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The effective prepotential of $N = 2$ supersymmetric $SO(N_c)$ and $Sp(N_c)$ gauge theories ⋆

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Abstract

We calculate the effective prepotentials for $N = 2$ supersymmetric $SO(N_c)$ and $Sp(N_c)$ gauge theories, with an arbitrary number of hypermultiplets in the defining representation, from restrictions of the prepotentials for suitable $N = 2$ supersymmetric gauge theories with unitary gauge groups. (This extends previous work in which the prepotential for $N = 2$ supersymmetric $SU(N_c)$ gauge theories was evaluated from the exact solution constructed out of spectral curves.) The prepotentials have to all orders the logarithmic singularities of the one-loop perturbative corrections, as expected from non-renormalization theorems. We evaluate explicitly the contributions of 1- and 2-instanton processes. © 1997 Elsevier Science B.V.

1. Introduction

Powerful techniques are now available for the evaluation of the effective prepotential of $N = 2$ supersymmetric Yang-Mills theories in their Abelian Coulomb phase (where the gauge group is broken down to an Abelian subgroup). The effective prepotential, as well as the masses of the BPS states, are determined from a spectral curve, together with a meromorphic 1-form $d\lambda$, both of which are parametrized by the vacuum expectation values of the scalar fields (also called order parameters). The original developments

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for an SU(2) gauge group are in Ref. [1], the spectral curve and meromorphic 1-form were determined for other gauge groups in Refs. [2–5], and the effect of N_f hypermultiplets in the fundamental representation were also included for $SU(N_c)$ gauge groups in Refs. [6,7].

In a recent paper [8], we developed methods for determining the prepotential from the spectral curves for an arbitrary $SU(N_c)$ gauge group and arbitrary numbers of hypermultiplets $N_f < 2N_c$, in the regime where the renormalization scale $\bar{\Lambda}$ is small. We explicitly calculated the full expansion of the renormalized order parameters (obtained from the A-periods of $d\lambda$) using the method of residues, and provided a simple and systematic algorithm for the evaluation of the renormalized dual order parameters (obtained from the B-periods of $d\lambda$). Using these methods, we confirmed $N = 2$ supersymmetry non-renormalization theorems and worked out explicitly the perturbative corrections as well as the 1- and 2-instanton contributions to the effective potential. These results were found to agree with those of Ref. [1] for SU(2), with those of Ref. [9] for SU(3), as well as with direct field theory calculations in Ref. [10] for SU(2) with $N_f < 4$ hypermultiplets in the fundamental representation and in Ref. [11] for $SU(N_c)$ with $N_f = 0$, both to 1-instanton order. We also showed that the different models [6,7,12] for the spectral curves that were proposed for the cases $N_f \geqslant N_c + 2$ all give rise to the same effective prepotential.

In the present paper, we extend the above results to the cases of all classical groups, including $SO(N_c)$ and $Sp(N_c)$, with any number of hypermultiplets so as to keep the theory asymptotically free. We make use of the fact that the spectral curves associated with the classical groups $SO(N_c)$ and $Sp(N_c)$ are hyperelliptic, and may be viewed as restrictions of the spectral curves for $SU(N_c)$.[4] Analogously, we show that the homology cycles, the meromorphic 1-form, and thus the entire effective prepotential may be obtained by simple restriction from the unitary case. These results imply that, to all orders in the instanton expansion, all logarithmic singularities of the prepotential are just those of one-loop perturbation theory, thereby confirming the $N = 2$ supersymmetry non-renormalization theorems. Also, they show that the prepotential is unchanged under analytic redefinitions of the classical order parameters, just as we showed for the case of $SU(N_c)$ in Ref. [8].

For the gauge groups $SO(N_c)$ and $Sp(N_c)$, we shall work out explicitly the perturbative corrections as well as the contributions of 1- and 2-instanton processes to the prepotential and arbitrary numbers of hypermultiplets in the defining representation of the color group (see however footnote 4), with the restriction that the theory remains asymptotically free.

[4] For the gauge group $Sp(N_c)$, the identification of its spectral curve with a restriction of a curve for a unitary group appears possible only when there is at least one exactly massless hypermultiplet in the defining representation of $Sp(N_c)$. As we shall see, this condition appears in our work for purely technical reasons; it is unclear to us at this point whether it is in any way fundamental.

2. Spectral curves, 1-forms and homology cycles

We consider $N = 2$ supersymmetric gauge theories with the classical gauge groups $SU(r+1)$, $SO(2r+1)$, $Sp(2r)$ and $SO(2r)$, which are all of rank r, and numbers of colors $N_c = r+1$, $2r+1$, $2r$ and $2r$, respectively. We also assume that there are N_f hypermultiplets, transforming under the defining representation of the gauge group, of dimension N_c, and with bare masses m_j, $j = 1, \dots, N_f$. The $N = 2$ chiral multiplet contains a complex scalar field ϕ in the adjoint representation of the gauge group. The flat directions in the potential correspond to $[\phi, \phi^\dagger] = 0$, so that the classical moduli space of vacua is r-dimensional, and can be parametrized by the eigenvalues $\bar{a}_k$, $k = 1, \dots, r$ of ϕ, in the following way:

$$
\begin{aligned}
&SU(r+1) && \phi = \text{diagonal } [\bar{a}_1, \dots, \bar{a}_r, \bar{a}_{r+1}], && \bar{a}_1 + \dots + \bar{a}_r + \bar{a}_{r+1} = 0, \\
&SO(2r+1) && \phi = \text{diagonal } [\mathcal{A}_1, \dots, \mathcal{A}_r, 0], && \\
&Sp(2r) && \phi = \text{diagonal } [\bar{a}_1, -\bar{a}_1, \dots, \bar{a}_r, -\bar{a}_r], && \\
&SO(2r) && \phi = \text{diagonal } [\mathcal{A}_1, \dots, \mathcal{A}_r], && \mathcal{A}_k = \begin{pmatrix} 0 & \bar{a}_k \\ -\bar{a}_k & 0 \end{pmatrix}.
\end{aligned}
\tag{2.1}
$$

For generic $\bar{a}_k$, the gauge symmetry is broken down to $U(1)^r$ and the dynamics of the theory is that of an Abelian Coulomb phase. The Wilson effective Lagrangian of the quantum theory to leading order in the low momentum expansion in the Abelian Coulomb phase is of the form (in $N = 1$ superfield notation)

$$
\mathcal{L} = \operatorname{Im} \frac{1}{4\pi} \left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^k} \overline{A^k} + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^k \partial A^l} W^k W^l \right], \tag{2.2}
$$

where the A^k's are $N = 1$ chiral superfields whose scalar components correspond to the $\bar{a}_k$'s at the classical level, and $\mathcal{F}$ is the holomorphic prepotential.

The Seiberg-Witten ansatz for the effective prepotential $\mathcal{F}$ is based on the choice of a fibration of spectral curves over the space of vacua, and a meromorphic 1-form $d\lambda$ over each of these curves. The renormalized order parameters a_k of the theory, their duals $a_{D,k}$, and the prepotential $\mathcal{F}$ are then given by

$$
2\pi i a_k = \oint_{A_k} d\lambda, \qquad 2\pi i a_{D,k} = \oint_{B_k} d\lambda, \qquad a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k} \tag{2.3}
$$

with A_k, B_k a suitable set of homology cycles on the spectral curves.

For $SU(N_c)$ gauge theories, with $N_f < 2N_c$ hypermultiplets in the defining representation of the gauge group, general arguments based on the holomorphicity of $\mathcal{F}$, perturbative non-renormalization beyond 1-loop order, the nature of instanton corrections and the restrictions of $U(1)_R$ invariance, suggest that $\mathcal{F}$ should have the following form: [5]

[5] We shall omit contributions to $\mathcal{F}$ that are of the form of a Λ-independent constant times the classical prepotential $\sum_k a_k^2$ throughout this paper.

$$
\begin{aligned}
&\mathcal{F}_{\mathrm{SU}(N_c);N_f}(a_1,\dots a_{N_c};m_1,\dots,m_{N_f};\Lambda)\\
&= -\frac{1}{8\pi i}\left(\sum_{k,l=1}^{N_c}(a_k-a_l)^2\log\frac{(a_k-a_l)^2}{\Lambda^2}\right.\\
&\qquad\left.-\sum_{k=1}^{N_c}\sum_{j=1}^{N_f}(a_k+m_j)^2\log\frac{(a_k+m_j)^2}{\Lambda^2}\right)\\
&\quad+\sum_{d=1}^{\infty}\mathcal{F}^{(d)}_{\mathrm{SU}(N_c);N_f}(a_1,\dots,a_{N_c};m_1,\dots,m_{N_f};\Lambda).
\end{aligned}
\tag{2.4}
$$

The terms on the right-hand side are respectively the contribution of perturbative one-loop effects (higher loops do not contribute in view of non-renormalization theorems), and the contributions of d-instanton processes. The results for $d = 1$ and $d = 2$ were computed explicitly in Ref. [8], and we shall record them here for later reference,

$$
\begin{aligned}
\mathcal{F}^{(1)} &= \frac{1}{8\pi i}\Lambda^{2N_c-N_f}\sum_{k=1}^{N_c}S_k(a_k),\\
\mathcal{F}^{(2)} &= \frac{1}{32\pi i}\Lambda^{2(2N_c-N_f)}\left[\sum_{k\neq l}\frac{S_k(a_k)S_l(a_l)}{(a_k-a_l)^2}+\frac{1}{4}\sum_{k=1}^{N_c}S_k(a_k)\left.\frac{\partial^2 S_k(x)}{\partial x^2}\right|_{x=a_k}\right],
\end{aligned}
\tag{2.5}
$$

where the fundamental function $S_k(x)$ is defined by

$$
S_k(x) = \frac{\prod_{j=1}^{N_f}(x+m_j)}{\prod_{l\neq k}(x-a_l)^2}.
\tag{2.6}
$$

By construction, these contributions to $\mathcal{F}$ are invariant under the group of permutations of the variables a_k, i.e. under the Weyl group of $\mathrm{SU}(N_c)$. It is of course possible, though in general cumbersome, to re-express these results in terms of symmetric polynomials in the variables a_k.

2.1. Spectral curves and associated meromorphic 1-form

The spectral curves for the classical gauge groups were derived in Ref. [1] for SU(2), in Refs. [2,6,7,12] for general $\mathrm{SU}(N_c)$, in Refs. [4,12] for $\mathrm{SO}(2r+1)$ in Refs. [5,12] for $\mathrm{SO}(2r)$, and in Ref. [12] for $\mathrm{Sp}(2r)$. All these curves are hyperelliptic. In some cases, different curves have been proposed for the same gauge group and the same hypermultiplet contents. For example, in the case of $\mathrm{SU}(N_c)$ gauge group and $N_f > N_c + 1$ hypermultiplets, the curves proposed in Refs. [6,7,12] are all different. However, we have shown in Ref. [8], by general arguments and confirmed by explicit calculations up to 2-instanton processes, that the corresponding effective prepotentials are the same for each of these different models of curves. This equivalence results from the fact that the effective prepotential is unchanged under analytic reparametrizations of the classical order parameters. Also, we note that for non-simply laced groups, like Sp$(2r)$, non-hyperelliptic curves were proposed in Ref. [3].

For all $N=2$ supersymmetric gauge theories based on classical groups, and with N_f hypermultiplets in the defining representation of the gauge group, hyperelliptic spectral curves with associated meromorphic 1-forms have been proposed as follows:

$$y^2 = A^2(x) - B(x),$$
$$d\lambda = \frac{x\,dx}{y}\left(A' - \frac{1}{2}A\frac{B'}{B}\right). \tag{2.7}$$

Here, $A(x)$ and $B(x)$ are polynomials in x, whose coefficients vary with the physical parameters of the theory, and are given by

$$\begin{aligned}
&\mathrm{SU}(r+1) \qquad A(x) = \prod_{k=1}^{r+1}(x-\bar{a}_k), \qquad B(x) = \Lambda^{2r+2-N_f}\prod_{j=1}^{N_f}(x+m_j),\\
&\mathrm{SO}(2r+1) \qquad A(x) = \prod_{k=1}^{r}(x^2-\bar{a}_k^2), \qquad B(x) = \Lambda^{4r-2N_f-2}x^2\prod_{j=1}^{N_f}(x^2-m_j^2),\\
&\mathrm{Sp}(2r) \qquad A(x) = x^2\prod_{k=1}^{r}(x^2-\bar{a}_k^2) + A_0,\\
&\qquad\qquad B(x) = \Lambda^{4r-2N_f+4}\prod_{j=1}^{N_f}(x^2-m_j^2),\\
&\mathrm{SO}(2r) \qquad A(x) = \prod_{k=1}^{r}(x^2-\bar{a}_k^2), \qquad B(x) = \Lambda^{4r-2N_f-4}x^4\prod_{j=1}^{N_f}(x^2-m_j^2),
\end{aligned} \tag{2.8}$$

where $A_0 = \Lambda^{2r-N_f+2}\prod_{j=1}^{N_f} m_j$.

Note that the differential $d\lambda$ only depends upon the ratio $B(x)/A(x)^2$, so that simultaneous *rescaling* of $A(x)$ by a function $f(x)$ and $B(x)$ by the function $f(x)^2$ leaves the variables a_k and $a_{D,k}$, and hence the effective prepotential $\mathcal{F}$ invariant.

2.2. The case of Sp(N_c) gauge theories

It is apparent from the form of the functions $A(x)$ above that the case of Sp(N_c) gauge group is special: there appears an extra constant A_0 that was not present for the other classical groups. The methods that we shall present do not seem to extend easily to the case when $A_0 \neq 0$, because there is no natural map onto the curve for unitary groups. Thus, in this paper, we shall restrict analysis to the case where at least one of the hypermultiplets of the Sp(N_c) supersymmetric gauge theory has exactly zero mass. We shall denote this restricted case by Sp$(N_c)'$. Under this assumption, $A_0 = 0$ and using the rescaling property of the prepotential explained in the previous paragraph, we find that the curve for $\mathrm{Sp}(N_c)'$, i.e. $\mathrm{Sp}(N_c)$ with at least one hypermultiplet of exactly zero mass is given by

$$\begin{aligned} &\mathrm{Sp}(2r)' && A(x) = x\prod_{k=1}^{r}(x^2 - \bar{a}_k^2), \\ &(m_{N_f} = 0) && B(x) = \Lambda^{4r-2N_f+4}\prod_{j=1}^{N_f-1}(x^2 - m_j^2). \end{aligned} \tag{2.9}$$

Henceforth, we shall specialize to this case for the gauge group $\mathrm{Sp}(N_c)$.

Actually, we further notice that when two hypermultiplets are exactly massless, the rescaled curves for $\mathrm{Sp}(N_c)$ gauge groups admit an even simpler form, which we shall record here. We denote this case by $\mathrm{Sp}(N_c)''$.

$$\begin{aligned} &\mathrm{Sp}(2r)'' && A(x) = \prod_{k=1}^{r}(x^2 - \bar{a}_k^2), \\ &(m_{N_f-1} = m_{N_f} = 0) && B(x) = \Lambda^{4r-2N_f+4}\prod_{j=1}^{N_f-2}(x^2 - m_j^2). \end{aligned} \tag{2.10}$$

These curves have the same genera as the ones for the $\mathrm{SO}(N_c)$ gauge groups, and their treatment will be carried out completely in parallel to that of the orthogonal groups.

2.3. Homology cycles

The hyperelliptic curves for $\mathrm{SO}(2r+1)$, $\mathrm{Sp}(2r)''$ and $\mathrm{SO}(2r)$ all have genus $2r-1$. To each classical root $\bar{a}_k$, $k = 1, \dots, r$, there correspond two branch points $x_k^{\pm}$, which define a quadratic branch cut and an associated homology cycle A_k surrounding the cut joining the two branch points. (Due to $\mathbb{Z}_2$ symmetry of the curves, under which $x \longrightarrow -x$, there correspond to the negative roots $-a_k$, $k = 1, \dots, r$, two negative branch points $-x_k^{\pm}$, which define a quadratic branch cut and an associated homology cycle A'_k.) For the B_k cycle, we choose the cycle going from $-x_k^-$ to x_k^- in the first sheet, completed by its counterpart in the second sheet. We note that $\#(A_k \cap A_l) = \#(B_k \cap B_l) = 0$, $\#(A_k \cap B_l) = \delta_{kl}$, although B_k intersects also A'_k. The cycles A_k and B_k thus defined are the ones we shall take for the Seiberg-Witten ansatz (2.3).

Taking into account the fact that the differential $d\lambda$ is itself odd under the $\mathbb{Z}_2$ symmetry, under which $x \longrightarrow -x$, the normalized periods of the differential $d\lambda$ obtained in this way are

$$a_k = \frac{1}{\pi i}\int_{x_k^-}^{x_k^+} d\lambda, \qquad a_{D,k} = \frac{1}{\pi i}\int_{-x_k^-}^{x_k^-} d\lambda, \qquad k = 1, \dots, r. \tag{2.11}$$

This normalization is clearly in agreement with the classical limit, where $\Lambda \longrightarrow 0$, and $a_k \longrightarrow \bar{a}_k$.

3. Restricting prepotentials for unitary gauge groups

From the form of the curves for the different gauge groups in (2.7) and (2.8) and restrictions with massless hypermultiplets for the symplectic groups in (2.9) and (2.10), we see that the curves for the orthogonal and symplectic gauge groups can be viewed as natural restrictions of the curves for unitary groups. The precise correspondences are as follows.

The curves for $SO(2r+1)$, $Sp(2r)''$ and $SO(2r)$ can be obtained from those of $SU(2r)$, where the $2r$ classical order parameters of $SU(2r)$ are chosen to be $\bar{a}_1, \ldots, \bar{a}_r, -\bar{a}_1, \ldots, -\bar{a}_r$. As a result of $\mathbb{Z}_2$ symmetry, the quantum order parameters a_k then also come in pairs of opposites: $a_1, \ldots, a_r, -a_1, \ldots, -a_r$. The correspondences of the number of hypermultiplets, N_f, in these theories and their masses is slightly more involved. For orthogonal groups, the presence of a power of x^2 for $SO(2r+1)$, and a factor of x^4 for $SO(2r)$ in the function $B(x)$ in (2.8), forces us to make identifications with unitary groups with $2N_f+2$ and $2N_f+4$ hypermultiplets of $SU(2r)$ respectively. For symplectic groups with at least two massless hypermultiplets, i.e. the case $Sp(2r)''$, the correspondence is with a theory of $2N_f-4$ hypermultiplets in $SU(2r)$.

The curves for $Sp(2r)$ without massless hypermultiplets (this includes the case with no hypermultiplets at all) can be obtained from those of $SU(2r+2)$, where the classical order parameters of $SU(2r+2)$ are chosen to be $0, 0, \bar{a}_1, \ldots, \bar{a}_r, -\bar{a}_1, \ldots, -\bar{a}_r$, and the number of $SU(2r+2)$ hypermultiplets is $2N_f$. The appearance of the double zero at $\bar{a}=0$ implies that the corresponding $SU(2r+2)$ theory has an unbroken $SU(2)$ invariance and is not in the Abelian Coulomb phase at the classical level. The expansion methods developed in Ref. [8] for the effective prepotential do not apply to this case, and we shall not consider it again in this paper.

3.1. Restriction of the quantum order parameters a_k and $a_{D,k}$

Given the above restrictions of the curves of unitary gauge groups to $SO(N_c)$ and $Sp(N_c)$, and the fact that the functional form of the meromorphic 1-form is the same for the various groups, we obtain the following relations between the quantum order parameters a_k and $a_{D,k}$. For maximum clarity, we make all dependences completely explicit, and we let the range of k and l be $1 \leqslant k, l \leqslant r$. For $SO(2r+1)$, we have

$$\begin{aligned}
&a_k|_{SO(2r+1);N_f}(\bar{a}_l; m_1, \ldots, m_{N_f}; \Lambda) \\
&\quad = a_k|_{SU(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \ldots, m_{N_f}, -m_1, \ldots, -m_{N_f}, 0, 0; \Lambda), \\
&a_{D,k}|_{SO(2r+1);N_f}(\bar{a}_l; m_1, \ldots, m_{N_f}; \Lambda) \\
&\quad = a_{D,k}|_{SU(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \ldots, m_{N_f}, -m_1, \ldots, -m_{N_f}, 0, 0; \Lambda) \\
&\qquad - a_{D,k+r}|_{SU(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \ldots, m_{N_f}, -m_1, \ldots, -m_{N_f}, 0, 0; \Lambda).
\end{aligned} \tag{3.1}$$

For $Sp(2r)''$, we have

$$a_k|_{Sp(2r);N_f}(\bar{a}_l; m_1, \ldots, m_{N_f-2}, 0, 0; \Lambda)$$

$$= a_k|_{\mathrm{SU}(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \dots, m_{N_f-2}, -m_1, \dots, -m_{N_f-2}; \Lambda),$$
$$a_{D,k}|_{\mathrm{Sp}(2r);N_f}(\bar{a}_l; m_1, \dots, m_{N_f-2}, 0, 0; \Lambda)$$
$$= a_{D,k}|_{\mathrm{SU}(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \dots, m_{N_f-2}, -m_1, \dots, -m_{N_f-2}; \Lambda)$$
$$- a_{D,k+r}|_{\mathrm{SU}(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \dots, m_{N_f-2}, -m_1, \dots, -m_{N_f-2}; \Lambda). \quad (3.2)$$

For $\mathrm{SO}(2r)$, we obtain

$$a_k|_{\mathrm{SO}(2r);N_f}(\bar{a}_l; m_1, \dots, m_{N_f}; \Lambda)$$
$$= a_k|_{\mathrm{SU}(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \dots, m_{N_f}, -m_1, \dots, -m_{N_f}, 0, 0, 0, 0; \Lambda),$$
$$a_{D,k}|_{\mathrm{SO}(2r);N_f}(\bar{a}_l; m_1, \dots, m_{N_f}; \Lambda)$$
$$= a_{D,k}|_{\mathrm{SU}(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \dots, m_{N_f}, -m_1, \dots, -m_{N_f}, 0, 0, 0, 0; \Lambda)$$
$$- a_{D,k+r}|_{\mathrm{SU}(2r)}(\bar{a}_l, -\bar{a}_l; m_1, \dots, m_{N_f}, -m_1, \dots, -m_{N_f}, 0, 0, 0, 0; \Lambda). \quad (3.3)$$

In Ref. [8], an exact formula was derived for the relation between the quantum order parameters a_k as a function of the classical order parameters $\bar{a}_k$ for gauge group $\mathrm{SU}(N_c)$. Using the above identifications, we easily extend these exact results to the case of $\mathrm{SO}(N_c)$ and $\mathrm{Sp}(N_c)$ gauge groups. The result is given in the form of infinite power series expansions in the renormalization scale Λ,

$$a_k = \bar{a}_k + \sum_{m=1}^{\infty} \frac{\bar{\Lambda}^{2m}}{2^{2m}(m!)^2} \left(\frac{\partial}{\partial x}\right)^{2m-1} \bar{\Sigma}_k(x)^m \bigg|_{x=\bar{a}_k} \quad (3.4)$$

with the following results:

$$\mathrm{SU}(r+1) \qquad \bar{\Lambda} = \Lambda^{r+1-N_f/2}, \qquad \bar{\Sigma}_k(x) = \prod_{j=1}^{N_f}(x+m_j)\prod_{l\neq k}(x-\bar{a}_l)^{-2},$$
$$\mathrm{SO}(2r+1) \qquad \bar{\Lambda} = \Lambda^{2r-1-N_f},$$
$$\bar{\Sigma}_k(x) = x^2(x+\bar{a}_k)^{-2}\prod_{j=1}^{N_f}(x^2-m_j^2)\prod_{l\neq k}(x^2-\bar{a}_l^2)^{-2},$$
$$\mathrm{Sp}(2r)'' \qquad \bar{\Lambda} = \Lambda^{2r+2-N_f},$$
$$\bar{\Sigma}_k(x) = (x+\bar{a}_k)^{-2}\prod_{j=1}^{N_f}(x^2-m_j^2)\prod_{l\neq k}(x^2-\bar{a}_l^2)^{-2},$$
$$\mathrm{SO}(2r) \qquad \bar{\Lambda} = \Lambda^{2r-2-N_f},$$
$$\bar{\Sigma}_k(x) = x^4(x+\bar{a}_k)^{-2}\prod_{j=1}^{N_f}(x^2-m_j^2)\prod_{l\neq k}(x^2-\bar{a}_l^2)^{-2}. \quad (3.5)$$

In the above expressions, the range of k is just over the independent variables, and is thus restricted to $k = 1, \dots, r$.

3.2. The effective prepotential

Since the renormalized order parameters a_k and $a_{D,k}$ for $SO(N_c)$ and $Sp(N_c)$ gauge groups may both be obtained as restrictions from the unitary case, it is natural to expect that also the effective prepotential may be viewed as such a restriction. The restriction rules for the prepotential turn out to be particularly simple in view of the fact that the differences $a_{D,k} - a_{D,k+r}$ are naturally produced by a straightforward restriction of the $\mathcal{F}_{SU(2r)}$ to the $\mathbb{Z}_2$ symmetric arrangements for the gauge groups $SO(N_c)$ and $Sp(N_c)$. As a result, we readily deduce the correct prepotentials for the orthogonal and symplectic groups. For $SO(2r+1)$, we have

$$\begin{aligned}
&\mathcal{F}_{SO(2r+1);N_f}(a_1,\dots,a_r;m_1,\dots,m_{N_f};\Lambda) \\
&\quad = \mathcal{F}_{SU(2r);2N_f+2}(a_1,\dots,a_r,-a_1,\dots,-a_r;m_1,\dots,m_{N_f},-m_1, \\
&\qquad\qquad \dots,-m_{N_f},0,0;\Lambda),
\end{aligned}$$

for $Sp(2r)$, with at least two massless hypermultiplets, i.e. the case $Sp(2r)''$, we have

$$\begin{aligned}
&\mathcal{F}_{Sp(2r);N_f}(a_1,\dots,a_r;m_1,\dots,m_{N_f-2},0,0;\Lambda) \\
&\quad = \mathcal{F}_{SU(2r);2N_f-4}(a_1,\dots,a_r,-a_1,\dots,-a_r;m_1,\dots,m_{N_f-2},-m_1, \\
&\qquad\qquad \dots,-m_{N_f-2};\Lambda)
\end{aligned}$$

and, finally, for $SO(2r)$, we have

$$\begin{aligned}
&\mathcal{F}_{SO(2r);N_f}(a_1,\dots,a_r;m_1,\dots,m_{N_f};\Lambda) \\
&\quad = \mathcal{F}_{SU(2r);2N_f+4}(a_1,\dots,a_r,-a_1,\dots,-a_r;m_1,\dots,m_{N_f},-m_1, \\
&\qquad\qquad \dots,-m_{N_f},0,0,0,0;\Lambda).
\end{aligned}$$

From the above restriction rules, it follows that for each of the gauge groups, the prepotential may be decomposed in a sum over the number of instantons contributing to the process, just as was the case for unitary gauge groups in (2.3). We shall denote by $\mathcal{F}^{(d)}$ the contribution arising from d-instanton processes, and, for $d \geqslant 1$, these functions depend on Λ through a factor of $\bar{\Lambda}^{2d}$ where $\bar{\Lambda}$ was defined for each group in (3.5). The contribution from zero instantons, i.e. classical plus perturbative corrections, is denoted by $\mathcal{F}^{(0)}$. Using the results from Ref. [8], and the above restriction rules, we now have the following results for the effective prepotential.

The perturbative contributions $\mathcal{F}^{(0)}$ are given as follows. For gauge groups $G = SO(2r+1)$, $Sp(2r)$ with at least two massless hypermultiplets, i.e. the case $Sp(2r)''$, and $SO(2r)$ we have the following formula:

$$\mathcal{F}_{G;N_f}(\bar{a}_l;m_1,\dots,m_{N_f};\bar{\Lambda}) = \frac{i}{4\pi}\left\{\sum_{k\neq l}^{r}\sum_{\epsilon=\pm1}(a_k+\epsilon a_l)^2\log\frac{(a_k+\epsilon a_l)^2}{\Lambda^2}\right.$$

$$+\xi\sum_{k=1}^{r}a_k^2\log\frac{a_k^2}{\Lambda^2}-\sum_{k=1}^{r}\sum_{j=1}^{N_f}\sum_{\epsilon=\pm1}(a_k+\epsilon m_j)^2\log\frac{(a_k+\epsilon m_j)^2}{\Lambda^2}\Bigg\}, \tag{3.6}$$

where the constant ξ takes on the values $\xi = 2$, 4 and 0 for $G = SO(2r+1)$, $Sp(2r)''$ ($Sp(2r)$ with at least two massless hypermultiplets), and $SO(2r)$, respectively. We readily recognize these numbers from the structure of the corresponding Dynkin diagrams.

The 1-instanton contributions are also readily deduced from the results of Ref. [8], combined with the restriction rules above. The results are most easily cast in terms of the parameters $\bar{\Lambda}$ and the functions $\Sigma_k(x)$ defined for each group gauge group $G = SO(2r+1)$, $Sp(2r)''$ ($Sp(2r)$ with at least two massless hypermultiplets), and $SO(2r)$ as in (3.5), but with *the classical order parameters $\bar{a}_k$ replaced by their renormalized counterparts a_k*,

$$\begin{aligned}
&SU(r+1) \qquad \bar{\Lambda}=\Lambda^{r+1-N_f/2}, \qquad \Sigma_k(x)=\prod_{j=1}^{N_f}(x+m_j)\prod_{l\neq k}(x-a_l)^{-2},\\
&SO(2r+1) \qquad \bar{\Lambda}=\Lambda^{2r-1-N_f},\\
&\qquad \Sigma_k(x)=x^2(x+a_k)^{-2}\prod_{j=1}^{N_f}(x^2-m_j^2)\prod_{l\neq k}(x^2-a_l^2)^{-2},\\
&Sp(2r)'' \qquad \bar{\Lambda}=\Lambda^{2r+2-N_f},\\
&\qquad \Sigma_k(x)=(x+a_k)^{-2}\prod_{j=1}^{N_f}(x^2-m_j^2)\prod_{l\neq k}(x^2-a_l^2)^{-2},\\
&SO(2r) \qquad \bar{\Lambda}=\Lambda^{2r-2-N_f},\\
&\qquad \Sigma_k(x)=x^4(x+a_k)^{-2}\prod_{j=1}^{N_f}(x^2-m_j^2)\prod_{l\neq k}(x^2-a_l^2)^{-2}.
\end{aligned} \tag{3.7}$$

Then we have

$$\mathcal{F}^{(1)}_{G;N_f}=\frac{1}{4\pi i}\bar{\Lambda}^2\sum_{k=1}^{r}\Sigma_k(a_k). \tag{3.8}$$

(Note: this formula does not apply to $SU(N_c)$ as written, and would require an extra factor of $\frac{1}{2}$.)

Similarly, the 2-instanton contributions may also be worked out, and we have

$$\mathcal{F}^{(2)}_{G;N_f}=\frac{1}{16\pi i}\bar{\Lambda}^2\left[\sum_{k\neq l}^{r}\sum_{\epsilon=\pm1}\frac{\Sigma_k(a_k)\Sigma_l(a_l)}{(a_k+\epsilon a_l)^2}+\frac{1}{4}\sum_{k=1}^{r}\Sigma_k(a_k)\frac{\partial^2\Sigma_k(x)}{\partial x^2}\bigg|_{x=a_k}\right]. \tag{3.9}$$

4. Special cases and discussion

We compare briefly now our results with various special cases discussed in the literature either directly from the quantum field theory point of view, using instanton calculations or from the Seiberg–Witten type approach.

The literature on the effective prepotential for $SO(N)$ and $Sp(N)$ gauge groups is not nearly as extensive as that for $SU(N)$. In Ref. [16], Ito and Sasakura evaluate the prepotential, up to 1-instanton order, from both instanton calculations and the Seiberg–Witten approach in the case of *pure* $N = 2$ supersymmetric Yang–Mills (no hypermultiplets). Using instanton calculations, they propose a formula for the 1-instanton correction $\mathcal{F}^{(1)}$ for any simple Lie group. Using the Seiberg–Witten approach, they derive explicitly Picard–Fuchs equations in the case of rank $\leqslant 3$, and rely on the scaling equations of Ref. [17]. For $SO(2r+1)$ and $SO(2r)$ gauge groups, our results for $\mathcal{F}^{(1)}$ do specialize to theirs if we set N_f to be 0. For $Sp(2r)$, it is of course not possible at the present time to compare the two results, since in the case they consider, there are no hypermultiplets, while in ours, we require at least two massless ones. It is however intriguing that there is no obvious way of interpolating between the two types of expressions that have been put forth.

A few days ago, another preprint [18] appeared, which also deals with the Seiberg–Witten approach for classical gauge groups, up to 1-instanton order.

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