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TOPICAL REVIEW

Periodic and almost-periodic potentials in inverse problems

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Abstract. An updated and detailed survey of basic ideas of the finite-gap theory is presented. That theory, developed to construct periodic and quasi-periodic solutions of the soliton equations, combines the Bloch–Floquet spectral theory of linear periodic operators, the theory of completely integrable Hamiltonian systems, the classical theory of Riemann surfaces, and theta-functions.

1. Introduction

To begin with we are going to consider the inverse spectral problem for a one-dimensional Schrödinger operator with periodic potential. In the late 1960s the famous discovery of the inverse scattering transform for the Korteweg–de Vries (KdV) equation was made. A periodic analogue of this transform was found in 1974. It is based on the solution of the following inverse spectral problem:

> *to describe effectively the 'isospectral manifold' of all the potentials with a given spectrum on the line (i.e. the spectrum of a Schrödinger operator acting in the Hilbert space of square integrable complex-valued functions on the line R).*

As everybody working in quantum solid-state physics knows, this spectrum is generically a union of an infinite number of intervals (allowed bands) on the energy line ϵ. The complementary part on the energy line is also a union of an infinite number of intervals (gaps or forbidden bands) whose lengths tend to zero for $\epsilon \to +\infty$.

The periodic problem was solved in 1974–5 for the so-called *finite-gap potentials*. Any periodic potential can be approximated by the finite-gap ones. This solution involves a combination of the theory of Riemann surfaces and their θ-functions, Hamiltonian dynamics of special completely integrable systems and the spectral theory of the Schrödinger operator. The mathematical technique used was (and still is) unusual for the community of physicists. Later, the necessity to use this kind of mathematics also appeared in other branches of mathematical and theoretical physics (for example, in string theory, matrix models, and supersymmetric Yang–Mills theory [1–15]). The authors believe this technique will be needed in future by the broad community of theoretical physicists.

Integrability of the famous KdV equation, $u_t = 6uu_x + u_{xxx}$, was discovered in 1965–8 (see [16–18]) for rapidly decreasing initial data on the line x. Exact solutions for the KdV equation expressing $u(x, t)$ through the *inverse scattering data* of the Schrödinger operator $L = -\partial_x^2 + u(x, 0)$ were found. This procedure has been called the inverse scattering transform

(IST). It was extended later for some other highly nontrivial (1 + 1)-systems including such famous systems as the nonlinear Schrödinger, $NS_{\pm}$: $i\psi_t = -\psi_{xx} \pm |\psi|^2\psi$ and sine(sinh)–Gordon equations, SG: $u_{xt} = \sin u$ or $u_{xt} = \sinh u$. Note that for the SG equation a large family of exact solutions had already been constructed in the 19th century by Bianchi, Lie and Bäcklund (see [19–21]). Since 1974, several (2 + 1)-dimensional physically interesting systems have been discovered as integrable by the IST procedure. The most famous of them is the Kadomtsev–Petviashvilii (KP) system (see [22, 23]).

It is necessary to emphasize that the IST procedure in its original form cannot be applied to the solution of the periodic problem (i.e. $u(x, t)$ is periodic in the variable x). This problem was solved on the basis of the new approach proposed in [24] and in [25–31] (see also the surveys [32, 35, 37]). An extension of this method to (2 + 1)-systems was found in [33–35]. A new development of this approach associated with two-dimensional Schrödinger operators began in 1976 (see [36–39, 41–43]).

A complete detailed description of the solution of the periodic problem can be found in the surveys [35, 37, 45], encyclopedia articles [40], and in the book [44]. We are going to present here the basic ideas of this theory in the simplest form possible. Let us point out that the KdV system, as well as other nontrivial *completely integrable by IST* partial differential equation (PDE) systems, are indeed completely integrable in any reasonable sense for rapidly decreasing or periodic (quasi-periodic) boundary conditions only. In fact, even that is well established for few of them. For example, for the KdV any periodic solution can be approximated by the finite-gap solutions. This statement easily follows from the theory of finite-gap potentials if we do not try to preserve the period, i.e. in the class of all quasi-periodic finite-gap potentials. The approximation of any periodic potential by the finite-gap potentials with exactly the same period, was constructed on the basis of another approach developed in [46]. The extension of the theory of Riemann surfaces and θ-functions to the specific class of surfaces of the infinite genus associated with the periodic Schrödinger operator was done in [47]. This theory is a beautiful description of the infinite limit. However, it seems that all fundamental properties of θ-functions associated with the complex continuation of variables are lost in this limit. It is interesting to point out that an analogous (but more complicated) theory of Riemann surfaces of the infinite genus was developed later in [41] for the periodic two-dimensional Schrödinger operators.

Outside these functional classes almost no effective information is known. Beautiful methods have also been developed for the studies of the special self-similar and 'string-type' solutions, but in most cases they lead to the very hard analytical problems associated with the famous Painlevé equations and their generalizations [48–55].

2. Rapidly decreasing potentials and the Gardner–Green–Kruskal–Miura (GGKM) procedure. Bäcklund transformations

Let us recall the basic information about the IST method for KdV. We start from the so-called *Lax representation* for KdV (see [18]). The Heisenberg-type equation for the Schrödinger operator L

$$L_t = [L, A] = LA - AL, \tag{2.1}$$

$$L = -\partial_x^2 + u, \qquad A = -4\partial_x^3 + 3(u\partial_x + \partial_x u), \tag{2.2}$$

is equivalent to the identity (KdV equation)

$$u_t = 6uu_x + u_{xxx}. \tag{2.3}$$

For this reason, any KdV-type equations admitting some analogue of the Lax representation are called *isospectral deformations*. The existence of such deformations indicates the possibility of an effective solution of the inverse scattering problem for the operator L.

For the rapidly enough decreasing functions $u(x,t) \to 0$, $x \to \pm\infty$ we define two bases of solutions (t is fixed):

$$\phi_\pm(x,t;k) \sim \exp^{\pm ikx}, \qquad x \to -\infty, \quad L\phi_\pm = \lambda\phi_\pm, \tag{2.4}$$

$$\psi_\pm(x,t;k) \sim \exp^{\pm ikx}, \qquad x \to +\infty, \quad L\psi_\pm = \lambda\psi_\pm, \quad k^2 = \lambda. \tag{2.5}$$

By definition, monodromy matrix T connects these two bases, $T\phi = \psi$ for the column vectors $\phi = (\phi_+, \phi_-)$, $\psi = (\psi_+, \psi_-)$:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \psi_+ = a\phi_+ + b\phi_-, \qquad \psi_- = c\phi_+ + d\phi_-. \tag{2.6}$$

A conservation of the Wronskian implies that $\det T = ad - bc = 1$. For the real values of k or $\lambda > 0$ we have $a = \bar{d}, c = \bar{b}$. Therefore, $|a|^2 - |b|^2 = 1$. The whole set of the so-called inverse scattering data can be extracted from the monodromy matrix T if it is well-defined for all complex values of k. The so-called *scattering matrix* is constructed from T for the real k. Its entries are the *transmission* coefficient $1/a$ and the *reflection* coefficient $\bar{b}/a$. The property $b = 0$ for all real values of k characterizes reflectionless or multisoliton potentials. For all rapidly decreasing potentials the matrix element $a(k)$ is well-defined for complex k such that $\operatorname{Im} k > 0$ and $a \to 1$ for $k \to \infty$, $\operatorname{Im} k > 0$. There is only a finite number of purely imaginary zeros $a(k_n) = 0$ in this domain. They correspond to the discrete spectrum $\lambda_n = k_n^2 < 0$.

The famous result of [17] (the GGKM procedure) easily follows from the Lax representation, which implies the equation:

$$T_t = [T, \Lambda], \qquad \Lambda = \begin{pmatrix} -4ik & 0 \\ 0 & 4ik \end{pmatrix}. \tag{2.7}$$

This result was formulated as a set of the following GGKM formulae:

$$a_t = 0, \qquad b_t = -8(ik)^3 = -c_t, \qquad d_t = 0. \tag{2.8}$$

The latter equations give a full description of the KdV dynamics in these variables because any rapidly decreasing potential can be reconstructed from the inverse scattering data. A special family of the reflectionless potentials where $b = 0$ for real values of k, leads to the so-called multisoliton solutions for the KdV equation (see [44]).

The multisoliton solutions can also be directly obtained with the help of the elementary substitutions (Bäcklund transformations) transforming any solution of the KdV into another solution: let u be a solution of the KdV equation and v be a solution of the Ricatti equation $\alpha + u = v_x + v^2$ with the initial value independent of time. The new function $\tilde{u} = -v_x + v^2$ satisfies the KdV equation. Starting from the trivial solution $u = u_0 = 0$ we construct a sequence of potentials $\tilde{u}_{n-1} = u_n$, $n > 0$, given by the Bäcklund transformation. We choose parameters α_n, $\alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots$, and take the real nonzero functions $f_n \to \infty$, $x \to \pm\infty$, $-f_{nxx} + u_{n-1}f_n = \alpha_n f_n$, which define $v_n = (\log f_n)_x$. Every such sequence leads to the multisoliton reflectionless potential:

$$u_0 = 0, \qquad u_1 = -\frac{2\alpha}{ch^2(\sqrt{\alpha}(x - x_0) + \beta t)}, \ldots. \tag{2.9}$$

In terms of the Schrödinger operator this transformation (invented by Euler in 1742) is called the Darboux transformation. The operator L can be factorized

$$L = -\partial^2 + u = -(\partial + v)(\partial - v).$$

Using the noncommutativity of these factors, we define the Darboux transformation for the operator and its eigenfunction in the following way:

$$\tilde{L} = -\partial^2 + \tilde{u} = -(\partial - v)(\partial + v), \qquad \tilde{\psi} = (\partial + v)\psi. \tag{2.10}$$

These transformations can be considered as some kind of *discrete spectral symmetries* for Schrödinger operators. They preserve a spectrum of the operator L (except maybe for one eigenfunction).

3. KdV hierarchy. Integrals of motion. Hamiltonian formalism

The local integrals for the KdV equation can be constructed with the help of the Schrödinger operator. Consider the associated Riccati equation $v_x + v^2 = u - k^2$ and find the solution for it as a formal series in the variable k:

$$v(x,k) = \mathrm{i}k + \sum_{n=1}^{\infty} v_n(x)(\mathrm{i}k)^{-n}, \tag{3.1}$$

where all v_n are polynomials in the variables $u, u_x, \ldots$. The integral along the line x is a k-dependent constant of motion for the KdV equation

$$\partial_t\left(\int v(x,k)\,\mathrm{d}x\right) = 0, \qquad u_t = 6uu_x + u_{xxx}. \tag{3.2}$$

For the real potentials $u(x,t)$ and real k we can see that the imaginary part of $v(x,k) - \mathrm{i}k$ is a total derivative. The remaining quantities in the expansion,

$$\int (v(x,k) - \mathrm{i}k)\,\mathrm{d}x = \sum_{n\geqslant 1} \int v_n(x)(\mathrm{i}k)^{-n}\,\mathrm{d}x, \tag{3.3}$$

define local integrals of motion

$$I_n = c_n \int v_{2n+3}(x)\,\mathrm{d}x, \qquad n = -1, 0, 1, 2, \ldots, \tag{3.4}$$

where c_n are constants. After a proper choice of the constants c_n we have

$$I_{-1} = \int u\,\mathrm{d}x, \qquad I_0 = \int u^2\,\mathrm{d}x, \qquad I_1 = \int (u_x^2/2 + u^3)\,\mathrm{d}x, \ldots, \tag{3.5}$$

$$I_{nt} = 0, \qquad u_t = 6uu_x + u_{xxx}. \tag{3.6}$$

Let us introduce a Gardner–Zakharov–Faddeev (GZF) Poisson bracket [56, 57] on the space of functions

$$\{u(x), u(y)\} = \delta'(x - y)). \tag{3.7}$$

Then any functional H (Hamiltonian) defines the corresponding Hamiltonian system

$$u_t = \partial_x\left(\frac{\delta H}{\delta u(x)}\right). \tag{3.8}$$

For the case $H = I_{-1}$ we get a trivial flow (i.e. this integral is a Casimir for the GZF bracket). For $H = I_0$ we arrive at the x-translations, $u_t = u_x$. Let us call this equation KdV_0. For the case $H = I_1$ we have the ordinary $KdV = KdV_1$. Higher integrals give us the equations KdV_n of the order $2n+1$ admitting the Lax representations with the same Schrödinger operator L but with the differential operators $A_n = (\mathrm{const})\partial_x^{2n+1} + \cdots$:

$$u_{t_n} = \partial_x\left(\frac{\delta I_n}{\delta u(x)}\right) = [L, A_n]. \tag{3.9}$$

In particular, $A_0 = \partial_x$, $A_1 = A$. A nice formula for all operators A_n can be extracted from [58].

Let $L = -\mathcal{L}^2$, where $\mathcal{L} = \partial_x + \sum_{k\geqslant 1} a_k(u, u_x, \ldots)\partial_x^{-k}$. Here all a_k are polynomials in the variables $u, u_x, \ldots$ and $\partial_x^{-1}a = \sum_{n\geqslant 0}(-1)^n a^{(n)}\partial_x^{-n-1}$ for the composition of the operator ∂^{-1} and multiplication operator by a. By definition,

$$A_n = (\mathcal{L}^{2n+1})_+ = (L^{2n+1/2})_+, \tag{3.10}$$

where the sign + means omitting all strictly negative powers of ∂_x.

All higher KdV_n systems can be integrated by the same IST procedure for the class of rapidly decreasing functions. In particular, GGKM equations for the scattering data (or monodromy matrix) have the form

$$T_{t_n} = [T, \Lambda_n], \qquad \Lambda_n = (\text{const}) \begin{pmatrix} (\mathrm{i}k)^{2n+1} & 0 \\ 0 & (-\mathrm{i}k)^{2n+1} \end{pmatrix}. \tag{3.11}$$

The latter result also implies that all these flows commute with each other. Hence, we get the following conclusion without any calculation:

integrals I_n have zero Poisson brackets,

$$\{I_n, I_m\} = \int \frac{\delta I_n}{\delta u(x)} \partial_x \frac{\delta I_m}{\delta u(x)}\, \mathrm{d}x = 0. \tag{3.12}$$

A generalization of the GZF Poisson bracket for the isospectral deformations of the higher-order (scalar) Lax operators L was found in [59].

It should be emphasized that there exists a family of local field-theoretical Poisson brackets (Lenhart–Magri (LM) brackets [60]) describing the KdV theory:

$$B = B_{(\lambda,\mu)} = \lambda\partial_x + \mu(-\partial_x^3 + 4u\partial_x + 2u_x) = \lambda B_0 + \mu B_1, \tag{3.13}$$

$$\{u(x), u(y)\}_{\lambda,\mu} = B\delta(x - y), \tag{3.14}$$

$$\{I_n, I_m\}_{\lambda,\mu} = 0. \tag{3.15}$$

These brackets were generalized for the higher-order operators L in [61].

The recurrence operator $B_1 B_0^{-1} = C$ generates all the right-hand sides of all higher KdV systems:

$$C(0) = u_x, \qquad C^2(0) = 6uu_x - u_{xxx}, \ldots, \qquad C^n(0) = \partial_x \left(\frac{\delta I_{n-1}}{\delta u(x)}\right). \tag{3.16}$$

It gives also a simple proof of a very useful identity [62]:

$$\int \frac{\delta I_n}{\delta u(x)}\, \mathrm{d}x = (\text{const}) I_{n-1}. \tag{3.17}$$

All these identities are local and can be used for the class of periodic functions as well. However, we shall see in the next section that the direct analogue of the GGKM procedure does not lead to the integration procedure. We are going to use a different approach.

4. Spectral theory of periodic Schrödinger operators. Finite-gap potentials

The spectral theory for the periodic potentials on the whole line x is based on the monodromy matrix as in case of the scattering theory. However, in the periodic case with a period $T < \infty$, we have nothing like the selected point $x = \infty$ for the definition of the monodromy matrix (as it was for the rapidly decreasing case ($T = \infty$)—see section 2 above). Any point x_0 can be used. Let us fix an initial point x_0 and choose a special basis of the solutions $C(x, x_0, \epsilon)$, $S(x, x_0, \epsilon)$ for the spectral equation $LC = \epsilon C$, $LS = \epsilon S$ such that for $x = x_0$ we have

$$\begin{pmatrix} C & S \\ C_x & S_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

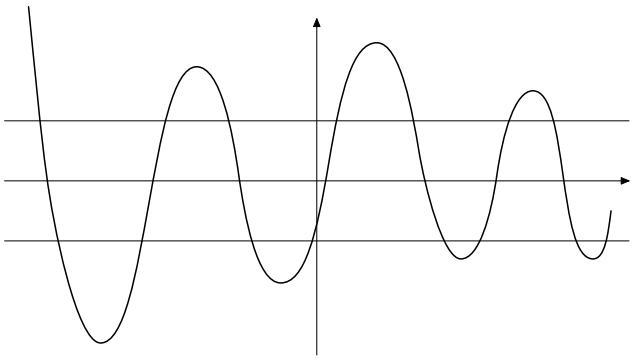

Figure 1.

The shift operator $T : x \to x + T$ in the basis C, S defines the monodromy matrix

$$\hat{T}(x_0, \epsilon) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.2}$$

$$C(x + T) = aC(x) + bS(x), \qquad S(x + T) = cC(x) + dS(x). \tag{4.3}$$

The key element of the periodic spectral theory is a notion of the so-called Bloch waves or Bloch–Floquet eigenfunctions. We present here some essential properties of these functions without any proofs. An exposition of this theory may be found in the encyclopedia article [40], where the main ideas of the proofs are clearly presented for the difference Schrödinger operator (it is much simpler).

By definition, the Bloch–Floquet functions are solutions of the Schrödinger equation that are at the same time eigenfunctions of the shift operator, i.e.

$$L\psi = \epsilon\psi, \qquad T\psi(x) = \psi(x + T, \epsilon) = \exp(\pm \mathrm{i} p(\epsilon)T)\psi(x). \tag{4.4}$$

We uniquely normalize ψ by the condition $\psi|_{x=x_0} = 1$.

For any complex number ϵ the eigenvalues, $w_\pm(\epsilon) = \exp(\pm p(\epsilon)T)$, of the shift operator are defined by the characteristic equation for the monodromy matrix. From the Wronskian property it follows that $\det \hat{T} = 1$. Therefore, the characteristic equation has the form

$$w^2 - (\operatorname{tr} \hat{T})w + 1 = 0. \tag{4.5}$$

The multivalued function $p(\varepsilon)$ is called quasi-momentum.

The spectrum of the Schrödinger operator on the whole line is a union of spectral zones which are segments of the real line of the variable ϵ, where the quasi-momentum is real. The latter condition is equivalent to the inequality:

$$|\operatorname{Tr} T| = 2\cos(pT) \leqslant 2. \tag{4.6}$$

The typical graph of the function $f(\epsilon) = \cos(pT)$ is shown in figure 1. In particular, its extreme points $f'(\epsilon) = 0$ are'generically' located inside the gaps (i.e. for the open and everywhere dense set of periodic potentials we have $|f| > 1$ at the extreme points, and there is only one extremal point in each gap). For some special cases we may have $f = \pm 1$ at the extreme point. Such a point lies inside the spectral zone. However, generic perturbations create a new small gap near this point—see figure 2. This point is a double point of periodic or anti-periodic spectrum with the boundary conditions $\psi(x) = \pm\psi(x + T)$.

The Riemann surface of the Bloch–Floquet functions is defined by the equation

$$z^2 = \cos(p(\epsilon)T), \tag{4.7}$$

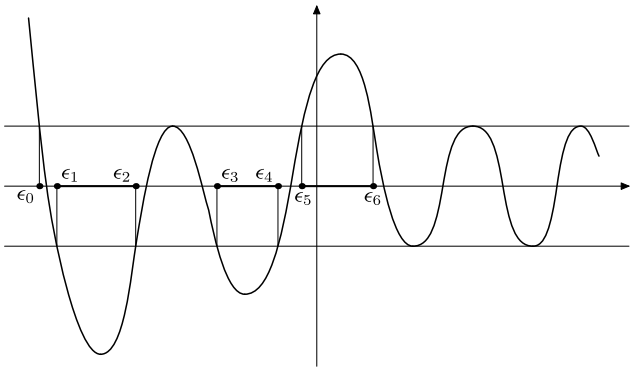

Figure 2.

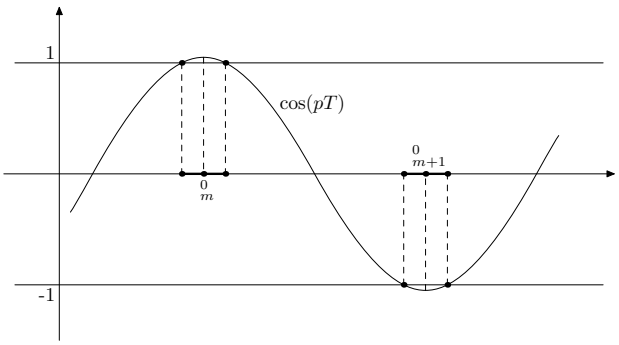

Figure 3.

but this surface is nonsingular only for the generic case (when there are no double points of the periodic or anti-periodic problem for the Schrödinger operator). For large $\epsilon \to +\infty$ the following asymptotics are valid for the gaps (see figure 3):

(1) The length of gaps tends to zero with its rate depending on the smoothness of the potential (this rate is exponential for the analytic potentials).
(2) All gaps are located near the points $\epsilon_m = 4\pi^2 m^2 T^{-2}$, and the distance of gaps from these point tends to zero.

A nonsingular Riemann surface of the Bloch–Floquet solutions is defined with the help of the equation

$$z^2 = (\epsilon - \epsilon_0)(\epsilon - \epsilon_1)\dots(\epsilon - \epsilon_n)\dots, \tag{4.8}$$

where $\epsilon_0 < \epsilon_1 < \cdots$, are simple eigenvalues of the periodic and anti-periodic spectral problems.

As follows from the Lax representation for all higher KdV_n systems (3.9), these boundary spectral points for the periodic operator L in the Hilbert space $L^2(R)$ of the square integrable complex-valued functions on the x-line are integrals of motion for the KdV hierarchy.

We can say that the Riemann surface of the Bloch–Floquet solutions as a whole is an integral of motion for the KdV hierarchy. How many potentials correspond to the same Riemann surface (i.e. have the same spectrum in $L^2(R)$)?

The original idea to introduce a special class of potentials for which this problem can be solved effectively was proposed in [24] and is based on the KdV hierarchy. This idea naturally combines two different ways to describe the corresponding isospectral manifold of potentials. Both are in fact closely related to each other.

The first approach—use of the KdV hierarchy and Hamiltonian dynamics. Let us consider a stationary equation for some linear combination of higher KdV_n flows (3.9). It is an ordinary differential equation (ODE) that can be written in the form:

$$\delta(I_n + c_1 I_{n-1} + \cdots + c_n I_0 + c_{n+1} I_{-1}) = 0. \tag{4.9}$$

This is a finite-dimensional Hamiltonian system with n degrees of freedom depending on $n+1$ parameters $c_1, \ldots, c_{n+1}$. It is a completely integrable Hamiltonian system because there is a family of the commuting flows. This family coincides with the restriction of all higher KdV_m on this stationary subset of functions given by the equation (4.9). Therefore, its generic nonsingular solution is expected to be a periodic or quasiperiodic function of x.

In the next section we shall construct for (4.9) some kind of Lax representation

$$\Lambda_x = [Q(x, \lambda), \Lambda(x, \lambda)], \qquad \Lambda = \Lambda_n + \sum_{k=1}^{n+1} c_k \Lambda_k, \tag{4.10}$$

using 2×2 traceless matrices depending on the parameter λ and $(u, u_x, \ldots)$ polynomially. In the terminology used today some people call them *loop groups*. Note that in all soliton systems these λ-loops have very specific λ-dependence (polynomial, rational and in some exotic examples—elliptic functions).

Lax representation (4.10) implies that an algebraic Riemann surface defined by the characteristic equation

$$\det(\Lambda(\lambda) - zI) = P(\lambda, z) = 0, \tag{4.11}$$

does not depend on x and is, therefore, an integral of (4.9). The same Riemann surface can be extracted from the *commutativity equation* $[L, A] = 0$, which according to (3.9) is equivalent to (4.9).

It should be emphasized that the latter form of (4.9), i.e. the commutativity condition for two ODEs

$$[L, A_n + \sum c_k A_{n-k}] = [L, A] = 0, \qquad L = -\partial_x^2 + u, \tag{4.12}$$

was considered formally (i.e. locally in the variable x without any periodicity assumptions) as a pure algebraic problem in the 1920s (see [63, 64]). Even the formal algebraic Riemann surface (4.11) appeared as a relation $P(L, A) = 0$.

According to our logic however, the corresponding system of equations in the variable x is Hamiltonian and completely integrable. Therefore, its generic solution is quasiperiodic in x, containing a dense family of periodic solutions. We may ask about the spectrum of the corresponding operators L in $L^2(R)$ and boundaries of gaps. Remarkably, they exactly coincide with the branching points of the Riemann surface defined by equation (4.11). Therefore, for the periodic potential which satisfies equation (4.9), the nonsingular spectral Riemann surface of the Bloch–Floquet solutions is an algebraic Riemann surface of genus $g = n$. The spectrum of such an operator contains only a finite number of gaps $[\epsilon_{2j-1}, \epsilon_{2j}]$, $j = 1, \ldots, n$.

This key step unifies the first approach with the second one (below). So the solution of the inverse spectral problem can be identified with the process of a solution of some special families of completely integrable systems using Riemann surfaces.

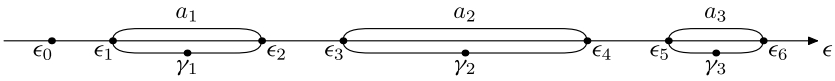

Figure 4.

The second approach—periodic spectral theory in the Hilbert space $L^2(R)$. For any periodic Schrödinger operator $L = -\partial^2 + u(x = t_0, t_1, \ldots)$ we have already defined the Riemann surface Γ of Bloch–Floquet solutions with the help of the monodromy matrix $\widehat{T}$. For the finite-gap potentials a graph of the function $f = \cos(p(\epsilon)T) = \frac{1}{2}\,\mathrm{tr}\,\hat{T}$ is highly degenerate (see figure 2 and compare with figure 1). For all real and large enough ϵ we have $|f| \leqslant 1$.

Once again, we ask how to describe all the potentials with the same finite-gap spectrum and what are the additional variables which uniquely define a finite-gap potential? It turns out that these additional spectral data are the poles of the Bloch–Floquet functions. It can by shown that the Bloch–Floquet solutions defined by (4.4) and normalized by the condition $\psi|_{x=x_0} = 1$ have exactly one simple pole $\gamma_j(x_0)$ inside each gap or on its boundary. A point of the hyperelliptic surface Γ (4.8) can be represented as the complex number ϵ and a sign (or a branch) of the radical $z = \sqrt{(\epsilon - \epsilon_0)(\epsilon - \epsilon_1)\ldots}$. The branches of the radical coincide at the boundaries of gaps. Therefore, each gap should be considered as a closed cycle a_j, $j = 1, 2, \ldots n$ (see figure 4). Any pole of ψ should be considered as a point γ_j on the cycle a_j. The above-mentioned statement that there is only one pole in a gap means, in particular, that ψ has no pole at the point $(\gamma_j, +)$ if it already has a pole at $(\gamma_j, -)$ and vice versa. Geometrically, the total set of poles $(\gamma_1, \ldots, \gamma_n)$ represents a point on the real torus $T^g = a_1 \times \cdots \times a_n$. As we shall see later, this set of poles completely determines the original potential. Different points of the real torus lead to the different potentials (if normalization point x_0 is fixed).

5. Periodic analogue of the GGKM. Zero-curvature representation for the KdV hierarchy and corollaries

As was emphasized above, the definition of the monodromy matrix in the periodic case depends on the choice of initial point x_0. For different choices of the initial point the corresponding monodromy matrices are conjugated (because they represent the same linear transformation in different bases). Therefore, the dependence of the monodromy matrix with respect to the choice of the initial point x_0 can be described by the equation:

$$T_{x_0} = [Q(\epsilon, x_0), T], \qquad Q = \begin{pmatrix} 0 & 1 \\ \epsilon - u(x_0) & 0 \end{pmatrix}. \tag{5.1}$$

In the same way, for the isospectral deformations corresponding to the KdV hierarchy (i.e., all higher KdV_n systems) we can establish the following equations (the periodic analogues of GGKM):

$$T_{t_j} = [\Lambda_j, T], \qquad \Lambda_0 = Q, \qquad t_0 = x_0. \tag{5.2}$$

A compatibility condition of equations (5.2) for any pair of variables t_i, t_j implies the following *zero-curvature representation* for the KdV hierarchy (where periodic boundary conditions are already inessential)

$$[\partial_{t_i} - \Lambda_i, \partial_{t_j} - \Lambda_j] = 0. \tag{5.3}$$

Using the Lax representations (3.9) for all higher KdV_n, we can express all the matrices Λ_j as polynomials in the variable ϵ and variables $u(x_0), u_{x_0}(x_0), \ldots$. For example, for the ordinary

KdV we have (replacing x_0 by x):

$$\Lambda_1 = \begin{pmatrix} -u_x & 2u + 4\epsilon \\ -4\epsilon^2 + 2\epsilon u + 2u^2 - u_{xx} & u_x \end{pmatrix}. \tag{5.4}$$

The matrices Λ_k can be completely reconstructed from equations (5.3) and from the following properties of matrix elements:

$$\Lambda_k = (\text{const}) \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}, \tag{5.5}$$

$$d_k + a_k = \operatorname{Tr} \Lambda_k = 0, \qquad b_k = \epsilon^k + u/2\epsilon^{k-1} + \cdots, \tag{5.6}$$

$$\det \Lambda_k = -(a^2 + bc) = (\text{const}) R_{2k+1}(\epsilon) = (\text{const})(\epsilon^{2k+1} + \cdots), \tag{5.7}$$

$$2a_k = -b_{kx}, \qquad (R_{2k+1})_x = -b_k. \tag{5.8}$$

For $i = 0$, equations (5.3), where $t_0 = x$, give the zero-curvature representation for the KdV_n system:

$$\partial_x \Lambda_n - \partial_{t_n} Q = [Q, \Lambda_n]. \tag{5.9}$$

From this representation and from the periodic analogue of GGKM we arrive at the following results for the stationary higher KdV_n system (4.10):

(1) The monodromy matrix T commutes with Λ

$$[T, \Lambda] = 0. \tag{5.10}$$

Therefore, they have common eigenvectors. It implies, in particular, that the Bloch–Floquet function $\psi_\pm$ is single-valued on the algebraic Riemann surface associated with the matrix Λ.

(2) The stationary higher KdV admits an ϵ-parametric Lax-type representation in the variable x (4.10):

$$(\Lambda)_x = [Q, \Lambda], \qquad \Lambda = \Lambda_n + \sum c_k \Lambda_k. \tag{5.11}$$

Therefore, we have a full set of conservation laws organized in the form of the Riemann surface (i.e. all coefficients of the polynomial P are x-independent):

$$\det(\Lambda(\epsilon) - zI) = P(\epsilon, z) = 0, \tag{5.12}$$

$$P(\epsilon, z) = z^2 - R_{2n+1}(\epsilon) = (\text{const})(\epsilon - \epsilon_0) \ldots (\epsilon - \epsilon_{2n}). \tag{5.13}$$

These points ϵ_j are exactly the boundaries of gaps for periodic potentials because the Riemann surfaces of the matrices T and Λ coincide. The matrix element b_{12} for the matrix Λ determines another set of points

$$b_{12}(\epsilon) = (\text{const})(\epsilon - \gamma_1) \ldots (\epsilon - \gamma_n). \tag{5.14}$$

These points coincide with projections of the zeros of Bloch wave $\psi_\pm$ as functions of the variable x or poles as functions of x_0 (see the next section).

An original approach to the solution of the inverse spectral problem was based on the use of the following trace-type formula for the potential:

$$-u(x)/2 + \text{const} = \gamma_1(x) + \ldots \gamma_n(x). \tag{5.15}$$

From (5.11) and (5.14), 'Dubrovin equations' defining dynamics in x of the points γ_j can be derived. They have the form

$$\gamma_{jx} = \frac{\sqrt{R_{2n+1}(\gamma_j)}}{\prod_{k \neq j}(\gamma_j - \gamma_k)}. \tag{5.16}$$

These equations (see [27, 32]) can be linearized by the so-called Abel transformation.

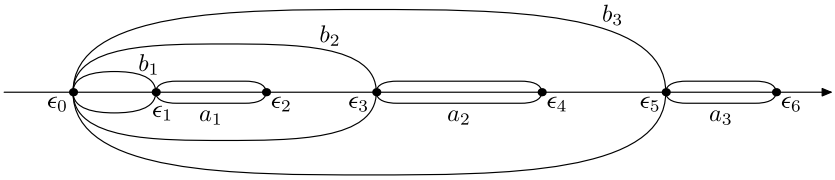

Figure 5.

Consider the first-kind differentials on the Riemann surface (i.e. holomorphic one-forms without poles anywhere—even at infinity). The basic first-kind forms are

$$\omega_j = \frac{\epsilon^{j-1}\,\mathrm{d}\epsilon}{\sqrt{R_{2n+1}(\epsilon)}}, \qquad j = 1, \ldots, n. \tag{5.17}$$

It is convenient to choose a normalized basis taking linear combinations $\Omega_s = \sum_j v_{sj}\omega_j$ such that

$$\oint_{a_j} \Omega_s = \delta_{sj},$$

for all the gaps a_j.

Fix a set of paths κ_j from the point $P_0 = \infty$ to the points P_j on the Riemann surface. The Abel transformation is defined by the formula

$$A_p(P_1, \ldots, P_n) = \sum_j \int_{\kappa_j} \Omega_p. \tag{5.18}$$

Equations (5.16) after the Abel transform become linear:

$$A_{px} = U_q = \frac{1}{2\pi} \oint_{b_q} \mathrm{d}p. \tag{5.19}$$

Here the closed paths b_q are 'canonically conjugated' to the paths a_j (see figure 5). It means that the intersection numbers are $a_j \circ b_q = \delta_{jq}$. The differential $\mathrm{d}p$ is equal to $p_\epsilon\,\mathrm{d}\epsilon$ where $p(\epsilon)$ is a multi-valued quasi-momentum. It is a second-kind differential (i.e. a meromorphic one-form on the surface Γ with a pole of order two at the point ∞ with negative part $\mathrm{d}k = -(\mathrm{d}w)w^{-2}$ in the local coordinate $w = k^{-1}$ near ∞), and therefore has the form

$$\mathrm{d}p = \frac{(\epsilon^{n+1} + \sum_{j>0} v_j \epsilon^{n-1-j})\,\mathrm{d}\epsilon}{\sqrt{R_{2n+1}(\epsilon)}}. \tag{5.20}$$

All the coefficients v_j can be found from the normalization condition:

$$\oint_{a_j} \mathrm{d}p = 0, \qquad j = 1, \ldots, n. \tag{5.21}$$

Following the classical 19th century theory of Riemann surfaces, formulae (5.15)–(5.21) lead to some expression of the potential through the θ-functions avoiding the calculation of the eigenfunction $\psi_\pm$ (see [25, 27, 29, 31]). The most beautiful θ-functional formula for the potential was obtained in [29].

However, in the next section we shall not follow the approach for solving the inverse problem which has been outlined above, but use another approach proposed for the first time in [32] (see the appendix, based on the idea of Its): it is possible to calculate all the family of the Bloch–Floquet functions. The original approach has been used later in the problems, where the effective calculation of the eigenfunction is impossible (as in the cases of *higher rank* commuting ordinary differential (OD) linear operators (see for example [37]).

6. Solution of the periodic and quasiperiodic inverse spectral problems. Baker–Akhiezer functions

We are going to solve the inverse spectral problem for the finite-gap potentials following the scheme proposed in [33–35] and based on the concept of the Baker–Akhiezer functions. These functions are uniquely defined by their analytical properties on the spectral Riemann surface. As we shall see later, this scheme is evenly applicable to the solution of inverse problems in the two-dimensional case where the corresponding analytical properties naturally generalize the analytical properties of the Bloch–Floquet solutions for finite-gap Schrödinger operators.

Let us start with the following real inverse spectral data:

(1) Riemann surface Γ given in the form

$$z^2 = (\epsilon - \epsilon_0)\dots(\epsilon - \epsilon_{2n}),$$

where all numbers ϵ_j are real;

(2) a set of n real points $\gamma_j \in \Gamma$ such that there is exactly one point γ_j on the cycle a_j.

Below we shall, for brevity, identify the point γ_j as a point on Γ with its projection on the complex ϵ-plane. The condition that there is only one point on each cycle means that projection of the points satisfies the restriction:

$$\epsilon_{2j-1} \leqslant \gamma_j \leqslant \epsilon_{2j}, \qquad j = 1, \dots n, \tag{6.1}$$

By definition, the complex inverse spectral data are the same data where the Riemann surface (i.e. its branching points ϵ_k) and the points $\gamma_j \in \Gamma$ are arbitrary complex points.

As we shall see below, a generic set of algebraic–geometric spectral data leads to the explicit solution of the inverse spectral problem in terms of the Riemann θ-function. The corresponding potentials are complex meromorphic *quasi-periodic* functions of the variable x. For the real data described above, we are coming to the smooth (even analytic) quasiperiodic potentials with their periods expressed through the hyperelliptic integrals (see below). It is necessary to mention that there is no way to find simple and effective criteria for the potential to be periodic in terms of these data. The periods depend on the Riemann surface only. Of course, we may write the condition that all the corresponding hyperelliptic integrals are commensurable; however, this condition is transcendental. Recently, based on the results of [7, 65], the effective numerical approach has been developed for the solution of this problem. It is based on the discovery of some specific dynamic systems on the set of potentials which preserve all the periods but change the spectrum. Following [65], we start from one periodic potential and create all others using these dynamic systems.

In the last section we shall present some examples of finite-gap potentials written in terms of elliptic functions. They are periodic in the variable x (even double-periodic as functions of the complex variable x). The first nontrivial examples different from the classical Lamé potentials $u(x) = n(n+1)\wp(x)$ were found in [25]. This subject was developed in [66–70].

We define the Baker–Akhiezer function $\psi = \psi(x, t_1, \dots t_n; P)$ for the parameters x, t_j and the point $P = (\epsilon, \pm) \in \Gamma$, by its analytical properties on Γ with respect to the variable P. For the case $t_1 = t_2 = \dots = 0$ these analytical properties are just the same as the analytical properties of the Bloch–Floquet solutions of the periodic finite-gap operator.

From pure algebraic–geometric arguments it follows that there exists a unique function ψ such that

(1) it is meromorphic on Γ outside infinity and has at most simple poles at the points γ_j, $j = 1, \dots, n$;

(2) in the neighbourhood of infinity the function ψ has the form

$$\psi = \exp[xk + t_1k^3 + \cdots + k^{2n+1}t_n](1 + \xi_1(x,t)k^{-1} + \cdots),$$

where $k^2 = \epsilon$ and, therefore, k^{-1} is a local coordinate on the Riemann surface near infinity.

A proof of this statement is identical to the proof of the existence and uniqueness of general Baker–Akhiezer functions that were introduced in [33] for construction of exact solutions of the two-dimensional KP equation and all associated Zakharov–Shabat hierarchies.

The general Baker–Akhiezer function is defined with the help of an arbitrary Riemann surface of finite genus n instead of special hyperelliptic (i.e. two-sheeted) surfaces. We fix an arbitrary 'infinity' point P_0 on it; a local coordinate $k^{-1} = w$ where $w(P_0) = 0$; a generic set of points $(\gamma_1, \ldots, \gamma_n)$, and numbers $(x = \tau_1, \tau_2, \ldots, \tau_k, \ldots)$. The corresponding Baker–Akhiezer function has the same analytical properties as above but (2) is replaced by the following:

$$\psi = \exp[xk + \tau_1k^2 + \tau_2k^3 + \cdots](1 + \xi_1(x,\tau)k^{-1} + \cdots). \tag{6.2}$$

For the KdV hierarchy we have a hyperelliptic Riemann surface, $k^2 = \epsilon$, and $\tau_{2k} = 0$, $\tau_{2j+1} = t_j$.

The use of such functions (dependent only on a single parameter x) was proposed by Baker [71] for the common eigenfunctions of two commuting OD linear operators. He expected that this construction would improve the results of [63, 64]. He also made the very interesting conjecture that this approach might seriously improve the classical theory of θ-functions. Unfortunately, this proposal was not realized and was forgotten. The soliton theory appeared many decades later absolutely independently. The 1970s saw the start of the use of such functions on Riemann surfaces in the process of solution of a periodic problem for the KdV-type systems and inverse spectral periodic problems. In the classical spectral theory, Akhiezer [72] was the first to use some special cases of this function type for the construction of some examples of operators on the half-line $x \geqslant 0$ with interesting spectral properties. No author before the 1970s had associated anything like that with periodic problems. Indeed, this type of functional construction on the Riemann surface was extracted in 1974 from [72].

We shall prove the existence and uniqueness of the Baker–Akhiezer function and present its exact expression through the θ-functions later. At this moment we would like to show how the uniqueness of this function leads to the proof that ψ is an eigenfunction for the Schrödinger operator for our special case. We apply the operator $\partial = \partial_x$ twice to this function and use analytical properties. After elementary calculation, we arrive at the formulae:

$$\begin{aligned} \partial\psi &= k\psi + \exp[kx + \cdots](\xi_{1x}k^{-1} + \cdots), \\ \partial^2\psi &= k^2\psi + 2\xi_{1x}\psi + \mathrm{O}(k^{-1})\exp[kx + \cdots]. \end{aligned} \tag{6.3}$$

From the latter equations, we get the equality

$$(\partial^2 - k^2 - 2\xi_{1x})\psi = \mathrm{O}(k^{-1})\exp[kx + \cdots].$$

The left-hand side is a globally well-defined function on the same Riemann surface because $k^2 = \epsilon$. It has the same poles independent of parameters. Up to the same exponential factor, it is of the order $\mathrm{O}(k^{-1})$ at the infinity. Therefore, it is equal to zero due to the uniqueness of the Baker–Akhiezer function.

So we come to the conclusion that

$$L\psi = \epsilon\psi, \qquad L = -\partial^2 + u, \qquad u = -2\xi_{1x}. \tag{6.4}$$

We can apply the operators $\partial_j = \partial_{t_j}$ to the function ψ:

$$\partial_j\psi = k^{2j+1}\psi + \exp[kx + \cdots](\xi_{1j}k^{-1} + \xi_{2j}k^{-2}). \tag{6.5}$$

As in the previous case, we can easily construct a linear operator $A_j = \partial_x^{2j+1} + \cdots$ with the coefficients independent of k such that $(\partial_j - A_n)\psi = \mathrm{O}(k^{-1})\exp(kx + \cdots)$. Using global analytical properties on the Riemann surface, we deduce from this that the left-hand side is equal to zero, as before. The compatibility condition of these pairs of equations is exactly a KdV_n system for the potential $u(x, t_1, \ldots, t_m)$.

Following [33–35], we can prove in the same way that for the general Baker–Akhiezer function $\psi(x, \tau, P)$ associated with an arbitrary Riemann surface with fixed local coordinate near a puncture, the following equations are valid:

$$(\partial_{\tau_k} - L_k)\psi = 0, \qquad k = 2, 3, \ldots. \tag{6.6}$$

Here $L_k = \partial_x^k + \cdots$ are linear OD operators acting on the variable x with coefficients depending on parameters τ. These coefficients are differential polynomials in the coefficients of the expansion of the regular factor of the Baker–Akhiezer function at the puncture (infinity). They are uniquely defined by the condition that the expansion of the left-hand side of (6.6) at the puncture has the form $\mathrm{O}(k^{-1})\psi$.

The compatibility conditions

$$[\partial_{\tau_j} - L_j, \partial_{\tau_k} - L_k] = 0. \tag{6.7}$$

are equivalent to nonlinear PDEs for the coefficients of the operators L_k. For the case $j = 2, k = 3$, $\tau_2 = y$, $\tau_3 = z$, the operators L_2 and L_3 have the form

$$L_2 = \partial_x^2 - u(x, y, t), \qquad L_3 = \partial_x^3 - \tfrac{3}{2}u\partial_x + w(x, y, t), \tag{6.8}$$

where we consider the dependence of the coefficients with respect to the first three variables $\tau_1 = x$, $\tau_2 = y$, $\tau_3 = t$, only. The coefficient $u(x, y, t)$ is equal to $u = 2\xi_{1x}(x, y, t)$, where ξ_1 is the first coefficient of the expansion (6.2).

From (6.7) we get a system of two equations for two coefficients $u(x, y, t)$ and $w(x, y, t)$ which can be reduced for an equation for $u(x, y, t)$. The reduced equation is the famous KP equation

$$3u_{yy} = (4u_t - 6uu_x + u_{xxx})_x. \tag{6.9}$$

It appeared in the physics literature for the investigation of the transversal stability of the KdV solitons (see [73]) and is one of the most natural physical two-dimensional analogues of KdV. The Lax representation for it was found in [22, 23]. We call the whole set of higher systems (6.7) the KP hierarchy. We shall discuss the periodic problem for this equation at greater length in the next section.

For the special choice of the hyperelliptic Riemann surface and $k = \pm\sqrt{\epsilon}$ we have $k^{2s} = \epsilon^s$, where the function ϵ is well-defined globally as a meromorphic function on the Riemann surface Γ. Therefore, we may represent globally the corresponding Baker–Akhiezer function in the form

$$\psi = \tilde{\psi}\exp(\tau_2 k^2 + \tau_4 k^4 + \cdots), \tag{6.10}$$

where $\tilde{\psi}$ does not depend on the parameters τ_{2j}. So in this case all the KP hierarchy reduces to the KdV hierarchy.

The relationship of this construction with commuting OD linear operators is as follows. Let $f(P)$ be a meromorphic function on the Riemann surface Γ with one pole at the puncture P_0. Its negative part written in the parameter k is some polynomial $q(k) = q_1^l + q_2 k^{l-1} + \cdots + q_l k$. Apply the operator $A_f = q_1 L_l + q_2 L_{l-1} + \cdots + q_1\partial_x$ to the function ψ. By the definition of the operators L_k, we can see that

$$(A_f - f)\psi = \mathrm{O}(k^{-1})\psi. \tag{6.11}$$

We conclude, as before, that the difference $A_f\psi - f(P)\psi = 0$.

For any pair of functions f and g on the Riemann surface Γ with poles at the point ∞, we get a pair of the commuting OD scalar linear operators A_f, A_g such that $A_f A_g = A_g A_f$. In a special case of the hyperelliptic Riemann surface $z^2 = (\epsilon - \epsilon_0)\dots(\epsilon - \epsilon_{2n})$, we have a pair of functions $f = \epsilon$, $g = z$, leading to the Schrödinger operator $L = -\partial_x^2 + u$ commuting with the second operator of order $2n + 1$, because $z = k^{2n+1} + \cdots$.

Now we return to the problem of existence and uniqueness of the Baker–Akhiezer function. The simplest way to prove this existence is to define this function by an exact formula in terms of the θ-function and meromorphic differentials. Let us first recall the necessary information.

As in the previously discussed case of hyperelliptic curves, we introduce a basis of cycles a_j, b_j, $j = 1, \dots, g$, on a Riemann surface Γ of genus g with canonical matrix of intersections, $a_i \cdot b_j = \delta_{ij}$, and a basis ω_i of holomorphic differentials normalized by the condition

$$\oint_{a_i} \omega_j = \delta_{ij}. \tag{6.12}$$

The matrix $B = (B_{ij})$ of b-periods of these differentials

$$B_{ij} = \oint_{b_i} \omega_j, \tag{6.13}$$

is symmetric and has a positively defined imaginary part. The Riemann θ-function is a function defined with the help of this matrix by the formula

$$\theta(z|B) = \sum_{m \in Z^g} \mathrm{e}^{2\pi \mathrm{i}(z,m)+(Bm,m)} \tag{6.14}$$

where $z = (z_1, \dots, z_g)$ is a complex g-dimensional vector, (m, z) stands for a standard scalar product and summation is taken over all integer vectors $m = (m_1, \dots, m_g)$. The θ-function is an entire periodic function of g variables z_j and has the following monodromy properties with respect to the shifts defined by vectors B_k which are columns of the matrix of b-periods:

$$\theta(z + B_k) = \mathrm{e}^{-2\pi \mathrm{i} z_k - \pi \mathrm{i} B_{kk}} \theta(z). \tag{6.15}$$

The basic vectors e_k and the vectors B_k define a lattice $\mathcal{L}$ in C^g which determines the g-dimensional complex torus $J(\Gamma) = C^g/\mathcal{L}$ called the Jacobian of the curve. The Abel map $A : \Gamma \to J(\Gamma)$ is defined by the formula

$$A_k(P) = \int_{P_0}^{P} \omega_k. \tag{6.16}$$

Note that the vector $A(P)$ with coordinates $A_k(P)$ depends on the choice of path of integration but its ambiguity just coincides with shifts by vectors of the lattice $\mathcal{L}$.

From the monodromy properties of the θ-function it follows that zeros of the multivalued function $\theta(A(P) + Z)$ considered as a function on Γ are well defined. For a generic vector Z this function has exactly g zeros $(\gamma_1, \dots, \gamma_g)$. The vector Z can be expressed in terms of Abel transforms of these points by the formula

$$Z = -\sum_{j=1}^{g} A(\gamma_j) + K, \tag{6.17}$$

where K is a vector of the Riemann constants.

Let us introduce a set of meromorphic differentials $\mathrm{d}\Omega_i$ that are holomorphic on Γ outside the puncture where they have poles of the form

$$\mathrm{d}\Omega_i = \mathrm{d}k^i(1 + \mathrm{O}(k^{-i-1})), \tag{6.18}$$

and are normalized in the usual way by the condition

$$\oint_{a_i} \mathrm{d}\Omega_j = 0. \tag{6.19}$$

The Abelian integrals

$$\Omega_i(P) = \int^P \mathrm{d}\Omega_i \tag{6.20}$$

are multivalued functions on Γ.

Let U_j be a vector with the coordinates

$$U_{jk} = \frac{1}{2\pi \mathrm{i}} \oint_{b_k} \mathrm{d}\Omega_j. \tag{6.21}$$

Then, from the statements presented above, it follows directly that the formula

$$\psi(\tau, P) = \exp\left(\sum_i \tau_i \Omega_i(P)\right) \frac{\theta((A(P) + \sum_i U_i\tau_i + Z)\theta(Z)}{\theta(A(P) + Z)\theta(\sum_i U_i\tau_i + Z)} \tag{6.22}$$

correctly defines a function on Γ which satisfies all the properties of the Baker–Akhiezer function.

Suppose now that there exists another Baker–Akhiezer function ψ_1. From the definition of the Baker–Akhiezer functions it follows that the ratio ψ_1/ψ is a meromorphic function on Γ which is equal to one at the puncture and with the only possible poles at the zeros of the function ψ. According to (6.22) the zeros of ψ are zeros of the function $\theta\left(A(P) + \sum_i U_i\tau_i + Z\right)$. Therefore, ψ has g zeros. The simplest form of the Riemann–Roch theorem (which can be considered as a generalization of the Liouville theorem for Riemann surfaces) implies that a function on Γ with at most g poles at a generic set of points is a constant. Therefore, $\psi_1 = \psi$ and the existence and uniqueness of the Baker–Akhiezer function is proved.

Now, according to the previously established formula $u = 2\partial_x\xi_1$, in order to get an exact formula for a solution of the KP hierarchy it is enough to take the first coefficient $\xi_1(\tau)$ of the expansion of the pre-exponential factor in (6.2) at the puncture. Finally, we obtain the expression

$$u(\tau) = 2\partial_x^2 \log\theta\left(\sum_i U_i\tau_i + Z\right) + \text{const} \tag{6.23}$$

for the finite-gap solutions of the whole KP hierarchy.

If we consider only the KP equation, we get the formula

$$u(x, y, t) = 2\partial_x^2 \log\theta(Ux + Vy + Wt + Z) + \text{const}, \tag{6.24}$$

where we redenote $x = \tau_1$, $y = \tau_2$, $t = \tau_3$ and $U = U_1$, $V = U_2$, $W = U_3$.

For the case of the hyperelliptic curve the vector $V = 0$ and we get the Its–Matveev formula for the finite-gap solutions of the KdV equation.

The formula (6.24) derived in [34, 35] has led to one of the most important pure mathematical applications of the theory of nonlinear integrable systems. This is the solution of the famous Riemann–Shottky problem.

According to the Torrelli theorem, the matrix of b-periods of normalized holomorphic differentials uniquely defines the corresponding algebraic curve. The Riemann–Shottky problem is: to describe symmetric matrices with the positive imaginary part which are the matrices of b-periods of normalized holomorphic differentials on algebraic curves. One of the authors conjected that the function $u(x, y, t)$ given by (6.24) is a solution of the KP equation if and only if the matrix B that defines the θ-function is the matrix of b-periods of normalized holomorphic differentials on an algebraic curve and U, V, W are vectors of b-periods of corresponding normalized meromorphic differentials with the only pole at a point of this curve. This conjecture was proved in [74].

7. Spectral theory of two-dimensional periodic operators. KP hierarchy

A general algebraic–geometric construction of the finite-gap potentials for the Schrödinger operators and for solutions of the KP hierarchy that was presented in the previous section has been developed extensively over the years. It is applicable for all soliton systems which are equivalent to various types of compatibility conditions for over-determined systems of auxiliary linear problems. In its algebraic form it is in some sense local and is a sort of inverse transform: from a set of algebraic–geometrical data to solutions of the integrable nonlinear partial differential equations

$$\{\text{algebraic–geometrical data}\} \longmapsto \{\text{solutions of NLPDE}\}. \tag{7.1}$$

In a generic case the space of algebraic–geometrical data is a union for all g of the spaces

$$\tilde{M}_{g,N} = \{\Gamma_g, P_\alpha, k_\alpha^{-1}(Q), \gamma_1, \ldots, \gamma_g\}, \qquad \alpha = 1, \ldots, N, \tag{7.2}$$

where Γ_g is an algebraic curve of genus g with fixed local coordinates $k_\alpha^{-1}(Q), k_\alpha^{-1}(P_\alpha) = 0$, in neighbourhoods of N punctures P_α, and $\gamma_1, \ldots, \gamma_g$ are points of Γ_g in a general position. (It is to be mentioned that $\tilde{M}_{g,N}$ are 'universal' data. For the given nonlinear integrable equation the corresponding subset of data has to be specified.)

A posteriori it can be shown that these solutions can be expressed in terms of the corresponding Riemann θ-functions and are quasi-periodic functions of all variables. Within this approach it is absolutely impossible to give an answer to the basic question: 'How many algebraic–geometrical solutions are there? And what is their role in the solution of the periodic Cauchy problem for two-dimensional equations of the KP type?'

The answer to the corresponding question in lower dimensions is as follows. For finite-dimensional (0 + 1) systems a typical Lax representation has the form

$$\partial_t U(t, \lambda) = [U(t, \lambda), V(t, \lambda)], \tag{7.3}$$

where $U(t, \lambda)$ and $V(t, \lambda)$ are matrix functions that are rational (or sometimes elliptic) functions of the spectral parameter λ. In that case *all* the general solutions are algebraic–geometrical and can be represented in terms of the Riemann θ-functions.

For spatial one-dimensional evolution equations of the KdV type ((1 + 1)-systems) the existence of a direct and inverse spectral transform allows one to prove (although it is not always a rigorous mathematical statement) that algebraic–geometrical solutions are dense in the space of all periodic (in x) solutions.

It turns out that the situation for two-dimensional integrable equations is much more complicated. For one of the real forms of the KP equation that is called the KP-2 equation and coincides within (6.9), the algebraic–geometrical solutions are dense in the space of all periodic (in x and y) solutions [41]. It seems that the same statement for the KP-1 equation which can be obtained from (6.9) by replacing $y \to \mathrm{i}y$ is wrong. One of the most important problems in the theory of two-dimensional integrable systems which are still unsolved is: 'in what sense' is the KP-1 equation, which has the operator representation (6.7) and for which a wide class of periodic solution was constructed, a '*non-integrable*' system.

The proof of the integrability of the periodic problem for the KP-2 equation is based on the spectral Floquet theory of the parabolic operator

$$M = \partial_y - \partial_x^2 + u(x, y), \tag{7.4}$$

with periodic potential $u(x + l_1, y) = u(x, y + l_2) = u(x, y)$. We are now going to present the most essential points of this theory, which was developed in [41]. It is the natural generalization of the spectral theory of the periodic Sturm–Liouville operator. We would like to mention that, despite its application to the theory of nonlinear equations and related topics, the structure of

the Riemann surface of Bloch solutions of the corresponding linear equation that was found in [41] has been used as a starting point for an abstract definition of the Riemann surfaces of the infinite genus [75].

Solutions $\psi(x, y, w_1, w_2)$ of the nonstationary Schrödinger equation

$$(\sigma\partial_y - \partial_x^2 + u(x, y))\psi(x, y, w_1, w_2) = 0 \tag{7.5}$$

with a periodic potential $u(x, y) = u(x + a_1, y) = u(x, y + a_2)$ are called Bloch solutions if they are eigenfunctions of the monodromy operators, i.e.

$$\psi(x + a_1, y, w_1, w_2) = w_1\psi(x, y, w_1, w_2), \tag{7.6}$$

$$\psi(x, y + a_2, w_1, w_2) = w_2\psi(x, y, w_1, w_2). \tag{7.7}$$

The Bloch functions will always be assumed to be normalized so that $\psi(0, 0, w_1, w_2) = 1$. The set of pairs $Q = (w_1, w_2)$, for which there exists such a solution, is called the Floquet set and will be denoted by Γ. The multivalued functions $p(Q)$ and $E(Q)$ such that

$$w_1 = \mathrm{e}^{\mathrm{i}pa_1}, \qquad w_2 = \mathrm{e}^{\mathrm{i}Ea_2}$$

are called quasi-momentum and quasi-energy, respectively.

The gauge transformation $\psi \to \mathrm{e}^{h(y)}\psi$, where $\partial_y h(y)$ is a periodic function, transfers the solutions of (7.5) into solutions of the same equation but with another potential $\tilde{u} = u - \sigma\partial_y h$. Consequently, the spectral sets corresponding to the potentials u and $\tilde{u}$ are isomorphic. Therefore, in what follows we restrict ourselves to the case of periodic potentials such that $\int_0^{a_1} u(x, y)\,\mathrm{d}x = 0$.

To begin with let us consider as a basic example the 'free' operator $M_0 = \sigma\partial_y - \partial_x^2$ with zero potential $u(x, y) = 0$. The Floquet set of this operator is parametrized by the points of the complex plane of the variable k $w_1^0 = \mathrm{e}^{\mathrm{i}ka_1}$, $w_2^0 = \mathrm{e}^{-\sigma^{-1}k^2a_2}$, and the Bloch solutions have the form $\psi(x, y, k) = \mathrm{e}^{\mathrm{i}kx-\sigma^{-1}k^2y}$. The functions $\psi^+(x, y, k) = \mathrm{e}^{-\mathrm{i}kx+\sigma^{-1}k^2y}$ are Bloch solutions of the formal adjoint operator $(\sigma\partial_y + \partial_x^2)\psi^+ = 0$.

An image of the map $k \in C \longmapsto (w_1^0, w_2^0)$ is the Floquet set for the free operator M_0. It is the Riemann surface with self-intersections. The self-intersections correspond to the pairs $k \neq k'$ such that $w_i^0(k) = w_i^0(k')$, $i = 1, 2$. The latter conditions imply the equations

$$k - k' = \frac{2\pi N}{a_1}, \qquad k^2 - (k')^2 = \frac{\sigma 2\pi\mathrm{i}M}{a_2}, \tag{7.8}$$

where N and M are integers. Hence, all the resonant points have the form

$$k = k_{N,M} = \frac{\pi N}{a_1} - \frac{\sigma\mathrm{i}Ma_1}{Na_2}, \qquad N \neq 0, \qquad k' = k_{-N,-M}. \tag{7.9}$$

The basic idea of the construction of the Riemann surface of Bloch solutions of the equation (7.5) that was proposed in [41] is to consider (7.5) as a perturbation of the free operator, assuming that the potential $u(x, y)$ is formally small.

For any $k_0 \neq k_{N,M}$ it is easy to construct a formal Bloch solution. It turns out that the corresponding formal series converges and defines a holomorphic function of k_0 for $|k_0| > M$ big enough and lies outside small neighbourhoods of the resonant points. Moreover, it can be shown that this function can be extended on the Riemann surface that can be thought of as a surface obtained from the complex plane by some kind of surgery that creates *gaps* in places of the resonant points.

More precisely, if $u(x, y)$ is a smooth real potential that has analytical continuation in some neighbourhood of the real values of x and y, then the corresponding Riemann surface of Bloch–Floquet solutions can be described in the following way.

Let us fix some finite or infinite subset S of integer pairs $(N > 0, M)$. The set of pairs of complex numbers $\pi = \{p_{s,1}, p_{s,2}\}$ where $s \in S$ would be called 'admissible', if

$$\operatorname{Re} p_{s,i} = \frac{\pi N}{a_1}, \qquad |p_{s,i} - k_s| = o(|k_s|^{-1}), \qquad i = 1, 2, \tag{7.10}$$

and the intervals $[p_{s,1}, p_{s,2}]$ do not intersect. (Here k_s, $s = (N, M)$ are resonant points.)

Let us define the Riemann surface $\Gamma(\pi)$ for any admissible set π. It is obtained from the complex plane of the variable k by cutting it along the intervals $[p_{s,1}, p_{s,2}]$ and $[-\bar{p}_{s,1}, -\bar{p}_{s,2}]$ and then by sewing the left side of the first cut with the right side of the second cut and vice versa. (After this surgery for each cut $[p_{s,1}, p_{s,2}]$ corresponds to a nontrivial cycle a_s on $\Gamma(\pi)$.)

For any real periodic potential $u(x, y)$ which can be analytically extended into some neighbourhood of the real values x, y, the Bloch solutions of the equation (7.5) are parametrized by points Q of the Riemann surface $\Gamma(\pi)$, corresponding to some admissible set π. The function $\psi(x, y, Q)$ which is normalized by the condition $\psi(0, 0, Q) = 1$ is meromorphic on Γ and has a simple pole γ_s on each cycle a_s. If the admissible set π contains only a finite number of pairs, then $\Gamma(\pi)$ has finite genus and is compactified by only one point P_1 $(k = \infty)$, in the neighbourhood of which the Bloch function ψ has the form (6.2).

The potentials u for which $\Gamma(\pi)$ has finite genus are called finite gap. They coincide with the algebraic–geometrical potentials. The direct spectral transform for periodic operators (7.4) allows us to prove that as in the one-dimensional case the finite-gap potentials are dense in the space of all periodic smooth functions in two variables [41].

8. Spectral theory of the two-dimensional Schrödinger operator for a fixed energy level and two-dimensional Toda lattice

In this section we discuss a spectral theory of the two-dimensional periodic Schrödinger operator. Unlike the one-dimensional case, spectral data for the two-dimensional linear operator are over-determined and therefore for generic operators there are no nontrivial isospectral flows. As was noted in [77], deformations that preserve spectral data for *one fixed* energy level do exist. An analogue of the Lax representation for such a system has the form

$$H_t = [A, H] + BH, \tag{8.1}$$

where H, A, B are two-dimensional operators with coefficients depending on x, y, t. Equation (8.1) is equivalent to the condition that operators H and $(\partial_t - A)$ commute on the space of solutions of the equation $H\psi = 0$. Therefore, (8.1) describes deformations preserving all the spectral data associated with the zero energy level of the operator H.

It should be mentioned that until the moment when equations (8.1) were proposed in the framework of the soliton theory the spectral problem associated with one energy level of two-dimensional periodic operators had never been considered.

For the first time an inverse algebraic–geometric spectral problem for a two-dimensional Schrödinger operator in the magnetic field

$$H = (\mathrm{i}\partial_x - A_x(x, y))^2 + (\mathrm{i}\partial_y - A_y(x, y)^2 + u(x, y) \tag{8.2}$$

was formulated and solved in [36].

Consider the Bloch solutions of the equation $H\psi = \epsilon\psi$, which by definition are solutions that at the same time are eigenfuctions for the shift operators:

$$\psi(x + T_1, y) = \mathrm{e}^{\mathrm{i}p_x T_1}\psi(x, y), \qquad \psi(x, y + T_2) = \mathrm{e}^{\mathrm{i}p_y T_2}\psi(x, y). \tag{8.3}$$

Here T_1, T_2 are periods of the operator H, i.e. periods of the potential $u(x, y)$ and periods of the magnetic field $B(x, y)$, which is defined by the vector potential (A_x, A_y) by the formula $B = \partial_y A_x - \partial_x A_y$.

Multivalued quantities p_x, p_y are components of the two-dimensional quasi-momentum. For fixed values of p_x, p_y a spectrum of the operator H restricted on the space of functions satisfying (8.3) is discrete and defines different branches of dispersion relations $\epsilon_j(p_x, p_y)$, $j = 1, \ldots$. Level lines $\epsilon_j(p_x, p_y) = \epsilon_0$ in the space of variables p_x, p_y define the so-called *Fermi curves*. Of course, in solid-state physics all the considerations were restricted by real values of quasi-momentum.

In [36] it was suggested to consider operators for which a *complex* Fermi curve does exist and is the Riemann surface of finite genus for some energy level ϵ_0. Moreover, it was assumed that this curve is compactified by two *infinity* points $P_\pm$ in the neighbourhoods of which the corresponding Bloch solutions have the form

$$\psi = \mathrm{e}^{k_\pm(x \pm \mathrm{i}y)}\left(\sum_{s=0}^{\infty} \xi_s^\pm(x, y) k_\pm^{-s}\right), \tag{8.4}$$

where $k_\pm^{-1}$ are local coordinates in the neighbourhoods of the punctures $P_\pm$. It was also assumed that outside the punctures the function $\psi(x, y, P)$ considered as a function of the variable P (which is a point of the complex Fermi curve Γ) is a meromorphic function with g poles independent of the variables x and y.

We present here a solution to this inverse spectral problem in a more general form which is necessary for the construction of exact solutions to the two-dimensional Toda lattice which has deep connections with the theory of two-dimensional Schrödinger operators. After that we return to the spectral problems.

Let Γ be a smooth algebraic curve of genus g with fixed local coordinates $w_\pm(P)$ in the neighbourhoods of the points $P^\pm$, $w_\pm(P^\pm) = 0$. Then for each set of g points $\gamma_1, \ldots, \gamma_g$ in general position there exists a unique function $\psi_n(T, P)$, $T = \{t_i^\pm, i = 1, \ldots, \infty, \}$ such that:

(1^0) The function ψ_n of the variable $P \in \Gamma$ is meromorphic outside the punctures and has at most simple poles at the points γ_s (if all of them are distinct);

(2^0) In a neighbourhood of the point $P^\pm$ it has the form

$$\psi_n(T, P) = w_\pm^{\mp n}\left(\sum_{s=0}^{\infty} \xi_s^\pm(n, T) w_\pm^s\right) \exp\left(\sum_{i=1}^{\infty} w_\pm^{-i} t_i^\pm\right), \qquad w_\pm = w_\pm(P), \tag{8.5}$$

$$\xi_0^+(x, T) \equiv \delta_{\alpha j}. \tag{8.6}$$

The proof of this statement, as well as the explicit formula for ψ_n in terms of Riemann θ-functions, is almost identical to the method of solution of the inverse problem for finite-gap Schrödinger operators discussed in the previous sections.

Let $\mathrm{d}\Omega_j^{(\pm)}$ be a unique meromorphic differential holomorphic on Γ outside the point $P^\pm$, which has the form

$$\mathrm{d}\Omega_j^{(\pm)} = \mathrm{d}(w_\pm^{-j} + \mathrm{O}(w_\pm)), \tag{8.7}$$

near the point $P^\pm$, and normalized by the conditions $\oint_{a_k} \mathrm{d}\Omega_i^{(\pm)} = 0$. It defines a vector $U_j^{(\pm)}$ with coordinates

$$U_{jk}^{(\pm)} = \frac{1}{2\pi \mathrm{i}} \oint_{b_k} \mathrm{d}\Omega_j^{(\pm)}. \tag{8.8}$$

Further, let us define the normalized differential $\mathrm{d}\Omega^{(0)}$, which is holomorphic outside the points $P^\pm$ where it has simple poles with residues ± 1, respectively. From Riemann's bilinear relations it follows that the vector of b-periods of this differential equals $2\pi \mathrm{i} U^{(0)}$, where

$$U^{(0)} = A(P^-) - A(P^+). \tag{8.9}$$

As in the previous case, one can directly check that the function $\psi_n(T, P)$ given by the formula:

$$\psi_n(T, P) = \frac{\theta(A(P) + U^{(0)}n + \sum U_i^{(\pm)}t_i^{\pm} + Z)\theta(A(P^+) + Z))}{\theta(A(P) + Z)\theta(A(P^+) + U^{(0)}n + \sum U_i^{(\pm)}t_i^{\pm} + Z)} \mathrm{e}^{(n\Omega^{(0)}(P) + \sum t_i^{\pm}\Omega_i^{(\pm)}(P))}, \quad (8.10)$$

$$\Omega_i^{(\pm)}(P) = \int^P \mathrm{d}\Omega_i^{(\pm)}, \quad (8.11)$$

is well defined and has all the properties of the Baker–Akhiezer function.

Note, that for $n = 0$ and $t_1^{\pm} = x \pm \mathrm{i}y$, $t_i^{\pm} = 0$, $i > 1$, the analytical properties of this function coincide with the properties that were described above as analytical properties of the Bloch solutions for finite-gap two-dimensional Schrödinger operators.

From the uniqueness of the Baker–Akhiezer functions $\psi_n(T, P)$ it follows that they satisfy the linear equations

$$\partial_+\psi_n = \psi_{n+1} + v_n\psi_n, \qquad \partial_-\psi_n = c_n\psi_{n-1}, \qquad \partial_{\pm} = \frac{\partial}{\partial t_1^{\pm}}, \quad (8.12)$$

where

$$v_n = \partial_+\varphi_n(T), \qquad c_n = \mathrm{e}^{\varphi_n(T) - \varphi_{n-1}(T)}, \qquad \mathrm{e}^{\varphi_n} = \xi_0^-(T), \quad (8.13)$$

and $\xi_0^-(T)$ is a leading term of the expansion of ψ_n at the puncture P^-. From (8.10) we get the formula

$$\varphi_n = \log \frac{\theta(U^{(0)}(n+1) + \sum U_i^{(\pm)}t_i^{\pm} + Z_0)}{\theta(U^{(0)}n + \sum U_i^{(\pm)}t_i^{\pm} + Z_0)} + \mathrm{const}, \qquad Z_0 = Z + A(P^+). \quad (8.14)$$

for algebraic–geometric solutions of the two-dimensional Toda lattice which was obtained in [76].

Note that (8.12) imply that ψ_0 satisfies the equation

$$\partial_+\partial_-\psi_0 + v_0\partial_-\psi_0 + (c_1 + \partial_- v_0)\psi_0, \quad (8.15)$$

which is gauge equivalent to (8.2) and, therefore, we do get a solution of the inverse problem that was introduced above.

The next important step was done in [38, 39] where algebraic–geometric spectral data corresponding to potential two-dimensional Schrödinger operators (i.e. operators with zero magnetic field) were found.

Let Γ be a smooth genus g algebraic curve with fixed local coordinates $k_{\pm}^{-1}(Q)$, $k_{\pm}^{-1}(P_{\pm}) = 0$, in the neighbourhoods of two punctures $P_{\pm}$. Let us assume that there exists a holomorphic involution of the curve $\sigma : \Gamma \longmapsto \Gamma$ such that $P_{\pm}$ are its only fixed points, i.e. $\sigma(P_{\pm}) = P_{\pm}$. The local parameters are to be 'odd', i.e. $k_{\pm}(\sigma(Q)) = -k_{\pm}(Q)$. The factor curve will be denoted by Γ_0. The projection

$$\pi : \Gamma \longmapsto \Gamma_0 = \Gamma/\sigma \quad (8.16)$$

represents Γ as a two-sheet covering of Γ_0 with the two branch points $P_{\pm}$. In this realization the involution σ is a permutation of the sheets. As there are only two branching points $g = 2g_0$, where g_0 is the genus of Γ_0. Let us consider a meromorphic differential $\mathrm{d}\Omega(Q)$ of the third kind on Γ_0 with residues ∓ 1 at the points $P_{\pm}$. The differential $\mathrm{d}\Omega$ has g zeros that will be denoted by $\hat{\gamma}_i$, $i = 1, \ldots, 2g_0 = g$. Let us choose for each i a point γ_i on Γ such that

$$\pi(\gamma_i) = \hat{\gamma}_i, \qquad i = 1, \ldots, g. \quad (8.17)$$

In [38, 39] it was shown that the Baker–Akhiezer function corresponding to algebraic–geometric data which have been just defined satisfies the equation

$$(\partial_+\partial_- + u(x, y))\psi(x, y, Q) = 0, \quad (8.18)$$

where

$$u = -\partial_- \xi_1^+ = -\partial_+ \xi_1^-, \tag{8.19}$$

and $\xi_1^\pm = \xi_1^\pm(x_+, x_-)$ are the first coefficients in the expansion (8.4).

It should emphasized that although general formula (8.10) in terms of the Riemann θ-functions is valid for the Baker–Akhiezer functions corresponding to the potential Schrödinger operator H, in [38, 39] another more effective representation in terms of the so-called Prim θ-function was found.

A space of holomorphic differentials on Γ splits into two g_0-dimensional subspaces of even and odd (with respect to the involution σ) differentials. A matrix of b-periods of odd differentials defines the function $\theta_{Pr}(z)$ by the same formula (6.14). Then

$$\psi = \frac{\theta_{Pr}(A^{od}(Q) + U^+ x_+ + U^- x_- - Z)\theta_{Pr}(A^{od}(Z))}{\theta_{Pr}(A^{od}(Q) - Z)\theta_{Pr}(U^+ x_+ + U^- x_- - Z)} \mathrm{e}^{\mathrm{i}(p^+(Q)x_+ + p^-(Q)x_-)}. \tag{8.20}$$

Here $p^\pm(Q)$ are Abelian integrals of the second-kind normalized differential $\mathrm{d}p^\pm$ on Γ that have poles of the second order at points $P_\pm$, respectively; vectors $2\pi U^\pm$ are the vectors of b-periods of these differentials.

As was mentioned above, the inverse algebraic–geometric spectral problem on one energy level for a two-dimensional Schrödinger operator was posed and solved at the time when no direct spectral theory was known. This theory was developed much later in [41], where it was shown that the Bloch solutions for (8.18) with analytical periodic potential are parametrized by points of an infinite-genus Riemann surface. It was proved that if this surface has finite genus, then the Bloch functions have all the analytical properties suggested in the inverse problem and are, therefore, just the Baker–Akhiezer functions. Moreover, it was proved that algebraic–geometric (finite-gap) potentials are dense in the space of all periodic potentials.

9. Spectral theory of operators with elliptic coefficients

In this section we are going to discuss a specific spectral problem for operators with elliptic coefficients, i.e. with coefficients that are meromorphic functions of a variable x and have two periods 2ω, $2\omega'$, $\mathrm{Im}\,(\omega'/\omega) > 0$.

Since Hermite's time, it has been known that a one-dimensional Schrödinger operator with potential of the form $n(n+1)\wp(x)$ (where $\wp(x) = \wp(x|\omega, \omega')$ is a Weierstrass $\wp$-function corresponding to an elliptic curve with periods 2ω, $2\omega'$, and n is an integer) has only n gaps in the spectrum. These Lamé potentials had been the only known examples with the finite-gap property before the finite-gap theory was constructed in the framework of the soliton theory (see above). As we have already shown, a generic algebraic–geometric potential can be expressed in terms of a higher-genus Riemann θ-function. Sometimes, a higher-genus formula can be reduced to an exact expression in terms of the elliptic function.

The first example of this type which is different from the Lamé potentials was proposed in [25]. Later, a theory of elliptic finite-gap potentials attracted particular interest due to the remarkable observation made in [78] with regards to a connection with the elliptic Calogero–Moser model. The most recent burst of interest is due to the unexpected connections of these systems to the Seiberg–Witten solution of $N = 2$ supersymmetric gauge theories [14, 15]. It turns out that the low-energy effective theory for the $SU(N)$ model with matter in the adjoint representation (identified first in [1] with the $SU(N)$ Hitchin system) is isomorphic to the elliptic CM system. Using this connection, quantum order parameters were found in [79].

The elliptic CM system [80, 81] is a system of N identical particles on a line interacting

with each other via the potential $V(x) = \wp(x)$. Its equations of motion have the form

$$\ddot{x}_i = 4 \sum_{j \neq i} \wp'(x_i - x_j). \tag{9.1}$$

The CM system is a completely integrable Hamiltonian system, i.e. it has N independent integrals H_k in involution [82, 83]. The second integral H_2 is the Hamiltonian of (9.1).

In [78] it was shown that the elliptic solutions of the KdV equations have the form

$$u(x, t) = 2 \sum_{i=1}^{N} \wp(x - x_i(t)) \tag{9.2}$$

and the poles $x_i(t)$ of the solutions satisfy the constraint $\sum_{j \neq i} \wp'(x_i - x_j) = 0$, which is the *locus* of the stationary points of the CM system. Moreover, it turns out that the dependence of the poles with respect to t coincides with the Hamiltonian flow corresponding to the third integral H_3 of the system. In [84,85] it was found that this connection becomes an isomorphism in the case of the elliptic solutions of the KP equation. Moreover, in [66] it was revealed that the connection of the CM systems with the KP equation is in some sense secondary and is a corollary of more fundamental connections with the spectral theory of *linear* operators with elliptic potentials. The corresponding approach has been developed extensively in [67–69].

Let $\mathcal{L}$ be a linear differential or difference operator in two variables x, t with coefficients which are scalar or matrix elliptic functions of the variable x. We do not assume any special dependence of the coefficients with respect to the second variable. Then it is natural to introduce a notion of *double-Bloch* solutions of the equation

$$\mathcal{L}\psi = 0. \tag{9.3}$$

We call a *meromorphic* vector-function $f(x)$, which satisfies the following monodromy properties:

$$f(x + 2\omega_\alpha) = B_\alpha f(x), \qquad \alpha = 1, 2, \tag{9.4}$$

a *double-Bloch function*. The complex numbers B_α are called *Bloch multipliers*. (In other words, f is a meromorphic section of a vector bundle over the elliptic curve.)

In the most general form a problem that we are going to address is to *classify* and to *construct* all the operators $\mathcal{L}$ such that equation (9.3) has *sufficient* double-Bloch solutions.

It turns out that the existence of the double-Bloch solutions is so restrictive that only in exceptional cases do such solutions exist. A simple and general explanation of that is due to the Riemann–Roch theorem. Let D be a set of points x_i, $i = 1, \ldots, m$, on the elliptic curve Γ_0 with multiplicities d_i and let $V = V(D; B_1, B_2)$ be a linear space of the double-Bloch functions with the Bloch multipliers B_α that have poles at x_i of order less or equal to d_i and holomorphic outside D. Then the dimension of D is equal to

$$\dim D = \deg D = \sum_i d_i.$$

Now let x_i depend on the variable t. Then for $f \in D(t)$ the function $\mathcal{L}f$ is a double-Bloch function with the same Bloch multipliers but in general with higher orders of poles, because taking derivatives and multiplication by the elliptic coefficients increase these orders. Therefore, the operator $\mathcal{L}$ defines a linear operator

$$\mathcal{L}|_D : V(D(t); B_1, B_2) \longmapsto V(D'(t); B_1, B_2), \qquad N' = \deg D' > N = \deg D,$$

and (9.3) is *always* equivalent to an *over-determined* linear system of N' equations for N unknown variables which are the coefficients $c_i = c_i(t)$ of expansion of $\Psi \in V(t)$ with respect to a basis of functions $f_i(t) \in V(t)$. With some exaggeration one may say that in the

soliton theory the representation of a system in the form of the compatibility condition of an over-determined system of the linear problems is considered equivalent to integrability.

In all of the basic examples $N' = 2N$ and the over-determined system of equations has the form

$$LC = kC, \qquad \partial_t C = MC, \tag{9.5}$$

where L and M are $N \times N$ matrix functions depending on a point z of the elliptic curve as on a parameter. A compatibility condition of (9.5) has the standard Lax form $\partial_t L = [M, L]$, and is equivalent to a finite-dimensional integrable system.

The basis in the space of the double-Bloch functions can be written in terms of the fundamental function $\Phi(x, z)$ defined by the formula

$$\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} \mathrm{e}^{\zeta(z)x}. \tag{9.6}$$

Note, that $\Phi(x, z)$ is a solution of the Lamé equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2\wp(x)\right)\Phi(x, z) = \wp(z)\Phi(x, z). \tag{9.7}$$

From the monodromy properties it follows that Φ considered as a function of z is double-periodic:

$$\Phi(x, z + 2\omega_\alpha) = \Phi(x, z),$$

although it is not elliptic in the classical sense due to an essential singularity at $z = 0$ for $x \neq 0$.

As a function of x the function $\Phi(x, z)$ is a double-Bloch function, i.e.

$$\Phi(x + 2\omega_\alpha, z) = T_\alpha(z)\Phi(x, z), \qquad T_\alpha(z) = \exp(2\omega_\alpha\zeta(z) - 2\zeta(\omega_\alpha)z).$$

In the fundamental domain of the lattice defined by $2\omega_\alpha$ the function $\Phi(x, z)$ has a unique pole at the point $x = 0$:

$$\Phi(x, z) = x^{-1} + \mathrm{O}(x). \tag{9.8}$$

The gauge transformation $f(x) \longmapsto \tilde{f}(x) = f(x)\mathrm{e}^{ax}$, where a is an arbitrary constant does not change poles of any function and transforms a double-Bloch function into another double-Bloch function. If B_α are Bloch multipliers for f then Bloch multipliers for $\tilde{f}$ are equal to

$$\tilde{B}_1 = B_1\mathrm{e}^{2a\omega_1}, \qquad \tilde{B}_2 = B_2\mathrm{e}^{2a\omega_2}. \tag{9.9}$$

The two pairs of Bloch multipliers that are connected with each other through the relation (9.9) for some a are called equivalent. Note that for all equivalent pairs of Bloch multipliers the product $B_1^{\omega_2}B_2^{-\omega_1}$ is a constant depending on the equivalence class, only.

From (9.8) it follows that a double-Bloch function $f(x)$ with simple poles x_i in the fundamental domain and with Bloch multipliers B_α (such that at least one of them is not equal to one) may be represented in the form:

$$f(x) = \sum_{i=1}^{N} c_i \Phi(x - x_i, z)\mathrm{e}^{kx}, \tag{9.10}$$

where c_i is a residue of f at x_i and z, k are parameters related by $B_\alpha = T_\alpha(z)\mathrm{e}^{2\omega_\alpha k}$. (Any pair of Bloch multipliers may be represented in this form with an appropriate choice of the parameters z and k.)

Let us consider as an example the equation

$$\mathcal{L}\psi = (\partial_t - \partial_x^2 + u(x, t))\psi = 0, \tag{9.11}$$

where $u(x,t)$ is an elliptic function. Then, as shown in [66], equation (9.11) has N linear independent double-Bloch solutions with equivalent Bloch multipliers and N simple poles at points $x_i(t)$ if and only if $u(x,t)$ has the form

$$u(x,t) = 2\sum_{i=1}^{N} \wp(x - x_i(t)) \tag{9.12}$$

and $x_i(t)$ satisfy the equations of motion of the elliptic CM system (9.1).

The assumption that there exist N linear independent double-Bloch solutions with equivalent Bloch multipliers implies that they can be written in the form

$$\psi = \sum_{i=1}^{N} c_i(t,k,z)\Phi(x - x_i(t), z)\mathrm{e}^{kx+k^2t}, \tag{9.13}$$

with the same z but different values of the parameter k.

Let us substitute (9.13) into (9.11). Then (9.11) is satisfied if and only if we get a function holomorphic in the fundamental domain. First of all, we conclude that u has poles at x_i, only. The vanishing of the triple poles $(x - x_i)^{-3}$ implies that $u(x,t)$ has the form (9.12). The vanishing of the double poles $(x - x_i)^{-2}$ gives the equalities that can be written as a matrix equation for the vector $C = (c_i)$:

$$(L(t,z) - kI)C = 0, \tag{9.14}$$

where I is the unit matrix and the Lax matrix $L(t,z)$ is defined as follows:

$$L_{ij}(t,z) = -\tfrac{1}{2}\delta_{ij}\dot{x}_i - (1 - \delta_{ij})\Phi(x_i - x_j, z). \tag{9.15}$$

Finally, the vanishing of the simple poles gives the equations

$$(\partial_t - M(t,z))C = 0, \tag{9.16}$$

where

$$M_{ij} = \left(\wp(z) - 2\sum_{j\neq i}\wp(x_i - x_j)\right)\delta_{ij} - 2(1 - \delta_{ij})\Phi'(x_i - x_j, z). \tag{9.17}$$

The existence of N linear independent solutions for (9.11) with equivalent Bloch multipliers implies that (9.14) and (9.16) have N independent solutions corresponding to different values of k. Hence, as a compatibility condition we get the Lax equation $\dot{L} = [M, L]$ which is equivalent to (9.1). Note that the last system does not depend on z. Therefore, if (9.14) and (9.16) are compatible for some z, then they are compatible for all z. As a result we conclude that if (9.11) has N linear independent double-Bloch solutions with equivalent Bloch multipliers then it has infinitely many of them. All the double-Bloch solutions are parametrized by points of an algebraic curve Γ defined by the characteristic equation

$$R(k,z) \equiv \det(kI - L(z)) = k^N + \sum_{i=1}^{N} r_i(z)k^{N-i} = 0. \tag{9.18}$$

Equation (9.18) can be seen as a dispersion relation between two Bloch multipliers and defines Γ as N-sheet cover of Γ_0.

As was shown in [66] expansion of the characteristic equation (9.14) at $z = 0$ has the form:

$$R(k,z) = \prod_{i=1}^{N}(k + \nu_i z^{-1} + h_i + \mathrm{O}(z)), \qquad \nu_1 = 1 - N, \quad \nu_i = 1, \quad i > 1. \tag{9.19}$$

We call the sheet of Γ at $z = 0$ corresponding to the branch $k = z^{-1}(N-1) + O(1)$, an upper sheet and mark the point P_1 on this sheet among the pre-images of $z = 0$. From (9.19) it follows that in the general position when the curve Γ is smooth, its genus equals N.

Further consideration of analytical properties of the function ψ given by (9.13) where c_i are components of the eigenvector to the matrix L shows that this function is just the Baker–Akhiezer function introduced in section 6. Combined with expression (6.22) for ψ in terms of the θ-function this result leads directly to the main statement of [66]:

the coordinates of the particles $x_i(t)$ are roots of the equation

$$\theta(Ux + Vt + Z_0) = 0, \tag{9.20}$$

where $\theta(\xi) = \theta(\xi|B)$ is the Riemann θ-function corresponding to the matrix of b-periods of holomorphic differentials on Γ; the vectors U and V are the vectors of b-periods of normalized meromorphic differentials on Γ, with poles of order two and three at the point P_1.

Among other examples of integrable systems that can be generated in a similar way is the Ruijesenaars–Schneider system [86]

$$\ddot{x}_i = \sum_{s\neq i} \dot{x}_i\dot{x}_s(V(x_i - x_s) - V(x_s - x_i)), \qquad V(x) = \zeta(x) - \zeta(x+\eta), \tag{9.21}$$

and the nested Bethe ansatz equations [87]

$$\prod_{j\neq i} \frac{\sigma(x_i^n - x_j^{n+1})\sigma(x_i^n - \eta - x_j^n)\sigma(x_i^n - x_j^{n-1} + \eta)}{\sigma(x_i^n - x_j^{n-1})\sigma(x_i^n + \eta - x_j^n)\sigma(x_i^n - x_j^{n+1} - \eta)} = -1. \tag{9.22}$$

As shown in [68, 69], they are generated by spectral problems for equations

$$\mathcal{L}\psi = \partial_t\psi(x,t) - \psi(x+\eta,t) - v(x,t)\psi(x,t) = 0, \tag{9.23}$$

and

$$\psi(x, m+1) = \psi(x+\eta) + v(x,m)\psi(x,m), \tag{9.24}$$

respectively. (Here η is a complex number and $v(x,t)$ is an elliptic function.)

Strange as it seems, the inverse spectral problem which is discussed here is simpler for two-dimensional operators than for one-dimensional stationary operators. For example, a family of spectral curves corresponding to operators (9.11) that have double-Bloch solutions can be described explicitly. A nice formula was found in [79]:

$$R(k,z) = f(k - \zeta(z), z), \qquad f(k,z) = \frac{1}{\sigma(z)}\sigma\left(z + \frac{\partial}{\partial k}\right)H(k), \tag{9.25}$$

where $H(k)$ is a polynomial. Note that (9.25) may be written as

$$f(k,z) = \frac{1}{\sigma(z)}\sum_{n=1}^{N}\frac{1}{n!}\partial_z^n\sigma(z)\left(\frac{\partial}{\partial k}\right)^n H(k).$$

The coefficients of the polynomial $H(k)$ are free parameters of the spectral curve of the CM system.

The spectral curves corresponding to the Schrödinger operator with the same property are a special case of the curves (9.25) but their explicit description is unknown. In particular, the exact formula for branching points for Lamé potentials is unknown. As was mentioned above, the first example of elliptic finite-gap potentials different from the Lamé potentials was found in [25]. A wide class of such potentials was found in [70].

In [88] it was noted that the problem of classification of Schrödinger operators with elliptic potentials that have two double-Bloch solutions for almost all energy levels was posed by Picard, although this had not been solved until very recently. In [88], using Floquet spectral theory for the Schrödinger operator it was proved that all such potentials are finite gap. This result is an essential step in the Picard problem although its complete and effective solution is still an open problem.

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