eta,z) = -\Phi(x-\eta,z)(V(-x) + \zeta(\eta) - A) - \Phi(-\eta,z)\Phi(x,z), \tag{2.44}$$

where the constant A is defined in (2.13). The first relation is equivalent to the three-term functional equation for the Weierstrass σ-function, and the second relation follows from the first one as $y \to 0$.

Using (2.43) and (2.44) it is easy to prove by direct computation the following lemma, which completes the proof of the theorem.

Lemma 2.1. *Let the matrices L and M be defined by* (2.32) *and* (2.36), *respectively, where a_i and b_i^+ satisfy* (2.23) *and* (1.25). *Then the Lax equations* (2.42) *are valid if and only if the $x_i(t)$ satisfy* (1.21).

Proof of Theorem 2.2. As we proved above, equations (2.19), (2.20) have N linear independent solutions provided that the equations (2.23), (2.24), the constraints (3.9) and the Lax equations (2.42) are satisfied for some value of the spectral parameter z. Therefore, by Lemma 2.1, the Lax equations are satisfied for all values of the spectral parameter z. Thus, for each value of z the double-Bloch solutions of equation (2.19) are defined by (2.26) and (2.29), where the c_i are components of the common solution of equations (2.31), (2.35).

It follows from (2.31) that all admissible pairs of the spectral parameters z and k satisfy the characteristic equation

$$R(k,z) \equiv \det\bigl(kI - L(t,z)\bigr) = 0.$$

At the beginning of § 3 we show that this equation defines an algebraic curve $\widehat{\Gamma}$ of finite genus. This completes the proof of the theorem.

Remark 1. In the Abelian case ($l = 1$) one can use the 'gauge' transformation with matrix $U_{ij} = a_i\delta_{ij}$ to represent the operators of the Lax pair in the form

$$L_{ij}^{(l=1)} = \dot{x}_i\Phi(x_i - x_j - \eta, z), \tag{2.45}$$

$$M_{ij}^{(l=1)} = \Bigl(\bigl(\zeta(\eta)-A\bigr)\dot{x}_i - \sum_{s\neq i} V(x_i-x_s)\dot{x}_s\Bigr)\delta_{ij} - (1-\delta_{ij})\dot{x}_i\Phi(x_i-x_j,z). \tag{2.46}$$

These operators are equivalent to the Lax pair derived in [29], [30].

Remark 2. In the Abelian case it is sufficient to require that only one of the equations (2.19), (2.20) has N linear independent double-Bloch solutions with Bloch multipliers satisfying condition (2.22).

§ 3. The direct problem

It follows from the Lax equation (2.42) that the coefficients of the characteristic equation

$$R(k,z) \equiv \det\bigl(kI - L(t,z)\bigr) = 0. \tag{3.1}$$

are independent of time. Note that they are invariant with respect to the symmetries (1.24) and (1.26).

Theorem 3.1. *The coefficients $r_i(z)$ of the characteristic equation* (3.1)

$$R(k,z) = k^N + \sum_{i=1}^{N} r_i(z)k^{N-i} \tag{3.2}$$

are independent of t and are given by

$$r_i(z) = \phi_i(z)\Bigl(I_{i,0} + (1-\delta_{l,1})I_{i,1}\widetilde{s}_i(z) + \sum_{s=2}^{m_i} I_{i,s}\partial_z^{s-2}\wp(z+\eta)\Bigr), \tag{3.3}$$

where

$$m_i = i-1 \textit{ for } i=1,\ldots,l, \qquad m_i = l-1 \textit{ for } i = l+1,\ldots,N, \tag{3.4}$$

$$\phi_i(z) = \frac{\sigma(z+\eta)^{(i-2)/2}\sigma(z-(i-1)\eta)}{\sigma(z-\eta)^{i/2}}, \tag{3.5}$$

$$\widetilde{s}_i(z) = \frac{\sigma(z-\eta)\sigma(z-(i-3)\eta)}{\sigma(z+\eta)\sigma(z-(i-1)\eta)}. \tag{3.6}$$

In a neighbourhood of the point $z=-\eta$ the function $R(k,z)$ can be represented in the form

$$R(k,z) = \prod_{i=1}^{l}\left(k+(z+\eta)^{-1/2}h_i(z+\eta)\right)\prod_{i=l+1}^{N}\left(k+(z+\eta)^{1/2}h_i(z+\eta)\right), \tag{3.7}$$

where the functions $h_i(z)$ are regular in a neighbourhood of zero.

Proof. It follows from (2.10) that the matrix elements L_{ij} are doubly-periodic functions of z. Therefore, one can consider them as single-valued functions on the original elliptic curve Γ_0 with a branch cut between the points $z=\pm\eta$. First we show that the coefficients $r_i(z)$ of the characteristic polynomial (3.1) are meromorphic functions on the Riemann surface $\widehat{\Gamma}_0$ of the function $E(z)$ defined by (2.9). (This means that $r_i(z)$ are two-valued functions of z with square root branching at the points $z=\pm\eta$.)

This assertion follows from the fact that $L(t,z)$ can be written in the 'gauge equivalent' form

$$L(t,z) = G(t,z)\widetilde{L}(t,z)G^{-1}(t,z), \qquad G_{ij} = \delta_{ij}\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{x_i(t)/(2\eta)}, \tag{3.8}$$

where the matrix elements of $\tilde{L}(t,z)$ have square root branching at the points $z=\pm\eta$. Moreover, using the explicit formulae

$$\widetilde{L}_{ij}(t,z) = \frac{(b_i^+a_j)}{[\sigma(z-\eta)\sigma(z+\eta)]^{1/2}}\,\frac{\sigma(z+x_i-x_j)}{\sigma(x_i-x_j-\eta)}, \tag{3.9}$$

one can conclude that $r_{2i}(z)$ are single-valued meromorphic functions of z, that is, elliptic functions. Further, $r_{2i+1}(z)$ are meromorphic functions on $\widehat{\Gamma}_0$, which are odd with respect to the involution $\widehat{\tau}_0 : \widehat{\Gamma}_0 \to \widehat{\Gamma}_0$ interchanging sheets of the covering $\widehat{\Gamma}_0 \to \Gamma_0$ (this involution corresponds to the change of sign $E(z)\to -E(z)$ of the square root). Thus, the curve $\widehat{\Gamma}$ is invariant with respect to the involution

$$\widehat{\tau}\colon \widehat{\Gamma} \mapsto \widehat{\Gamma}, \qquad \widehat{\tau}(k,E) \mapsto (-k,-E), \tag{3.10}$$

which covers the involution $\widehat{\tau}_0$.

We denote by Γ the factor-curve

$$\Gamma := \{\widehat{\Gamma}/\widehat{\tau}\}. \tag{3.11}$$

This curve is an N-fold ramified covering of the initial elliptic curve

$$\Gamma \mapsto \Gamma_0 \tag{3.12}$$

and is defined by the equation

$$\widehat{R}(K,z) = K^N + \sum_{i=1}^{N} R_i(z) K^{N-i} = 0, \tag{3.13}$$

where

$$K = k\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{1/2}, \qquad R_i(z) = r_i(z)\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{i/2}. \tag{3.14}$$

We now dwell on the sense of this assertion in more detail. The coefficients $R_j(z)$ of (3.13) are meromorphic functions of z having the following monodromy properties:

$$R_j(z+2\omega_\alpha) = R_j(z)e^{-2j\zeta(\omega_\alpha)\eta}. \tag{3.15}$$

The Riemann surface $\tilde{\Gamma}$ defined by (3.13) is an N-fold covering of the complex plane. It follows from (3.15) that this surface is invariant with respect to the transformations

$$z \mapsto z + 2\omega_\alpha, \qquad K \mapsto Ke^{-2\zeta(\omega_\alpha)\eta}. \tag{3.16}$$

The corresponding factor-surface of $\tilde{\Gamma}$ over the transformations (3.16) is an algebraic curve Γ, which is the covering of some elliptic curve with periods $2\omega_\alpha$.

We now prove equality (3.7). The proof is based on the fact that for the expansion of $\tilde{L}(t,z)$ in a neighbourhood of the point $z = -\eta$,

$$\widetilde{L}_{ij}(t,z) = \frac{(b_i^+ a_j)}{[\sigma(-2\eta)(z+\eta)]^{1/2}} + O\big((z+\eta)^{1/2}\big), \tag{3.17}$$

the rank of the leading term is equal to l. The corresponding $(N-l)$-dimensional subspace of eigenvectors $C = (c_1, \dots, c_N)$ with zero eigenvalue is defined by the equations

$$\sum_{j=1}^{N} c_j a_{j,\alpha} = 0, \qquad \alpha = 1, \dots, l. \tag{3.18}$$

We now go back to the determination of the coefficients in the characteristic equation (3.1). Since the matrix elements of $\tilde{L}(t,z)$ have simple poles at the point $z = \eta$, we see that in the case of general position the function $r_i(z)$ has a pole of order i at this point. It follows from (3.7) that at the point $z = -\eta$ the function r_i has a pole of order i for $i = 1, \dots, l$, a pole of order $(2l-i)$ for $i = l+1, \dots, 2l$, and a pole of order $i-2l$ for $i=2l+1,\dots,N$, that is,

$$r_i(z)=(z+\eta)^{-i/2}\rho_i(z+\eta), \qquad i=1,\dots,l, \tag{3.19}$$

$$r_i(z)=(z+\eta)^{i/2-l}\rho_i(z+\eta), \qquad i=l+1,\dots,N, \tag{3.20}$$

where $\rho_i(z)$ are regular functions. Using these properties of the functions $r_i(z)$, as well as their evenness properties, we arrive at the representation (3.3). (Note that the function $\phi_i(z)$ defined by (3.5) has a pole of order i at the point $z=\eta$ and a zero of order $i-2$ at the point $z=-\eta$.)

Important remark. It should be emphasized that the representation (3.7) implies that the characteristic equation (3.1) defines a singular algebraic curve. Indeed, (3.7) implies that $(N-l)$ sheets of the corresponding ramified covering intersect at the point $(z=-\eta,\ k=0)$. In what follows we keep the same notation Γ for an algebraic curve with the resolved singularity at this point.

The coefficients $I_{i,s}$ in (3.3) are integrals of motion for the system considered. The total number of integrals is equal to $Nl-l(l-1)/2$, which is exactly half the dimension of the reduced phase space. (It follows from the results of § 4 that these integrals are independent.)

Lemma 3.1. *In the case of general position the genus g of the spectral curve Γ defined by* (3.13) *is equal to* $Nl-l(l+1)/2+1$.

Proof. First we determine the genus $\widehat{g}$ of the curve $\widehat{\Gamma}$ defined by (3.1). By the Riemann–Hurwitz formula we have $2\widehat{g}-2=2N+\nu$, where ν is the number of branch points of $\widehat{\Gamma}$ over $\widehat{\Gamma}_0$, that is, the number of points z at which the equation $R(k,z)=0$ has a double root. This number is equal to the number of zeros of the function $\partial_k R(k,z)$ on the surface $R(k,z)=0$ outside preimages of the point $z=-\eta$ (due to the above-mentioned singularity of the initial curve). The function $\partial_k R(k,z)$ has poles of order $N-1$ at preimages of the point $z=\eta$; moreover, it also has poles of the same order at l preimages of the point $z=-\eta$, which correspond to the first l factors in (3.7). At the other $N-l$ preimages of the point $z=-\eta$ this function has zeros of order $(N-l)(N-2l-1)$. Therefore, $\nu=4lN-2l(l+1)$. The curve $\widehat{\Gamma}$ is a two-fold branched covering of the spectral curve Γ, and the number of branch points at preimages of the points $z=\pm\eta$ is equal to $2N$. In this case the Riemann–Hurwitz formula leads to the relation $2\widehat{g}-2=2(2g-2)+2N$, which proves the lemma.

Not only does the characteristic equation (3.1) define the spectral curve (3.13), but it also defines two sets of distinguished points on the spectral curve. Namely, using the factorization (3.7) of $R(k,z)$ we see that the function k has poles at l points of the set of preimages of the point $z=-\eta$ (these poles correspond to the first l factors in (3.7)); we denote them by P_i^+, $i=1,\dots,l$. Since a meromorphic function has as many zeros as poles, it follows from (3.14) and (3.7) that on the unreduced spectral curve $\widehat{\Gamma}$ the function k has $2l$ zeros outside preimages of the point $z=-\eta$. These zeros correspond to l points P_i^-, $i=1,\dots,l$, lying on the spectral curve Γ,

$$k(P_i^-)=0. \tag{3.21}$$

In the case of general position there is such a point P_i^- above each zero z_i^- of the function $r_N(z)$, different from its apparent zero $z=-\eta$, so that

$$r_N(z)=\widetilde{I}_{N,0}\frac{\sigma^{(N-l)/2}(z+\eta)}{\sigma^{N/2}(z-\eta)}\prod_{i=1}^{l}\sigma(z-z_i^-). \tag{3.22}$$

(For $l=1$ the second marked point P_1^- lies above the point $z=(N-1)\eta$.)

Theorem 3.2. *Let $\Psi(x,t,P)$ be a solution of equation* (2.19). *Then its components $\Psi_\alpha(x,t,P)$ are defined on an N-fold covering Γ of the initial elliptic curve cut between the points P_i^+ and P_i^-, $i=1,\dots,l$, and they are meromorphic outside these branch cuts. In the case when the initial conditions are of general position the curve Γ is smooth, its genus is equal to $g=Nl-\frac{l(l+1)}{2}+1$, and the function Ψ_α has $g-1$ poles $\gamma_1,\dots,\gamma_{g-1}$, whose position is independent of x,t. In a neighbourhood of the points P_i^+, $i=1,\dots,l$, the function Ψ_α has the form*

$$\Psi_\alpha(x,t,P)=\left(\chi_0^{\alpha i}+\sum_{s=1}^{\infty}\chi_s^{\alpha i}(x,t)(z+\eta)^s\right)\left(\varkappa_i(z+\eta)^{-1}\right)^{x/\eta}e^{\varkappa_i(z+\eta)^{-1}t}\Psi_1(0,0,P), \tag{3.23}$$

where $\chi_0^{\alpha i}$ are constants independent of x, t, and $\varkappa_i$ are non-zero eigenvalues of the matrix $(b_i^+a_j)$. In a neighbourhood of the points P_i^- the function Ψ_α has the form

$$\Psi_\alpha(x,t,P)=(z-z_i^-)^{x/\eta}\left(\sum_{s=0}^{\infty}\widetilde{\chi}_s^{\alpha i}(x,t)(z-z_i^-)^s\right)\Psi_1(0,0,P), \tag{3.24}$$

where z_i^- are projections of the points P_i^- onto the initial elliptic curve; these projections are defined by (3.22). *The boundary values $\Psi_\alpha^{(\pm)}$ of the function Ψ_α on opposite sides of the cuts are connected by the relation*

$$\Psi_\alpha^{(+)}=\Psi_\alpha^{(-)}e^{2\pi i x/\eta}. \tag{3.25}$$

Proof. First we study analytic properties of the eigenvectors of the Lax matrix defined by (2.31), (2.35).

We denote by $\widehat{\Gamma}^*$ the curve $\widehat{\Gamma}$ with cuts between the preimages P_i^+ of the point $z=-\eta$ and the preimages Q_i^- of the point $z=\eta$, $i=1,\dots,N$. Let $\widehat{P}$ be a generic point of the curve $\widehat{\Gamma}$, that is, a pair $(k,z)=\widehat{P}$ that satisfies (3.1); then there exists a unique eigenvector $C(0,\widehat{P})$ of the matrix $L(0,z)$ normalized by the condition $c_1(0,P)=1$. All other components $c_i(0,\widehat{P})$ are equal to $\Delta_i(0,\widehat{P})/\Delta_1(0,\widehat{P})$, where $\Delta_i(0,P)$ are suitable minors of the matrix $kI-L(0,z)$, and therefore they are meromorphic on $\widehat{\Gamma}^*$. The poles of $c_i(0,\widehat{P})$ are zeros of the first principal minor

$$\Delta_1(0,\widehat{P})=\det\bigl(k\delta_{ij}-L_{ij}(0,z)\bigr)=0,\qquad i,j>1, \tag{3.26}$$

on $\widehat{\Gamma}^*$. Therefore, the position of these poles depends only on the initial data of the Cauchy problem considered.

Lemma 3.2. *The coordinates $c_j(0,\widehat{P})$ of the eigenvector $C(0,\widehat{P})$ are meromorphic functions on $\widehat{\Gamma}^*$. The boundary values $c_j^{\pm}$ of the functions $c_j(0,\widehat{P})$ on opposite sides of the cuts satisfy the relation*

$$c_j^+ = c_j^- e^{\pi i(x_j(0)-x_1(0))/\eta}. \tag{3.27}$$

In a neighbourhood of the points P_i^+ the functions $c_j(0,\widehat{P})$ have the form

$$c_j(0,\widehat{P}) = \left(c_j^{(i,+)}(0) + O(z+\eta)\right)(z+\eta)^{(x_1(0)-x_j(0))/(2\eta)}, \tag{3.28}$$

where $c_j^{(i,+)}(t)$ are eigenvectors of the residue of the matrix $\tilde{L}(0,z)$ at $z=-\eta$, that is,

$$\sum_{j=1}^{N}(b_k^+ a_j)c_j^{(i,+)}(t) = -\varkappa_i c_k^{(i,+)}(t). \tag{3.29}$$

In a neighbourhood of the points Q_i^- the functions $c_j(0,\widehat{P})$ have the form

$$c_j(0,\widehat{P}) = \left(c_j^{(i,-)}(0) + O(z-\eta)\right)(z-\eta)^{(x_j(0)-x_1(0))/(2\eta)}, \tag{3.30}$$

where $c_j^{(i,-)}(t)$ are eigenvectors of the residue of the matrix $\tilde{L}(0,z)$ at $z=\eta$.

The proof is based on the following representation (see (3.8)):

$$C(0,\widehat{P}) = G(0,z)\widetilde{C}(0,P), \tag{3.31}$$

where $G(t,z)$ is defined in (3.8), and $\tilde{C}(0,P)$ is an eigenvector of the matrix $\tilde{L}(0,z)$. The matrix elements of $\tilde{L}(0,z)$ are analytic on the cuts between $z=\pm\eta$, and therefore $\tilde{C}(0,P)$ has no discontinuity on the cuts. This proves the relation (3.27). The representations (3.28) and (3.30) are direct consequences of the fact that $\tilde{L}(0,z)$ has simple poles at the points $z=\pm\eta$.

Remark. The vector $\tilde{C}(0,P)$ is invariant under the involution (3.10). That is why its argument is a point P of the spectral curve Γ rather than a point $\widehat{P}$ of the covering $\widehat{\Gamma}$. However, this notation is somewhat misleading, since both factors in (3.31) are multivalued on $\widehat{\Gamma}$, and only their product is well defined.

Lemma 3.3. *The poles of $C(0,\widehat{P})$ are invariant under the involution $\widehat{\tau}$. The number of them is equal to $2Nl - l(l+1)$.*

The proof of the lemma is based on standard arguments of finite-gap integration theory. Namely, let us consider the following function of z:

$$F(z) = \left(\operatorname{Det}|c_i(0,M_j)|\right)^2,$$

where M_j, $j=1,\dots,N$, are preimages of z. Since this function does not depend on the ordering of the preimages, it is well defined as a function of z. Using the analytic properties of c_j, we see that $F(z)$ can be represented in the form

$$F(z) = \widetilde{F}(z)\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{\sum(x_i(0)-x_1(0))}, \tag{3.32}$$

where $\tilde{F}$ is a meromorphic function. This means that $F(z)$ has as many zeros as poles. The number of its poles is twice the number of zeros of the vector $C(0,\widehat{P})$, whereas the number of zeros of $F(z)$ is equal to the number ν of branch points of the covering $\widehat{\Gamma}$ over $\widehat{\Gamma}_0$ defined by (3.1). In the proof of Lemma 3.1 we showed that $\nu = 4Nl - 2l(l+1)$. The invariance of poles of $C(0,\widehat{P})$ under the involution $\widehat{\tau}$ follows from the $\widehat{\tau}$-invariance of equation (3.26), which determines the positions of the poles. This completes the proof.

Let $\gamma_1,\dots,\gamma_{g-1}$ be points of the spectral curve Γ such that their preimages are poles of $C(0,\widehat{P})$. (Note that if Γ is smooth, then the number $g = Nl - l(l+1)/2 + 1$ is its genus.)

We now consider the vector $C(t,\widehat{P})$ obtained from $C(0,\widehat{P})$ by the time evolution given by (2.35).

Lemma 3.4. *The coordinates $c_j(t,\widehat{P})$ of the vector $C(t,\widehat{P})$ are meromorphic on $\widehat{\Gamma}^*$. Their poles are preimages of the points $\gamma_1,\dots,\gamma_{g-1}$ and do not depend on t. The boundary values $c_j^\pm$ of $c_j(t,\widehat{P})$ on opposite sides of the cuts satisfy the relation*

$$c_j^+ = c_j^- e^{\pi i(x_j(t)-x_1(0))/\eta}. \tag{3.33}$$

In a neighbourhood of the points P_i^+ the functions $c_j(t,\widehat{P})$ have the form

$$c_j(0,\widehat{P}) = \big(c_j^{(i,+)}(t) + O(z+\eta)\big)(z+\eta)^{(x_1(0)-x_j(t))/(2\eta)}\exp\big(\varkappa_i(z+\eta)^{-1}t\big), \tag{3.34}$$

where $\varkappa_i$ and $c_j^{(i,+)}(t)$ are defined in (3.29). In a neighbourhood of the points Q_i^- the functions $c_j(t,\widehat{P})$ have the form

$$c_j(t,\widehat{P}) = \big(c_j^{(i,-)}(t) + O(z-\eta)\big)(z-\eta)^{(x_j(t)-x_1(0))/(2\eta)}. \tag{3.35}$$

Proof. The fundamental matrix $S(t,z)$ of solutions of the equation

$$\big(\partial_t + M(t,z)\big)S(t,z) = 0, \qquad S(0,z) = 1, \tag{3.36}$$

is a holomorphic function of z outside the cut connecting the points $z = \pm\eta$. By the Lax equation we have $L(t,z) = S(t,z)L(0,z)S^{-1}(t,z)$. Therefore, the vector $C(t,z)$ is equal to $C(t,z) = S(t,z)C(0,z)$, whence it has the same poles as $C(0,P)$.

We consider the vector $\widetilde{C}(t,\widehat{P})$ defined by the relation

$$C(t,\widehat{P}) = G(t,z)\widetilde{C}(t,\widehat{P}), \tag{3.37}$$

where $G(t,z)$ is the same diagonal matrix as in (3.8). This vector is an eigenvector of the matrix $\tilde{L}(t,z)$ and satisfies the equation

$$\big(\partial_t + \widetilde{M}(t,z)\big)\widetilde{C}(t,P) = 0, \qquad \widetilde{M} = G^{-1}\partial_t G + G^{-1}MG. \tag{3.38}$$

The matrix elements of $\tilde{M}$ are analytic at the cut between the points $z = \pm\eta$. Therefore, $\tilde{C}(t,\widehat{P})$ is analytic at the cuts on $\widehat{\Gamma}$. Thus, the multivaluedness of $C(t,\widehat{P})$ is completely caused by the multivaluedness of $G(t,z)$, which proves (3.33). Equality (3.35) follows from the analyticity of $\tilde{M}$ at the point $z=\eta$. In a neighbourhood of $z=-\eta$ we have

$$\widetilde{M}_{ij}(t,z)=\frac{(b_i^+a_j)}{(z+\eta)}+O\big((z+\eta)^0\big). \tag{3.39}$$

Therefore, in a neighbourhood of the points P_i^+ the following relation holds:

$$\partial_t\widetilde{C}(t,\widehat{P})=\big(\mu_i(t,z)+O(z^0)\big)\widetilde{C}(t,\widehat{P}),$$

where

$$\mu_i(t,z)=\varkappa_i(z+\eta)^{-1}+O(1) \tag{3.40}$$

are eigenvalues of the matrix $\tilde{M}$. This proves (3.34).

Now we proceed directly to the proof of the theorem. From the initial definition

$$\Psi(x,t,\widehat{P})=\sum_{j=1}^{N}s_j(t,\widehat{P})\Phi\big(x-x_j(t),z\big)k^{x/\eta},\qquad s_j(t,\widehat{P})=c_j(t,\widehat{P})a_j(t),$$

it follows that the solutions Ψ of the generating linear problem (2.19) are defined on the curve $\widehat{\Gamma}$. To show that Ψ is well defined on the spectral curve Γ, we use the equality

$$c_j(t,\widehat{P})\Phi(x-x_j(t),z)k^{x/\eta}=\tilde{c}_j(t,P)\frac{\sigma(z+x-x_j+\eta)}{\sigma(z+\eta)\sigma(x-x_j)}\left[k\left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right)^{1/2}\right]^{x/\eta}. \tag{3.41}$$

We recall that the components of the vector $\tilde{C}$ are even with respect to the involution $\widehat{\tau}$ given by (3.10). The factor

$$K(P)=k\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{1/2} \tag{3.42}$$

is $\widehat{\tau}$-invariant too. Thus, $\Psi(x,t,P)$ is well defined as a vector-valued function on the spectral curve Γ. At the same time we see that the poles of $\Psi(x,t,P)$ coincide with the poles of $\tilde{C}(0,P)$, that is, they are located at the points $\gamma_1,\dots,\gamma_{g-1}$.

Note that $K(P)$ is a multivalued meromorphic function on Γ with zeros and poles (which are nevertheless well defined) at the points P_i^- and P_i^+, $i=1,\dots,l$, respectively. Therefore, by cutting Γ between the points $P_i^{\pm}$, $i=1,\dots,l$, we can choose the branch of the third factor in (3.42) in such a way that Ψ becomes single-valued outside these cuts, and its boundary values at the sides of the cuts satisfy (3.25). We consider the behaviour of Ψ in neighbourhoods of the points P_i^+. In a neighbourhood of $z=-\eta$ we have

$$\frac{\sigma(z+x+\eta)}{\sigma(z+\eta)\sigma(x)}=\frac{1}{z+\eta}+O(1). \tag{3.43}$$

Therefore, in a neighbourhood of the points P_i^+ the following relation holds:

$$\Psi_\alpha = \sum_{j=1}^{N}\left(\frac{a_{j,\alpha}c_j^{(i,+)}(t)}{z+\eta} + O(1)\right)\left[k_i(z)\left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right)^{1/2}\right]^{x/\eta}, \tag{3.44}$$

where $k_i(z)$ is the branch of $k(P)$ defined by the i-th factor of the representation (3.7). For $i > l$ the product of the second and third factors in (3.44) is regular in a neighbourhood of P_i^+. Since the eigenvalues $\varkappa_i$ in (3.29) are equal to zero for $i > l$, it follows that the first factor is also regular in a neighbourhood of P_i^+, $i > l$. Therefore, the functions Ψ_α are regular at these points. Similar arguments for $i = 1, \dots, l$ prove (3.23). Note also that $\Psi_1(0,0,P)$ in (3.23) has simple poles at the points P_i^+, $i = 1, \dots, l$.

We consider the term $\chi_0^{\alpha i}$ in (3.23). By the construction, it does not depend on x. Substituting the series (3.23) in (2.19), we see that this term does not depend on t either.

The following theorem can be proved by the same arguments.

Theorem 3.3. *Let $\Psi^+(x,t,P)$ be a solution of equation (2.20). Then its components $\Psi^{+,\alpha}(x,t,P)$ are defined on an N-fold covering Γ of the initial elliptic curve with branch cuts between the points P_i^+, P_i^-, $i = 1, \dots, l$, and are meromorphic outside these cuts. In the case of general position the curve Γ is a smooth algebraic curve, its genus is equal to $g = Nl - \frac{l(l+1)}{2} + 1$, and $\Psi^{+,\alpha}$ has $(g-1)$ poles $\gamma_1^+, \dots, \gamma_{g-1}^+$, which do not depend on x, t. In a neighbourhood of the points P_i^+, $i = 1, \dots, l$, the function $\Psi^{+,\alpha}$ has the form*

$$\Psi^{+,\alpha}(x,t,P) = \left(\chi_0^{+,\alpha i} + \sum_{s=1}^{\infty}\chi_s^{+,\alpha i}(x,t)(z+\eta)^s\right) \times \left(\varkappa_i(z+\eta)^{-1}\right)^{-x/\eta} e^{-\varkappa_i(z+\eta)^{-1}t}\Psi^{+,1}(0,0,P), \tag{3.45}$$

where $\chi_0^{\alpha i}$ are constants independent of x, t. In a neighbourhood of the points P_i^- the function $\Psi^{+,\alpha}$ has the form

$$\Psi^{+,\alpha}(x,t,P) = (z - z_i^-)^{-x/\eta}\left(\sum_{s=0}^{\infty}\tilde{\chi}_s^{\alpha i}(x,t)(z-z_i^-)^s\right)\Psi^{+,1}(0,0,P). \tag{3.46}$$

The boundary values $\Psi^{+,\alpha;(\pm)}$ of the function $\Psi^{+,\alpha}$ on opposite sides of the cuts are connected by the relation

$$\Psi^{+,\alpha;(+)} = \Psi^{+,\alpha;(-)}e^{-2\pi i x/\eta}. \tag{3.47}$$

Remark. From Theorem 3.2 it follows, in particular, that the solution Ψ of equation (2.19) is a Baker–Akhiezer function to within normalization. In § 4 we show that this function is uniquely defined by the curve Γ, the poles γ_s, the matrix χ_0, and the value $x_1(0)$. All these quantities are defined by the initial Cauchy data and are independent of t. However, it is necessary to emphasize that some of them depend on the choice of the normalization point t_0, chosen above as $t_0 = 0$. Given a set $\{x_i, \dot{x}_i, a_i, b_i^+ | (b_i^+, a_i) = \dot{x}_i\}$ of initial data, the matrix L is defined by (2.32), the curve Γ is defined by the characteristic equation (3.1), and equation (3.26) defines a set of $g-1$ points γ_s on Γ. Therefore, the following map is well defined:

$$\{x_i, \dot{x}_i, a_i, b_i^+ \mid (b_i^+, a_i) = \dot{x}_i\} \mapsto \{\Gamma,\ D \in J(\Gamma)\}, \tag{3.48}$$

$$D = \sum_{s=1}^{g-1} A(\gamma_s) + x_1 U^{(0)}, \tag{3.49}$$

where $A : \Gamma \to J(\Gamma)$ is the Abel map and $U^{(1)}$ is a vector depending only on Γ (see (4.13)). The coefficients of (3.1) are integrals of the system (1.21)–(1.23). In § 4 we show that the second part of the data (3.48) defines angle-type variables, that is, the vector $D(t)$ evolves linearly, $D(t) = D(t_0) + (t - t_0)U^{(+)}$, if the point of the phase space evolves according to equations (1.21)–(1.23). The motion equations for the system considered have the following obvious symmetries:

$$a_i,\ b_i^+ \to q_i a_i,\ q_i^{-1} b_i^+, \qquad a_i,\ b_i^+ \to W^{-1} a_i,\ b_i^+ W, \tag{3.50}$$

where q_i are constants and W is an arbitrary constant matrix. In § 4 we prove that the data Γ, D uniquely define a point of the phase space to within the symmetry transformations (3.50).

§ 4. Finite-gap solutions of the non-Abelian Toda chain

Finite-gap solutions of the non-Abelian Toda chain were constructed in [31] by one of the authors. Preparatory to constructing the inverse spectral transformation for the spin generalization of the Ruijsenaars–Schneider model, we recall the main points of this theory. Since we are working with a continuous variable x rather than with the discrete variable n, some minor modifications of the construction are introduced.

Theorem 4.1. *Let Γ be a smooth algebraic curve of genus g with fixed local coordinates $w_{j,\pm}(P)$ in neighbourhoods of $2l$ points $P_j^\pm$, $w_{j,\pm}(P_j^\pm) = 0$, $j = 1, \dots, l$, and with fixed cuts between the points $P_j^\pm$. Let $\gamma_1, \dots, \gamma_{g+l-1}$ be a set of $g+l-1$ points in general position. Then there exists a unique function $\psi_\alpha(x, T, P)$, $\alpha = 1, \dots, l$, $T = \{t_{i,j;\pm},\ i = 1, \dots, \infty;\ j = 1, \dots, l\}$, such that*

1⁰. *the function ψ_α, as a function of $P \in \Gamma$, is meromorphic outside the cuts and has at most simple poles at the points γ_s (if all of them are distinct);*

2⁰. *the boundary values $\psi_\alpha^{(\pm)}$ of this function on opposite sides of the cuts satisfy the relation*

$$\psi_\alpha^{(+)}(x, T, P) = \psi_\alpha^{(-)}(x, T, P) e^{2\pi i x/\eta}; \tag{4.1}$$

3^0. *in a neighbourhood of the point $P_j^{\pm}$ the function ψ_α has the form*

$$\psi_\alpha(x,T,P) = w_{j,\pm}^{\mp x/\eta}\left(\sum_{s=0}^{\infty}\xi_s^{\alpha j;\pm}(x,T)w_{j,\pm}^s\right)\exp\left(\sum_{i=1}^{\infty}w_{j,\pm}^{-i}t_{i,j;\pm}\right), \quad w_{j,\pm} = w_{j,\pm}(P), \tag{4.2}$$

$$\xi_0^{\alpha j;+}(x,T) \equiv \delta_{\alpha j}. \tag{4.3}$$

The proof of theorems of this kind, as well as the explicit formula for ψ_α in terms of Riemann theta-functions, are standard in finite-gap integration theory. To give the corresponding formulae we use the notation of [19].

By the Riemann–Roch theorem, if $D = \gamma_1 + \cdots + \gamma_{g+l-1}$ is a divisor in general position, then there is a unique meromorphic function $h_\alpha(P)$ such that the divisor of its poles coincides with D and

$$h_\alpha(P_j^+) = \delta_{\alpha j}. \tag{4.4}$$

If we fix the basis of cycles a_i^0, b_i^0 on Γ with canonical intersection matrix, then this function may be written as follows:

$$h_\alpha(P) = \frac{f_\alpha(P)}{f_\alpha(P_\alpha^+)}; \qquad f_\alpha(P) = \theta\big(A(P)+Z_\alpha\big)\frac{\prod_{j\neq\alpha}\theta(A(P)+R_j)}{\prod_{i=1}^{l}\theta(A(P)+S_i)}, \tag{4.5}$$

where the Riemann theta-function $\theta(z_1,\dots,z_g) = \theta(z_1,\dots,z_g|B)$ is defined by the matrix $B = (B_{ik})$ of periods of holomorphic differentials on Γ, $A(P)$ is the Abel map, $A : P \in \Gamma \to J(\Gamma)$, and

$$R_j = -\mathcal{K} - A(P_j^+) - \sum_{s=1}^{g-1}A(\gamma_s), \qquad j = 1,\dots,l, \tag{4.6}$$

$$S_i = -\mathcal{K} - A(\gamma_{g-1+i}) - \sum_{s=1}^{g-1}A(\gamma_s), \tag{4.7}$$

$$Z_\alpha = Z_0 - A(P_\alpha^+), \qquad Z_0 = -\mathcal{K} - \sum_{i=1}^{g+l-1}A(\gamma_i) + \sum_{j=1}^{l}A(P_j^+), \tag{4.8}$$

where $\mathcal{K}$ is the vector of Riemann constants. (The derivation of these formulae can be found in [19].)

Let $d\Omega^{(i,j;\pm)}$ be the unique meromorphic differential that is holomorphic on Γ outside the points $P_j^{\pm}$, $j = 1,\dots,l$, and has the following form in a neighbourhood of these points:

$$d\Omega^{(i,j;\pm)} = d\big(w_{j,\pm}^{-i} + O(w_{j;\pm})\big) \tag{4.9}$$

with the normalization condition

$$\oint_{a_k^0} d\Omega^{(i,j;\pm)} = 0. \tag{4.10}$$

This differential defines the vector $U^{(i,j;\pm)}$ with coordinates

$$U_k^{(i,j;\pm)} = \frac{1}{2\pi i} \oint_{b_k^0} d\Omega^{(i,j;\pm)}. \tag{4.11}$$

Also let $d\Omega^{(0)}$ be the differential with zero a-periods that is holomorphic outside the points $P_j^{\pm}$ and has the following form in a neighbourhood of these points:

$$d\Omega^{(0)} = \pm \frac{dw_{j,\pm}}{\eta w_{j,\pm}} + O(1)\, dw_{j,\pm}. \tag{4.12}$$

This differential defines the vector $U^{(0)}$ with coordinates

$$U_k^{(0)} = \frac{1}{2\pi i} \oint_{b_k^0} d\Omega^{(0)}. \tag{4.13}$$

By the bilinear Riemann relations for periods of differentials of the third kind we have

$$U^{(0)} = \eta^{-1} \sum_{j=1}^{l} \big(A(P_j^-) - A(P_j^+)\big). \tag{4.14}$$

Theorem 4.2. *The components ψ_α of the Baker–Akhiezer function $\psi(x,T,P)$ are given by the formula*

$$\psi_\alpha = h_\alpha(P) \frac{\theta(A(P) + U^{(0)}x + \sum_A U^{(A)} t_A + Z_\alpha)\theta(Z_0)}{\theta(A(P) + Z_\alpha)\theta(U^{(0)}x + \sum_A U^{(A)} t_A + Z_0)} e^{\left(x\Omega^{(0)}(P) + \sum_A t_A \Omega^{(A)}(P)\right)}, \tag{4.15}$$

$$\Omega^{(A)}(P) = \int_{q_0}^{P} d\Omega^{(A)}, \qquad A = (i,j;\pm). \tag{4.16}$$

Remark. Note that a single-valued branch of ψ can be defined only after cutting the curve Γ between the points $P_j^{\pm}$, since the Abelian integral $\Omega^{(0)}$ has logarithmic singularities at the marked points.

We now define the dual Baker–Akhiezer function. For each set of $g+l-1$ points in general position there exists a unique differential $d\Omega$ such that it is holomorphic outside the points $P_j^{\pm}$ and has simple poles at these points with residues ± 1, that is,

$$d\Omega = \pm \frac{dw_{j,\pm}}{w_{j,\pm}} + O(1)\, dw_{j,\pm}. \tag{4.17}$$

As well as being zero at the points γ_s,

$$d\Omega(\gamma_s) = 0, \tag{4.18}$$

this differential is also zero at the other $g+l-1$ points denoted by γ_s^+.

The function $\psi^+(x,T,P)$ is the dual Baker–Akhiezer function if its components $\psi^{+,\alpha}(x,t,P)$ satisfy the following conditions.

1^0. The function $\psi^{+,\alpha}$, as a function of the variable $P \in \Gamma$, is meromorphic outside the cuts and has at most simple poles at the points γ_s^+ (if all of them are distinct).

2^0. The boundary values $\psi^{+;\alpha,(\pm)}$ of this function on opposite sides of the cuts satisfy the relation

$$\psi^{+;\alpha,(+)}(x,T,P) = \psi^{+;\alpha,(-)}(x,T,P)e^{-2\pi ix/\eta}. \tag{4.19}$$

3^0. In a neighbourhood of the points $P_j^{\pm}$ the function $\psi^{+,\alpha}$ has the form

$$\psi^{+,\alpha}(x,T,P) = w_{j,\pm}^{\pm x/\eta}\left(\sum_{s=0}^{\infty}\xi_s^{+;\alpha j;\pm}(x,T)w_{j,\pm}^s\right)\exp\left(-\sum_{i=1}^{\infty}w_{j,\pm}^{-i}t_{i,j;\pm}\right), \tag{4.20}$$

$$\xi_0^{+;\alpha j;+}(x,T) \equiv \delta_{\alpha j}. \tag{4.21}$$

Let $h_\alpha^+(P)$ be a function having poles at the points of the dual divisor $\gamma_1^+,\dots,\gamma_{g+l-1}^+$ and normalized by the condition $h_\alpha^+(P_j^+) = \delta_{\alpha j}$. This function can be written in the form (4.5), where γ_s are replaced by γ_s^+. By the definition of dual divisors we have

$$\sum_{s=1}^{g+l-1} A(\gamma_s) + \sum_{s=1}^{g+l-1} A(\gamma_s^+) = K_0 + \sum_{j=1}^{l}\big(A(P_j^+) + A(P_j^-)\big), \tag{4.22}$$

where K_0 is the canonical class, that is, the equivalence class of the divisor of zeros of a holomorphic differential. Consequently, the vector Z_0^+ in the formulae for h_α^+ is connected with Z_0 by the relation

$$Z_0 + Z_0^+ = -2\mathcal{K} - K_0 + \sum_{j=1}^{l}\big(A(P_j^+) - A(P_j^-)\big) = -2\mathcal{K} - K_0 - U^{(0)}\eta. \tag{4.23}$$

Theorem 4.3. *The components $\psi^+(x,T,P)$ of the dual Baker–Akhiezer function are given by*

$$\psi^{+;\alpha} = h_\alpha^+(P)\frac{\theta(A(P)-U^{(0)}x-\sum_A U^{(A)}t_A+Z_\alpha^+)\theta(Z_0^+)}{\theta(A(P)+Z_\alpha^+)\theta(U^{(0)}x+\sum_A U^{(A)}t_A-Z_0^+)}e^{-(x\Omega^{(0)}(P)+\sum_A t_A\Omega^{(A)}(P))}, \tag{4.24}$$

where

$$Z_0^+ = -Z_0 - 2\mathcal{K} - K_0 - U^{(0)}\eta, \qquad Z_\alpha^+ = Z_0^+ - A(P_\alpha^+). \tag{4.25}$$

The above-mentioned results are valid for any algebraic curve having two sets of marked points. We now consider the class of special curves corresponding to the spin generalizations of the Ruijsenaars–Schneider model.

Theorem 4.4. *Let $\tilde{\Gamma}$ be a smooth algebraic curve defined by the equation*

$$\widehat{R}(K,z) = K^N + \sum_{i=1}^{N} R_i(z)K^{N-i} = 0, \tag{4.26}$$

and suppose that the coefficients $R_j(z)$ are meromorphic functions of z, satisfy the condition

$$R_j(z+2\omega_\alpha) = R_j(z)e^{-2j\zeta(\omega_\alpha)\eta}, \tag{4.27}$$

and are holomorphic in the fundamental domain of the lattice with periods $2\omega_\alpha$ outside the point $z=-\eta$. Suppose also that in a neighbourhood of $z=-\eta$ the polynomial $\widehat{R}$ has the following factorization:

$$\widehat{R}(K,z) = \prod_{i=1}^{l}\big(K+(z+\eta)^{-1}H_i(z+\eta)\big)\prod_{i=l+1}^{N}\big(K+(z+\eta)H_i(z+\eta)\big), \tag{4.28}$$

where the functions $H_i(z)$ are regular in a neighbourhood of $z=-\eta$. Let ψ be the Baker–Akhiezer function corresponding to: (i) *the curve Γ, which is the factor of $\tilde{\Gamma}$ with respect to the transformation group*

$$z \mapsto z+2\omega_\alpha, \qquad K \mapsto Ke^{-2\zeta(\omega_\alpha)\eta}, \tag{4.29}$$

(ii) *the local coordinates $w_{j,+}=(z+\eta)H_j^{-1}(0)$ near the poles $P_j^+, j=1,\dots,l$, of the multivalued function $K=K(P)$, and arbitrary local coordinates $w_{j,-}$ near the zeros P_j^- of $K=K(P)$. Then the function ψ satisfies the following relation:*

$$\psi(x+2\omega_\alpha,T,P) = \varphi_\alpha(P)\psi(x,T,P), \tag{4.30}$$

where

$$\varphi_\alpha(P) = K(P)^{2\omega_\alpha/\eta}e^{\zeta(\omega_\alpha)z}. \tag{4.31}$$

The proof is based on the following evident facts. By the monodromy properties (4.29), the values of $\varphi_\alpha(P)$ do not change under shifts of z by periods of the elliptic curve, that is, $\varphi_\alpha(P)$ is well defined on Γ. Relation (4.30) follows from the fact that its left- and right-hand sides have the same analytic properties.

Corollary 4.1. *Let $\psi_\alpha(x,T,P)$ be components of the Baker–Akhiezer function $\psi(x,T,P)$ defined in Theorem 4.4. Then the following representation is valid:*

$$\psi(x,T,P) = \sum_{i=1}^{m} s_i(T,P)\Phi\big((x-x_i(T)),z\big)k^{x/\eta}, \qquad k = K\left[\frac{\sigma(z+\eta)}{\sigma(z-\eta)}\right]^{1/2}. \tag{4.32}$$

Moreover, the dual Baker–Akhiezer function has the form

$$\psi^+(x,T,P) = \sum_{i=1}^{m} s_i^+(T,P)\Phi\big((-x+x_i(T)-\eta),z\big)k^{-x/\eta}. \tag{4.33}$$

The proof of the corollary is identical to the proof of the corresponding assertion in [19]; in point of fact, this proof has already been presented at the beginning of § 2, because Theorem 4.4 implies that ψ and ψ^+ are double-Bloch functions. (Note that from (4.23), (4.24) it follows that ψ^+ has poles at the points $x_i - \eta$.)

So far it has been assumed that $t_{i,\alpha;\pm}$ are arbitrary parameters, and ψ depends on these parameters through the form of essential singularity at the points $P_\alpha^\pm$. We now fix the values of these parameters for $i > 1$ as $t_{i,\alpha;\pm} = t^0_{i,\alpha;\pm}$, whereas for $i = 1$ we put

$$t_{1,\alpha;\pm} = t_\pm + t^0_{1,\alpha;\pm}. \tag{4.34}$$

The corresponding Baker–Akhiezer function now depends on the variables (x, t_+, t_-). For brevity, we denote it by $\psi(x, t_+, t_-, P)$, skipping the dependence on the constants T^0.

Theorem 4.5. *For any choice of the constants T^0, the Baker–Akhiezer function $\psi(x, t_+, t_-, P)$ satisfies the equations*

$$\partial_+\psi(x, t_+, t_-, P) = \psi(x + \eta, t_+, t_-, P) + v(x, t_+, t_-)\psi(x, t_+, t_-, P), \tag{4.35}$$

$$\partial_-\psi(x, t_+, t_-, P) = c(x, t_+, t_-)\psi(x - \eta, t_+, t_-, P), \qquad \partial_\pm = \partial/\partial t_\pm, \tag{4.36}$$

where

$$v(x, t_+, t_-) = \partial_+ g(x, t_+, t_-)g^{-1}(x, t_+, t_-) = \xi_1^+(x, t_+, t_-) - \xi_1^+(x + \eta, t_+, t_-), \tag{4.37}$$

$$c(x, t_+, t_-) = g(x, t_+, t_-)g^{-1}(x - \eta, t_+, t_-) = \partial_-\xi_1^+(x, t_+, t_-), \tag{4.38}$$

and the matrices g and ξ_1^+ are defined by the coefficients of expansion (4.2)

$$g^{\alpha,j}(x, t_+, t_-) = \xi_0^{\alpha,j;-}(x, t_+, t_-), \qquad \xi_1^+(x, t_+, t_-) = \{\xi_1^{\alpha j;+}\}. \tag{4.39}$$

Proof. Equation (4.35) follows from the fact that the function

$$\partial_+\psi_\alpha(x, t_+, t_-, P) - \psi_\alpha(x + \eta, t_+, t_-, P)$$

has the same analytic properties as ψ, except for the normalization condition (4.3). Therefore, we can write it as a linear combination of the basis functions ψ_β with coefficients $v^{\alpha\beta}$. These coefficients are determined by comparing the coefficients in the left- and right-hand sides of (4.35) at the points P_j^-. Hence we arrive at the first equality in (4.37). On the other hand, expanding ψ near the points P_j^+ we arrive at the second equality in (4.37). Equalities (4.36) and (4.38) are obtained in a similar way.

Corollary 4.2. *The matrix function $g_n(t_+, t_-) = g(n\eta + x_0, t_+, t_-)$ corresponding (according to the definition of the Baker–Akhiezer functions) to the curve Γ with fixed local coordinates near the marked points $P_j^\pm$ and to the set of points $\gamma_1, \dots, \gamma_{g+l-1}$, is a solution of the 2D Toda chain equations* (2.1).

Remark. The dependence of g_n on the variables $t_{i,j}$, $i = 1, \dots, \infty$, $j = 1, \dots, l$, corresponds to higher flows of the 2*D* Toda chain hierarchy.

Theorem 4.6. *The dual Baker–Akhiezer function satisfies the equations*

$$-\partial_+\psi^+(x,t_+,t_-,P) = \psi^+(x-\eta,t_+,t_-,P) + \psi^+(x,t_+,t_-,P)v(x,t_+,t_-), \tag{4.40}$$

$$-\partial_-\psi^+(x,t_+,t_-,P) = \psi^+(x+\eta,t_+,t_-,P)c(x+\eta,t_+,t_-), \tag{4.41}$$

where $c(x,t_+,t_-)$, $v(x,t_+,t_-)$ *are the same as in* (4.35), (4.36).

Proof. The same arguments as in the proof of Theorem 4.5 show that ψ^+ satisfies equations of form (4.40), (4.41) with the coefficients c^+, v^+ given by

$$c^+(x,t_+,t_-) = [\xi_0^{+;-}(x+\eta,t_+,t_-)]^{-1}\xi_0^{+;-}(x,t_+,t_-), \tag{4.42}$$

$$v^+(x,t_+,t_-) = -[\xi_0^{+;-}(x,t_+,t_-)]^{-1}\partial_+\xi_0^{+;-}(x,t_+,t_-), \tag{4.43}$$

where the matrix elements $\xi_0^{+;-} = \{\xi_0^{+;\alpha j;-}\}$ are determined from (4.20). The coincidence of the coefficients of equations for ψ and ψ^+ is a consequence of the relation

$$[\xi_0^{+;-}]^{-1} = \xi_0^-(x,t_+,t_-) = g(x,t_+,t_-), \tag{4.44}$$

which follows from the definition of the dual Baker–Akhiezer function. To prove (4.44), we consider the differential $\psi_\alpha\psi^{+\beta}d\Omega$, where $d\Omega$ is the same as in the definition of the dual divisor γ_s^+. It is a meromorphic differential on Γ with the only poles at $P_j^\pm$, and its residue at the point P_j^+ is given by

$$\operatorname*{res}_{P_j^+}\psi_\alpha\psi^{+\beta}\,d\Omega = \delta_{\alpha,j}\delta_{\beta,j}. \tag{4.45}$$

Further, the residue of this differential at P_j^- is given by

$$\operatorname*{res}_{P_j^+}\psi_\alpha\psi^{+\beta}\,d\Omega = -\xi_0^{\alpha,j;-}\xi_0^{+;\beta,j;-}. \tag{4.46}$$

For a meromorphic differential the sum of the residues should be zero; therefore, relation (4.44) follows from (4.45) and (4.46).

Theorem 4.7. *Let the curve Γ, the marked points $P_j^\pm$ and the local coordinates in their neighbourhoods be the same as in Theorem* 4.4. *Then the corresponding algebraic-geometrical 'potentials' v and c in equations* (4.35), (4.36) *are elliptic functions of x. In the case of general position they have the form*

$$v(x,T) = \sum_{i=1}^{N} a_i(T)b_i^+(T)V\big(x-x_i(T)\big), \tag{4.47}$$

$$c(x,T) = \partial_-\left(S_0(T) + \sum_{i=1}^{N} a_i(T)b_i^+(T)\zeta\big(x-x_i(T)\big)\right), \tag{4.48}$$

where a_i and b_i^+ are vectors, and S_0 is a matrix-valued function independent of x.

Proof. By (4.30) the potentials in (4.35) and (4.36) are elliptic functions. It follows from (4.15) that the poles $x = x_i(T)$ of the Baker–Akhiezer function correspond to solutions of the equation

$$\theta\Big(U^{(0)}x + \sum_A U^{(A)}t_A + Z_0\Big) = 0. \tag{4.49}$$

Further, it follows from (4.8) that for the corresponding solutions $(x_i(T),\ T)$ the first factor in the numerator of (4.15) is equal to zero for $P = P_\alpha^+$. On the other hand, the function $h_\alpha(P)$ vanishes at the points P_β^+, $\beta \neq \alpha$. Therefore, the residues $\psi^0_{\alpha,i}(T,P)$ of the function $\psi_\alpha(x,T,P)$ at the points $x = x_i(T)$ have the following analytic properties as functions of P:

1^0. they are meromorphic functions on Γ outside the cuts between the points $P_j^\pm$ and have the same poles as ψ;

2^0. their boundary values $\psi^{0;(\pm)}_{\alpha,i}(T,P)$ on opposite sides of the cuts satisfy the relation

$$\psi^{0;(+)}_{\alpha,j}(T,P) = \psi^{0;(-)}_{\alpha,j}(T,P)e^{2\pi i x_j(T)/\eta}; \tag{4.50}$$

3^0. in a neighbourhood of the points $P_j^\pm$

$$\psi^0_{\alpha,i}(T,P) = w_{j,\pm}^{\mp x_i(T)/\eta} \exp\Big(\sum_{s=1}^\infty w_{j,\pm}^{-s} t_{s,j;\pm}\Big) F^\pm_{i,j,\alpha}(w_{j,\pm}), \tag{4.51}$$

where $F^\pm_{i,j,\alpha}$ are regular functions and

$$F^+_{i,j,\alpha}(0) = 0. \tag{4.52}$$

This means that $\psi^0_{\alpha,i}$ has the same analytic properties as the Baker–Akhiezer function, except the following one. The regular factor in the expansion of this function near *all* the points P_j^+ has vanishing leading term. For general x, t_A there is no such function, and for the special values $(x = x_i(T), T)$ such a function $\psi_{i0}(T,P)$ does exist and is *unique* to within a constant (with respect to P) factor; moreover, this function is uniquely defined in the general case when $x_i(T)$ is a simple root of (4.49). Thus, we can represent ψ_α in the form

$$\psi_\alpha(x,T,P) = \frac{\phi_\alpha(T)\,\psi_{i0}(T,P)}{x - x_i(T)} + O\big((x - x_i(T))^0\big). \tag{4.53}$$

It follows from (4.53) that for the matrix $\xi_1^+(x,T)$ with matrix elements $\xi_1^{\alpha j;+}(x,T)$ the residues $\rho_i(T)$ of $\xi_1^+(x,T)$, which are defined by the formula

$$\xi_1^+(x,T) = \frac{\rho_i(T)}{x - x_i(T)} + O\big((x - x_i(t))^0\big), \tag{4.54}$$

have rank 1. Therefore, there exist vectors $a_i(T)$ and covectors $b_i^+(T)$ such that $\rho_i = a_i(T)b_i^+(T)$. Then it follows from (4.30) that the matrix ξ_1^+ has the following monodromy properties:

$$\xi_1^+(x + 2\omega_l) = \xi_1^+(x,T) + 2\zeta(\omega_l)r, \tag{4.55}$$

where r is a constant. Thus, by (4.54) and (4.55) we see that ξ_1^+ can be written in the form

$$\xi_1^+ = S_0(T) + \sum_{i=1}^{N} a_i(T) b_i^+(T) \zeta\big(x - x_i(T)\big). \tag{4.56}$$

Taking into account the second equalities in (4.37) and (4.38), we arrive at the conclusion of the theorem.

Remark. In the Abelian case ($l = 1$) there is an equivalent representation for $c(x)$ in the form of the product of pole factors (see also (1.17))

$$c(x,T) = \prod_{i=1}^{N} \frac{\sigma(x - x_i(T) + \eta)\sigma(x - x_i(T) - \eta)}{\sigma^2(x - x_i(T))}. \tag{4.57}$$

Comparing the coefficients of the pole terms in (4.48) and (4.57), we obtain the relations

$$\partial_+\partial_- x_i(T) = -\sigma^2(\eta) \prod_{k\neq i} \frac{\sigma(x_i - x_k + \eta)\sigma(x_i - x_k - \eta)}{\sigma^2(x_i - x_k)}, \tag{4.58}$$

$$\partial_+\partial_- x_i(T) = -\partial_+ x_i(T)\partial_- x_i(T) \sum_{k\neq i} \big(V(x_i - x_k) - V(x_k - x_i)\big). \tag{4.59}$$

It is easy to check that if the dynamics with respect to $t_\pm$ is given by the Hamiltonians $\sigma(\pm\eta)H_\pm$, where $H_\pm$ are defined by (1.20), then these relations follow from the motion equations.

For equations (4.35) and (4.36) the connection between algebraic-geometrical potentials corresponding to equivalent divisors is well known in the theory of finite-gap integration. Let $D = \gamma_1 + \cdots + \gamma_{g+l-1}$ and $D^{(1)} = \gamma_1^{(1)} + \cdots + \gamma_{g+l-1}^{(1)}$ be two equivalent divisors, that is, there exists a meromorphic function $h(P)$ on Γ such that D is the divisor of its poles, and $D^{(1)}$ is the divisor of its zeros.

Corollary 4.3. *The algebraic-geometrical potentials $v(x,T)$, $c(x,T)$ and $v^{(1)}(x,T)$, $c^{(1)}(x,T)$ corresponding to the set $\{\Gamma, P_j^\pm, w_{j,\pm}(P)\}$ and to the equivalent divisors D and $D^{(1)}$ are gauge equivalent, that is,*

$$v^{(1)}(x,T) = Hv(x,T)H^{-1}, \quad c^{(1)}(x,T) = Hc(x,T)H^{-1}, \quad H^{\alpha j} = h(P_j^+)\delta^{\alpha j}. \tag{4.60}$$

Corollary 4.4. *In the case of general position the curve Γ satisfies the conditions of Theorem 4.4 if and only if it is the spectral curve (3.13) of the Lax matrix L defined by (2.32), for which x_i, $\dot{x}_i$ are arbitrary constants, and the vectors a_i, b_i^+ satisfy conditions (1.25).*

By Theorem 3.2, the Baker–Akhiezer function $\Psi_\alpha(x,t,P)/\Psi_1(0,0,P)$ is connected with the normalized Baker–Akhiezer function $\psi_\alpha(x,t,P)$ by the relation

$$\frac{\Psi_\alpha(x,t,P)}{\Psi_1(0,0,P)} = \sum_\beta \chi_0^{\alpha\beta} \psi_\beta(x,t,P), \tag{4.61}$$

where $\psi(x,t,P)$ is the Baker–Akhiezer function defined at the beginning of this section, which corresponds to the following values of the parameters $T = \{t_{i,j;\pm}\}$:

$$t_{1,j;+} = t, \quad t_{i,j,\pm} = 0, \quad (i,j,\pm) \neq (1,j,+). \tag{4.62}$$

Equality (4.61) yields the following corollary.

Corollary 4.5. *Let $a_i(t)$, $b_i(t)$, $x_i(t)$ be solutions of the motion equations* (1.21)–(1.23). *Then*

$$\sum_{i=1}^{N} a_i(t) b_i^+(t) V\big(x - x_i(t)\big) = \chi_0 v(x,t) \chi_0^{-1}, \tag{4.63}$$

where $v(x,t) = v(x, t_+ = t, t_- = 0)$ is the algebraic-geometrical potential corresponding (according to Theorem 4.5) *to the normalized Baker–Akhiezer function $\psi(x,t,P)$ constructed from the data (a curve with marked points) obeying the conditions of Theorem* 4.4.

Corollary 4.6. *The map*

$$a_i(t),\ b_i^+(t),\ x_i(t) \mapsto \{\Gamma, [D]\}, \tag{4.64}$$

where $[D]$ is the equivalence class of the divisor D, is an isomorphism to within the transformations (3.50).

The corresponding curve Γ does not depend on time, whereas $[D]$ depends on the choice $t_0 = 0$ of the initial point. The following theorem shows that this dependence of $[D(t_0)]$ is linearized on the Jacobian.

Theorem 4.8. *Let Γ be the curve defined by equation* (4.28), *and let $D = \gamma_1, \dots, \gamma_{g+l-1}$ be a set of points in general position. Then the solutions of the system* (1.21), (2.23), (2.24) *are given by the formulae*

$$\theta(U^{(0)} x_i(t) + U^{(+)} t + Z_0) = 0, \qquad U^{(+)} = \sum_j U^{(1,j,+)}, \tag{4.65}$$

$$a_{i,\alpha}(t) = Q_i^{-1}(t) h_\alpha(q_0) \frac{\theta(U^{(0)} x_i(t) + U^{(+)} t + Z_\alpha)}{\theta(Z_\alpha)}, \tag{4.66}$$

$$b_i^\alpha(t) = Q_i^{-1}(t) h_\alpha^+(q_0) \frac{\theta(U^{(0)} x_i(t) + U^{(+)} t - Z_\alpha^+)}{\theta(Z_\alpha^+)}, \tag{4.67}$$

where

$$Q_i^2(t) = \frac{1}{2} \sum_{\alpha=1}^{l} h_\alpha^+(q_0) h_\alpha(q_0) \frac{\theta(U^{(0)} x_i(t) + U^{(+)} t - Z_\alpha)\theta(U^{(0)} x_i(t) + U^{(+)} t - Z_\alpha^+)}{\theta(Z_\alpha)\theta(Z_\alpha^+)} \tag{4.68}$$

and q_0 is an arbitrary point of Γ. Moreover, any solution in general position may be obtained from solutions (4.65)–(4.67) *by means of the symmetries* (3.50).

Remark. If the vector Z_0 in (4.65)–(4.67) is replaced by

$$Z_0 \mapsto Z_0 + \sum_A U^{(A)} t_A, \tag{4.69}$$

then the corresponding quantities $x_i(T)$, $a_i(T)$, $b_i(T)$, $T = \{t_A\}$ depend on t_A in accordance with the higher commuting flows of the system (1.21)–(1.23). It should be emphasized that the points $P_j^{\pm}$ enter symmetrically. For this reason the dependence of $x_i(T)$, $a_i(T)$, $b_i(T)$ on the variable $t_- = l^{-1} \sum_{j=1}^{l} t_{1,j;-}$ is described by the same equations as for $t = t_+$.

§ 5. Difference analogues of Lamé operators

We consider the operator S_0 defined by formula (1.9) for integer ℓ. Taking into account the obvious symmetry $-\ell \leftrightarrow \ell - 1$, we see that it is sufficient to restrict the consideration by the assumption $\ell \in \mathbb{Z}_+$. The finite-gap property of S_0 means that the Bloch solutions of the equation

$$(S_0 f)(x) = \varepsilon f(x) \tag{5.1}$$

are parametrized by points of a hyperelliptic curve of genus 2ℓ.

Any solution $f(x)$ of (5.1) may be represented in the form

$$f(x) = \Psi(x)\big(\theta_1(\eta/2)\big)^{-x/\eta} \prod_{j=1}^{\ell} \theta_1(x - j\eta), \tag{5.2}$$

where $\Psi(x)$ satisfies the equation

$$(\widetilde{S}_0 \Psi)(x) \equiv \Psi(x+\eta) + c_\ell(x)\Psi(x-\eta) = \varepsilon \Psi(x), \tag{5.3}$$

$$c_\ell(x) = \theta_1^2(\eta/2) \frac{\theta_1(x+\ell\eta)\theta_1(x-(\ell+1)\eta)}{\theta_1(x)\theta_1(x-\eta)}. \tag{5.4}$$

The transformation (5.2) sends Bloch solutions of the first equation to Bloch solutions of the second one. To construct the solutions of (5.3), we use the ansatz similar to the one used for constructing solutions of the linear equation (2.2):

$$\Psi = \sum_{j=1}^{\ell} s_j(z, k, \varepsilon) \Phi_\ell(x - j\eta, z) k^{x/\eta}, \tag{5.5}$$

where

$$\Phi_\ell(x, z) = \frac{\theta_1(z + x + N\eta)}{\theta_1(z + N\eta)\theta_1(x)} \left[\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right]^{x/(2\eta)}, \qquad N = \frac{\ell(\ell+1)}{2}. \tag{5.6}$$

(Note that Φ_1 coincides with the function $\Phi(x, z)$ given by (2.6), and $c_1(x)$ coincides with the function $c(x - \eta)$ defined by (2.8) to within a constant factor.)

The function $\Phi_\ell(x,z)$ is doubly-periodic in z,

$$\Phi_\ell(x, z+2\omega_\alpha) = \Phi_\ell(x,z). \tag{5.7}$$

(In what follows we take the periods to be $2\omega_1 = 1$ and $2\omega_2 = \tau$.) For values of x such that $x/2\eta$ is a half-integer, the function Φ_ℓ is single-valued on the Riemann surface $\widehat{\Gamma}_0$ of the function $E(z)$ defined by (2.9). For general values of x one can select a single-valued branch of $\Phi_\ell(x,z)$ by cutting the elliptic curve Γ_0 between the points $z = \pm\eta$.

The function $\Phi_\ell(x,z)$, as a function of x, is a double-Bloch function, that is,

$$\Phi_\ell(x+2\omega_\alpha, z) = T_\alpha^{(\ell)}(z)\Phi_\ell(x,z), \tag{5.8}$$

where

$$T_1^{(\ell)}(z) = \left(\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right)^{1/(2\eta)}, \tag{5.9}$$

$$T_2^{(\ell)}(z) = \exp\bigl(-2\pi i(z+N\eta)\bigr)\left(\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right)^{\tau/(2\eta)}. \tag{5.10}$$

In the fundamental domain of the lattice the function $\Phi_\ell(x,z)$ has a unique pole at the point $x = 0$,

$$\Phi_\ell(x,z) = \frac{1}{\theta_1'(0)x} + O(1). \tag{5.11}$$

Substituting (5.5) in (5.3) and comparing the residues of the left- and right-hand sides of the equation at the points $z = j\eta$, $j = 0,\dots,\ell$, we obtain $\ell+1$ linear equations

$$\sum_{j=1}^{\ell} L_{i,j}s_j = 0, \qquad i = 0,\dots,\ell, \tag{5.12}$$

for ℓ unknown parameters $s_j = s_j(z,k,\varepsilon)$. The matrix elements $L_{i,j}$ of this system are given by

$$L_{0,1} = k + h\Phi_\ell(-2\eta,z)k^{-1},\ \ L_{0,j} = h\Phi_\ell(-(j+1)\eta,z)k^{-1}, \quad j = 2,\dots,\ell; \tag{5.13}$$

$$\begin{gathered} L_{1,1} = -\varepsilon - h\Phi_\ell(-\eta,z)k^{-1}, \qquad L_{1,2} = k - h\Phi_\ell(-2\eta,z)k^{-1}, \\ L_{1,j} = -h\Phi_\ell(-j\eta,z)k^{-1}, \qquad j>2; \end{gathered} \tag{5.14}$$

$$L_{i,j} = \delta_{i,j+1}c_ik^{-1} - \varepsilon\delta_{i,j} + \delta_{i,j-1}k, \qquad i>1, \tag{5.15}$$

where

$$h = \theta_1'(0)\operatorname*{res}_{x=0} c_\ell(x) = \frac{\theta_1^2(\eta/2)}{\theta_1(\eta)}\theta_1(\ell\eta)\theta_1\bigl((\ell+1)\eta\bigr), \tag{5.16}$$

$$c_j = c_\ell(j\eta) = \theta_1^2(\eta/2)\frac{\theta_1((j+\ell)\eta)\theta_1((j-\ell-1)\eta)}{\theta_1(j\eta)\theta_1((j-1)\eta)}. \tag{5.17}$$

The overdetermined system (5.12) has non-trivial solutions if and only if the rank of the rectangular matrix $L_{i,j}$ is less than ℓ. We denote by $L^{(0)}$ and $L^{(1)}$ the $(\ell \times \ell)$-matrices obtained from L by deleting the rows with $i = 0$ and $i = 1$, respectively. Then the set of parameters z, k, ε for which equation (5.3) has solutions of the form (5.12) is defined by the system of two equations

$$\det L^{(i)} \equiv R^{(i)}(z,k,\varepsilon) = 0, \qquad i = 0,1. \tag{5.18}$$

Expanding the determinants with respect to the upper row, we obtain

$$R^{(0)}(z,k,\varepsilon) = r_\ell^{(0)}(\varepsilon) + \sum_{j=1}^{\ell} k^{-j}\Phi_\ell(-j\eta,z)r_{\ell-j}^{(0)}(\varepsilon), \tag{5.19}$$

$$R^{(1)}(z,k,\varepsilon) = k\widetilde{r}_{\ell-1}(\varepsilon) + \sum_{j=1}^{\ell} k^{-j}\Phi_\ell(-(j+1)\eta,z)r_{\ell-j}^{(1)}(\varepsilon), \tag{5.20}$$

where $\tilde{r}_{\ell-1}$ and $r_{\ell-j}^{(0)}, r_{\ell-j}^{(1)}$ are polynomials in ε of degrees $\ell-1$ and $\ell-j$, respectively.

These equations define an algebraic curve $\widehat{\Gamma}$ realized as an $\ell(\ell+1)/2$-fold ramified covering of genus 2 for the curve $\widehat{\Gamma}_0$, on which the functions $\Phi_\ell(-j\eta,z)$ are single-valued. This curve has the obvious symmetry

$$(z,k,\varepsilon) \mapsto (z,-k,-\varepsilon), \tag{5.21}$$

which is a direct consequence of the following general property of equation (5.3). Let $\Psi(x)$ be a solution of (5.3) with an eigenvalue ε; then $\Psi(x)\exp(\pi i x/\eta)$ is a solution of the same equation with the eigenvalue $-\varepsilon$, and this transformation corresponds to the change of sign for k.

Note that the function $\Psi(x,z,k)$ is invariant under the transformation $k \to -k$ in (5.5) accompanied by the simultaneous interchanging of sheets of $E(z)$. Therefore, $\widehat{\Gamma}$ may be considered as an $\ell(\ell+1)$-fold ramified covering of Γ_0.

We now show that this curve is also invariant with respect to the following involution:

$$(z,k,\varepsilon) \mapsto (-z, k^{-1}\theta_1^2(\eta/2), \varepsilon). \tag{5.22}$$

In fact, let $\Psi(x)$ be a solution of (5.3). Then the function

$$\widetilde{\Psi}(x) = \Psi(-x)A(x), \tag{5.23}$$

where

$$A(x) = \theta_1(\eta/2)^{2x/\eta}\prod_{j=1}^{\ell}\frac{\theta_1(x+j\eta)}{\theta_1(x-j\eta)}, \tag{5.24}$$

is also a solution of (5.3). It is easy to see that if Ψ is a double-Bloch solution, then $\tilde{\Psi}$ is also a double-Bloch solution; moreover, if the Bloch multipliers of Ψ are parametrized by the pair (z,k), then the Bloch multipliers of $\tilde{\Psi}$ correspond to the pair $(-z, k^{-1}\theta_1^2(\eta/2))$.

The variable ε, as a function of $P \in \widehat{\Gamma}$, is a meromorphic function on the curve $\widehat{\Gamma}$. It cannot take any value more than twice, because for a given value of ε the second order difference equation (5.3) has at most two different Bloch solutions. Since the involution (5.22) is non-trivial, it follows that the function ε does take a generic value twice. Therefore, the algebraic curve $\widehat{\Gamma}$ is a hyperelliptic curve of finite genus g. Because of the symmetry (5.21) it can be parametrized by the equation

$$y^2 = \prod_{i=1}^{g+1}(\varepsilon^2 - \varepsilon_i^2). \tag{5.25}$$

The involution (5.22) is the hyperelliptic involution that interchanges the sheets of the ramified covering (5.25).

Now we prove that $g = 2\ell$. From (5.25) it follows that there are $2g + 2$ fixed points of the hyperelliptic involution. On the other hand, the number of fixed points of the hyperelliptic involution (5.22) is equal to the number of preimages of the second-order points $\omega_a \in \Gamma_0$, $a = 0, \dots, 3$ (which are fixed points with respect to the involution $z \to -z$ on Γ_0) on $\widehat{\Gamma}$ such that the corresponding value of k is equal to $\pm i^{\delta_{a,0}}\theta_1(\eta/2)$. Explicitly, from now on we adopt the following notation:[1]

$$\omega_0 = 0, \quad \omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1+\tau}{2}, \quad \omega_3 = \frac{\tau}{2}. \tag{5.26}$$

Note that the values of k corresponding to preimages of the point $z = 0$ and other semiperiods, which are fixed under involution (5.22), are different, since the involution $z \to -z$ sends the point $z = 0$ to a point on the other side of the cut between the points $z = \pm\eta$.

For each fixed point of the hyperelliptic involution there is only one Bloch solution; moreover, this solution has a definite parity with respect to the involution (5.23), that is,

$$\Psi(x) = \nu\Psi(-x)A(x), \qquad \nu = \pm 1. \tag{5.27}$$

We now prove that $\nu = (-1)^\ell$.

In fact, equality (5.27) implies that for $x = \eta$

$$s_1 = \nu k^{-1}\Psi(-\eta)(-1)^{\ell-1}\frac{\theta_1^2(\eta/2)}{\theta_1(\eta)}\theta_1(\ell\eta)\theta_1((\ell+1)\eta). \tag{5.28}$$

Comparing this equality with equality (5.3) taken at $x = 0$, we see that $\nu = (-1)^\ell$ if $s_1 \neq 0$. Otherwise, $s_1 = 0$ and $\Psi(-\eta) = 0$. From (5.3) it follows that the coefficients s_1 and s_2 in (5.5) cannot be equal to zero simultaneously. Indeed, let j be a minimal index such that $s_j \neq 0$. Assume that $j > 2$. Then the left-hand side of (5.3) has a pole at the point $z = (j-1)\eta$, but the right-hand side has no pole at this point. Therefore, $s_2 \neq 0$ if $s_1 = 0$, whence equation (5.3) implies that $\Psi(0) \neq 0$. At the same time from equality (5.27) taken at $x = 0$ it follows that

$$\Psi(0) = (-1)^\ell\nu\Psi(0). \tag{5.29}$$

This proves the relation $\nu = (-1)^\ell$.

[1]This notation differs from that adopted in § 2.

In the case when $z = \omega_a$ and $k = i^{\delta_{a,0}}\theta_1(\eta/2)$ the representation of double-Bloch functions in the the form (5.5) can be written as follows:

$$\Psi(x) = \left(\theta_1\left(\frac{\eta}{2}\right)\right)^{x/\eta} \exp\bigl(\pi i(\delta_{2,a} + \delta_{3,a})x\bigr) \prod_{j=1}^{\ell} \frac{\theta_1(x + x_j)}{\theta_1(x - j\eta)}, \tag{5.30}$$

where[2]

$$\sum_{j=1}^{\ell} x_j = \omega_a. \tag{5.31}$$

Lemma 5.1. *The hyperelliptic involution of the curve $\widehat{\Gamma}$ has $2d$ fixed points, where d is equal to the sum of the dimensions of the functional subspaces consisting of functions that have the form* (5.30) *and satisfy* (5.27) *with $\nu = (-1)^\ell$.*

Proof. As was shown above, given a pair of fixed points of the hyperelliptic involution of the curve $\widehat{\Gamma}$ that are invariant with respect to the involution (5.21) corresponding to the change of sign for k, there exists a unique solution of (5.3), which has the form (5.30) and satisfies (5.20) for $\nu = (-1)^\ell$. On the other hand, the space of such functions is invariant with respect to the operator $\tilde{S}_0$. In fact, equality (5.28) for $\nu = (-1)^\ell$ (in which case it is a consequence of (5.27)) implies that $\tilde{S}_0\Psi$ has no pole at $z = 0$. At the same time, $\tilde{S}_0$ commutes with the linear operator (5.23). Therefore, the number of solutions of (5.3) that have the form (5.30) and satisfy (5.27) is equal to the number of eigenvalues ε_i, $i = 1, \dots, d$, of the operator $\tilde{S}_0$ on these finite-dimensional spaces, that is, it is equal to the sum of their dimensions.

It is easy to see that for $\omega_0 = 0$ the dimension of the space defined in Lemma 5.1 is equal to $(\ell - 1)/2$ if ℓ is odd, and $\ell/2 + 1$ if ℓ is even. For the other three points of second order the corresponding dimension is equal to $(\ell + 1)/2$ if ℓ is odd, and $\ell/2$ if ℓ is even. Therefore, the number of fixed points is equal to $4\ell + 2$, which proves the relation $g = 2\ell$.

Lemma 5.2. *The direct sum of the functional subspaces consisting of functions that have the form* (5.30) *and satisfy relation* (5.27) *with $\nu = (-1)^\ell$ is invariant with respect to the operators $\tilde{S}_a$ defined by*

$$(\widetilde{S}_a\Psi)(x) = H_a\Biggl[\frac{\theta_{a+1}(x - \ell\eta)}{\theta_1(x - \ell\eta)}\Psi(x + \eta) - \theta_1^2\left(\frac{\eta}{2}\right)\frac{\theta_1(x - (\ell + 1)\eta)\theta_{a+1}(-x - \ell\eta)}{\theta_1(x)\theta_1(x - \eta)}\Psi(x - \eta)\Biggr], \tag{5.32}$$

$$H_a = (i)^{\delta_{a,2}}\frac{\theta_{a+1}(\eta/2)}{\theta_1(\eta/2)}; \tag{5.33}$$

moreover, these operators are gauge equivalent to the operators (1.9).

[2]This condition is similar to the 'sum rule' in [35].

We omit the detailed proof of the theorem, because it is in perfect analogy with the proof of Lemma 5.1. Equality (5.28) implies that the operators $\tilde{S}_a \Psi$ have no pole at $x = 0$, as before. At the same time these operators commute with the transformation (5.23) and keep invariant the set of Bloch multipliers corresponding to the spaces of functions of the form (5.30). Note that for $a \neq 0$ the coefficients of $\tilde{S}_a$ are not elliptic; hence they do not preserve each space but only their direct sum.

Remark. The above-described invariant spaces of $\tilde{S}_a$ coincide (after the gauge transformation (5.2)) with the finite-dimensional representation spaces of the Sklyanin algebra found in [8]. More information on invariant functional subspaces for Sklyanin operators is contained in § 6.

The variables k and ε are defined by equations (5.18) as multivalued functions of z. We now consider analytic properties of the double-Bloch solutions $\Psi(x, Q)$, $Q = (z,\ k,\ \varepsilon) \in \widehat{\Gamma}$, on the hyperelliptic curve $\widehat{\Gamma}$.

Theorem 5.1. *Let $\widehat{\Gamma}$ be an algebraic curve of genus 2ℓ defined by the equation*

$$y^2 = \prod_{i=1}^{2\ell+1} (\varepsilon^2 - \varepsilon_i^2), \tag{5.34}$$

where ε_i are eigenvalues of $\tilde{S}_0$ on the finite-dimensional invariant space of functions of the form (5.30), (5.31). *Then the Bloch solution $\psi(x, P) = \Psi(x, P)\Psi^{-1}(0, P)$ of equation* (5.3) *is a meromorphic function on $\widehat{\Gamma}$ outside the cut between the points $P^{\pm}$ at infinity* (*that are preimages of $\varepsilon = \infty$ on $\widehat{\Gamma}$*). *Outside the cut, the function $\psi(x, P)$ has 2ℓ poles, which are independent of x and are invariant with respect to the involution of $\widehat{\Gamma}$ covering the involution $\varepsilon \to -\varepsilon$ (see* (5.21)*). The boundary values of $\psi^{(\pm)}(x, P)$ on opposite sides of the cut are connected by the relation*

$$\psi^{(+)} = \psi^{(-)} e^{2\pi i/\eta}. \tag{5.35}$$

In a neighbourhood of the points $P^{\pm}$ the function $\psi(x, P)$ has the form

$$\psi = \varepsilon^{\pm x/\eta} \left(\sum_{s=0}^{\infty} \xi_s^{\pm}(x) \varepsilon^{-s} \right), \qquad \xi_0^{+} \equiv 1. \tag{5.36}$$

Proof. The coefficients s_j in (5.5) are solutions of the linear system (5.12). We normalize them by the condition $s_1 = 1$. Then s_j are meromorphic functions on $\widehat{\Gamma}$, whence Ψ (as a function of $Q \in \widehat{\Gamma}$) is well defined on $\widehat{\Gamma}$ with cuts between the zeros and poles of the function

$$K = \left(\frac{\theta_1(z - \eta)}{\theta_1(z + \eta)} \right)^{1/2} k(z). \tag{5.37}$$

To within a factor of order $O(1)$, at the edges of these cuts the function Ψ has a singularity of the form

$$\Psi \sim K^{x/\eta}. \tag{5.38}$$

From (5.3) it follows directly that a function Ψ with such a type of singularity can be a solution of (5.3) only if $\varepsilon = \infty$ at the singular points. On the other hand, in neighbourhoods of these points we have $K \sim \varepsilon^{\pm 1}$, which proves (5.36). It is easy to see that the number of poles of ψ is equal to 2ℓ. In fact, the poles of ψ do not depend on x, because the poles of s_j are independent of x. Further, let us consider the following meromorphic function of ε:

$$F(\varepsilon) = \big(\psi(\eta, P_1(\varepsilon)) - \psi(\eta, P_2(\varepsilon))\big)^2, \tag{5.39}$$

where $P_i(\varepsilon)$ are two preimages of ε on $\widehat{\Gamma}$. This function does not depend on the ordering of these points, that is, F is a meromorphic function of ε. It has double poles at projections of poles of ψ, has a second-order pole at infinity, and has simple zeros at the branch points. Since a meromorphic function has as many zeros as poles, it follows that the function ψ has g poles.

Note that Theorem 5.1 could be proved by a direct study of analytic properties of Ψ on the curve $\widehat{\Gamma}$ represented in the form (5.18). Inasmuch as this alternative proof is very similar to the proof of Theorem 3.2, we skip it but present some arguments explaining why the multivalued function K has only one pole and one zero.

For $\ell > 1$ the function $\Phi_\ell(-j\eta, z)$ has a pole and a zero of order j at the points $z = \pm\eta$, respectively. Therefore, k has zeros and poles at all preimages of $z = -\eta$ (respectively, $z = \eta$) on $\widehat{\Gamma}$. Hence K is regular at all these points. The function Φ_ℓ has a simple pole at the point $z = -N\eta$. It turns out that at one of the preimages of this point the function ε (as well as k) has a pole. The corresponding point is one of the points at infinity on the curve $\widehat{\Gamma}$ represented in the form (5.18). The other point at infinity is one of the preimages of the point $z = N\eta$.

Remark. Formulae (1.9) give one particular series of representations of the Sklyanin algebra found in [8] (called series (a) there). In this series the operator S_0 has the finite-gap property. It should be noted that in each of the other series there is again an operator having the finite-gap property. In general, the other three operators in each series do not have this property. Trigonometric degenerations of the difference operators (1.9) were studied in [32]. In this case S_0 corresponds to soliton solutions of the Toda chain.

§ 6. Representations of the Sklyanin algebra

In this section we construct representations of the Sklyanin algebra from 'vacuum vectors' of the L-operator (1.4). The notions of vacuum vector and the vacuum curve of an L-operator were introduced by one of the authors [33] in studying the Yang–Baxter equation by the methods of algebraic geometry. We recall the main definitions.

To begin with, we consider an *arbitrary* L-operator $\mathcal{L}$ with two-dimensional auxiliary space $\mathbb{C}^2$, that is, an arbitrary $(2n \times 2n)$-matrix represented as a (2×2)-matrix, whose matrix elements are $(n \times n)$-matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$:

$$\mathcal{L} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{6.1}$$

The operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ act in a linear space $\mathcal{H} \cong \mathbb{C}^n$, which is called the *quantum space* of the L-operator. We emphasize that so far no conditions on $\mathcal{L}$ are imposed. In particular, we do not impose relation (5) and do not imply any specific parametrization of the matrix elements.

We consider a vector $X \otimes U \in \mathcal{H} \otimes \mathbb{C}^2$ ($X \in \mathcal{H}$, $U \in \mathbb{C}^2$) such that

$$\mathcal{L}(X \otimes U) = Y \otimes V, \tag{6.2}$$

where $Y \in \mathcal{H}$ and $V \in \mathbb{C}^2$. Relation (6.2) means that the indecomposable tensor $X \otimes U$ is transformed by $\mathcal{L}$ into another indecomposable tensor. Written in components, this relation has the form

$$\mathcal{L}^{i\alpha}_{j\beta} X_j U_\beta = Y_i V_\alpha, \tag{6.3}$$

where the indices α, β (respectively, i, j) enumerate the basis vectors of $\mathbb{C}^2$ (respectively, $\mathcal{H}$), and summation over repeated indices is implied.

Assume that (6.2) holds. Then the vector X is called the *vacuum vector* of the L-operator $\mathcal{L}$. Multiplying (6.2) from the left by the covector $\tilde{V} = (V_2, -V_1)$ orthogonal to V, we obtain

$$(\tilde{V}\mathcal{L}U)X = 0. \tag{6.4}$$

Here $\tilde{V}\mathcal{L}U$ is an operator in $\mathcal{H}$ with matrix elements $\tilde{V}_\alpha \mathcal{L}^{i\alpha}_{j\beta} U_\beta$. Conversely, assume that (6.4) holds. Then relation (6.2) holds, where the vector Y is uniquely determined by the vectors U, V, X.

In the particular case $\mathcal{H} \cong \mathbb{C}^2$ relation (6.2) was the starting point for Baxter [14] in constructing a solution of the eight-vertex model. In the papers on integrable lattice models of statistical physics this relation is called 'pair propagation through a vertex'. In the context of the quantum inverse scattering method [2] the equivalent condition (6.4) is more customary. It defines the local vacuum of the gauge-transformed L-operator (this explains the terminology introduced above). In general form the relation (6.2) first appeared in [33].

From (6.4) it follows that the necessary and sufficient condition for the existence of vacuum vectors is written as

$$\det(\tilde{V}\mathcal{L}U) = 0. \tag{6.5}$$

Putting $U_2 = V_2 = 1$ for simplicity and using (6.1), one may write (6.5) in the more explicit form

$$\det(U_1\mathcal{A} + \mathcal{B} - U_1V_1\mathcal{C} - V_1\mathcal{D}) = 0, \qquad U_2 = V_2 = 1. \tag{6.6}$$

This equation defines an algebraic curve in $\mathbb{C}^2$ called the *vacuum curve* of the L-operator $\mathcal{L}$. Thus, the family of vacuum vectors is parametrized by points of the vacuum curve, that is, by pairs (U_1, V_1) satisfying (6.6). In general the space of vacuum vectors corresponding to each point of the curve is one-dimensional.

Now assume that $\mathcal{H} \cong \mathbb{C}^2$ and the operator $\mathcal{L}$ satisfies (1.5) with some matrix R. In this case the vacuum curve has genus 1, that is, it is an elliptic curve $\mathcal{E}_0$.

This curve is parametrized by points z of a one-dimensional complex torus with periods 1 and τ.

Fixing a suitable normalization (for example, putting second components of all the vectors equal to 1), one may consider the components of the vectors $U(z)$, $V(z)$, $X(z)$, $Y(z)$ as meromorphic functions on $\mathcal{E}_0$ having at most two simple poles. With this normalization the right-hand side of (6.2) must be multiplied by a scalar meromorphic function $h(z)$. By the Yang–Baxter equation we have [33]

$$Y(z) = X\Big(z + \frac{\eta}{2}\Big), \qquad V(z) = U\Big(z - \frac{\eta'}{2}\Big), \tag{6.7}$$

where η and η' are some constants. Therefore, the basic relation (6.2) can be written in the form [33]

$$\mathcal{L}\big(X(z) \otimes U(z)\big) = h(z)Y(z) \otimes V(z) = h(z)X\Big(z + \frac{\eta}{2}\Big) \otimes U\Big(z - \frac{\eta'}{2}\Big). \tag{6.8}$$

Let D_X (respectively, D_U) be the divisor of poles of the meromorphic vector $X(z)$ (respectively, $U(z)$), and let $\mathcal{M}(D)$ be the space of functions associated with an effective divisor D, that is, functions having poles at points of D of order no higher than the multiplicity of the corresponding point in D. For divisors of degree 2 on elliptic curves, this space is two-dimensional in general position, so that $\dim \mathcal{M}(D_X) = \dim \mathcal{M}(D_U) = 2$, and components of the vectors X, U form bases in these spaces. Further, the functions $X_i(z)U_\alpha(z)$, $i, \alpha = 1, 2$, form a basis in the space $\mathcal{M}(D_X + D_U)$. Equality (6.8) implies that the functions $h(z)X_i(z + \frac{\eta}{2})U_\alpha(z - \frac{\eta'}{2})$ form another basis in this space, and the matrix $\mathcal{L}$ connects the two bases. The divisors of poles of the left- and right-hand sides of (6.8) must be equivalent, that is, must be equal modulo periods of the lattice. Since under the shift by $\eta'/2$ the divisor of poles of a function having two poles is shifted by η', it follows that

$$\eta' - \eta = M + N\tau, \qquad M, N \in \mathbb{Z}. \tag{6.9}$$

The vectors $X(z)$, $U(z)$ are doubly-periodic; therefore, there are four different cases:

$$\eta' = \eta + 2\omega_a, \tag{6.10}$$

where $\omega_0 = 0$, $\omega_1 = 1/2$, $\omega_2 = (\tau + 1)/2$, $\omega_3 = \tau/2$ (see (5.6)).

The Baxter parametrization of the L-operator is based on the relation (6.8). In fact, the equivalence class of the pole divisor of $X(z)$ may differ from that of $U(z)$ only by a shift on $\mathcal{E}_0$. The value of this shift is identified with the spectral parameter of the L-operator. By means of a 'gauge' transformation one may represent $\mathcal{L}$ in the form (1.4). With this parametrization one can write (6.8) in terms of θ-functions.

To do this, it is convenient to use another normalization, specifically, the one in which the vectors are entire functions of z (in this case they are cross-sections of certain line bundles on $\mathcal{E}_0$).

We introduce the vector

$$\Theta(z) = \begin{pmatrix} \theta_4\big(z \mid \frac{\tau}{2}\big) \\ \theta_3\big(z \mid \frac{\tau}{2}\big) \end{pmatrix}. \tag{6.11}$$

Its components form a basis in the space of second-order θ-functions $\theta(z)$ with monodromy properties $\theta(z+1)=\theta(z)$, $\theta(z+\tau)=\exp(-2\pi i\tau-4\pi iz)\theta(z)$. They have two zeros in the fundamental domain of the lattice with periods 1, τ. Further, we put

$$X(z)=\Theta(z), \tag{6.12}$$

$$U(z)=U^{\pm}(z)=\Theta\Bigl(z\pm\frac{1}{2}\Bigl(u+\frac{\eta}{2}\Bigr)\Bigr) \tag{6.13}$$

(for each choice of the sign). Then we may write (6.2) in the form [14], [2]:

$$\begin{aligned}L^{(a)}(u)\Theta(z)\otimes\Theta\Bigl(z\pm\frac{1}{2}\Bigl(u+\frac{\eta}{2}\Bigr)\Bigr)& \\ =2g_a^{\pm}\,\frac{\theta_1(u+\frac{\eta}{2}\mid\tau)}{\theta_1(\eta\mid\tau)}\Theta\Bigl(z+\frac{\eta}{2}\Bigr)&\otimes\Theta\Bigl(z\pm\frac{1}{2}\Bigl(u-\frac{\eta}{2}\Bigr)\pm\omega_a\Bigr),\end{aligned} \tag{6.14}$$

where

$$g_0^{\pm}=g_1^{\pm}=1,\qquad -ig_2^{\pm}=g_3^{\pm}=-\exp\Bigl(\pm2\pi iz+\pi i\Bigl(u+\frac{\tau-\eta}{2}\Bigr)\Bigr), \tag{6.15}$$

$$L^{(a)}(u)=\sum_{b=0}^{3}\frac{\theta_{b+1}(u\mid\tau)}{\theta_{b+1}(\frac{\eta}{2}\mid\tau)}\sigma_b\otimes(\sigma_a\sigma_b) \tag{6.16}$$

(see (1.4) and (1.7)). Note that the operator $L^{(a)}(u)=\sigma_aL(u)$, where $L(u)$ is given by (1.4) with $S_a=\sigma_a$ and the matrix product is performed in the auxiliary space, satisfies the relation (1.5) '$RLL=LLR$' with the *same* R-matrix (1.7) for each $a=0,\dots,3$. The scalar factor in the right-hand side of (6.14) is determined from the condition that $L(\eta/2)$ is proportional to the permutation operator in $\mathbb{C}^2\otimes\mathbb{C}^2$. One may verify (6.14) directly, by using identities for θ-functions (see the Appendix to this section).

Remark. Given an elliptic curve, one can consider the vector $X(z)$ to be an even function on this curve, $X(-z)=X(z)$. This is why we introduce the parametrization (6.11), for which the function $X(z)$ is even from the very beginning. Then the equality corresponding to the minus sign in (6.14) follows from the similar equality with the plus sign (it is enough to make the change $z\to-z$). However, in Baxter's approach it is useful to deal with both equalities.

We now turn to the case of arbitrary spin. Consider the L-operator of the form (1.4)

$$L(u)=\begin{pmatrix}W_0(u)S_0+W_3(u)S_3 & W_1(u)S_1-iW_2(u)S_2\\ W_1(u)S_1+iW_2(u)S_2 & W_0(u)S_0-W_3(u)S_3\end{pmatrix}, \tag{6.17}$$

where

$$W_a(u)=\frac{\theta_{a+1}(u\mid\tau)}{\theta_{a+1}(\frac{\eta}{2}\mid\tau)}, \tag{6.18}$$

and S_a are generators of some algebra (at this stage the commutation relations (1.1), (1.2) are not imposed). In what follows we generalize formulae (6.8) to obtain an explicit functional realization of these generators.

Preparatory to formulating the main result of this section, we need some more preliminaries.

We are interested in the general form of representations in terms of difference operators. For this reason in what follows we take for the quantum space of the L-operator (6.17) the space of *meromorphic functions of one complex variable z*. The generators S_a act on elements of this space (that is, on functions $X(z)$) as follows:

$$S_a\colon X(z) \mapsto (S_a X)(z). \tag{6.19}$$

We consider the following generalization of (6.8) and (6.14):

$$U^{\pm}(z)^T \big(\sigma_a L(u)\big) X(z) = g_a(u) U^{\pm}(z \mp \ell\eta \mp \omega_a)^T X\Big(z \pm \frac{\eta}{2}\Big), \qquad a = 0, \dots, 3. \tag{6.20}$$

Here $z \in \mathcal{E}_0$, $U^T = (U_1, U_2)$, ℓ is a parameter, $g_a(u)$ are scalar functions independent of z, and

$$U^{\pm}(z) = \Theta\left(z \pm \frac{u + \ell\eta}{2}\right) \tag{6.21}$$

(note that $U^{\pm}$ coincides with (6.13) for $\ell = 1/2$). As before, $L(u)$ acts on $U^{\pm}$ as a (2×2)-matrix, while the action of each matrix element of $L(u)$ on $X(z)$ is defined by (6.19). We have written (6.20) in terms of the *covector* U^T, since the action of the operator (6.19) on a function *from the left* is equivalent to the action of the corresponding matrix (representing this operator in a fixed basis) *from the right* on the covector formed by components of the function with respect to the basis. In what follows we show that for $\ell = 1/2$ relation (6.20) coincides with the *conjugated* equality (6.14).

At present we cannot suggest any explicit description of the vacuum curve of the L-operator (6.17). Moreover, we do not know any direct argument establishing the equivalence between (6.20) and the 'intertwining' relation (1.5) for $L(u)$ (taken together with the Yang–Baxter equation for $R(u)$). However, it follows from Theorem 6.1 proved below that relations (6.20) imply Sklyanin's commutation relations on S_a, and therefore $L(u)$ should satisfy (1.5). We stress once again that our arguments are in a sense inverse to Sklyanin's original approach (see also the papers [34], [35], where some formulae for vacuum vectors of the higher spin XYZ model are obtained). Namely, the starting point is relation (6.20), and no conditions on S_a are implied. It turns out that within this approach the Sklyanin algebra for S_a (together with its functional realization) is reconstructed.

The main result of this section is the following assertion.

Theorem 6.1. *Let the operator $L(u)$ be given by* (6.17), *where S_a are some operators in the space of meromorphic functions $X(z)$ of one complex variable z. Suppose that relations* (6.20) *hold for $a = 0$, that is,*

$$U^{\pm}(z)^T L(u) X(z) = g_0(u) U^{\pm}(z \mp \ell\eta)^T X\Big(z \pm \frac{\eta}{2}\Big), \tag{6.22}$$

where $U^{\pm}(z)$ is defined in (6.21), *ℓ is a parameter, and $g_0(u)$ is a scalar function independent of z. Then S_a are difference operators of the following form*:

$$(S_aX)(z) = \lambda \frac{(i)^{\delta_{a,2}}\theta_{a+1}(\frac{\eta}{2} \mid \tau)}{\theta_1(2z \mid \tau)} \times \left(\theta_{a+1}(2z-\ell\eta \mid \tau)X\left(z+\frac{\eta}{2}\right) - \theta_{a+1}(-2z-\ell\eta \mid \tau)X\left(z-\frac{\eta}{2}\right)\right), \tag{6.23}$$

where λ is an arbitrary constant. Conversely, if the S_a are defined by (6.23), *then* (6.22) *holds, where*

$$g_0(u) = 2\lambda\theta_1(u+\ell\eta \mid \tau). \tag{6.24}$$

Remark. (a) Using transformation properties of the vector $\Theta(z)$ under shifts by half-periods ω_a, it is easy to see that for $a \neq 0$ relations (6.20) follow from (6.22); (b) if $X(z)$ is an even function, then the two relations (6.22) are equivalent.

Proof. Relations (6.22) give a system of four linear equations for four unknown functions $(S_aX)(z)$ occurring in the left-hand side of (6.22). More explicitly, we have

$$U_1^{\pm}(z)\big(L_{11}(u)X\big)(z) + U_2^{\pm}(z)\big(L_{21}(u)X\big)(z) = g_0(u)U_1^{\pm}(z \mp \ell\eta)X\left(z \pm \frac{\eta}{2}\right), \tag{6.25}$$

$$U_1^{\pm}(z)\big(L_{12}(u)X\big)(z) + U_2^{\pm}(z)\big(L_{22}(u)X\big)(z) = g_0(u)U_2^{\pm}(z \mp \ell\eta)X\left(z \pm \frac{\eta}{2}\right), \tag{6.26}$$

where $(L_{\alpha\beta}(u)X)(z)$ are expressed in terms of $(S_aX)(z)$ by (6.17). We fix $g_0(u)$ as given by (6.24). Solving this system, we arrive at formulae (6.23) (all necessary identities for θ-functions are presented in the Appendix to this section). This proves that the representation (6.23) is equivalent to (6.22), (6.24).

Corollary 6.1. *Suppose that $L(u)$ is an operator of the form* (6.17) *that satisfies* (6.22). *Then $L(u)$ satisfies the 'intertwining' relation* (1.5), *where the R-matrix is given by* (1.7).

The proof follows from the identification of (6.23) with formulae (1.9) for the representations of the Sklyanin algebra by putting $2z \equiv x$, $X(x/2) \equiv f(x)$. The constant λ is not essential, since the commutation relations (1.1), (1.2) are homogeneous.

Remark. From the technical point of view, solving system (6.22) is much simpler than the direct substitution of (6.23) in the commutation relations. The amount of computations in the former case is comparable with that in the latter one if we identify only the coefficients in front of $f(x \pm 2\eta)$.

We now consider equality (6.20) for $a = 1, 2, 3$.

Lemma 6.1. *Define the transformation $S_b \longmapsto \mathcal{Y}_a(S_b)$, $a, b = 0, \ldots, 3$, of the generators by the following relation*:

$$\sigma_a L(u+\omega_a) = h_a(u) \sum_{b=0}^{3} \frac{\theta_{b+1}(u \mid \tau)}{\theta_{b+1}(\frac{\eta}{2} \mid \tau)} \mathcal{Y}_a(S_b) \otimes \sigma_b, \tag{6.27}$$

where $h_0(u) = h_1(u) = 1$, $h_2(u) = -ih_3(u) = \exp(-\frac{i\pi\tau}{4} - i\pi u)$. *Then the operators* $\mathcal{Y}_a$, $a = 0, \ldots, 3$, *are automorphisms of the algebra generated by* S_b.

The proof follows from the fact that the operators $\sigma_a L(u + \omega_a)$, $a = 0, \ldots, 3$, satisfy the 'intertwining' relation (1.5) with the R-matrix given by (1.7), since the matrices σ_a are c-number solutions of (1.5). Comparing the left- and right-hand sides in (6.27), we see that the explicit form of $\mathcal{Y}_a$ is given by (here $\theta_a(\frac{\eta}{2}) \equiv \theta_a(\frac{\eta}{2}|\tau)$)

$$\mathcal{Y}_1\colon (S_0, S_1, S_2, S_3) \mapsto \left(-\frac{\theta_1(\frac{\eta}{2})}{\theta_2(\frac{\eta}{2})}S_1,\ \frac{\theta_2(\frac{\eta}{2})}{\theta_1(\frac{\eta}{2})}S_0,\ -\frac{i\theta_3(\frac{\eta}{2})}{\theta_4(\frac{\eta}{2})}S_3,\ \frac{i\theta_4(\frac{\eta}{2})}{\theta_3(\frac{\eta}{2})}S_2\right), \tag{6.28}$$

$$\mathcal{Y}_2\colon (S_0, S_1, S_2, S_3) \mapsto \left(\frac{\theta_1(\frac{\eta}{2})}{\theta_3(\frac{\eta}{2})}S_2,\ \frac{i\theta_2(\frac{\eta}{2})}{\theta_4(\frac{\eta}{2})}S_3,\ \frac{\theta_3(\frac{\eta}{2})}{\theta_1(\frac{\eta}{2})}S_0,\ -\frac{\theta_4(\frac{\eta}{2})}{\theta_2(\frac{\eta}{2})}S_1\right), \tag{6.29}$$

$$\mathcal{Y}_3\colon (S_0, S_1, S_2, S_3) \mapsto \left(\frac{\theta_1(\frac{\eta}{2})}{\theta_4(\frac{\eta}{2})}S_3,\ -\frac{\theta_2(\frac{\eta}{2})}{\theta_3(\frac{\eta}{2})}S_2,\ \frac{\theta_3(\frac{\eta}{2})}{\theta_2(\frac{\eta}{2})}S_1,\ \frac{\theta_4(\frac{\eta}{2})}{\theta_1(\frac{\eta}{2})}S_0\right), \tag{6.30}$$

and $\mathcal{Y}_0$ is the identity transformation. These automorphisms were considered by Sklyanin in [8].

Shifting $u \to u + \omega_a$ in (6.20) and applying these automorphisms to (6.23), we obtain for $b = 0, \ldots, 3$:

$$\begin{aligned}(S_a X)(z) = {} & \frac{(i)^{\delta_{a,2}}\theta_{a+1}(\frac{\eta}{2})}{\theta_1(2z)} \\ & \times \left(\theta_{a+1}(2z - \ell\eta - \omega_b)X\Big(z + \frac{\eta}{2}\Big) - \theta_{a+1}(-2z - \ell\eta - \omega_b)X\Big(z - \frac{\eta}{2}\Big)\right).\end{aligned} \tag{6.31}$$

Note that if ℓ is an arbitrary complex parameter, then formulae (6.31) are reduced to (6.23) by the formal change $\ell \to \ell + \omega_b/\eta$. However, as is shown below, if the operators (6.31) are restricted to invariant subspaces, then they yield non-equivalent finite-dimensional representations of the Sklyanin algebra.

Now assume that $\ell \in \frac{1}{2}\mathbb{Z}_+$. In this case one may identify $X(z)$ with a cross-section of some linear bundle on the initial elliptic curve $\mathcal{E}_0$. Repeating the arguments presented after formula (6.8), we conclude that the degree of this bundle is equal to 4ℓ. By the Riemann–Roch theorem for elliptic curves, we see that in the case of general position the dimension of the space of holomorphic cross-sections of this bundle is equal to 4ℓ. It is convenient to identify these cross-sections with θ-functions of order 4ℓ. We consider the space $\mathcal{T}^+_{4\ell}$ of *even* θ-functions of order 4ℓ, that is, the space of entire functions $F(z)$, $z \in \mathbb{C}$, such that $F(-z) = F(z)$ and

$$\begin{aligned}F(z+1) &= F(z), \\ F(z+\tau) &= \exp(-4\ell\pi i\tau - 8\ell\pi iz)F(z).\end{aligned} \tag{6.32}$$

By analogy with [8] it is easy to verify that the space $\mathcal{T}^+_{4\ell}$ is invariant under the action of the operators (6.31) for $b = 0$ and $b = 1$, while for $b = 2$ and $b = 3$ the invariant space is $\exp(-\pi iz^2/\eta)\mathcal{T}^+_{4\ell}$. It is known that $\dim \mathcal{T}^+_{4\ell} = 2\ell + 1$ provided that $\ell \in \frac{1}{2}\mathbb{Z}_+$. Restricting the difference operators (6.31) to these invariant subspaces, we find four series of finite-dimensional representations.

Generally speaking, these representations are mutually non-equivalent. This follows from the analysis of values of the central elements

$$\boldsymbol{K}_0 = \sum_{a=0}^{3} S_a^2, \qquad \boldsymbol{K}_2 = \sum_{i=1}^{3} J_i S_i^2 \tag{6.33}$$

of the Sklyanin algebra, where the constants J_i are defined in (1.3) and (1.10). Their values for representations (6.31) at $b = 0, \ldots, 3$ are given by

$$\boldsymbol{K}_0 = 4\theta_1^2\Big(\Big(\ell + \frac{1}{2}\Big)\eta + \omega_b \mid \tau\Big), \tag{6.34}$$

$$\boldsymbol{K}_2 = 4\theta_1\big((\ell+1)\eta + \omega_b \mid \tau\big)\theta_1(\ell\eta + \omega_b \mid \tau). \tag{6.35}$$

The arguments given above lead to the following assertion.

Theorem 6.2. *The Sklyanin algebra* (1.1), (1.2) *has four different series of finite-dimensional representations indexed by the parameter* $b = 0,\ 1,\ 2,\ 3$. *Representations of each series are indexed by the discrete ('spin') parameter* $\ell \in \frac{1}{2}\mathbb{Z}_+$. *They are obtained by restriction of the operators* (6.31) *to the invariant* $(2\ell+1)$*-dimensional functional subspaces* $\mathcal{T}_{4\ell}^+$ *for* $b = 0,\ 1$ *and* $\exp(-\pi i z^2/\eta)\mathcal{T}_{4\ell}^+$ *for* $b = 2,\ 3$. *For general values of the parameters these representations are mutually non-equivalent.*

Let us compare these representations with those obtained by Sklyanin in [8]. For $b = 0$ and $b = 3$ we reproduce series (a) and (c), respectively. These representations are self-adjoint with respect to the real form of the algebra studied in [8]. The other two series (corresponding to $b = 1$ and $b = 2$) in general are not self-adjoint. We first consider the case $b = 1$. For rational values of η, $\eta = p/q$, and special values of ℓ, $\ell = (q-1)/2 \bmod q$, these representations are self-adjoint and are equivalent to some subset of representations of series (b).[3] To the best of our knowledge, the series corresponding to the case $b = 2$ was never mentioned in the literature (though, in a sense, it is implicitly contained in Sklyanin's paper). Another outcome of our approach is the natural correspondence between different series of representations and points of order 2 on the elliptic curve.

It is natural to surmise that representations of the last two series become self-adjoint with respect to other real forms of the algebra. A real form is defined by an anti-involution ($*$-operation) on the algebra. It should be noted that classification of non-equivalent real forms of the Sklyanin algebra and its generalizations is an interesting open problem.

Concluding this section, we would like to note that, after a suitable discretization, the variable z in (6.20) and (6.13) may be identified with the statistical variable ('height') in IRF-type ('Interaction Round a Face') models [36]. This follows from the well-known correspondence between the vertex- and IRF-type models, if we take into account that the transformation connecting these models is constructed

[3] The whole family of representations of series (b) found in [8] has three continuous parameters. These representations are self-adjoint and exist only if $\eta = p/q$. In this case all of them have dimension q. They are obtained by restriction of the operator (6.31) to a finite discrete uniform lattice.

in terms of the vacuum vectors (for the explicit form of this transformation in the case of higher spin models see [35]).

Finally, it seems instructive to carry out a detailed analysis of the trigonometric and rational limits of the constructions presented in this section. Some particular related problems have already been discussed in the literature. In the recent paper [37] the vacuum vectors for the higher spin XXZ-type quantum spin chains are constructed. The vacuum curves of trigonometric L-operators have been described in [38]. In the simplest case they are collections of rational curves intersecting at two points. Trigonometric degenerations of the Sklyanin algebra, which are in a sense 'intermediate' between the initial algebra and the standard quantum deformation of the algebra gl_2, are studied in [32].

Appendix to § 6

We use the following definition of the θ-functions:

$$\theta_1(z \mid \tau) = \sum_{k\in\mathbb{Z}} \exp\Big(\pi i\tau\Big(k+\frac{1}{2}\Big)^2 + 2\pi i\Big(z+\frac{1}{2}\Big)\Big(k+\frac{1}{2}\Big)\Big), \tag{6.36}$$

$$\theta_2(z \mid \tau) = \sum_{k\in\mathbb{Z}} \exp\Big(\pi i\tau\Big(k+\frac{1}{2}\Big)^2 + 2\pi i z\Big(k+\frac{1}{2}\Big)\Big), \tag{6.37}$$

$$\theta_3(z \mid \tau) = \sum_{k\in\mathbb{Z}} \exp(\pi i\tau k^2 + 2\pi i z k), \tag{6.38}$$

$$\theta_4(z \mid \tau) = \sum_{k\in\mathbb{Z}} \exp\Big(\pi i\tau k^2 + 2\pi i\Big(z+\frac{1}{2}\Big)\Big). \tag{6.39}$$

For the reader's convenience we recall here the definition of the σ-function used in §§ 2–4:

$$\sigma(z \mid \omega, \omega') = \frac{2\omega}{\theta_1'(0)} \exp\Big(\frac{\zeta(\omega)z^2}{2\omega}\Big)\theta_1\Big(\frac{z}{2\omega}\,\Big|\,\frac{\omega'}{\omega}\Big). \tag{6.40}$$

We now cite the identities used in the computations.

The first group of identities (the addition theorems) is given by:

$$\theta_4(x \mid \tau)\theta_3(y \mid \tau) = \theta_4(x+y \mid 2\tau)\theta_4(x-y \mid 2\tau) + \theta_1(x+y \mid 2\tau)\theta_1(x-y \mid 2\tau), \tag{6.41}$$

$$\theta_4(x \mid \tau)\theta_4(y \mid \tau) = \theta_3(x+y \mid 2\tau)\theta_3(x-y \mid 2\tau) - \theta_2(x+y \mid 2\tau)\theta_2(x-y \mid 2\tau), \tag{6.42}$$

$$\theta_3(x \mid \tau)\theta_3(y \mid \tau) = \theta_3(x+y \mid 2\tau)\theta_3(x-y \mid 2\tau) + \theta_2(x+y \mid 2\tau)\theta_2(x-y \mid 2\tau), \tag{6.43}$$

$$\theta_2(x \mid \tau)\theta_2(y \mid \tau) = \theta_3(x+y \mid 2\tau)\theta_2(x-y \mid 2\tau) + \theta_2(x+y \mid 2\tau)\theta_3(x-y \mid 2\tau), \tag{6.44}$$

$$\theta_1(x \mid \tau)\theta_1(y \mid \tau) = \theta_3(x+y \mid 2\tau)\theta_2(x-y \mid 2\tau) - \theta_2(x+y \mid 2\tau)\theta_3(x-y \mid 2\tau). \tag{6.45}$$

It is convenient to write out explicitly their simple consequences

$$\theta_4(x \mid \tau)\theta_3(y \mid \tau) + \theta_4(y \mid \tau)\theta_3(x \mid \tau) = 2\theta_4(x+y \mid 2\tau)\theta_4(x-y \mid 2\tau), \quad (6.46)$$

$$\theta_4(x \mid \tau)\theta_3(y \mid \tau) - \theta_4(y \mid \tau)\theta_3(x \mid \tau) = 2\theta_1(x+y \mid 2\tau)\theta_1(x-y \mid 2\tau), \quad (6.47)$$

$$\theta_3(x \mid \tau)\theta_3(y \mid \tau) + \theta_4(y \mid \tau)\theta_4(x \mid \tau) = 2\theta_3(x+y \mid 2\tau)\theta_3(x-y \mid 2\tau), \quad (6.48)$$

$$\theta_3(x \mid \tau)\theta_3(y \mid \tau) - \theta_4(y \mid \tau)\theta_4(x \mid \tau) = 2\theta_2(x+y \mid 2\tau)\theta_2(x-y \mid 2\tau). \quad (6.49)$$

The second group of identities is given by

$$2\theta_1(x \mid 2\tau)\theta_4(y \mid 2\tau) = \theta_1\Big(\frac{x+y}{2} \mid \tau\Big)\theta_2\Big(\frac{x-y}{2} \mid \tau\Big) + \theta_2\Big(\frac{x+y}{2} \mid \tau\Big)\theta_1\Big(\frac{x-y}{2} \mid \tau\Big), \quad (6.50)$$

$$2\theta_3(x \mid 2\tau)\theta_2(y \mid 2\tau) = \theta_1\Big(\frac{x+y}{2} \mid \tau\Big)\theta_1\Big(\frac{x-y}{2} \mid \tau\Big) + \theta_2\Big(\frac{x+y}{2} \mid \tau\Big)\theta_2\Big(\frac{x-y}{2} \mid \tau\Big), \quad (6.51)$$

$$2\theta_3(x \mid 2\tau)\theta_3(y \mid 2\tau) = \theta_3\Big(\frac{x+y}{2} \mid \tau\Big)\theta_3\Big(\frac{x-y}{2} \mid \tau\Big) + \theta_4\Big(\frac{x+y}{2} \mid \tau\Big)\theta_4\Big(\frac{x-y}{2} \mid \tau\Big), \quad (6.52)$$

$$2\theta_2(x \mid 2\tau)\theta_2(y \mid 2\tau) = \theta_3\Big(\frac{x+y}{2} \mid \tau\Big)\theta_3\Big(\frac{x-y}{2} \mid \tau\Big) - \theta_4\Big(\frac{x+y}{2} \mid \tau\Big)\theta_4\Big(\frac{x-y}{2} \mid \tau\Big). \quad (6.53)$$

Particular cases of these identities are given by

$$2\theta_1(z \mid \tau)\theta_4(z \mid \tau) = \theta_2\Big(0 \mid \frac{\tau}{2}\Big)\theta_1\Big(z \mid \frac{\tau}{2}\Big), \quad (6.54)$$

$$2\theta_2(z \mid \tau)\theta_3(z \mid \tau) = \theta_2\Big(0 \mid \frac{\tau}{2}\Big)\theta_2\Big(z \mid \frac{\tau}{2}\Big). \quad (6.55)$$

Two more identities are given by

$$\theta_1\Big(z \mid \frac{\tau}{2}\Big)\theta_2\Big(z \mid \frac{\tau}{2}\Big) = \theta_4(0 \mid \tau)\theta_1(2z \mid \tau), \quad (6.56)$$

$$\theta_4\Big(z \mid \frac{\tau}{2}\Big)\theta_3\Big(z \mid \frac{\tau}{2}\Big) = \theta_4(0 \mid \tau)\theta_4(2z \mid \tau). \quad (6.57)$$

§ 7. Concluding remarks

This work elaborates upon the following three subjects.

(I) The dynamics of poles for elliptic solutions to the $2D$ non-Abelian Toda chain.

(II) Difference analogues of Lamé operators.

(III) Representations of the Sklyanin algebra in terms of difference operators.

We now outline the results.

- The poles move according to equations of motion for spin generalizations of the Ruijsenaars-Schneider model; the action-angle variables for the latter are constructed in terms of some algebraic-geometrical data.
- One of the generators of the Sklyanin algebra, represented as a difference operator with elliptic coefficients, has the 'finite-gap' property that is a motivation for the analogy with Lamé operators.
- Starting from the notion of vacuum vectors of an L-operator, a general simple scheme for constructing functional realizations of the Sklyanin algebra is suggested.

We would like to explain why these three themes are more intimately connected than seems possible at a first glance.

With each problem (I)–(III) a distinguished class of algebraic curves has been associated. In case (I) these are spectral curves Γ for the L-operators of the Ruijsenaars–Schneider-type models; in case (II) we deal with the spectral curve Γ' for the Lamé difference operator S_0 (a generator of the Sklyanin algebra); in case (III) the representations are defined on cross-sections of certain line bundles on a vacuum curve $\mathcal{E}$ of the elliptic higher spin L-operator (1.4). It has been shown that the curves Γ and Γ' are ramified coverings of the initial elliptic curve. The characteristic property (6.20) for the vacuum vectors suggests that the same should be true for $\mathcal{E}$, that is, $\mathcal{E}$ is a ramified covering of the initial elliptic curve $\mathcal{E}_0$ (that is, of the vacuum curve of the spin-1/2 L-operator).

The connection between points (I) and (II) is similar to the relation between elliptic solutions of KP and KdV equations. Specifically, the elliptic solutions of the Abelian $2D$ Toda chain, which are *stationary with respect to the time flow* $t_+ + t_-$, correspond to *isospectral deformations* of the Lamé difference operator S_0, which is the Lax operator for the $1D$ Toda chain. In other words, the hyperelliptic curves Γ' form a specific subclass of the curves Γ. A similar reduction in the non-Abelian case yields spin generalizations of Lamé difference operators. Their properties and a possible relation to the Sklyanin-type quadratic algebras remain to be figured out.

Apart from the apparent result that the construction proposed in § 6 provides a natural source of Lamé-like difference operators, we expect deeper connection between points (II) and (III). Specifically, the spectral curves Γ' are expected to be very close to the vacuum curves $\mathcal{E}$. Conjecturally, they may even coincide, at least in some particular cases. At the moment we cannot present any more arguments and leave this as a further problem.

Finally, we would like to note an intriguing similarity between the basic ansatz (2.26) for a double-Bloch solution of the generating linear problem and the functional Bethe ansatz [39]. Indeed, in the latter case the wave function (with separated variables) is sought in the form of an 'elliptic polynomial' $\prod \sigma(z - z_j)$, where the roots z_j are subject to Bethe equations. Similarly, in the former case we deal with *a ratio* of two 'elliptic polynomials' (see (5.30)). The only difference is that we parametrize this function by residues at the poles rather than zeros of the numerator. This may indicate a non-trivial interplay between Calogero–Moser-type models (and more general Hitchin systems) and quantum integrable models solved by means of the Bethe ansatz.[4]

Acknowledgements

We are grateful to A. Gorsky and T. Takebe for discussions. I.K. is very grateful to Technion University at Haifa, Rome-1 University, Scoula Normale at Pisa, and Berlin Technische University for their hospitality during periods when this work was done.

[4]The recently observed formal resemblance [40] between Bethe equations and motion equations for discrete time Calogero–Moser-like systems may be a particular aspect of this relation.

Bibliography

[1] E. K. Sklyanin, "On some algebraic structures related to the Yang–Baxter equation", *Funktsional. Anal. i Prilozhen.* **16**:4 (1982), 27–34; English transl. in *Functional Anal. Appl.* **16** (1982).

[2] L. A. Takhtadzhyan and L. D. Faddeev, "The quantum method of the inverse problem and the Heisenberg XYZ-model", *Uspekhi Mat. Nauk* **34**:5 (1979), 13–63; English transl. in *Russian Math. Surveys* **34**:5 (1979).

[3] P. P. Kulish and E. K. Sklyanin, "Quantum spectral transform method. Recent developments", *Lecture Notes in Phys.* **151** (1982), 61–119.

[4] A. G. Izergin and V. E. Korepin, "The quantum method of the inverse problem", *Fiz. Elementar. Chastits i Atom. Yadra* **13** (1982), 501–541; English transl. in *Soviet J. Particles and Nuclei* **13** (1982).

[5] A. A. Belavin, "Discrete groups and integrability of quantum systems", *Funktsional. Anal. i Prilozhen.* **14**:4 (1980), 18–26; English transl. in *Functional Anal. Appl.* **14** (1980).

[6] A. V. Odesskii and B. L. Feigin, "Elliptic Sklyanin algebras", *Funktsional. Anal. i Prilozhen.* **23**:3 (1989), 45–54; English transl. in *Functional Anal. Appl.* **23** (1989).

[7] M. Artin, J. Tate, and M. Van den Bergh, "Modules over regular algebras of dimension 3", *Invent. Math.* **106** (1991), 335–338; "Some algebras related to automorphisms of elliptic curves", Grothendieck Festschrift, Vol. 1, Birkhäuser Boston, Boston 1990, pp. 33–85.

[8] E. K. Sklyanin, "On some algebraic structures related to the Yang–Baxter equation. Representations of the quantum algebra", *Funktsional. Anal i Prilozhen.* **17**:4 (1983), 34–48; English transl. in *Functional Anal. Appl.* **17** (1983).

[9] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, "Non-linear equations of Korteweg-de Vries type, finite zone linear operators, and Abelian varieties", *Uspekhi Mat. Nauk* **31**:1 (1976), 55–136; English transl. in *Russian Math. Surveys* **31**:1 (1976).

[10] I. M. Krichever, "Algebraic curves and non-linear difference equations", *Uspekhi Mat. Nauk* **33**:4 (1978), 215–216; English transl. in *Russian Math. Surveys* **33**:4 (1978).

[11] I. M. Krichever, "Nonlinear equations and elliptic curves", in: *Current problems in mathematics,* Vol. 23, VINITI, Moscow 1983, pp. 79–136.

[12] E. Date and S. Tanaka, "Exact solutions of the periodic Toda lattice", *Prog. Theor. Phys.* **5** (1976), 457–465.

[13] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, *Theory of solitons. Inverse scattering method,* Nauka, Moscow 1980. (Russian)

[14] R. Baxter, "Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors", *Ann. Phys.* **76** (1973), 1–24.

[15] H. Airault, H. McKean, and J. Moser, "Rational and elliptic solutions of the KdV equation and related many-body problem", *Comm. Pure Appl. Math.* **30** (1977), 95–125.

[16] I. M. Krichever, "On rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line", *Funktsional. Anal i Prilozhen.* **12**:1 (1978), 76–78; English transl. in *Functional Anal. Appl.* **12** (1978).

[17] D. V. Chudnovsky and G. V. Chudnovsky, "Pole expansions of non-linear partial differential equations", *Nuovo Cimento B* (11) **40** (1977), 339–353.

[18] I. M. Krichever, "Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles", *Funktsional. Anal. i Prilozhen.* **14**:4 (1980), 45–54; English transl. in *Functional Anal. Appl.* **14** (1980).

[19] I. Krichever, O. Babelon, E. Billey, and M. Talon, "Spin generalization of the Calogero–Moser system and the matrix KP equation", Preprint LPTHE 94/42.

[20] S. N. M. Ruijsenaars and H. Schneider, "A new class of integrable systems and its relation to solitons", *Ann. Physics* **170** (1986), 370–405.

[21] M. A. Olshanetsky and A. M. Perelomov, "Classical integrable finite-dimensional systems related to Lie algebras", *Phys. Rep.* **71** (1981), 313–400.

[22] D. Kazhdan, B. Kostant and S. Sternberg, "Hamiltonian group actions and dynamical systems of Calogero type", *Comm. Pure Appl. Math.* **31** (1978), 481–507.

[23] A. Gorsky and N. Nekrasov, "Hamiltonian systems of Calogero type and two-dimensional Yang–Mills theory", *Nucl. Phys. B* **414** (1994), 213–238.

[24] A. Gorsky and N. Nekrasov, "Elliptic Calogero–Moser system from two-dimensional current algebra", Preprint ITEP-NG/1-94, hep-th/9401021.
[25] N. Hitchin, "Stable bundles and integrable systems", *Duke Math. J.* **54** (1987), 91–114.
[26] N. Nekrasov, "Holomorphic bundles and many-body systems", Preprint PUPT-1534 (1995), hep-th/9503157.
[27] M. Gaudin, *La fonction d'onde de Bethe*, Masson, Paris 1983; Russian transl., Mir, Moscow 1987.
[28] B. Enriquez and V. Rubtsov, "Hitchin systems, higher Gaudin operators and r-matrices", Preprint, alg-geom/9503010.
[29] S. N. M. Ruijsenaars, "Complete integrability of relativistic Calogero–Moser systems and elliptic function identities", *Comm. Math. Phys.* **110** (1987), 191–213.
[30] M. Bruschi and F. Calogero, "The Lax representation for an integrable class of relativistic dynamical systems", *Comm. Math. Phys.* **109** (1987), 481–492.
[31] I. Krichever, "The periodic non-Abelian Toda chain and its two-dimensional generalization", *Uspekhi Mat. Nauk* **36**:2 (1981), 72–77; English transl. in *Russian Math. Surveys* **36**:2 (1981)[5].
[32] A. S. Gorskii and A. V. Zabrodin, "Degenerations of Sklyanin algebra and Askey–Wilson polynomials", *J. Phys. A* **26** (1993), L635–L639.
[33] I. Krichever, "Baxter's equations and algebraic geometry", *Funktsional. Anal. i Prilozhen.* **15**:2 (1981), 22–35; English transl. in *Functional Anal. Appl.* **15** (1981).
[34] T. Takebe, "Generalized Bethe ansatz with the general spin representation of the Sklyanin algebra", *J. Phys. A* **25** (1992), 1071–1083.
[35] T. Takebe, "Bethe ansatz for higher spin eight vertex models", Univ. of Tokyo preprint 95-25, q-alg/950427 (April 1995).
[36] G. Andrews, R. Baxter, and P. Forrester, "Eight-vertex SOS model and Rogers–Ramanujan type identities", *J. Stat. Phys.* **35** (1984), 193–266.
[37] C. M. Yung and M. T. Batchelor, "Exact solution for the spin-s XXZ quantum chain with non-diagonal twists", Preprint MRR 014-95 (January 1995), hep-th/9502041.
[38] I. G. Korepanov, "Vacuum curves of L-operators associated with the six-vertex model", *Algebra i Analiz* **6** (1994), 176–194; English transl. in *St. Petersburg Math. J.* **6** (1995).
[39] E. K. Sklyanin, "Goryachev–Chaplygin top and inverse scattering method", *Zap. Nauchn. Sem. LOMI* **133** (1984), 236–257.
[40] F. Nijhoff, O. Ragnisco and V. Kuznetsov, "Integrable time-discretization of the Ruijsenaars–Schneider model", Univ. of Amsterdam preprint 94-27, hep-th/9412170 (1994).

Landau Institute for Theoretical Physics, Moscow
Joint Institute of Chemical Physics, Moscow

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

[5] This was an appendix to an article by B. A. Dubrovin.