Discrete analogs of the Darboux – Egoroff metrics

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Abstract

Discrete analogs of the Darboux – Egoroff metrics are considered. It is shown that the corresponding lattices in the Euclidean space are described by a set of functions $h_i^{\pm}(u)$, $u \in \mathbb{Z}^n$. Discrete analogs of the Lamé equations are found. It is proved that up to a gauge transformation these equations are necessary and sufficient for discrete analogs of rotation coefficients to exist. Explicit examples of the Darboux – Egoroff lattices are constructed by means of algebro-geometric methods.

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1 Introduction

Discrete analogs of various special coordinate systems on two-dimensional surfaces in three-dimensional Euclidean space, and discrete analogs of multi-dimensional conjugated coordinate nets have attracted great interest recently [1, 2, 3].

This interest has been motivated by revealed connections between corresponding problems of classical (continuous) differential geometry and modern problems of mathematical and theoretical physics. For example, it turned out that one of the central problems of the differential geometry of the previous century: the problem of constructing n-orthogonal curvilinear coordinate systems, or *flat* diagonal metrics

$$ds^2 = \sum_{i=1}^{n} H_i^2(u)(du^i)^2, \qquad u = (u^1, \dots, u^n), \tag{1.1}$$

is deeply connected to the theory of integrable quasilinear $(1+1)$-systems of the hydrodynamic type. These systems are the core of the Whitham approach to the perturbation theory of soliton equations [4, 5, 6]. Moreover, as it was noticed in [7], the classification problem of the so-called Darboux – Egoroff metrics, i. e. flat diagonal metrics such that

$$\partial_j H_i^2 = \partial_i H_j^2, \qquad \partial_i = \frac{\partial}{\partial u^i}, \tag{1.2}$$

is equivalent to the classification problem of massive topological quantum field models [8, 9, 10].

It should be emphasized that classical results [11] in the theory of n-orthogonal curvilinear coordinate systems were mainly of classification nature, and as a result the list of explicit examples of such coordinate systems was relatively short. In the remarkable paper [12] it was shown that a wide class of solutions to the Lamé equations which describe the rotation coefficients

$$\beta_{ij} = \frac{\partial_i H_j}{H_i}, \qquad i \neq j, \tag{1.3}$$

of flat diagonal metrics, can be obtained with the help of the “dressing” procedure which is well-known in the theory of solitons. Results of [12, 13] were a starting point of the work [14] by one of the authors, where a construction of algebro-geometric n-orthogonal coordinate systems was proposed, and a new type of solutions of the associativity equations was found. Explicit expressions in terms of the Riemann theta-functions associated with auxiliary algebraic curves were obtained.

Originally the main goal of this paper was to construct *discrete* analogs of the algebro-geometric n-orthogonal coordinate systems. Note, that in general the problem of finding an *integrable* discrete analog of integrable continuous system is ill-defined and has no universal answer. At the same time discretization methods developed in the theory of solitions are universal enough to be applicable to all the systems that are considered in the framework of the inverse method. They use natural discretization of auxiliary linear problems or even more natural change of analytical properties with respect to spectral parameter of common eigenfunctions of auxiliary linear problems.

It turns out that discretization of the algebro-geometric scheme of [14] leads to a construction of lattices of vectors $\mathbf{x}(u) = (x^1(u), \dots, x^n(u))$ in the Euclidean space parameterized by integer n-dimensional vectors $u = (u^1, \dots, u^n)$, $u^i \in \mathbb{Z}$, and satisfying *planar* and *circular* conditions. These conditions were proposed in [15] as discrete analogs of general n-orthogonal coordinate systems. Note, that planar and circular lattices in three-dimensional space were introduced for the first time in [16], based on the earlier works [17] and [18, 19].

The planar condition means that for each pair of indices i, j the corresponding elementary quadrilateral of the lattice, i.e. the polygon with vertices $\{\mathbf{x}(u), T_i\mathbf{x}(u), T_j\mathbf{x}(u), T_iT_j\mathbf{x}(u)\}$, is flat. Here and below T_i denotes the shift operator in the discrete variable u^i:

$$T_i\mathbf{x}(u^1, \dots, u^i, \dots, u^n) = \mathbf{x}(u^1, \dots, u^i + 1, \dots, u^n).$$

The circular condition means that each of the elementary quadrilatteral can be inscribed into a circle, i.e. that the sum of opposite angles of the polygon equals π.

The main goal of the paper is to prove integrability of the lattices satisfying more rigid constraint than the circular condition. Namely, that for each pair of indices $i \neq j$ the edges of the lattice

$$X_i^+(u) = T_i\mathbf{x}(u) - \mathbf{x}(u), \quad X_j^-(u) = T_j^{-1}\mathbf{x}(u) - \mathbf{x}(u) \tag{1.4}$$

with vertices $\{\mathbf{x}(u), T_i\mathbf{x}(u)\}$ and $\{\mathbf{x}(u), T_j^{-1}\mathbf{x}(u)\}$ are orthogonal to each other, i.e.

$$\langle X_i^+, X_j^- \rangle = 0. \tag{1.5}$$

Here and below $\langle \cdot\,, \cdot \rangle$ stands for Euclidean scalar product of n-dimensional vectors. Note that (1.5) implies that the two opposite angles of the polygon are right and therefore, the corresponding lattice satisfies the circular condition.

Lattices satisfying (1.5) will be called Darboux–Egoroff lattices, because (1.5) implies in particular, that there exists a potential function $\Phi(u)$ such that its discrete derivatives equal to the lengths of edges:

$$\Delta_i\Phi(u) = \langle X_i^+(u), X_i^+(u) \rangle. \tag{1.6}$$

Note that in the continuous case the definition (1.2) of the Darboux–Egoroff metric is equivalent to the existence of function Φ such that $\partial_i\Phi = H_i^2 = \langle \partial_i x, \partial_i x \rangle$.

Constraint (1.5) has naturally arisen from the discretization of the algebro-geometric scheme [14][1]. Moreover, the algebro-geometric construction of Darboux–Egoroff lattices suggests the possibility to introduce discrete analogs of the Lamé coefficients $h_i^{\pm}(u)$. It should be mentioned that unlike the continuous case the definition of such coefficients is local, but not ultra-local and requires scalar products of edges not from the same but also

[1]When the reduction (1.5) was found the authors became aware that the same reduction was proposed in [20].

from the nearest vertices. As a result, it turns out that the proof that these coefficients are well-defined is not evident.

This proof is presented in the next section of the paper. In the third section we derive a full set of equations which describe the discrete Lamé coefficients and prove their integrability. Algebro-geometric construction of the Darboux – Egoroff lattices is presented in the last section.

We use the following notation for the discrete derivatives:

$$\Delta_i F(u) = T_i F(u) - F(u), \qquad \Delta_i^- F(u) = T_i^- F(u) - F(u), \qquad T_i^- = T_i^{-1}.$$

Various objects in the paper has upper indices $+$ and $-$. Sometimes for the sake of brevity we omit the index $+$, i. e. we assume that $F(u) = F^+(u)$. We use also the following discrete analogs of the Leibnitz rule

$$\begin{aligned}\Delta_i(F(u)G(u)) &= \Delta_i F(u)\,G(u) + T_i F(u)\,\Delta_i G(u) = \\ &= \Delta_i F(u)\,G(u) + F(u)\,\Delta_i G(u) + \Delta_i F(u)\,\Delta_i G(u)\end{aligned}$$

and the formula for taking discrete derivatives of a ratio

$$\Delta_i \left(\frac{F(u)}{G(u)}\right) = \frac{\Delta_i F(u)\,G(u) - F(u)\,\Delta_i G(u)}{G(u)\,T_i G(u)}.$$

At the end of the introduction we would like to emphasize that our definition of the discrete Lamé coefficients extensively uses the properties of the Darboux – Egoroff lattices. The problem of a similar description of *intrinsic* geometry of general analogs of flat diagonal metrics is still open. We would like to consider this problem and more general problem of intrinsic geometry on graphs in future.

2 Discrete analogs of the Lamé coefficients

To begin with let us give an equivalent definition of the Darboux – Egoroff lattices. This definition uses only the orthogonality properties of edges of the lattice. Note, that all the lattices in this paper are assumed to be non-degenerate, i. e. for any u the vectors $X_i(u)$, $i = 1, \ldots, n$, are linearly independent.

Lemma 2.1 *The following definitions* $1^0 - 3^0$ *of the Darboux – Egoroff lattice* $\mathbf{x}(u)$ *are equivalent:*

1^0. *The lattice* $\mathbf{x}(u)$ *is planar and for any* $i \neq j$ *equation (1.5) holds:*

$$\langle X_i, X_j^- \rangle = 0.$$

2^0. *For any triple of indices* i, j, m *different from each other we have*

$$\langle X_i^+, X_j^- \rangle = 0, \qquad \langle T_m X_i^+, X_j^- \rangle = 0. \tag{2.1}$$

3^0. *For any $j \neq i$ and any set $\{m_1, m_2, \dots, m_s\}$ of distinct indices which does not contain i it follows*

$$\langle X_i^+, X_j^- \rangle = 0, \qquad \langle \Big(\prod_{k=1}^{s} T_{m_k} \Big) X_i^+, X_j^- \rangle = 0. \tag{2.2}$$

Proof. Since $\mathbf{x}(u)$ is planar we see that the vector $T_m X_i$ is a linear combination of the vectors X_i and X_m. From (1.5) it follows that $T_m X_i$ is orthogonal to X_j^-. Therefore, the equality (2.1) follows from the definition 1^0.

In the reverse case, from (2.1) it follows that the vectors X_i, X_j and $T_i X_j$ are orthogonal to all the vectors X_m^-, $m \neq i, j$. Since the lattice is non-degenerate due to our assumption we obtain that X_i, X_j and $T_i X_j$ belong to the two-dimensional plane, which is orthogonal to all vectors X_m^-, $m \neq i, j$. This implies that the lattice is planar.

We have proved that definitions 1^0 and 2^0 are equivalent. In the same way one can prove the equivalence of definitions 2^0 and 3^0. □

Theorem 2.1 *For any Darboux–Egoroff lattice $\mathbf{x}(u)$ there exists a unique set of functions $h_i^{\pm}(u)$, $i = 1, \dots, n$, normalized by the condition $h_i^+(0, \dots, 0) = 1$, and such that the following equalities for the scalar products of edges of the lattice hold*

$$\langle X_i, X_i \rangle = 2(T_i h_i^+) \cdot h_i^-; \tag{2.3}$$

$$\langle X_i, X_i^- \rangle = -(T_i h_i^+) \cdot (T_i^- h_i^-); \tag{2.4}$$

$$\langle T_j X_i, X_i^- \rangle = -(T_i T_j h_i^+) \cdot (T_i^- h_i^-), \quad i \neq j. \tag{2.5}$$

Note that equalities (2.3−2.5) are invariant under the gauge transformation $h_i^{\pm}(u) \mapsto a_i^{\pm 1} h_i^{\pm}$, where $\{a_i\}$ is a set of arbitrary nonzero constants. Initial conditions $h_i^+(0) = 1$ are chosen to fix the gauge.

Proof. Let us fix some index i. First we shall show that equations (2.3−2.5) imply Pfaff-type system of partial difference equations for the function h_i^+. Using (2.3, 2.4) we obtain two different expressions for h_i^-:

$$h_i^- = \frac{\langle X_i, X_i \rangle}{2 T_i h_i^+} = \frac{\langle T_i X_i, X_i \rangle}{T_i T_i h_i^+}. \tag{2.6}$$

(Here and later we use equality $T_i X_i^- = -X_i$, which follows directly from the definition of $X_i^{\pm}$). It follows from (2.6) that h_i^+ satisfies the difference equation

$$T_i h_i^+ = -2 h_i^+ \frac{\langle X_i, X_i^- \rangle}{\langle X_i^-, X_i^- \rangle} = -2 h_i^+ \cdot A. \tag{2.7}$$

If we replace h_i^- in (2.5) by the middle term of (2.6) we obtain

$$T_j T_i h_i^+ = -2 h_i^+ \frac{\langle T_j X_i, X_i^- \rangle}{\langle X_i^-, X_i^- \rangle} = -2 h_i^+ \cdot B_j. \tag{2.8}$$

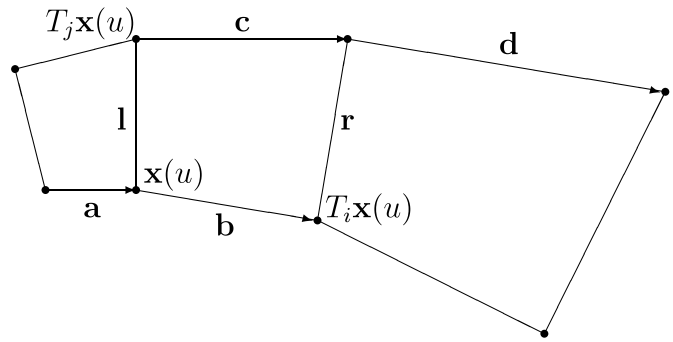

Figure 1:

Let us show that equations (2.7) and (2.8) are compatible. The compatibility conditions are equivalent to the equation $T_jA \cdot T_i^{-1}B_j = B_j \cdot T_i^{-1}A$, that is

$$T_j\left(\frac{\langle X_i, X_i^-\rangle}{\langle X_i^-, X_i^-\rangle}\right) \cdot T_i^{-1}\left(\frac{\langle T_jX_i, X_i^-\rangle}{\langle X_i^-, X_i^-\rangle}\right) = \frac{\langle T_jX_i, X_i^-\rangle}{\langle X_i^-, X_i^-\rangle} \cdot T_i^{-1}\left(\frac{\langle X_i, X_i^-\rangle}{\langle X_i^-, X_i^-\rangle}\right). \tag{2.9}$$

Applying T_i to the both sides we get

$$\frac{\langle T_jT_iX_i, T_jX_i\rangle \cdot \langle T_jX_i, X_i^-\rangle}{|T_jX_i|^2} = \frac{\langle T_jT_iX_i, X_i\rangle \cdot \langle X_i, X_i^-\rangle}{|X_i|^2}. \tag{2.10}$$

Let us denote the vectors X_i^-, X_i, T_jX_i, $T_iT_jX_i$ by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively (see fig. 1). Consider orthogonal projections $\mathbf{a}'$, $\mathbf{d}'$ of the vectors $\mathbf{a}$, $\mathbf{d}$ on the plane spanned by the edges X_i and X_j (the plane of the central plaquet in fig. 1). Replacing $\mathbf{a}$, $\mathbf{d}$ by $\mathbf{a}'$, $\mathbf{d}'$ in equation (2.10), we change no scalar product in it. Dividing both sides by $|\mathbf{a}'||\mathbf{d}'|$, we obtain

$$\frac{\langle \mathbf{a}', \mathbf{c}\rangle\,\langle \mathbf{c}, \mathbf{d}'\rangle}{|\mathbf{a}'||\mathbf{c}|^2|\mathbf{d}'|} = \cos(\mathbf{a}', \mathbf{c})\cos(\mathbf{c}, \mathbf{d}') = \cos(\mathbf{a}', \mathbf{b})\cos(\mathbf{b}, \mathbf{d}') = \frac{\langle \mathbf{a}', \mathbf{b}\rangle\,\langle \mathbf{b}, \mathbf{d}'\rangle}{|\mathbf{a}'||\mathbf{b}|^2|\mathbf{d}'|}.$$

It follows from the definition 1^0 of the Darboux–Egoroff lattice that the vectors $\mathbf{a}'$ and $\mathbf{c}$ are both orthogonal to the vector $\mathbf{l} = X_j$. Similarly, $\mathbf{b}$ and $\mathbf{d}'$ are both orthogonal to $\mathbf{r} = T_iX_j$. Consequently,

$$\cos(\mathbf{a}', \mathbf{c}) = \cos(\mathbf{b}, \mathbf{d}') = 1, \qquad \cos(\mathbf{a}', \mathbf{b}) = \cos(\mathbf{c}, \mathbf{d}').$$

The latter equalities prove compatiblity of (2.7) and (2.8).

Let us consider now two equations (2.8) for indices j and k such that $i \neq j \neq k$. In terms of the coefficients B_j and B_k of these equations their compatibility is equivalent to the equality $B_k \cdot T_kB_j = B_j \cdot T_jB_k$, which can be written as follows

$$\frac{\langle T_kX_i^-, T_i^-X_i^-\rangle \cdot \langle T_kT_jX_i, T_kX_i^-\rangle}{|T_kX_i^-|^2} = \frac{\langle T_jX_i^-, T_i^-X_i^-\rangle \cdot \langle T_jT_kX_i, T_jX_i^-\rangle}{|T_jX_i^-|^2}\ . \tag{2.11}$$

Let us introduce the following notations: $\mathbf{a} = T_i^-X_i^-$, $\mathbf{b} = T_kX_i^-$, $\mathbf{c} = T_jX_i^-$, $\mathbf{d} = T_kT_jX_i$ (see fig. 2). Let $\mathcal{C}$ be the three-dimensional cube with edges $T_i^-X_i$, $T_i^-X_j$ and $T_i^-X_k$ (the central cube in fig. 2). Let $\mathbf{a}'$ and $\mathbf{d}'$ be orthogonal projections of the vectors $\mathbf{a}$ and $\mathbf{d}$ on the three-dimensional space spanned by the edges of $\mathcal{C}$. One can replace $\mathbf{a}$, $\mathbf{d}$ by $\mathbf{a}'$, $\mathbf{d}'$,

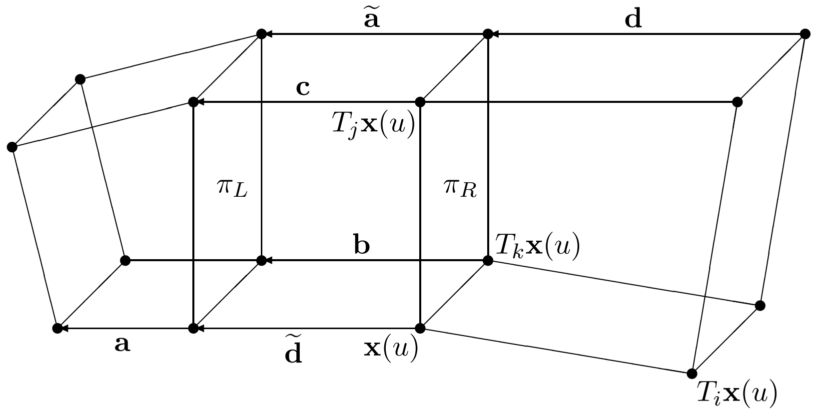

Figure 2:

respectively, in (2.11) without changing scalar products in it. Moreover, equality (2.11) remains valid if the vectors $\mathbf{a}'$ and $\mathbf{d}'$ are multiplied by any non-zero constants.

From the orthogonality properties of the Darboux–Egoroff lattices it follows that $\mathbf{a}'$ is proportional to $\widetilde{\mathbf{a}} = T_j T_k X_i^-$, because both of them are orthogonal to the plane π_L, spanned by $T_i^- X_j$ and $T_i^- X_k$ (left face of $\mathcal{C}$). In the same way, the vector $\mathbf{d}'$ is proportional to $\widetilde{\mathbf{d}} = X_i^-$, because they are orthogonal to the plane π_R, spanned by X_j and X_k (right face of $\mathcal{C}$). Hence, in (2.11) we may replace $\mathbf{a}$ and $\mathbf{d}$ by the vectors $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{d}}$. If we divide it by $|\widetilde{\mathbf{a}}||\widetilde{\mathbf{d}}|$, it takes the form

$$\frac{\langle \widetilde{\mathbf{a}}, \mathbf{c}\rangle \langle \mathbf{c}, \widetilde{\mathbf{d}}\rangle}{|\widetilde{\mathbf{a}}||\mathbf{c}|^2|\widetilde{\mathbf{d}}|} = \cos(\widetilde{\mathbf{a}}, \mathbf{c}) \cos(\mathbf{c}, \widetilde{\mathbf{d}}) = \cos(\widetilde{\mathbf{a}}, \mathbf{b}) \cos(\mathbf{b}, \widetilde{\mathbf{d}}) = \frac{\langle \widetilde{\mathbf{a}}, \mathbf{b}\rangle \langle \mathbf{b}, \widetilde{\mathbf{d}}\rangle}{|\widetilde{\mathbf{a}}||\mathbf{b}|^2|\widetilde{\mathbf{d}}|} .$$

To prove the last equality we use the following result, which is well-known in spherical geometry:

Lemma 2.2 *Let one of the dihedral angles of a three-edged piramid be a right angle. Let α_1, α_2 and α_3 be the plane angles at the vertex, and α_3 corresponds to the face opposite to the right dihedral angle. Then*

$$\cos\alpha_3 = \cos\alpha_1 \cos\alpha_2.$$

The upper face of the central cube $\mathcal{C}$ (the face with edges $T_j T_i^- X_i$ and $T_j T_i^- X_k$) is perpendicular to the front face (the face with edges $T_i^- X_i$ and $T_i^- X_j$). At the same time, the lower face of $\mathcal{C}$ (the face with edges $T_i^- X_i$ and $T_i^- X_k$) is perpendicular to the rear face (the face with edges $T_k T_i^- X_i$ and $T_k T_i^- X_j$). Therefore, the above presented lemma implies

$$\cos(\widetilde{\mathbf{a}}, \mathbf{b}) \cos(\mathbf{b}, \widetilde{\mathbf{d}}) = \cos(\widetilde{\mathbf{a}}, \widetilde{\mathbf{d}}) = \cos(\widetilde{\mathbf{a}}, \mathbf{c}) \cos(\mathbf{c}, \widetilde{\mathbf{d}}).$$

Hence, the relation (2.11), and, consequently, compatibility of (2.5) for distinct i and j are proved.

Equations (2.7) and (2.8) uniquely define $h_i^+(u)$, if we fix the normalization $h_i^+(0) = 1$. Note, that the functions $h_i^\pm(u)$ with given i are defined independently of $h_j^\pm(u)$ with other indices. Theorem is proved. $\square$

The definition of $h_i^\pm$ uses a minimal set of scalar products. It turns out that this set is *complete* in the following sense. All the other scalar products of the edges can be expressed through the scalar products from the minimal set, and therefore, through the functions $h_i^\pm$, $i = 1, \ldots, n$. Let us present some of the corresponding formulae which will be used later.

Lemma 2.3 *Let $h_i^\pm(u)$, $i = 1, \ldots, n$, be the functions defined by Theorem 2.1. Then*

$$\langle X_i, X_j \rangle = -2 \frac{(\Delta_i T_j h_j^+)\,(T_i h_i^+)\,h_j^-}{T_i T_j h_i^+}, \qquad i \neq j, \tag{2.12}$$

$$\langle X_i^-, X_j^- \rangle = -2 \frac{(\Delta_i^- T_j^- h_j^-)\,(T_i^- h_i^-)\,h_j^+}{T_i^- T_j^- h_i^-}, \qquad i \neq j. \tag{2.13}$$

Proof. Due to planar property of the lattice, $T_j X_i$ is a linear combination of X_i and X_j, i. e.

$$T_j X_i = \alpha X_i + \beta X_j. \tag{2.14}$$

Let us find the coefficients of the sum. Taking scalar product of (2.14) with X_i^-, we get a relation which implies

$$\alpha = \frac{\langle T_j X_i, X_i^- \rangle}{\langle X_i, X_i^- \rangle} = \frac{T_i T_j h_i^+}{T_i h_i^+}.$$

If we take scalar product of (2.14) with X_j^- and use the equality $T_j X_i = T_i X_j + X_i - X_j$, then we get the formula

$$\beta = \frac{\langle T_i X_j, X_j^- \rangle - \langle X_j, X_j^- \rangle}{\langle X_j, X_j^- \rangle} = \frac{\Delta_i T_j h_j^+}{T_j h_j^+}.$$

Let us multiply (2.14) by the vector X_j, which is orthogonal to $T_j X_i$. We get $\alpha \langle X_i, X_j \rangle + \beta \langle X_j, X_j \rangle = 0$, which proves (2.12):

$$\langle X_i, X_j \rangle = \left(\frac{\langle T_i X_j, X_j^- \rangle}{\langle X_j, X_j^- \rangle} - 1 \right) \cdot \frac{\langle X_i, X_i^- \rangle}{\langle T_j X_i, X_i^- \rangle} \cdot |X_j^2| = -2 \frac{(\Delta_i T_j h_j^+)\,(T_i h_i^+)\,h_j^-}{T_i T_j h_i^+}.$$

The second formula (2.13) can be proved in the same way. $\square$

It turns out that the scalar products $\langle X_i^\pm, X_j^\pm \rangle$ can be expressed through the Lamé coefficients by different formulae.

Lemma 2.4 *For any Darboux – Egoroff lattice the following formulae*

$$\langle X_i, X_j \rangle = -2 (T_i h_i^+)(\Delta_j h_i^-), \tag{2.15}$$

$$\langle X_i^-, X_j^- \rangle = -2 (T_i^- h_i^-)(\Delta_j^- h_i^+), \tag{2.16}$$

are valid. Here $h_i^\pm(u)$, $i = 1, \ldots, n$, are the functions defined by Theorem 2.1.

Proof. Let us express $\langle X_i, X_j\rangle$ through the scalar products from the minimal set $(2.3-2.5)$. From the definition of Darboux–Egoroff lattices it follows that $0 = \langle X_i, T_i X_j\rangle = \langle X_i, X_j\rangle - \langle X_i, X_i\rangle + \langle X_i, T_j X_i\rangle$. In order to express the last term in the right hand side of this equality through the minimal set, note that $T_j X_i$ is proportional to the orthogonal projection $\mathbf{d}$ of the vector X_i^- on the plane spanned by X_i and X_j. Therefore,

$$\langle X_i, T_j X_i\rangle = \langle X_i, X_i^-\rangle \frac{|T_j X_i|}{|\mathbf{d}|} = \langle X_i, X_i^-\rangle \frac{\langle T_j X_i, T_j X_i\rangle}{\langle T_j X_i, X_i^-\rangle} \ .$$

We obtain

$$\langle X_i, X_j\rangle = |X_i|^2 - \langle X_i, X_i^-\rangle \frac{|T_j X_i|^2}{\langle T_j X_i, X_i^-\rangle}.$$

Direct substitution of $(2.3-2.5)$ into the latter formula gives (2.15). Formula (2.16) can be proved in the same way. $\square$

Lemmas 2.3 and 2.4 imply constraints for the functions $h_i^{\pm}(u)$. Using the symmetry of scalar product we get from (2.15) and (2.16) the equations:

$$\frac{\Delta_i h_j^-}{T_i h_i^+} = \frac{\Delta_j h_i^-}{T_j h_j^+}, \qquad \frac{\Delta_i^- h_j^+}{T_i^- h_i^-} = \frac{\Delta_j^- h_i^+}{T_j^- h_j^-}, \qquad i \neq j. \tag{2.17}$$

From (2.12) and (2.15) it follows that

$$\beta_{ij}^+ = -\beta_{ji}^-, \tag{2.18}$$

where

$$\beta_{ij}^+ = \frac{\Delta_i h_j}{T_i h_i}, \qquad \beta_{ji}^- = \frac{\Delta_j^- h_i^-}{T_j^- h_j^-}, \qquad i \neq j. \tag{2.19}$$

The functions $\beta_{ij}^{\pm}(u)$ are discrete analogs of the rotation coefficients (1.3) of flat diagonal metrics. Below we will often omit upper index $+$ for the functions $\beta_{ij}^+(u)$.

Relations (2.17) and (2.18) are discrete analogs of symmetry conditions for Darboux–Egoroff metrics. For $n \geq 3$, there is another analog of symmetry conditions which can be written in terms of the functions β_{ij} only.

Lemma 2.5 *Let $\beta_{ij}(u)$ be the discrete rotation coefficients of Darboux–Egoroff lattice. Then for each triple of pairwise distinct indices i, j, k the equation*

$$(T_k\beta_{ik})\,(T_i\beta_{ji})\,(T_j\beta_{kj}) = (T_k\beta_{jk})\,(T_j\beta_{ij})\,(T_i\beta_{ki}). \tag{2.20}$$

is fulfilled.

Proof. From (2.12) and definition of the discrete rotation coefficients it follows that

$$\langle X_i, X_k\rangle = (T_k\beta_{ik}) \cdot \frac{h_k^-}{T_k h_k^+} \ .$$

The desired relation can be obtained if we divide the equality $\langle X_i, X_k\rangle\langle X_j, X_i\rangle\langle X_k, X_j\rangle = \langle X_j, X_k\rangle\langle X_i, X_j\rangle\langle X_k, X_i\rangle$ by symmetric expression

$$\frac{h_i^-}{T_i h_i^+}\cdot\frac{h_j^-}{T_j h_j^+}\cdot\frac{h_k^-}{T_k h_k^+}\,.$$

□

Of course, constraints (2.17) and (2.18) do not form a complete set of equations on the discrete Lamé coefficients. The main goal of the next section is to get such a set of equations.

3 Discrete analogs of the Lamé equations

Just as in the continuous case the discrete analog of the Lamé equations can be obtained as compatibility conditions of a system of linear equations satisfied by the edges $X_i^{\pm}(u)$ of the lattice. These linear equations become simplier after an appropriate rescaling of the edges.

For each Darboux – Egoroff lattice let us define vectors $Y_i^{\pm}(u)$ by the formulae

$$Y_i(u) = \frac{1}{T_i h_i^+(u)} X_i(u), \qquad Y_i^-(u) = \frac{1}{T_i^- h_i^-(u)} X_i^-(u), \tag{3.1}$$

where $h_i^{\pm}(u)$ are the functions constructed in Theorem 2.1. Equation (2.4) implies that these vectors form biorthogonal system:

$$\langle Y_i, Y_j^-\rangle = -\delta_{ij}. \tag{3.2}$$

For further needs we write down some other scalar products which can be obtained from the relations $(2.3 - 2.5)$:

$$\langle Y_i, Y_i\rangle = 2\frac{h_i^-}{T_i h_i^+}, \qquad \langle T_i Y_i, Y_i\rangle = \frac{h_i^-}{T_i h_i^+}, \qquad \langle T_j Y_i, Y_i^-\rangle = -1, \tag{3.3}$$

and $(2.12, 2.15)$:

$$\langle Y_i, Y_j\rangle = -2\frac{(\Delta_i T_j h_j^+)\, h_j^-}{(T_i T_j h_i^+)\,(T_j h_j^+)} = -2\frac{\Delta_j h_i^-}{T_j h_j^+}\ . \tag{3.4}$$

(The analogous formula can be written for $\langle Y_i^-, Y_j^-\rangle$.)

Theorem 3.1 *For any Darboux – Egoroff lattice the vectors $Y_i^{\pm}(u)$ satisfy the system of equations:*

$$\Delta_j^{\pm} Y_i^{\pm} = (T_j^{\pm}\beta_{ij}^{\pm}) Y_j^{\pm}, \qquad i \neq j, \tag{3.5}$$

$$\Delta_i^{\pm} Y_i^{\pm} = b_i^{\pm} Y_i^{\pm} - \sum_{j\neq i}\beta_{ij}^{\pm} Y_j^{\pm}, \tag{3.6}$$

where the coefficients $\beta_{ij}^{\pm}(u)$ are defined by (2.19), and the coefficients $b_i^{\pm}(u)$ are given by formulae

$$b_i^+ = -\frac{1}{2} - \sum_{j\neq i} \beta_{ij}(T_i\beta_{ji}), \qquad b_i^- = -\frac{1}{2} - \sum_{j\neq i} \beta_{ij}^-(T_i^-\beta_{ji}^-) \ . \tag{3.7}$$

Proof. Since the lattice is planar, the vector $\Delta_j Y_i$ $(i \neq j)$ is a linear combination of Y_i and Y_j, i. e. $\Delta_j Y_i = a_{ij} Y_i + d_{ij} Y_j$. To find the coefficient a_{ij}, we take scalar product of this equation with the vector Y_i^-. Using (3.2) and (3.3), we obtain $a_{ij} = \langle T_j Y_i - Y_i, Y_i^- \rangle = 0$. Therefore, $\Delta_j Y_i = d_{ij} Y_j$. Taking the scalar product of this equation and the vector Y_j we obtain

$$\langle \Delta_j Y_i, Y_j \rangle = d_{ij} \langle Y_j, Y_j \rangle = 2 d_{ij} \, \frac{h_j^-}{T_j h_j^+} \ .$$

On the other hand, (3.4) implies

$$\langle \Delta_j Y_i, Y_j \rangle = \langle T_j Y_i, Y_j \rangle - \langle Y_i, Y_j \rangle = -\langle Y_i, Y_j \rangle = 2 T_j \left(\frac{\Delta_i h_j^+}{T_i h_i^+} \right) \frac{h_j^-}{T_j h_j^+} \ .$$

Comparison of the right hand sides of the two last equations yields $d_{ij} = T_j \beta_{ij}$. In the same way we can prove (3.5) for the vectors Y_i^-.

We will show now that $Y_i(u)$ satisfy (3.6). For any u the vectors $Y_1(u), \ldots, Y_n(u)$ form a basis of $\mathbb{R}^n$ (since the lattice is non-degenerate). Thus, there exists a decomposition

$$\Delta_i Y_i = b_i Y_i + \sum_{j\neq i} c_{ij} Y_j. \tag{3.8}$$

Coefficient c_{ij} can be found by taking the scalar product of this equation with the vector Y_j^-. We get $\langle \Delta_i Y_i, Y_j^- \rangle = c_{ij} \langle Y_j Y_j^- \rangle = -c_{ij}$. On the other hand

$$\langle \Delta_i Y_i, Y_j^- \rangle = \Delta_i \langle Y_i, Y_j^- \rangle - \langle T_i Y_i, \Delta_i Y_j^- \rangle = T_i \langle Y_i, \Delta_i^- Y_j^- \rangle.$$

Using (3.5), which has been just proved, we obtain

$$-c_{ij} = \langle \Delta_i Y_i, Y_j^- \rangle = T_i \langle Y_i, (T_i^- \beta_{ji}^-) Y_i^- \rangle = -\beta_{ji}^- = \beta_{ij}.$$

To determine $b_i(u)$, we multiply (3.8) by Y_i:

$$\langle \Delta_i Y_i, Y_i \rangle = b_i \langle Y_i, Y_i \rangle - \sum_{i\neq j} \beta_{ij} \langle Y_i, Y_j \rangle. \tag{3.9}$$

Plugging (3.3) and (3.4) into equation (3.9) we establish formula (3.7) for $b_i(u)$. The second equation in (3.6) can be obtained in the same manner. $\square$

The compatibility conditions of the linear system (3.5, 3.6) are discrete analogs of the Lamé equations on the rotation coefficients of the Darboux – Egoroff metrics. Our next goal is to prove the inverse statement: any solution of these equations uniquely defines the rotation coefficients of some Darboux – Egoroff lattice.

Theorem 3.2 *Let functions $\beta_{ij}(u)$, $i \neq j$, $i, j \in \{1, \ldots, n\}$ $(n \geq 3)$, satisfy the relations (2.20)*

$$(T_k\beta_{ik})\,(T_i\beta_{ji})\,(T_j\beta_{kj}) = (T_k\beta_{jk})\,(T_j\beta_{ij})\,(T_i\beta_{ki})$$

and the equations

$$\Delta_k\beta_{ij} = (T_k\beta_{ik})\beta_{kj}, \qquad i \neq j \neq k, \tag{3.10}$$

$$\Delta_j v_i = \Delta_i(\beta_{ji}\,T_j\beta_{ij}), \qquad i \neq j, \tag{3.11}$$

$$\Delta_i\Delta_j\beta_{ij} + \Delta_i\beta_{ij} + \Delta_j\beta_{ij} = T_j\beta_{ij}(T_jv_i - v_j) - \sum_{k\neq i,j} T_j(\beta_{ik}\beta_{kj}), \quad i \neq j, \tag{3.12}$$

where $v_i(u)$ is defined by

$$v_i = -\sum_{p\neq i} \beta_{ip}(T_i\beta_{pi})\,. \tag{3.13}$$

Then there exists a unique constant η such that the functions

$$\widetilde{\beta}_{ij}(u) = (2\eta)^{u_j-u_i-1}\beta_{ij}(u) \tag{3.14}$$

are the rotation coefficients of some Darboux – Egoroff lattice.

Equations $(3.10 - 3.12)$ are equivalent to compatibility conditions of the linear system

$$\Delta_j\Psi_i = (T_j\beta_{ij})\Psi_j, \qquad i \neq j, \tag{3.15}$$

$$\Delta_i\Psi_i = (\mu + v_i)\Psi_i - \sum_{j\neq i} \beta_{ij}\Psi_j \tag{3.16}$$

where μ is an arbitrary complex constant. The transformation (3.14) sends solutions of $(3.10 - 3.13)$ to the solutions of the same system and preserves the relations (2.20), since it corresponds to the gauge transformation

$$Y_i = \widetilde{\Psi}_i = (2\eta)^{-u_i}\Psi_i \tag{3.17}$$

of the linear system $(3.15, 3.16)$. Therefore, the theorem gives the necessary and sufficient conditions for the functions $\beta_{ij}(u)$ to be the rotation coefficients of some Darboux – Egoroff lattice, up to gauge transformation.

Proof. Unlike the continuous case, the problem of reconstruction the Darboux – Egoroff lattice from the functions β_{ij} requires a highly nontrivial choice of the initial data for the solutions of system $(3.5, 3.6)$. We fix these data by defining the matrix of scalar products of the corresponding vectors. In the next two lemmas we shall construct functions $\beta^*_{ij}(u)$ by means of which we shall later determine these scalar products.

Lemma 3.1 *Let functions $\beta_{ij}(u)$ satisfy the conditions (2.20) and (3.10−3.13). Then there exists a unique solution $f_i(u)$, $i = 1, \ldots, n$, of the system*

$$\Delta_k f_i = -(T_k\beta_{ik})\,(T_i\beta_{ki})f_i, \qquad i \neq k, \tag{3.18}$$

$$(T_i\beta_{ki})f_i = (T_k\beta_{ik})f_k, \qquad i \neq k, \tag{3.19}$$

normalized by the condition $f_1(0) = 1$.

Proof. System (3.18, 3.19) is over-determined, but we will show that it is equivalent to a system of n compatible equations on the function $f_1(u)$. Note, that equations (3.19) are compatible due to (2.20).

First of all, let us prove the compatibility of a pair of equations (3.18) with distinct values of the index k, say p and q. It suffices to show that the following expression for $\Delta_p\Delta_q f_i$ is symmetric with respect to p and q:

$$\Delta_p\Delta_q f_i = -\Big[\Delta_p(T_q\beta_{iq}\cdot T_i\beta_{qi}) - T_p(T_q\beta_{iq}\cdot T_i\beta_{qi})\,(T_p\beta_{ip}\cdot T_i\beta_{pi})\Big]f_i.$$

Indeed, due to (3.10) the expression in brackets is equal to

$$\begin{aligned}[\ldots] &= T_q(T_p\beta_{ip}\cdot\beta_{pq})(T_i\beta_{qi}) + (T_pT_q\beta_{iq})\,T_i(T_p\beta_{qp}\cdot\beta_{pi}) - (T_pT_q\beta_{iq})\,T_p(T_i\beta_{qi}\cdot\beta_{ip})\,(T_i\beta_{pi}) = \\ &= (T_qT_p\beta_{ip})(T_q\beta_{pq})(T_i\beta_{qi}) + (T_pT_q\beta_{iq})\,[T_iT_p\beta_{qp} - T_p\Delta_i\beta_{qp}]\,(T_i\beta_{pi}).\end{aligned}$$

Obviously, the last formula has the symmetry needed.

Next, let us consider two equations (3.18) with distinct values of the index i (denote them again by p and q). We shall show that one of them implies the other provided f_p and f_q satisfy (3.19). Indeed,

$$\begin{aligned}\Delta_k f_p &= \Delta_k\left(\frac{T_q\beta_{pq}}{T_p\beta_{qp}}f_q\right) = \Delta_k\left(\frac{T_q\beta_{pq}}{T_p\beta_{qp}}\right)f_q - T_k\left(\frac{T_q\beta_{pq}}{T_p\beta_{qp}}\right)(T_k\beta_{qk})\,(T_q\beta_{kq})\,f_q = \\ &= \left(T_q(T_k\beta_{pk}\cdot\beta_{kq}) - \frac{T_q\beta_{pq}}{T_p\beta_{qp}}T_p(T_k\beta_{qk}\cdot\beta_{kp}) - T_k(T_q\beta_{pq}\cdot\beta_{qk})\,(T_q\beta_{kq})\right)\frac{f_q}{T_kT_p\beta_{qp}} = \\ &= \left((T_q\beta_{kq})(T_k\beta_{pk}) - \frac{T_q\beta_{pq}}{T_p\beta_{qp}}(T_pT_k\beta_{qk})(T_p\beta_{kp})\right)\frac{f_q}{T_kT_p\beta_{qp}}.\end{aligned}$$

Now, (2.20) implies $(T_q\beta_{kq})(T_k\beta_{pk})f_q = (T_k\beta_{qk})(T_p\beta_{kp})f_p$. Finally, we have

$$\Delta_k f_p = \Big((T_k\beta_{qk})(T_p\beta_{kp}) - (T_pT_k\beta_{qk})(T_p\beta_{kp})\Big)\frac{f_p}{T_kT_p\beta_{qp}} = -(T_k\beta_{pk})\,(T_p\beta_{kp})\,f_p,$$

which is the formula we need.

Now let us consider equation (3.18) for $\Delta_1 f_i$, $i > 1$. Plugging into both sides of this equation the expression $f_i = \alpha_{i1}f_1$, with $\alpha_{i1} = (T_1\beta_{i1})/(T_i\beta_{1i})$, we obtain the equation of the type $\Delta_1 f_1 = \mathcal{F}_i(u)f_1$, where the function $\mathcal{F}_i(u)$ is a rational combination of $\beta_{1i}(u)$ and $\beta_{i1}(u)$. In fact, the function $\mathcal{F}_i(u)$ and, therefore, the equation on $\Delta_1 f_1$ do not depend on the index i. In order to prove that, we choose index $j \neq i$ and consider equation (3.18) for $\Delta_1 f_j$. It was shown at the second step of this proof that the substitution $f_j = \alpha_{ji} f_i$ leads to (3.18) for $\Delta_1 f_i$. On the other hand, substitution $f_j = \alpha_{j1} f_1$ gives the equation $\Delta_1 f_1 = \mathcal{F}_j(u) f_1$. Therefore, the equation $\Delta_1 f_1 = \mathcal{F}_i(u) f_1$ can be obtained from $\Delta_1 f_1 = \mathcal{F}_j(u) f_1$ by substitution $f_1 = (\alpha_{j1})^{-1} \alpha_{ji} \alpha_{i1} f_1$. But due to (2.20), $(\alpha_{j1})^{-1} \alpha_{ji} \alpha_{i1} = \alpha_{1j} \alpha_{ji} \alpha_{i1} = 1$, and we conclude that $\mathcal{F}_i(u) = \mathcal{F}_j(u) = \mathcal{F}(u)$.

Thus we obtain n equations on the function $f_1(u)$, namely, the equations

$$\Delta_1 f_1 = \mathcal{F}(u) f_1, \qquad \Delta_i f_1 = -(T_1 \beta_{i1})(T_i \beta_{1i}) f_1, \quad i = 2, \ldots, n. \tag{3.20}$$

It was already shown that the equations from the second group are compatible. To establish compatibility of the equations $\Delta_1 f_1 = \mathcal{F}(u) f_1$ and $\Delta_i f_1 = -(T_1 \beta_{i1})(T_i \beta_{1i}) f_1$, $i > 1$, it suffices to note that they are gauge equivalent to compatible equations $\Delta_1 f_j = -(T_1 \beta_{j1})(T_j \beta_{1j}) f_j$ and $\Delta_i f_j = -(T_j \beta_{ij})(T_i \beta_{ji}) f_j$ for any index j not equal to 1 and i.

Summarizing all the facts, we see that the solution of system (3.18−3.19) can be obtained as follows. First, we define the function $f_1(u)$ from compatible system (3.20) with the initial condition $f_1(0) = 1$. Then, using (3.19), we obtain all the other functions $f_i(u)$, $i = 2, \ldots, n$. Then, as it was shown at the second step of the proof, all the equations of the system are fulfilled. The proof of lemma is now complete. □

Let us define $\beta^*_{ii}(u)$ by $\beta^*_{ii}(u) = f_i(u)$, $i = 1, \ldots, n$, where $f_i(u)$ were constructed in the previous lemma. For $i \neq j$ we define β^*_{ij} by the formula

$$-\beta^*_{ij} = (T_j \beta_{ij})\, \beta^*_{jj}, \qquad j \neq i. \tag{3.21}$$

Note, that (3.19) implies $\beta^*_{ij}(u) = \beta^*_{ji}(u)$.

Lemma 3.2 *The above-defined functions $\beta^*_{ij}(u)$, $i, j \in \{1, \ldots, n\}$, satisfy the following system of equations:*

$$\Delta_j \beta^*_{ik} = (T_j \beta_{ij})\, \beta^*_{jk}, \qquad j \neq i, k, \tag{3.22}$$

$$\Delta_i \beta^*_{ij} = (v_i - 1) \beta^*_{ij} - \sum_{k \neq i} \beta_{ik} \beta^*_{kj}, \qquad i \neq j. \tag{3.23}$$

Proof. Lemma is proved by straightforward calculations. Further in this proof we assume that $i \neq j \neq k$. Note, that lemma 3.1 and the definition of β^*_{ii} imply:

$$\Delta_j \beta^*_{ii} = -(T_j \beta_{ij})(T_i \beta_{ji}) \beta^*_{ii} = (T_j \beta_{ij}) \beta^*_{ji}.$$

Now, by definition $\Delta_j \beta^*_{ki} = -\Delta_j (T_i \beta_{ki} \cdot \beta^*_{ii}) = -(T_j T_i \beta_{ki}) \Delta_j \beta^*_{ii} - \Delta_j (T_i \beta_{ki}) \beta^*_{ii}$. Using the above established equalities and (3.10), we get

$$\Delta_j \beta^*_{ki} = (T_j T_i \beta_{ki})(T_j \beta_{ij})(T_i \beta_{ji}) \beta^*_{ii} - (T_j T_i \beta_{kj})(T_i \beta_{ji}) \beta^*_{ii} =$$

$$= (\Delta_i T_j \beta_{kj})(T_i \beta_{ji}) \beta^*_{ii} - (T_j T_i \beta_{kj})(T_i \beta_{ji}) \beta^*_{ii} = -(T_j \beta_{kj})(T_i \beta_{ji}) \beta^*_{ii} = (T_j \beta_{kj}) \beta^*_{ji}.$$

Thus, we establish equations (3.22). Let us prove (3.23). Using (3.12), we obtain

$$\begin{aligned}
\Delta_j\beta^*_{ji} &= -\Delta_j(T_i\beta_{ji}\cdot\beta^*_{ii}) = -(T_jT_i\beta_{ji})\Delta_j\beta^*_{ii} - (\Delta_jT_i\beta_{ji})\beta^*_{ii} = \\
&= (T_jT_i\beta_{ji})(T_j\beta_{ij})(T_i\beta_{ji})\beta^*_{ii} - \Big[T_i\beta_{ji}(-v_i+T_iv_j) - \sum_{k\neq i,j} T_i(\beta_{jk}\beta_{ki}) - \Delta_i\beta_{ji}\Big]\beta^*_{ii} = \\
&= \Big(T_i\beta_{ji}[v_i - T_iv_j + (T_jT_i\beta_{ji})T_j\beta_{ij}] + \sum_{k\neq i,j} T_i(\beta_{jk}\beta_{ki}) + \Delta_i\beta_{ji}\Big)\beta^*_{ii} = \\
&= -\beta^*_{ji}(v_i - T_iv_j + T_j(T_i\beta_{ji}\cdot\beta_{ij})) - \sum_{k\neq i,j}(T_i\beta_{jk})\beta^*_{ki} - \beta^*_{ji} - \beta_{ji}\beta^*_{ii}.
\end{aligned}$$

Plugging into the last formula $\Delta+1$ instead of the shift operators T and applying (3.11) we have

$$\begin{aligned}
\Delta_j\beta^*_{ji} &= -\beta^*_{ji}\Big(v_i - v_j + (T_i\beta_{ji})\beta_{ij}\Big) - \sum_{k\neq j}\beta_{jk}\beta^*_{ki} - \sum_{k\neq i,j}(\Delta_i\beta_{jk})\beta^*_{ki} - \beta^*_{ji} = \\
&= -\beta^*_{ji}(v_i + 1 - v_j + (T_i\beta_{ji})\beta_{ij}) - \sum_{k\neq j}\beta_{jk}\beta^*_{ki} - \sum_{k\neq i,j}(T_i\beta_{ji}\cdot\beta_{ik})\beta^*_{ki} = \\
&= \sum_{k\neq i,j}(T_i\beta_{ki}\cdot\beta_{ik})\beta^*_{ji} - \sum_{k\neq i,j}(T_i\beta_{ji}\cdot\beta_{ik})\beta^*_{ki} - \sum_{k\neq j}\beta_{jk}\beta^*_{ki} + (v_j-1)\beta^*_{ji} = \\
&= -\sum_{k\neq i,j}\left[(T_i\beta_{ki}\cdot\beta_{ik})(T_i\beta_{ji})\beta^*_{ii} - (T_i\beta_{ji}\cdot\beta_{ik})(T_i\beta_{ki})\beta^*_{ii}\right] - \sum_{k\neq j}\beta_{jk}\beta^*_{ki} + (v_j-1)\beta^*_{ji} = \\
&= -\sum_{k\neq j}\beta_{jk}\beta^*_{ki} + (v_j-1)\beta^*_{ji}.
\end{aligned}$$

Lemma is proved. □

The rotation coefficients of the Darboux−Egoroff metric satisfy the following condition: $\sum_k \partial_k\beta_{ij} = 0$. Our next goal is to prove the following discrete analog of this property.

Lemma 3.3 *Let β_{ij} be a solution of equations (3.10− 3.12), which satisfies (2.19). Then it also satisfies the following monodromy property:*

$$\widetilde{T}\beta_{ij}(u) = \beta_{ij}(u), \qquad \widetilde{T} = \prod_{k=1}^{n} T_k. \tag{3.24}$$

Proof. First of all, we prove by induction that if functions $\Psi_i(u)$ satisfy (3.15) then for any set I of pairwise distinct indices $I = \{i_1,\dots,i_s\}$ the following equation holds:

$$(T_I - 1)\Psi_j = \sum_{i\in I}(T_I\beta_{ji})\Psi_i, \quad j\notin I, \qquad T_I = T_{i_1}\dots T_{i_s}. \tag{3.25}$$

Indeed, if (3.25) is established for any set I of cardinality s, then for any $k\neq j$, $k\notin I$, we have

$$\begin{aligned}
(T_kT_I-1)\Psi_j &= T_I\Delta_k\Psi_j+(T_I-1)\Psi_j = T_I\big[(T_k\beta_{jk})\Psi_k\big]+\sum_{i\in I}(T_I\beta_{ji})\Psi_i = \\
&= (T_IT_k\beta_{jk})\Psi_k+(T_IT_k\beta_{jk})\big[(T_I-1)\Psi_k\big]+\sum_{i\in I}(T_I\beta_{ji})\Psi_i = \\
&= (T_IT_k\beta_{jk})\Psi_k+\sum_{i\in I}\big[T_I(T_k\beta_{jk}\beta_{ji}+\beta_{ji})\big]\Psi_j = \\
&= (T_IT_k\beta_{jk})\Psi_k+\sum_{i\in I}(T_kT_I\beta_{ji})\Psi_i.
\end{aligned}$$

The last equality proves the step of induction and completes the proof of equation (3.25). Note, that the compatibility conditions for equations (3.15) and (3.25) lead to the following formula for the "long" difference derivatives $\Delta_I=(T_I-1)$ of the discrete rotation coefficients:

$$\Delta_I\beta_{jk}=\sum_{s\in I}(T_I\beta_{js})\beta_{sk},\qquad j\neq k,\quad j,k\notin I.$$

Let us also note, that as, due to (3.22), the functions β^*_{ij} for any index $k\neq i,j$ satisfy the same equations as Ψ_i, we simultaneously prove the following equality:

$$\Delta_I\beta^*_{jk}=\sum_{s\in I}(T_I\beta_{js})\beta^*_{sk},\qquad j,k\notin I. \tag{3.26}$$

Equation (3.25) implies that

$$\left(T^{(i)}-1\right)\Psi_i=\sum_{j\neq i}\left(T^{(i)}\beta_{ij}\right)\Psi_j,\qquad T^{(i)}=\prod_{j\neq i}T_j.$$

Since $(\widetilde{T}-1)=T_i(T^{(i)}-1)+\Delta_i$, applying (3.16) we gain the equality

$$(\widetilde{T}-1)\Psi_i=(\mu+B_i(u))\Psi_i+\sum_{j=1}^{n}((\widetilde{T}-1)\beta_{ij})\Psi_j(u,\mu),$$

where $B_i(u)$ is some function whose explicit form is irrelevant now. Since the vectors Ψ_j are linearly independent, it now suffices to show that the vectors Ψ_i and $\widetilde{T}\Psi_i$ are parallel.

Let us fix an arbitrary point u_0 and consider the solution $\Psi_i(u)=\Psi_i(u;u_0)$ of equations (3.15, 3.16) with the following initial data at the point u_0:

$$\langle\Psi_j(u_0),\Psi_k(u_0)\rangle=\beta^*_{jk}(u_0),\qquad j,k=1,\ldots,n. \tag{3.27}$$

These relations define the vectors $\Psi_i(u_0)$ uniquely up to an orthogonal transformation of the whole space.

Let us prove by induction that relation (3.27) is satisfied at the point $u_I=T_{i_1}\ldots T_{i_s}u_0$ for any set of pairwise distinct indices $I=\{i_1,\ldots,i_s\}$ not containing j, i.e.

$$\langle\Psi_j(u_I),\Psi_k(u_I)\rangle=\beta^*_{jk}(u_I),\qquad j\notin I. \tag{3.28}$$

Suppose that $i \neq j, k$. Then

$$\begin{aligned}\Delta_i\langle\Psi_j,\ \Psi_k\rangle &= \langle\Delta_i\Psi_j,\Psi_k\rangle + \langle\Psi_j,\Delta_i\Psi_k\rangle + \langle\Delta_i\Psi_j,\Delta_i\Psi_k\rangle = \\ &= (T_i\beta_{ji})\langle\Psi_i,\Psi_k\rangle + (T_i\beta_{ki})\langle\Psi_j,\Psi_i\rangle + (T_i\beta_{ji})(T_i\beta_{ki})\langle\Psi_i,\Psi_i\rangle.\end{aligned}$$

If the induction hypothesis is true for the point u_I and $i \notin I$, then we obtain the following formula for the scalar product at the point T_iu_I:

$$\begin{aligned}\Delta_i\langle\Psi_j(u_I),\Psi_k(u_I)\rangle &= (T_i\beta_{ji})\beta^*_{ik} + (T_i\beta_{ki})\beta^*_{ij} + (T_i\beta_{ki})(T_i\beta_{ji})\beta^*_{ii} = \\ &= (T_i\beta_{ki})\beta^*_{ij} = \Delta_i\beta^*_{jk}\end{aligned} \tag{3.29}$$

(all the functions in the right hand side are evaluated at the point u_I). Analogously, using the induction hypothesis, we obtain

$$\begin{aligned}&\Delta_i\langle\Psi_j,\Psi_i\rangle(u_I) = (T_i\beta_{ji})\beta^*_{ii} + (\mu+v_i)\beta^*_{ij} - \textstyle\sum_{k\neq i}\beta_{ik}\beta^*_{kj} + \\ &+ (T_i\beta_{ji})\Big((\mu+v_i)\beta^*_{ii} - \sum_{k\neq i}\beta_{ik}\beta^*_{ki}\Big) = (v_i-1)\beta^*_{ij} - \sum_{k\neq i}\beta_{ik}\beta^*_{kj}.\end{aligned} \tag{3.30}$$

(again, all the functions in the RHS are evaluated at the point u_I). Comparing the RHS of the last equality to the RHS of (3.23), we obtain:

$$\Delta_i\langle\Psi_j,\Psi_i\rangle(u_I) = \Delta_i\beta^*_{ji}(u_I), \qquad j\neq i, \quad j\notin I. \tag{3.31}$$

Equations (3.29) and (3.31) imply (3.28) at the point T_iu_I.

Now we are ready to prove that the vectors $\Psi_i(u_0)$ and $\widetilde{T}\Psi_i(u_0)$ are parallel. First, we show that the vectors $\Psi_m(u_I)$ and $T_mT_J\Psi_k(u_I)$ with non-intersecting sets of indices $I, J \subset \{1,\dots,n\}$ are orthogonal provided $m,k \notin I$, $m \notin J$ ($m \neq k$). According to (3.25), we have

$$\langle\Psi_m(u_I),T_mT_J\Psi_k(u_I)\rangle = \langle\Psi_m,\Psi_k\rangle + \sum_{j\in J}(T_mT_J\beta_{kj})\langle\Psi_m,\Psi_j\rangle + (T_mT_J\beta_{km})\langle\Psi_m,\Psi_m\rangle \tag{3.32}$$

(all the functions in the RHS are evaluated at the point u_I). We can apply formula (3.28) to the scalar products in the RHS, so

$$\begin{aligned}\langle\Psi_m(u_I),T_mT_J\Psi_k(u_I)\rangle &= \beta^*_{mk} + \sum_{j\in J}(T_mT_J\beta_{kj})\beta^*_{jm} + (T_mT_J\beta_{km})\beta^*_{mm} = \\ &= \beta^*_{mm}T_m\Big(\Delta_J\beta_{km} - \sum_{j\in J}(T_J\beta_{kj})\beta_{jm}\Big) = 0.\end{aligned}$$

This fact implies in particular, that the vectors

$$\Psi_1(u_0),\ T_1\Psi_2(u_0),\ T_1T_2\Psi_3(u_0),\ \dots,\ T_1T_2\dots T_{n-1}\Psi_n(u_0) \tag{3.33}$$

form an orthogonal basis. On the other hand, considering the sets of indices $I = \{1,2,\dots,s\}$, $m = s+1$, $J = \{s+2,\dots,n\}$ and $k = 1$ for s running from 1 to $n-1$, we establish that $\widetilde{T}\Psi_1(u_0)$ is orthogonal to all the vectors of this basis, but for the first one. This implies that the vectors $\Psi_1(u_0)$ and $\widetilde{T}\Psi_1(u_0)$ are parallel. In the same manner we can show that the vectors $\Psi_i(u_0)$ and $\widetilde{T}\Psi_i(u_0)$ are parallel for $i \neq 1$. As it was noticed above, it implies that $\beta_{ij}(u_0) = \widetilde{T}\beta_{ij}(u_0)$. As the choice of the initial point u_0 was arbitrary, it completes the proof of the lemma. □

The definition of β^*_{ij} and (3.24) imply that

Corollary 3.1 *There exists a constant η^2 such that the following equations hold:*

$$\widetilde{T}\beta^*_{ij} = \eta^2\beta^*_{ij}. \tag{3.34}$$

Now we are ready to complete the proof of Theorem 3.2.

Lemma 3.4 *Let functions β_{ij} satisfy the conditions of Theorem 3.2 and let η^2 be the corresponding Bloch multiplier, defined by (3.34). Then for $\mu = \eta - 1$ there exists a solution $\Psi_i(u)$ of equations (3.15) and (3.16), satisfying relations (3.27) identically for u, i. e.*

$$\langle \Psi_j(u), \Psi_k(u)\rangle = \beta^*_{jk}(u), \qquad j, k = 1, \ldots, n. \tag{3.35}$$

Proof. Consider the solution $\Psi_i(u; u_0)$ for some point u_0. By definition of this solution, it satisfies the relations (3.35) at the point u_0. Let us show that if these relations are satisfied at the point u they are also satisfied at the point $T_i u$. Suppose, for instance, that $i = 1$. While proving lemma 3.3 we have shown that at the point $(T_1 u)$ equations (3.27) are satisfied for all the pairs of indices except for $(j = 1, k = 1)$. Therefore, we only have to show that

$$|T_1\Psi_1|^2 = \langle T_1\Psi_1, T_1\Psi_1\rangle = T_1\beta^*_{11}.$$

The fact that vectors (3.33) form an orthogonal basis implies that

$$|T_1\Psi_1|^2 = \frac{\langle T_1\Psi_1, \Psi_1\rangle^2}{|\Psi_1|^2} + \frac{\langle T_1\Psi_1, T_1\Psi_2\rangle^2}{|T_1\Psi_2|^2} + \ldots + \frac{\langle T_1\Psi_1, T_1\ldots T_{n-1}\Psi_n\rangle^2}{|T_1\ldots T_{n-1}\Psi_n|^2}. \tag{3.36}$$

Analogously to the derivation of equation (3.30), we obtain:

$$\langle T_1\Psi_1, \Psi_1\rangle = (v_1 + \mu + 1)\beta^*_{11} - \sum_{j\neq 1}\beta_{1i}\beta^*_{i1} = (\mu + 1)\beta^*_{11}. \tag{3.37}$$

From (3.28) it follows that

$$\langle T_1\Psi_1, T_1\Psi_2\rangle = T_1\beta^*_{12}. \tag{3.38}$$

Besides, repeating the proof of (3.32), we obtain

$$\langle T_1\Psi_1, T_1\ldots T_i\Psi_{i+1}\rangle = T_1\Big(\sum_{p=2}^{i}(T_2\ldots T_i\beta_{i+1,p})\beta^*_{p1} + \beta^*_{i+1,1}\Big) = T_1\ldots T_i\beta^*_{1,i+1} \tag{3.39}$$

(to get the last formula in (3.39) we use (3.26)). Plugging expressions (3.37 − 3.39) in (3.36), we have

$$|T_1\Psi_1|^2 = \frac{(\mu+1)^2(\beta_{11}^*)^2}{\beta_{11}^*} + \frac{(T_1\beta_{12}^*)^2}{T_1\beta_{22}^*} + \ldots + \frac{(T_1\ldots T_{n-1}\beta_{1n}^*)^2}{T_1\ldots T_{n-1}\beta_{nn}^*}.$$

Now, applying equations (3.21) and (3.22), we transform the last expression to the form

$$\begin{aligned} |T_1\Psi_1|^2 &= (\mu+1)^2\beta_{11}^* - T_1[(T_2\beta_{12})\beta_{12}^*] - \ldots - T_1\ldots T_{n-1}[(T_n\beta_{1n})\beta_{1n}^*] = \\ &= (\mu+1)^2\beta_{11}^* - T_1\Delta_2\beta_{11}^* - \ldots - T_1\ldots T_{n-1}\Delta_n\beta_{11}^* = \\ &= (\mu+1)^2\beta_{11}^* + T_1\beta_{11}^* - \widetilde{T}\beta_{11}^* = T_1\beta_{11}^*. \end{aligned}$$

Therefore, we prove that $\Psi_i(u;u_0)$ satisfy relations (3.35) in the positive octant with the origin at u_0.

Note, that the solutions $\Psi_i(u;u_0')$ and $\Psi_i(u;u_0'')$ coincide in the intersection of the corresponding octants up to a general orthogonal transformation. We kill this freedom by fixing the solution $\Psi(u,0)$ and choosing the initial conditions for $\Psi_i(u;u_0)$ with any u_0 so that $\Psi_i(u,u_0)$ coincides with $\Psi_i(u,0)$ in the intersection of their domains. Then the function $\Psi_i(u) = \Psi_i(u;u)$ is a well-defined solution of system (3.15, 3.16) over the whole space, satisfying conditions (3.35) for all u. Lemma is proved. □

Corollary 3.2 *The above-constructed vector-functions $\Psi_i(u)$ satisfy the relation*

$$\langle T_j\Psi_i(u), \Psi_j(u)\rangle = 0, \qquad i \neq j. \tag{3.40}$$

Let functions $\beta_{ij}(u)$ satisfy the conditions of the theorem. We define functions $h_i(u)$ as a solution to the system

$$\Delta_i h_j(u) = \beta_{ij}(u) T_i h_i(u), \qquad i \neq j. \tag{3.41}$$

These equations are compatible due to (3.10). A solution of (3.41) depends on n arbitrary functions of one variable, which are the initial data, i. e. functions $h_i(0,\ldots,0,u_i,0,\ldots,0)$.

Let us consider vector-functions

$$X_i(u) = T_i h_i(u)\Psi_i(u), \tag{3.42}$$

where $\Psi_i(u)$ were defined in the preceding lemma. Equation (3.15) implies

$$\Delta_j X_i = \left(T_i\frac{\Delta_j h_i}{h_i}\right) X_i + \left(T_j\frac{\Delta_i h_j}{h_j}\right) X_j = \Delta_i X_j. \tag{3.43}$$

Therefore, there exists a vector-function $\mathbf{x}(u)$ such that $X_i(u) = \Delta_i\mathbf{x}(u)$. Due to (3.43) the function $\mathbf{x}(u)$ defines the planar lattice which, according to (3.40), is a Darboux − Egoroff lattice.

To complete the proof of Theorem 3.2 we only need to show that the functions $\beta_{ij}(u)$ are gauge equivalent to the rotation coefficients of this lattice under transformation (3.14).

Let $\widetilde{h}_i(u)$ be the Lamé coefficients which, according to the results of the previous section, correspond to the constructed lattice. Equation (3.5) implies that X_i satisfy the same equation (3.43), where coefficients depend on the $\widetilde{h}_i$'s instead of the h_i's. Therefore,

$$\frac{\Delta_j h_i}{h_i} = \frac{\Delta_j \widetilde{h}_i}{\widetilde{h}_i}.$$

Hence, $h_i = f_i \widetilde{h}_i$, where $f_i = f_i(u_i)$ depends only on the variable u_i.

Plugging (3.42) into (3.16) and using (3.41), we obtain:

$$T_i X_i = \frac{T_i^2 h_i}{T_i h_i} \left((\eta + v_i) X_i - \sum_{j \neq i} \frac{\Delta_i h_j}{T_j h_j} X_j \right).$$

Equation (3.6) implies the same equality, where h_i are replaced by $\widetilde{h}_i$ and η by $1/2$. Therefore, comparing the coefficients of X_j we see that $f_i = c^{u_i}$, where c is a constant. The comparison of the coefficients at X_i defines this constant: $c = (2\eta)^{-1}$. Theorem 3.2 is proved. $\square$

In the next section we present an algebro-geometric construction of a wide class of the Darboux – Egoroff lattices, which can be written explicitly in terms of the theta-functions of auxiliary Riemann surfaces.

4 Algebro-geometric lattices

Let Γ_0 be a smooth genus g_0 algebraic curve on which there is a meromorphic function $E(P)$, $P \in \Gamma_0$, with n simple poles and n simple zeros. Let points $P_1, \ldots, P_n$ be poles, and $Q_1, \ldots, Q_n$ be zeros of $E(P)$. Consider the Riemann surface Γ of the function $\sqrt{E(P)}$. It is two-sheeted covering of Γ_0 with $2n$ branching points at the poles and the zeros of $E(P)$. According to the Riemann – Hurvitz formula genus g of Γ equals $g = 2g_0 + n - 1$. Let $\sigma \colon \Gamma \to \Gamma$ be the holomorphic involution of Γ which permutes sheets of the covering. The points P_i and Q_j are fixed points of the involution.

The function $E(P)$ on Γ_0 takes each value n times. Let us fix a constant c^2 and consider the points $P_i^c \in \Gamma_0$, $i = 1, \ldots, n$, such that $E(P_i^c) = c^2$. The function $\lambda = c^{-1}\sqrt{E(P)}$ is odd with respect to the involution σ, has simple poles at $P_1, \ldots, P_n$ and simple zeros at $Q_1, \ldots, Q_n$. Let $P_i^{\pm}$ be preimages on Γ of the point P_i^c. Then $\lambda(P_i^{\pm}) = \pm 1$ and $\sigma(P_i^+) = P_i^-$.

We choose $w_i^+ = \lambda - 1$, as a local coordinate on Γ near P_i^+ and $w_i^- = \lambda + 1$ as a local coordinate near P_i^-. Note that $\sigma(w_i^+) = -w_i^-$.

Let us fix an integer $l \geq 1$ and two sets of points in the general position on Γ. One of them is a set of l points $\mathcal{R} = (R_1, \ldots, R_l)$, and the other is a set of $g + l - 1$ points $\mathcal{D} = (\gamma_1, \ldots, \gamma_{g+l-1})$. A pair of the divisors $\mathcal{R}$, $\mathcal{D}$ is called admissible (see [14]), if there is a meromorphic differential $d\Omega_0$ on Γ_0 with the following properties:

(a^0) $d\Omega_0$ has $n + l$ simple poles at Q_i, $i = 1, \ldots, n$ and at the points $\widehat{R}_\alpha$, which are the projections of R_α on Γ_0, $\alpha = 1, \ldots, l$;

(b^0) the differential $d\Omega_0$ has $g+l-1$ zeros at the projections $\widehat{\gamma}_s$ of the points γ_s, $s=1,\ldots,g+l-1$.

The differential $d\Omega_0$ may be considered as an even, with respect to σ, differential on Γ, where it has:

(a) $n+2l$ simple poles at Q_i, $i=1,\ldots,n$, and at the points R_α and $\sigma(R_\alpha)$, $\alpha=1,\ldots,l$;

(b) $2(g+l-1)$ zeros at γ_s and $\sigma(\gamma_s)$, $s=1,\ldots,g+l-1$, and simple zeros at the points P_i, $i=1,\ldots,n$.

The Riemann – Roch theorem implies that for each pair of divisors $\mathcal{D},\mathcal{R}$ in the general position there exists a unique function $\psi(u,Q|\mathcal{D},\mathcal{R})$, $u=(u_1,\ldots,u_n)\in\mathbb{Z}^n$, $Q\in\Gamma$, such that:

(1) $\psi(u,Q|\mathcal{D},\mathcal{R})$ as a function of the variable Q is meromorphic on Γ. Outside the punctures $P_i^\pm$ it has at most simple poles at the points of the divisor $\mathcal{D}$ (if all of them are distinct);

(2) in the neighborhood of $P_i^\pm$ the function $\psi(u,Q|\mathcal{D},\mathcal{R})$ has the form

$$\psi=(w_i^\pm)^{\mp u_i}\left(\sum_{s=0}^{\infty}\xi^i_{s,\pm}(u)(w_i^\pm)^s\right); \tag{4.1}$$

(3) ψ satisfies the normalization conditions

$$\psi(u,R_\alpha|\mathcal{D},\mathcal{R})\equiv 1,\qquad \alpha=1,\ldots,l. \tag{4.2}$$

The function ψ is a discrete analog of the Baker – Akhiezer functions which are the core of algebro-geometric integration theory of soliton equations. Further on, we shall often omit indication of its explicit dependence on the divisors $\mathcal{D},\mathcal{R}$ and will simply denote it by $\psi(u,Q)$. The discrete Baker – Akhiezer function can be expressed in terms of the Riemann theta-function in a way almost identical to the continuous case (see [14]).

The Riemann theta-function, associated with an algebraic curve Γ of genus g is an entire function of g complex variables $z=(z_1,\ldots,z_g)$, and is defined by its Fourier expansion

$$\theta(z_1,\ldots,z_g)=\sum\nolimits_{m\in\mathbb{Z}^g}e^{2\pi i(m,z)+\pi i(Bm,m)},$$

where $B=B_{ij}$ is a matrix of b-periods, $B_{ij}=\oint_{b_i}\omega_j$, of normalized holomorphic differentials $\omega_j(P)$ on Γ: $\oint_{a_j}\omega_i=\delta_{ij}$. Here a_i,b_i is a basis of cycles on Γ with the canonical matrix of intersections: $a_i\circ a_j=b_i\circ b_j=0$, $a_i\circ b_j=\delta_{ij}$.

The theta-function has the following monodromy properties with respect to the lattice $\mathcal{B}$, spanned by the basis vectors $e_i\in\mathbb{C}^g$ and the vectors $B_j\in\mathbb{C}^g$ with coordinates B_{ij}:

$$\theta(z+l)=\theta(z),\qquad \theta(z+Bl)=\exp[-i\pi(Bl,l)-2i\pi(l,z)]\,\theta(z)$$

where l is an integer vector, $l\in\mathbb{Z}^g$. The complex torus $J(\Gamma)=\mathbb{C}^g/\mathcal{B}$ is called the Jacobian variety of the algebraic curve Γ. The vector $A(Q)$ with coordinates

$$A_k(Q)=\int_{q_0}^{Q}\omega_k$$

defines the so-called Abel transform: $\Gamma \mapsto J(\Gamma)$.

According to the Riemann – Roch theorem, for each divisors $\mathcal{D} = \gamma_1 + \ldots + \gamma_{g+l-1}$ and $\mathcal{R} = R_1 + \ldots + R_l$ in the general position there exists a unique meromorphic function $r_\alpha(Q)$ such that $\mathcal{D}$ is its poles' divisor and $r_\alpha(R_\beta) = \delta_{\alpha\beta}$. It can be written in the form (see details in [14]):

$$r_\alpha(Q) = \frac{f_\alpha(Q)}{f_\alpha(R_\alpha)}, \qquad f_\alpha(Q) = \theta(A(Q) + Z_\alpha)\frac{\prod_{\beta\neq\alpha}\theta(A(Q)+F_\beta)}{\prod_{m=1}^{l}\theta(A(Q)+S_m)},$$

where

$$F_\beta = -\mathcal{K} - A(R_\beta) - \sum_{s=1}^{g-1} A(\gamma_s), \qquad S_m = -\mathcal{K} - A(\gamma_{g-1+m}) - \sum_{s=1}^{g-1} A(\gamma_s),$$

$$Z_\alpha = Z_0 - A(R_\alpha), \quad Z_0 = -\mathcal{K} - \sum_{s=1}^{g+l-1} A(\gamma_s) + \sum_{\alpha=1}^{l} A(R_\alpha),$$

where $\mathcal{K}$ is the vector of Riemann constants.

Let $d\Omega_j$ be a unique meromorphic differential on Γ, which is holomorphic outside $P_j^{\pm}$, has simple poles at these punctures with residues ∓ 1, and is normalized by conditions

$$\oint_{a_k} d\Omega_j = 0.$$

It defines a vector $V^{(j)}$ with the coordinates

$$V_k^{(j)} = \frac{1}{2\pi i}\oint_{b_k} d\Omega_j.$$

The Baker – Akhiezer function $\psi(u, Q|\mathcal{D},\mathcal{R}))$ has the form:

$$\psi = \sum_{\alpha=1}^{l} r_\alpha(Q)\frac{\theta(A(Q) + \sum_{i=1}^{n}(u^i V^{(i)}) + Z_\alpha)\ \theta(Z_0)}{\theta(A(Q)+Z_\alpha)\ \theta(\sum_{i=1}^{n}(u^i V^{(i)}) + Z_0)} \exp\left(\sum_{i=1}^{n} u^i \int_{R_\alpha}^{Q} d\Omega_i\right)$$

Theorem 4.1 *The Baker – Akhiezer function $\psi(u,Q)$ satisfies the equation*

$$\Delta_i\Delta_j\psi(u,Q) = a_{ij}^i(u)\Delta_i\psi(u,Q) + a_{ij}^j(u)\Delta_j\psi(u,Q), \tag{4.3}$$

where

$$a_{ij}^i = \frac{\Delta_j T_i \xi_{0,+}^i}{T_i\xi_{0,+}^i}, \qquad a_{ij}^j = \frac{\Delta_i T_j \xi_{0,+}^j}{T_j\xi_{0,+}^j}.$$

and $\xi_{0,+}^s$ is the leading coefficient of expansion (4.1) of $\psi(u,Q)$ near the puncture P_s^+.

Proof. From the definition of the coefficients a^i_{ij} it follows that the difference of the right and left hand sides of (4.3) satisfies the first two defining properties of the Baker-Akhiezer function. At the same time this difference equals zero at the points R_α. The uniqueness of the Baker – Akhiezer function implies then that this difference equals zero identically. $\square$

Let $\eta_k^2 = \mathrm{Res}_{Q_k}\, d\Omega_0$. Then we define a lattice $\mathbf{x}(u) = (x_1(u), \ldots, x_n(u))$ by the formula

$$x^k(u) = \eta_k \psi(u, Q_k). \tag{4.4}$$

Edges of this lattice are vectors $X_i(u)$, with coordinates $(X_i)^k(u) = \eta_k \Delta_i \psi(u, Q_k)$. Let us define also vectors $X_i^-(u)$ by the formula $(X_i^-)^k = -T_i^-(X_i)^k = \eta_k \Delta_i^- \psi(u, Q_k)$. Evaluations of (4.3), at the points Q_k, show that the above-defined lattice is planar.

In a generic case the vector function $\mathbf{x}(u)$ is complex. Let us specify algebro-geometric data that are sufficient for getting real vectors.

Suppose that on Γ there exists an antiholomorphic involution τ, such that it commutes with σ and

$$\tau(\mathcal{D}) = \mathcal{D}, \quad \tau(\mathcal{R}) = \mathcal{R}, \quad \tau(Q_i) = Q_i, \quad \tau(P_j^+) = P_j^+. \tag{4.5}$$

From the definition of $d\Omega_0$ it follows that $\tau^* d\Omega_0 = \overline{d\Omega_0}$. Therefore, the residue η_k^2 of that differential at Q_k is a real number. Suppose in addition, that η_k is real (or η_k^2 is positive). From (4.5) it follows that defining analytical properties of $\psi(u, Q)$, coincide with the analytical properties of $\overline{\psi(u, \tau(Q))}$. Uniqueness of the Baker – Akhiezer function implies then that these functions coincide, i. e. $\overline{\psi(u, \tau(Q))} = \psi(u, Q)$. Hence, $\mathbf{x}(u) = \eta_k\big(\psi(Q_1, u), \ldots, \psi(Q_n, u)\big) = \overline{\mathbf{x}}(u)$, and therefore, the lattice constructed is a lattice in the real Euclidian space.

Lemma 4.1 *Let $X_i^\pm(u)$ be vectors, which are defined by the Baker – Akhiezer function. Then*

$$\langle X_i(u), X_j^-(u)\rangle = -\delta_{ij}\,(T_i h_i^+(u))\,(T_i^- h_i^-(u)),$$

where $h_i^\pm(u) = \varepsilon_i \xi^i_{0,\pm}(u)$, and ε_i^2 is the leading term of an expansion of $d\Omega_0$ in terms of local coordinate w_i^+ in the neighborhood of the puncture P_i^+.

Proof. Let us consider the differential

$$d\Omega_{ij} = (\lambda(Q) - 1)\,\Delta_i \psi(u, Q)\,\Delta_j^- \psi(u, \sigma(Q))\,d\Omega_0.$$

If $i \neq j$ then that differential has poles at Q_k, $k = 1, \ldots, n$ only. Indeed, poles of $\lambda(Q) - 1$ at P_k, $k = 1, \ldots, n$, cancel with zeros of $d\Omega_0$. At the points γ_s and $\sigma(\gamma_s)$ poles of the product $\Delta_i\psi\Delta_j^-\psi^\sigma$, cancel with zeros of $d\Omega_0$. At the points R_α and $\sigma(R_\alpha)$ the poles of $d\Omega_0$ cancel with zeros of the product. At the points $P_k^\pm$, $k \neq i, j$, the pole of one of the functions $\Delta_i\psi(u, Q)$, $\Delta_j^-\psi(u, \sigma(Q))$ cancel with the zero of the other one. The same is true for the points P_i^- and P_j^-. The product $\Delta_i\psi\Delta_j^-\psi^\sigma$ has poles at P_i^+ and P_j^+, but at these points the function $\lambda(Q) - 1$ has zeros.

The sum of residues of a meromorphic differential on a compact Riemann surface is equal to zero. Therefore,

$$\sum_{k=1}^{n} \mathrm{Res}_{Q_k} d\Omega_{ij} = \sum_{k=1}^{n} \Delta_i \mathbf{x}^k(u) \cdot \Delta_j^- \mathbf{x}^k(u) = \langle \Delta_i \mathbf{x}(u), \Delta_j^- \mathbf{x}(u) \rangle = 0.$$

Now let us consider the differential

$$d\Omega_{ii}(u, Q) = (\lambda(Q) - 1)\, \Delta_i \psi(u, Q)\, \Delta_i^- \psi(u, Q^\sigma)\, d\Omega_0.$$

It has additional pole at P_i^+ with the residue

$$\mathrm{Res}_{P_i^+} d\Omega_{ii} = \varepsilon_i^2\, (T_i \xi_{0,+}^i)\, (T_i^- \xi_{0,-}^i),$$

which is equal to the sum of the residues at the punctures Q_k. □

Summarizing, we conclude that the lattice defined by (4.4) is the Darboux−Egoroff lattice. In order to show a complete correspondence with the previous sections, let us find some other scalar products.

Lemma 4.2 *For scalar products of the vectors X_i formulae (2.3, 2.5)*

$$\langle X_i, X_i \rangle = 2(T_i h_i^+)\, h_i^-, \qquad \langle T_j X_i, X_i^- \rangle = -(T_i T_j h_i^+)\, (T_i^- h_i^-), \quad i \neq j,$$

where $h_i^\pm = \varepsilon_i \xi_{0,\pm}^i$, are valid.

Proof. Let us consider the differential

$$d\Omega_{ii}^+ = \Delta_i \psi(u, Q)\, \Delta_i \psi(u, \sigma(Q))\, d\Omega_0.$$

It is meromorphic on Γ and has only simple poles at $Q_1, \ldots, Q_n$, and at $P_i^\pm$. Therefore, the sum of its residues $Q_1, \ldots, Q_n$, which coincides with the left hand side of (2.3), is equal to the sum of residues at P_i^+ and P_i^-, taken with the negative sign. We have

$$\mathrm{Res}_{P_i^+} d\Omega_{ii}^+ = \mathrm{Res}_{P_i^-} d\Omega_{ii}^+ = (T_i h_i^+)(-h_i^-),$$

which implies (2.3).

The proof of (2.5) is almost identical. It is enough to apply the same arguments to the differential

$$d\Omega_{ij}^{(1)} = T_j \Delta_i \psi(u, Q)\, \Delta_i \psi(u, \sigma(Q))\, d\Omega_0.$$

The lemma is proved. □

Now let us define functions $\beta_{ij}^\pm(u)$ and $\beta_{ij}^*(u)$ by the formulae

$$\beta_{ij}^+ = \frac{\Delta_i h_j^+}{T_i h_i^+}, \qquad \beta_{ij}^- = \frac{\Delta_i^- h_j^-}{T_i^- h_i^-}, \qquad \beta_{ij}^* = \frac{\Delta_i h_j^-}{T_i h_i}.$$

Lemma 4.3 *The functions $\beta^{\pm}_{ij}(u)$ and $\beta^{*}_{ij}(u)$ satisfy the equalities:*

$$\beta^{+}_{ij}=-\beta^{-}_{ji}, \qquad \beta^{*}_{ij}=\beta^{*}_{ji}.$$

Proof. Above we have proved formulae $(2.3-2.5)$ for the vectors X_i and the functions $h_i^{\pm}$. Therefore, the statement of the lemma follows from the results of Section 2. Nevertheless, we would like to show that it can be directly proved within the framework of the algebro-geometric construction.

The differential

$$d\Omega^{(2)}_{ij}=\lambda(Q)\,\Delta_j^{-}\psi(u,Q)\,\Delta_i\psi(u,\sigma(Q))\,d\Omega_0$$

has simple poles at P_j^- and P_i^- only. The vanishing of the sum of its residues at these punctures implies the first equality of the Lemma. To prove the second equality it is enough to consider the residues of the differential

$$d\Omega^{(3)}_{ij}=\lambda(Q)\,\Delta_j\psi(u,Q)\,\Delta_i\psi(u,\sigma(Q))\,d\Omega_0,$$

which has only poles at P_j^+ and P_i^+. □

At the end, let us show that the functions

$$\Psi_i(u,Q)=\frac{1}{T_i\xi^i_{0,+}}\Delta_i\psi(u,Q)$$

satisfy equations which are gauge equivalent to (3.15) and (3.16). Note, that the vector-function Ψ_i is uniquely defined by the following analytical properties:

(1) $\Psi_i(u,Q)$ as a function of Q is meromorphic on Γ and for each u the divisor of its poles outside $P_i^{\pm}$ is less or equal to $\mathcal{D}$;

(2^+) in the neighborhood of P_j^+, the function $\Psi_i(u,Q)$ has the form

$$\Psi_i=(w_j^+)^{-u_j-1}\left(\delta_{ij}+\sum_{s=1}^{\infty}\zeta^j_{s,+}(u)(w_j^+)^s\right);$$

(2^-) in the neighborhood of P_j^-, the function $\Psi_i(u,Q)$ has the form

$$\Psi_i=(w_j^-)^{u_j}\left(\sum_{s=0}^{\infty}\zeta^j_{s,-}(u)(w_j^-)^s\right);$$

(3) Ψ_i satisfies the normalization condition

$$\Psi_i(u,R_\alpha)\equiv 0, \qquad \alpha=1,\ldots,l.$$

Theorem 4.2 *The functions $\Psi_i(u,Q)$ satisfy the equations:*

$$\Delta_j\Psi_i(u,Q)=(T_j\gamma_{ij}(u))\Psi_j(u,Q), \qquad i\neq j,$$
$$\Delta_i\Psi_i(u,Q)=(\mu+v_i)\Psi_i(u,Q)-\textstyle\sum_{j\neq i}\gamma_{ij}(u)\Psi_j(u,Q),$$

where

$$\gamma_{ij}(u) = \frac{\Delta_i \xi^j_{0,+}(u)}{T_i \xi^i_{0,+}(u)}, \qquad v_i = -\sum_{j\neq i} \gamma_{ij} T_i \gamma_{ji}, \qquad \mu(Q) = \frac{1}{\lambda(Q)-1}\,.$$

Proof. The proof of the Theorem is standard. The difference of the left and right hand sides of the first eqaulity satisfies the first two properties which define $\psi(u,Q)$, and equals zero at the normalization points. Therefore, it equals zero identically. In the same way, the difference of the left and right hand sides of the second equation is proportional to $T_i\psi(u,Q)$. The evaluation of this difference at the normalization points shows that it is identically zero. □

The coefficients γ_{ij}, defined in the theorem are connected with the functions β^+_{ij} by the gauge transformation

$$\beta^+_{ij}(u) = \frac{\varepsilon_j}{\varepsilon_i}\gamma_{ij}(u).$$

References

[1] A. Bobenko, U. Pinkall, *Discretization of Surfaces and Integrable Systems*, in: *Discrete Integrable Geometry and Physics,* eds. A. Bobenko and R. Seiler, Oxford University Press, 1999 (preprint Sfb 296 (1997), Berlin).

[2] A. Doliwa, P. M. Santini, *Multidimensional quadrilateral lattices are integrable*, Phys. Lett. **A** 233 (1997), 365 – 372.

[3] A. Doliwa, *Geometric discretization of the Toda system*, Phys. Lett. **A** 234 (1997), 187 – 192.

[4] B. Dubrovin, S. Novikov, *The Hamiltonian formalism of one dimensional systems of hydrodynamic type and Bogolyubov – Whitham averaging method*, Sov. Math. Doklady, 27, 654 – 665 (1983).

[5] B. Dubrovin, S. Novikov, *Hydrodynamics of of weakly deformed soliton lattices: Differential geometry and Hamiltonian theory*, Russian Math Surveys, 44, 35 – 124 (1989).

[6] S. Tsarev, *The geometry of Hamiltonian systems of hydrodynamic type. Generalized Hodograph method*, Math in the USSR Izvestiya, 37, no. 2, 397 – 419 (1991).

[7] B. Dubrovin, *Integrable systems in topological field theory*, Nucl. Phys. **B** 379 (1992), no. 3, 627 – 689.

[8] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Topological strings in $d<1$*, Nucl. Phys. **B** 352 (1991), 59 – 86.

[9] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Notes on topological string theory and 2D quantum gravity* in: *String theory and quantum gravity*, Proceedings of the Trieste Spring School 1990, ed. M. Green, World – Scientific, 1991.

[10] R. Dijkgraaf, E. Witten, *Mean field theory, topological field theory, and multi-matrix models*, Nucl. Phys. **B** 342 (1990), 486 – 522.

[11] G. Darboux, *Lecons sur le Systems Ortogonaux et les Coordones Curvilignes*, Paris, 1910.

[12] V. Zakharov, *Description of the n-ortogonal curvilinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type. Part 1. Integration of the Lamé equations*, Duke Math. Journal, v. 94, no. 1, 103 – 139.

[13] V.E. Zakharov, S.V. Manakov, *On reduction in systems integrable with the help of the inverse scattering method*, Doklady RAN, 360 (1998), no. 3, 324 – 327.

[14] I.M.Krichever, *Algebro-geometric n-orthogonal curvilinear coordinate sustems and solutions of the associativity equations*, Funkt. anal. i pril., 31 (1997), no. 1, 32 – 50.

[15] J. Cieśliński, A. Doliwa, and P. M. Santini, *The integrable discrete analogues of orthogonal coordinate systems are multi-dimensional circular lattices*, Phys. Lett. **A** 235 (1997), 480 – 488.

[16] A. Bobenko, *Discrete conformal maps and surfaces*, in: *Symmetries and Integrability of Difference Equations II*, eds. P. Clarkson and P. Nijhoff, Cambridge University Press, 1996.

[17] R. Sauer, *Differenzengeometrie*, Springer-Verlag, Berlin, 1970.

[18] A. W. Nutbourne, *The solution of a frame mathcing equation*, in: *The Mathematics of Surfaces*, ed. A. Gregory, Clarendon Press, Oxford, 1986, 233 – 252.

[19] R. R. Martin, J. de Pont, T. J. Sharrock, *Cyclic surfaces in computer aided design*, in: *The Mathematics of Surfaces*, ed. A. Gregory, Clarendon Press, Oxford, 1986, 253 – 268.

[20] P. M. Santini, A. Doliwa, *Discrete geometry and integrability: The quadrilateral lattice and its reductions*, Sabaudia, May 1998.