Elliptic Analog of the Toda Lattice

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1 Introduction

The main goal of this paper is to construct the action-angle variables for a finite-dimensional Hamiltonian system of equations

$$\ddot{x}_n = (\dot{x}_n^2 - 1)(V(x_n, x_{n+1}) + V(x_n, x_{n-1})), \quad x_{n+N} = x_n, \tag{1.1}$$

where

$$V(u, v) = \zeta(u - v) + \zeta(u + v) - \zeta(2u) = -\frac{1}{2} \frac{\wp'(u - v) - \wp'(u + v)}{\wp(u - v) - \wp(u + v)}, \tag{1.2}$$

and to identify it as an *elliptic analog of* N-*periodic Toda lattice*. Here $\wp(x) = \wp(x \mid 2\omega, 2\omega')$ and $\zeta(x) = \zeta(x \mid 2\omega, 2\omega')$ are classical Weierstrass functions.

Recently, finite-dimensional integrable soliton systems have attracted very special interest due to their unexpected relations to the theory of supersymmetric gauge models. The celebrated Seiberg-Witten ansatz [39], [40] identifies moduli space of physically nonequivalent vacua of the model with moduli space of a certain family of algebraic curves. In [3], [20] it was shown that the family of curves corresponding to 4-dimensional $N = 2$ supersymmetric $SU(N_c)$ theory is defined by the equation

$$w^2 - wP_{N_c}(E) + \Lambda^{2N_c} = 0, \quad P_{N_c}(E) = E^{N_c} + \sum_{i=0}^{N_c-1} u_i E^i. \tag{1.3}$$

In [14] it was noted that this family can be identified with the family of spectral curves of N_c-periodic Toda lattice, and the Seiberg-Witten ansatz was linked with the Whitham

perturbation theory of finite-gap solutions of soliton equations proposed in [23], [24]. Integrable systems related to various gauge models coupled with matter hypermultiplets in various representations were considered in [2], [4]–[13], [15], [16], [18], [19], [28], [29], [31]–[37], [41], and [42], where a more complete list of references can be found.

In [13], [32] the N_c-periodic spin chain related to an XYZ model was proposed as a soliton counterpart of $N = 1$ supersymmetric $SU(N_c)$ theory in six dimensions compactified in two directions and coupled with $N_f = 2N_c$ matter hypermultiplets. Spectral curves of the N_c-periodic homogeneous XYZ spin chain have the form

$$w^2 - wP^{el}_{N_c}(z) + Q^{el}_{2N_c}(z) = 0, \tag{1.4}$$

where $P^{el}_{N_c}(z)$ and $Q^{el}_{2N_c}(z)$ are elliptic polynomials, that is, elliptic functions with poles of order N_c and $2N_c$ at the point $z = 0$. Note that (1.4) is an elliptic deformation of the family of curves found in [17] for a 4-dimensional $N = 2$ supersymmetric $SU(N_c)$ model coupled with matter hypermultiplets.

A particular case of (1.4), when $Q^{el}_{2N_c}(z)$ is a constant, $Q^{el}_{2N_c}(z) = \Lambda^{2N_c}$ can be seen as an elliptic deformation of (1.3). The corresponding family of curves depends on N_c parameters, which can be chosen as Λ, and the coefficients u_i of the representation of $P^{el}_N(z)$ in the form

$$P^{el}_N(z) = \frac{(-1)^N}{(N-1)!}\partial_z^{N-2}\wp(z) + \sum_{i=1}^{N-2} u_i\partial_z^{i-1}\wp(z) + u_0. \tag{1.5}$$

An attempt to find a soliton system corresponding to the family of spectral curves defined by the equation

$$w^2 - wP^{el}_N(z) + \Lambda^{2N} = 0 \tag{1.6}$$

led us to (1.1). After the system was found, it turned out that, by itself, it is not new. Up to a change of variables $q_n = \wp(x_n)$, it coincides with one of the systems listed in [1], where the classification of all Toda-type chains that have Toda-type symmetries was obtained. The new results obtained in this work are an isomorphism of (1.1) with a pole system corresponding to elliptic solutions of a 2-dimensional Toda lattice, the construction of action-angle variables, and an explicit solution of the system in terms of the theta-functions.

In [28], [29] it was shown that a wide class of solutions of the Seiberg-Witten ansatz can be described in terms of a special foliation on the moduli space of curves with punctures. That allows us to consider such systems as reductions of 2-dimensional

soliton equations. Following this approach, let us note that (1.6) defines an algebraic curve Γ as a 2-sheeted cover of the elliptic curve Γ_0 with periods $2\omega, 2\omega'$. Let $P_\pm$ be preimages on Γ of $z = 0$. According to the construction of [22], any algebraic curve with two punctures generates a family of algebro-geometric solutions of the 2-dimensional Toda lattice

$$\left(\partial_{tt}^2 - \partial_{xx}^2\right)\varphi_n = 4\left(e^{\varphi_{n+1} - \varphi_n} - e^{\varphi_n - \varphi_{n-1}}\right), \tag{1.7}$$

parameterized by points of the Jacobian $J(\Gamma)$ of the curve.

In the next section we show that algebro-geometric solutions $\varphi_n(x, t)$ corresponding to Γ defined by (1.6) are periodic in n up to the shift, $\varphi_n = \varphi_{n+N} + 2N \ln \Lambda$, and have the form

$$\varphi_n(x, t) = \alpha_n(t) + \ln \frac{\sigma\left(x - x_{n+1}(t) + a\right)\sigma\left(x + x_{n+1}(t) + a\right)}{\sigma\left(x - x_n(t) + a\right)\sigma\left(x + x_n(t) + a\right)}. \tag{1.8}$$

Substitution of (1.8) into (1.7) leads to (1.1) for $x_n(t)$.

In Section 3 we construct a new Lax representation for (1.1) and show that the spectral curve defined by the Lax operator has the form (1.6). We also prove that if $x_n(t)$ is a solution of (1.1), then there exist functions $\alpha_n(t)$ (unique up to the transformation $\alpha_n(t) \to \alpha_n(t) + c_1 t + c_2$, $c_i = \text{const}$), such that the functions $\varphi_n(x, t)$ of the form (1.8) satisfy (1.7).

The last section is devoted to the Hamiltonian theory of system (1.1). Equations (1.1) are generated by the Hamiltonian

$$H = \sum_{n=0}^{N-1} \ln \text{sh}^{-2}\left(\frac{p_n}{2}\right) + \ln\left(\wp(x_n - x_{n-1}) - \wp(x_n + x_{n-1})\right) \tag{1.9}$$

and by the canonical Poisson brackets $\{p_m, x_n\} = \delta_{nm}$. We emphasize that although this Hamiltonian structure can be easily checked directly, it was found by the author using the algebro-geometric approach to Hamiltonian theory of the Lax equations proposed in [28], [29] and developed in [25]. The main advantage of this approach is that it allows us to simultaneously find the action-angle variables and a generating differential that defines low-energy effective prepotential.

Note that from the relation of system (1.1) to a 2-dimensional Toda lattice, it is clear that degeneration of the elliptic curve Γ_0 corresponds to a degeneration of this system to the Toda lattice. It would be very interesting to consider this degeneration explicitly on the level of the Hamiltonian structure. We consider this problem elsewhere.

2   Elliptic solutions of a 2-dimensional Toda lattice

Algebro-geometric solutions of a 2-dimensional Toda lattice were constructed in [22]. Let Γ be a smooth genus g algebraic curve with fixed local coordinates $z_\pm(Q)$ in neighborhoods of two punctures $P_\pm \in \Gamma$, $z_\pm(P_\pm) = 0$. Then for any set of g points $\gamma_1, \ldots, \gamma_g$ in general position, there exists a unique function $\psi_n(x, t, Q)$ such that the following conditions hold.

(1) As a function of the variable $Q \in \Gamma$, $\psi_n(x, t, Q)$ is meromorphic on Γ outside the punctures $P_\pm$ and has at most simple poles at the points γ_s (if all of them are distinct).

(2) In the neighborhoods of the punctures, the function ψ_n has the form

$$\psi_n = z_\pm^{\mp N} e^{(x \pm t) z^{-1}} \left(\sum_{s=0}^{\infty} \xi_s^\pm(x, t) z_\pm^s \right), \quad \xi_0^+ = 1. \tag{2.1}$$

Uniqueness of ψ_n implies that it satisfies the following system of linear equations:

$$(\partial_t + \partial_x)\psi_n(x, t, Q) = 2\psi_{n+1}(x, t, Q) + v_n(x, t)\psi_n(x, t, Q), \tag{2.2}$$

$$(\partial_t - \partial_x)\psi_n(x, t, Q) = 2c_n(x, t)\psi_{n-1}(x, t), \tag{2.3}$$

where the coefficients are defined by the leading coefficient ξ_0^- of expansion (2.1) with the help of the formulae

$$v_n = (\partial_t + \partial_x)\varphi_n(x, t), \qquad c_n = e^{\varphi_n(x,t) - \varphi_{n-1}(x,t)}, \qquad \varphi_n(x, t) = \ln \xi_0^-(x, t). \tag{2.4}$$

Compatibility of (2.2) and (2.3) implies that $\varphi_n(x, t)$ is a solution of the 2-dimensional Toda lattice (1.7).

The function $\psi_n(x, t, Q)$ is called the Baker-Akhiezer function and can be explicitly expressed in terms of the Riemann theta-function associated with a matrix of b-periods of holomorphic differentials on Γ. The corresponding formula for φ_n is as follows.

Let us fix a basis of cycles a_i, b_i, $i = 1, \ldots, g$, on Γ with the canonical matrix of intersections $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$. The basis of normalized holomorphic differentials $d\Omega_j^h(Q)$, $j = 1, \ldots, g$, is defined by conditions $\oint_{a_i} d\Omega_j^h = \delta_{ij}$. The b-periods of these differentials define the Riemann matrix $B_{kj} = \oint_{b_j} d\Omega_k^h$. The basic vectors e_k of $\mathbb{C}^g$ and the vectors B_k, which are columns of the matrix B, generate a lattice $\mathcal{B}$ in $\mathbb{C}^g$. The g-dimensional complex torus

$$J(\Gamma) = \frac{\mathbb{C}^g}{\mathcal{B}}, \qquad \mathcal{B} = \sum n_k e_k + m_k B_k, \quad n_k, m_k \in \mathbb{Z}, \tag{2.5}$$

is called the Jacobian variety of Γ. A vector with the coordinates

$$A_k(Q) = \int_{P_+}^{Q} d\Omega_k^h \tag{2.6}$$

defines the Abel map $A : \Gamma \to J(\Gamma)$.

The Riemann matrix has a positive-definite imaginary part. The entire function of g variables $z = (z_1, \ldots, z_g)$,

$$\theta(z) = \theta(z \mid B) = \sum_{m \in Z^g} e^{2\pi i(z,m) + \pi i(Bm,m)}$$

is called the Riemann theta-function. It has the following monodromy properties:

$$\theta(z + e_k) = \theta(z), \qquad \theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}} \theta(z). \tag{2.7}$$

The function $\theta(A(Q) + Z)$ is a multivalued function of Q, but according to (2.7), the zeros of this function are well defined. For Z in a general position, the equation $\theta(A(Q) + Z) = 0$ has g zeros $\gamma_1, \ldots, \gamma_g$. The vector Z and the divisor of these zeros are connected by the relation $Z = -\sum_s A(\gamma_s) + \mathcal{K}$, where $\mathcal{K}$ is the vector of Riemann constants.

Let us introduce normalized Abelian differentials $d\Omega^{(x)}$ and $d\Omega^{(t)}$ of the second kind. They are holomorphic on Γ except at the punctures $P_\pm$. In the neighborhoods of $P_\pm$, they have the form

$$d\Omega^{(x)} = d(z_\pm^{-1} + O(1)), \qquad d\Omega^{(t)} = d(\pm z_\pm^{-1} + O(1)).$$

Normalized means that they have zero a-periods. The vectors of b-periods of these differentials are denoted by $2\pi i V$ and $2\pi i W$, that is, the coordinates of the vectors V and W are equal to

$$V_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega^{(x)}, \qquad W_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega^{(t)}. \tag{2.8}$$

Let $d\Omega^{(n)}$ be a normalized Abelian differential of the third kind with simple poles at the punctures $P_\pm$ with residues ∓ 1. From the Riemann bilinear relations, it follows that the vector of its b-periods satisfies the relation

$$U_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega^{(n)} = A(P_-) - A(P_+). \tag{2.9}$$

If we choose a branch of the Abelian integral $\Omega^{(n)}$ near P_+ such that $\Omega^{(n)} = -\ln z_+ + O(z_+)$, then near P_- it has the form

$$\Omega^{(n)} = \ln z_- + I_0 + O(z_-).$$

Theorem 2.1 (see [22]). The Baker-Akhiezer function is equal to

$$\psi_n(x, t, Q) = \frac{\theta\big(A(Q) + nU + xV + tW + Z\big)\theta(Z)}{\theta\big(nU + xV + tW + Z\big)\theta\big(A(Q) + Z\big)} \exp\big(n\Omega^{(n)} + x\Omega^{(x)} + t\Omega^{(t)}\big).$$

(2.10)

The function $\varphi_n(x, t)$ given by the formula

$$\varphi_n(x, t) = nI_0 + \ln\frac{\theta\big((n + 1)U + xV + tW + Z\big)}{\theta\big(nU + xV + tW + Z\big)}$$

(2.11)

is a solution of a 2-dimensional Toda lattice.   □

For a generic set of algebro-geometric data, the function $\varphi_n(x, t)$ given by (2.11) is a quasi-periodic meromorphic function of all the variables (n, x, t). In [30] the solutions of a 2-dimensional Toda lattice which are *elliptic in the discrete variable* n were considered. It was found that dynamics of its poles coincide with the elliptic Ruijsenaars-Schneider system [38]. In this paper we consider solutions that are elliptic in the variable x and are periodic in n.

The condition that φ_n is elliptic in one of the variables is equivalent to the property that the complex linear subspace in $J(\Gamma)$ spanned by the corresponding directional vector is *compact*, that is, it is an elliptic curve Γ_0. In the case of the x-variable it means that the vectors $2\omega_\alpha V, \alpha = 1, 2$, belong to the lattice $\mathcal{B}$ defined by (2.5):

$$2\omega_\alpha V = \sum_k n_k^\alpha e_k + m_k^\alpha B_k, \quad n_k^\alpha, m_k^\alpha \in Z.$$

(2.12)

Here and below, $\omega_1 = \omega, \omega_2 = \omega'$ are half-periods of the elliptic curve Γ_0.

Theorem 2.2. Let Γ be a smooth curve defined by (1.6), and let $P_\pm$ be preimages on Γ of the point $z = 0 \in \Gamma_0$ with local coordinates in their neighborhoods defined by the local coordinate z on Γ_0. Then the corresponding algebro-geometric solutions given by (2.11) satisfy the relation

$$\varphi_{n+N}(x, t) = \varphi_n(x, t) + 2N\ln\Lambda,$$

(2.13)

and have the form (1.8), that is,

$$\varphi_n(x, t) = \alpha_n(t) + \ln\frac{\sigma\big(x - x_{n+1}(t) + a\big)\sigma\big(x + x_{n+1}(t) + a\big)}{\sigma\big(x - x_n(t) + a\big)\sigma\big(x + x_n(t) + a\big)}.$$

The functions $x_n(t)$ defined by this representation satisfy equations (1.1). The functions α_n satisfy the relation

$$4e^{\alpha_{n-1}(t)-\alpha_n(t)} = \left(1 - \dot{x}_n^2(t)\right)W(x_n, x_{n+1})W(x_n, x_{n-1}), \tag{2.14}$$

where

$$W(u, v) = \frac{\sigma(u-v)\sigma(u+v)}{\sigma(2u)}. \tag{2.15}$$
$\square$

Proof. The first statement of the theorem is a direct corollary of the uniqueness of the Baker-Akhiezer function. The projection $Q = (w, z) \in \Gamma \rightarrow w$ defines $w = w(Q)$ as a function on the curve. This function is holomorphic on Γ outside the puncture P_+, where it has the pole of order N, $w = z^{-N}(1 + O(z))$. At the point P_-, it has zero of order N, $w = \Lambda^{2N}z^N(1 + O(z))$. Therefore, we have the equality

$$\psi_{n+N}(x, t, Q) = w(Q)\psi_n(x, t, Q), \tag{2.16}$$

because the functions defined by its left- and right-hand sides have the same analytical properties.

Let us consider the functions

$$T_\alpha(z) = e^{2\zeta(z)\omega_\alpha - 2\eta_\alpha z}, \quad \eta_\alpha = \zeta(\omega_\alpha). \tag{2.17}$$

They are double-periodic and holomorphic on Γ_0 except at $z = 0$. Again, comparison of analytical properties of the left- and right-hand sides proves the equality

$$\psi_n(x + 2\omega_\alpha, t, Q) = T_\alpha(z)\psi_n(x, t, Q), \quad Q = (w, z). \tag{2.18}$$

The function e^{φ_n} is defined as a ratio of the leading coefficients of an expansion of ψ_n on two sheets of Γ. Therefore, it does not change under the shifts $x \rightarrow x + 2\omega_\alpha$, and consequently, it is an elliptic function of the variable x. From (2.11) it follows that if we denote roots of the equation $\theta(nU + xV + tW + Z) = 0$ in the fundamental domain of Γ_0 by $x_n^j(t)$, $j = 1, \ldots, D$, then

$$e^{\varphi_n(x,t)} = e^{\alpha_n(t)} \prod_{j=1}^{D} \frac{\sigma\left(x - x_{n+1}^j(t)\right)}{\sigma\left(x - x_n^j(t)\right)}. \tag{2.19}$$

Our next step is to show that e^{φ_n} has only two poles and zeros in Γ_0.

Lemma 2.1. The function $\theta(xV + \xi)$ corresponding to a smooth algebraic curve Γ, defined by (1.6) as a function of the variable x, is an elliptic theta-function of weight 2, that is, it can be represented in the form

$$\theta(xV + \xi) = r(\xi)\sigma(x - x^1(\xi))\sigma(x - x^2(\xi)). \tag{2.20}$$

$\square$

Proof. Let us find the coefficients of expansion (2.12). The branching points $z_i^{\pm}$ of Γ over Γ_0 are roots of the equations $P_N^{el}(z) = \pm\Lambda^N$. In a generic case, when they are distinct, the curve Γ is smooth. The Riemann-Hurwitz formula $2g - 2 = \nu$, which connects genus g of branching cover of an elliptic curve with a number ν of branching points, implies that Γ has genus $N + 1$. We choose a_i, b_i cycles on it as follows: $a_i, i = 1, \ldots, N - 1$, *are cycles around cuts between branching points* z_i^+, z_i^-, *and* a_N *and* a_{N+1} *are two preimages of* a-*cycle on* Γ_0. (We assume that a- and b-cycles on Γ_0 correspond to the periods 2ω and $2\omega'$, resp.)

From the definition of the differential $d\Omega^{(x)}$, it follows that

$$d\Omega^{(x)} = d\left(\zeta(z) - \frac{\eta}{\omega}z\right). \tag{2.21}$$

Therefore, the coordinates of the vector V defined by (2.8) are equal to

$$V_i = 0, \quad i = 1, \ldots, N - 1, \qquad V_N = V_{N+1} = \frac{1}{\pi i}\left(\eta' - \frac{\eta}{\omega}\omega'\right) = -\frac{1}{2\omega}. \tag{2.22}$$

Comparing the vector of b-periods of $d\Omega^{(x)}$ with the vector $(0, \ldots, 0, 2\omega', 2\omega')$ of b-periods of the differential dz, considered as a differential on Γ, we get

$$\oint_{b_i} d\Omega^{(x)} = -\frac{\pi i}{2\omega\omega'}\oint_{b_i} dz, \quad i = 1, \ldots, N + 1. \tag{2.23}$$

The a-periods of dz are equal: $(0, \ldots, 0, 2\omega, 2\omega)$. Therefore,

$$dz = 2\omega\left(d\Omega_N^h + d\Omega_{N+1}^h\right),$$

where $d\Omega_i^h$ are normalized holomorphic differentials. From (2.23), we finally obtain that

$$2\omega'V = -B_N - B_{N+1}, \tag{2.24}$$

where B_i is the vector of b-periods of $d\Omega_i^h$. The monodromy properties of θ-function imply

$$\theta\left((x + 2\omega)V + Z\right) = \theta(xV + Z), \qquad \theta\left((x + 2\omega')V + Z\right) = e^{l(x)}\theta(xV + Z), \tag{2.25}$$

where

$$l(x) = \pi i \big(2x(V_N + V_{N+1}) + B_{N+1,N+1}$$
$$+ B_{N,N} - B_{N,N+1} - B_{N+1,N} + 2Z_{N+1} + 2Z_N \big).$$

Using (2.22), we obtain

$$dl(x) = -\frac{2\pi i}{\omega}\, dx. \tag{2.26}$$

The number D of zeros of the function $\theta(xV + \xi)$ in the fundamental domain can be found by integrating the logarithmic derivative of this function over the boundary of the domain. From (2.25) and (2.26), it follows that

$$D = \frac{1}{2\pi i} \oint_{\partial G_0} d \ln \theta(xV + Z) = 2. \tag{2.27}$$

∎

The equality (2.20) implies that the index j in (2.19) takes values $j = 1, 2$. The sums of zeros and poles of an elliptic function are equal to each other (modulo periods of Γ_0). Hence, $x_n^j(t)$ can be represented in the form

$$x_n^1 = x_n(t) + a(t), \qquad x_n^2(t) = -x_n(t) + a(t). \tag{2.28}$$

To complete a proof of (1.8), we need only to show that $a(t)$ does not depend on t.

Let us substitute (2.19) into (1.7). A priori, the difference of the left- and right-hand sides of (1.7) is an elliptic function of x with poles of degree 2 at the points $x_n^j(t)$ and $x_{n+1}^j(t)$. Vanishing of the pole of degree 2 at x_n^i implies that

$$\left(\dot{x}_n^i\right)^2 - 1 = F_n^i\left(x_n^i\right), \tag{2.29}$$

where

$$F_n^i(x) = r_n \frac{\prod_j \sigma\left(x - x_{n+1}^j\right)\sigma\left(x - x_{n-1}^j\right)}{\prod_{j \neq i} \sigma^2\left(x - x_n^j\right)}, \quad r_n = -4e^{\alpha_n - \alpha_{n-1}}. \tag{2.30}$$

Vanishing of the pole of degree 1 at x_n^i implies that

$$\ddot{x}_n^i = \partial_x F_n^i\left(x_n^i\right) = F_n^i(x_n^i)\left(\partial_x \ln F_n^i\left(x_n^i\right)\right)$$
$$= \left(\left(\dot{x}_n^i\right)^2 - 1\right)\left(\sum_j \zeta\left(x_n^i - x_{n+1}^j\right) + \zeta\left(x_n^i - x_{n-1}^j\right) - 2\sum_{j \neq i} \zeta\left(x_n^i - x_n^j\right)\right). \tag{2.31}$$

Substitution of (2.28) in (2.30) shows that

$$F_n^1(x_n^1) = F_n^2(x_n^2) = r_n(t)W(x_n, x_{n+1})W(x_n, x_{n-1}).$$

Hence, we obtain the equality $(\dot{x}_n^1)^2 = (\dot{x}_n^2)^2$, which implies that $\dot{a} = 0$. Equalities (1.8) and (2.14) are proved. At the same time, substitution of (2.28) into (2.31) gives us (1.1).

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3   Generating problem and Lax representation

In this section we construct the Lax representation for (1.1) following an approach proposed in [21] and developed in [26], [27], and [30] (see the summary in [25]). According to this approach, pole dynamics can be obtained simultaneously with its Lax representation from a specific inverse problem for a *linear* operator with elliptic coefficients.

In the most general form the inverse problem is to find linear operators with elliptic coefficients that have *sufficient* double-Bloch solutions. A meromorphic function $f(x)$ is called double-Bloch if it has the following monodromy properties:

$$f(x + 2\omega_\alpha) = B_\alpha f(x), \quad \alpha = 1, 2. \tag{3.1}$$

The complex numbers B_α are called *Bloch multipliers*. (In other words, f is a meromorphic section of a vector bundle over the elliptic curve.) It turns out that existence of the double-Bloch solutions is so restrictive that only in exceptional cases do such solutions exist.

The basis in the space of double-Bloch functions can be written in terms of the fundamental function $\Phi(x, z)$ defined by the formula

$$\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}. \tag{3.2}$$

From the monodromy properties of the Weierstrass functions, it follows that Φ, considered as a function of z, is double-periodic: $\Phi(x, z+2\omega_\alpha) = \Phi(x, z)$, though it is not elliptic in the classical sense due to essential singularity at $z = 0$ for $x \neq 0$. As a function of x, the function $\Phi(x, z)$ is a double-Bloch function, that is,

$$\Phi(x + 2\omega_\alpha, z) = T_\alpha(z)\Phi(x, z),$$

where $T_\alpha(z)$ are given by (2.17). In the fundamental domain of the lattice defined by $2\omega_\alpha$, the function $\Phi(x, z)$ has a unique pole at the point $x = 0$:

$$\Phi(x, z) = x^{-1} + O(x). \tag{3.3}$$

Let $f(x)$ be a double-Bloch function with simple poles x_i in the fundamental domain and with Bloch multipliers B_α (such that at least one of them is not equal to 1). Then it can be represented in the form

$$f(x) = \sum_{i=1}^{N} c_i \Phi(x - x_i, z) e^{kx}, \tag{3.4}$$

where c_i is the residue of f at x_i, and (z, k) are parameters such that $B_\alpha = T_\alpha(z) \exp(2\omega_\alpha k)$.

Now we are in position to present the generating problem for (1.1).

Theorem 3.1. The equation

$$(\partial_t + \partial_x)\Psi_n = 2\Psi_{n+1} + v_n(x, t)\Psi_n \tag{3.5}$$

with an elliptic coefficient of the form

$$v_n(x, t) = \gamma_n(t) + \sum_{i=1}^{2} \left[h_n^i(t)\zeta(x - x_n^i(t)) - h_{n+1}^i(t)\zeta(x - x_{n+1}^i(t)) \right], \tag{3.6}$$

where

$$x_n^1(t) = x_n(t) + a, \qquad x_n^2(t) = -x_n(t) + a, \quad a = \text{const}, \tag{3.7}$$

has two linear independent double-Bloch solutions with Bloch multipliers $T_\alpha(z)$ (for some z), that is, solutions of the form

$$\Psi_n(x, t) = \sum_{i=1}^{2} c_n^i \Phi(x - x_n^i(t), z) \tag{3.8}$$

if and only if the functions $x_n(t)$ satisfy (1.1).

If (3.5) has two linear independent solutions of the form (3.8) for some z, then they exist for all values of z. □

Proof. Let us substitute (3.8) into (3.5). Both sides of the equation are double-Bloch functions with the same Bloch multipliers and with the pole of order 2 at x_n^i, and the simple pole at x_{n+1}^i. They coincide if and only if the coefficients of their singular parts at these points are equal to each other. The equality of the coefficients at $(x - x_n^i)^{-2}$ implies that

$$h_n^i = \dot{x}_n^i - 1. \tag{3.9}$$

The equality of residues at x_{n+1}^i is equivalent to the equation

$$c_{n+1}^i = 2^{-1}h_{n+1}^i \sum_j \Phi\big(x_{n+1}^i - x_n^j\big)c_n^j. \tag{3.10}$$

The equality of residues at x_n^i is equivalent to the equation

$$\partial_t c_n^i = M_n^i c_n^i + h_n^i \sum_{j\neq i} \Phi\big(x_{n+1}^i - x_n^j\big)c_n^j, \tag{3.11}$$

where

$$M_n^i = \gamma_n - \sum_j h_{n+1}^j \zeta\big(x_n^i - x_{n+1}^j\big) + \sum_{j\neq i} h_n^j \zeta\big(x_n^i - x_n^j\big). \tag{3.12}$$

Equations (3.10) and (3.11) are linear equations for c_n^i. Their compatibility is just a system of the equations

$$\partial_t\big(\ln h_{n+1}^i\big)\Phi\big(x_{n+1}^i - x_n^j\big) + \big(\dot{x}_{n+1}^i - \dot{x}_n^j\big)\Phi'\big(x_{n+1}^i - x_n^j\big)$$
$$= \big(M_{n+1}^i - M_n^j\big)\Phi\big(x_{n+1}^i - x_n^j\big) + \sum_{k\neq i} \Phi\big(x_{n+1}^i - x_{n+1}^k\big)h_{n+1}^k \Phi\big(x_{n+1}^k - x_n^j\big)$$
$$- \sum_{k\neq j} \Phi\big(x_{n+1}^i - x_n^k\big)h_n^k \Phi\big(x_n^k - x_n^j\big), \tag{3.13}$$

which can be written in the matrix form as

$$\partial_t L_n = M_{n+1}L_n - L_n M_n, \tag{3.14}$$

where L_n and M_n are matrices defined by the right-hand sides of (3.10) and (3.11). Equation (3.14) is a necessary and sufficient condition for the existence of solutions of (3.5) that have the form (3.8). Therefore, the following statement completes a proof of the theorem.

Lemma 3.1. Let $L_n = (L_n^{ij}(t,z))$ and $M_n = (M_n^{ij}(t,z))$ be defined by the formulae

$$L_n^{ij} = 2^{-1}h_{n+1}^i \Phi\big(x_{n+1}^i - x_n^j, z\big), \qquad M^{ii} = M_n^i, \qquad M_{ij} = h_n^i \Phi\big(x_n^i - x_n^j, z\big), \quad i\neq j, \tag{3.15}$$

where $x_n^1 = x_n$, $x_n^2 = -x_n$, $h_n^i = \dot{x}_n^i - 1$, and M_n^i is given by (3.12) with γ_n such that

$$\gamma_n - \gamma_{n-1} = d_t \ln\left(\frac{(\dot{x}_n^2 - 1)\sigma^2(2x_n)}{\sigma(x_n - x_{n+1})\sigma(x_n + x_{n+1})\sigma(x_n - x_{n-1})\sigma(x_n + x_{n-1})}\right). \tag{3.16}$$

Then they satisfy equation (3.14) if and only if the functions $x_n(t)$ solve (1.1). $\qquad\square$

Note that (3.16) defines $\gamma_n(t)$ up to a constant shift $\gamma_n(t) \to \gamma_n(t) + g(t)$, which corresponds to the gauge transformation $\Psi_n \to e^g \Psi_n$ of equation (3.5) and which does not effect equations for x_n.

Proof. The right- and left-hand sides of (3.13) are double-periodic functions of z that are holomorphic except at $z = 0$, where they have the form $O(z^{-2}) \exp((x_{n+1}^i - x_n^j)\zeta(z))$. Such functions are equal if and only if the corresponding coefficients at z^{-2} and z^{-1} are equal. The equality of the coefficients at z^{-2} gives

$$\left(\dot{x}_{n+1}^i - \dot{x}_n^j\right) = h_{n+1}^i - h_n^j + \sum_k \left(h_n^k - h_{n+1}^k\right) = h_{n+1}^i - h_n^j, \tag{3.17}$$

which is fulfilled due to (3.9). (The second equality in (3.17) holds because $v(x,t)$ is an elliptic function of x and, therefore, a sum of its residues is equal to zero.)

The equality of the coefficients at z^{-1} in the expansion of (3.13) at $z = 0$ gives

$$\begin{aligned}
\partial_t\left(\ln h_{n+1}^i\right) &- \left(\dot{x}_{n+1}^i - \dot{x}_n^j\right)\zeta\left(x_{n+1}^i - x_n^j\right) \\
&= M_{n+1}^i - M_n^j + \sum_{k \neq i} h_{n+1}^k \left[\zeta\left(x_{n+1}^i - x_{n+1}^k\right) + \zeta\left(x_{n+1}^k - x_n^j\right)\right] \\
&\quad - \sum_{k \neq j} h_n^k \left[\zeta\left(x_{n+1}^i - x_n^k\right) + \zeta\left(x_n^k - x_n^j\right)\right].
\end{aligned} \tag{3.18}$$

The second line in (3.18) is equal up to the sign to the sum of residues at x_n^k, $k \neq j$, and at x_{n+1}^k, $k \neq i$, of the elliptic function

$$\widetilde{v}_n(x,t) = v_n(x,t)\left[\zeta\left(x_{n+1}^i - x\right) + \zeta\left(x - x_n^j\right)\right].$$

Therefore, it equals to the sum of residues of this function at x_{n+1}^i and x_n^j. We have

$$\begin{aligned}
\operatorname*{res}_{x_{n+1}^i} \widetilde{v}_n(x,t) + \operatorname*{res}_{x_n^j} \widetilde{v}_n(x,t) &= \left(h_n^j - h_{n+1}^i\right)\zeta\left(x_{n+1}^i - x_n^j\right) + M_n^j - \gamma_n \\
&\quad - \sum_k h_n^k \zeta\left(x_{n+1}^i - x_n^k\right) \\
&\quad + \sum_{k \neq i} h_{n+1}^k \zeta\left(x_{n+1}^i - x_{n+1}^k\right).
\end{aligned} \tag{3.19}$$

Substitution of the right-hand side of the last equality into (3.18) implies (after the shift $n + 1 \to n$) that

$$\begin{aligned}
\frac{\dot{h}_n^i}{h_n^i} &= \gamma_n - \gamma_{n-1} + \sum_{k \neq i} 2h_n^k \zeta\left(x_n^i - x_n^k\right) \\
&\quad - \sum_k \left[h_{n+1}^k \zeta\left(x_n^i - x_{n+1}^k\right) + h_{n-1}^k \zeta\left(x_n^i - x_{n-1}^k\right)\right].
\end{aligned} \tag{3.20}$$

From (3.9), it follows that (3.20) can be rewritten in the form

$$\frac{\ddot{x}_n^i}{\dot{x}_n^i - 1} = \partial_x G_n^i(x_n^i) + \partial_t G_n^i(x_n^i),$$

(3.21)

where the function

$$G_n^i(x) = a_n + \ln\left(\frac{\prod_k \sigma(x - x_{n+1}^k)\sigma(x - x_{n-1}^k)}{\prod_{k \neq i} \sigma^2(x - x_n^k)}\right), \qquad \partial_t a_n = \gamma_n - \gamma_{n-1},$$

(3.22)

depends on t through the dependence on t of x_m^i and a_n, only. By the chain rule, we have

$$d_t\left(G_n^i(x_n^i)\right) = \dot{x}_n^i \partial_x G_n^i(x_n^i) + \partial_t G_n^i(x_n^i), \quad d_t = \frac{d}{dt}.$$

(3.23)

Therefore,

$$\frac{\ddot{x}_n^i}{\dot{x}_n^i - 1} = (1 - \dot{x}_n^i)\partial_x G_n^i(x_n^i) + d_t\left(G_n^i(x_n^i)\right).$$

(3.24)

From (3.7), it follows that

$$G_n^1(x_n^1) = G_n^2(x_n^2) = G_n(x_n), \qquad \partial_x G_n^1(x_n^1) = -\partial_x G_n^2(x_n^2) = \partial_x G_n(x_n),$$

where

$$G_n(x) = a_n + \ln\left(\frac{\sigma(x - x_{n+1})\sigma(x + x_{n+1})\sigma(x - x_{n-1})\sigma(x + x_{n-1})}{\sigma^2(x + x_n)}\right).$$

(3.25)

Therefore, for $i = 1, 2$, (3.24) has the form

$$\frac{\ddot{x}_n}{\dot{x}_n - 1} = d_t\left(G_n(x_n)\right) - (\dot{x}_n - 1)\partial_x G_n(x_n),$$

(3.26)

$$\frac{\ddot{x}_n}{\dot{x}_n + 1} = d_t\left(G_n(x_n)\right) - (\dot{x}_n + 1)\partial_x G_n(x_n).$$

(3.27)

Equations (3.26) and (3.27) are equivalent to the equations

$$\ddot{x}_n = \left(\dot{x}_n^2 - 1\right)\partial_x G_n(x_n), \qquad d_t\left(G_n(x_n)\right) = d_t \ln\left(\dot{x}_n^2 - 1\right).$$

(3.28)

The first among them coincides with (1.1) for x_n, and the second one (compare it with (2.14)) is equivalent to the definition of γ_n by (3.16). Lemma 3.1 and Theorem 3.1 are proved. ∎

4 Direct problem: Spectral curves

In this section we consider periodic n-solutions of (1.1).

Lemma 4.1. Let $x_n(t) = x_{n+N}(t)$ be a solution of (1.1). Then

$$I = \prod_{n=1}^{N} \left(\frac{\sigma(x_n - x_{n+1})\sigma(x_n + x_{n+1})\sigma(x_n - x_{n-1})\sigma(x_n + x_{n-1})}{(\dot{x}_n^2 - 1)\sigma^2(2x_n)} \right) \tag{4.1}$$

is an integral of motion, $I = \text{const}$, and the monodromy matrix

$$T(t,z) = \prod_{n=0}^{N-1} L_n(t,z) \tag{4.2}$$

satisfies the Lax equation

$$\partial_t T = [M_0, T]. \tag{4.3}$$
$\square$

Proof. If $x_n(t)$ is periodic in n, then the corresponding matrix functions $L_n(t,z)$ and $M_n(t,z)$ defined by (3.15) satisfy the relations

$$L_{n+N} = L_n, \qquad M_{n+N} = M_n - d_t(\ln I). \tag{4.4}$$

Therefore, (3.14) implies that

$$\partial_t T = -d_t(\ln I)T + [M_0, T]. \tag{4.5}$$

Note that if $\partial_t I = 0$, then (4.5) coincides with (4.3), and, therefore, the second statement of the Lemma follows from the first one.

Equation (4.5) implies that the function

$$P(z) = I(t)\big(\text{tr}\, T(t,z)\big) \tag{4.6}$$

is *time-independent*.

Matrix entries of L_n are double-periodic functions that are holomorphic on Γ_0 except at $z = 0$. Therefore, $(\text{tr}\, T)$ is also double-periodic and holomorphic on Γ_0 outside $z = 0$. In order to prove that this function is meromorphic on Γ_0, it is enough to note that L_n has the form

$$L_n(t,z) = g_{n+1}\widetilde{L}_n g_n^{-1}, \tag{4.7}$$

where

$$g_n = \begin{pmatrix} e^{x_n \zeta(z)} & 0 \\ 0 & e^{-x_n \zeta(z)} \end{pmatrix}.$$

From (3.2), it follows that in the neighborhood of $z = 0$,

$$\widetilde{L}_n = (z)^{-1} \widetilde{L}_n^0 + \widetilde{L}_n^1 + O(z), \tag{4.8}$$

where

$$\widetilde{L}_n^0 = \frac{1}{2} \begin{pmatrix} 1 - \dot{x}_{n+1} & 1 - \dot{x}_{n+1} \\ 1 + \dot{x}_{n+1} & 1 + \dot{x}_{n+1} \end{pmatrix} \tag{4.9}$$

and

$$\widetilde{L}_n^1 = \frac{1}{2} \begin{pmatrix} 1 - \dot{x}_{n+1} & 0 \\ 0 & 1 + \dot{x}_{n+1} \end{pmatrix} \begin{pmatrix} -\zeta(x_{n+1} - x_n) & -\zeta(x_{n+1} + x_n) \\ \zeta(x_{n+1} + x_n) & \zeta(x_{n+1} - x_n) \end{pmatrix}. \tag{4.10}$$

Therefore,

$$\operatorname{tr} T = \operatorname{tr} \left(\prod_{n=0}^{N-1} \widetilde{L}_n(t, z) \right) = z^{-N} \left(1 + 0(z) \right). \tag{4.11}$$

The last equality shows that $(\operatorname{tr} T)$ is a monic elliptic polynomial $P_N^{el}(z)$. Therefore, at $z = 0$, we have $P(z) = I(t) z^{-N}(1 + 0(z))$. Hence, $I(t)$ is an integral of (1.1) because $P(z)$ does not depend on z. Lemma 4.1 is proved. ∎

Due to (4.3) the spectral curve Γ defined by the characteristic equation

$$R(w, z) \equiv \det \left(w - T(t, z) \right) = w^2 - (\operatorname{tr} T) w + \det T = 0 \tag{4.12}$$

is time-independent.

Lemma 4.2. The characteristic equation (4.12) has the form (1.3). □

Proof. We have already proved that $(\operatorname{tr} T)$ has the form (1.5). The relation $\Phi(x, z)\Phi(-x, z) = \wp(z) - \wp(x)$, which is equivalent to the addition formula for the Weierstrass σ-function, implies that

$$\det L_n(t, z) = 2^{-2} \left(\dot{x}_{n+1}^2 - 1 \right) \left[\wp(x_{n+1} - x_n) - \wp(x_{n+1} + x_n) \right]. \tag{4.13}$$

Therefore, although $L_n(t, z)$ depends on z, its determinant does not depend on z. Hence, $(\det T)$ is also z-independent. As it does not depend on t, we identify Λ^{2N} in (1.6) with

$$\Lambda^{2N} = \det T(t, z) = 2^{-2N} \prod_{n=0}^{N-1} (\dot{x}_n^2 - 1)\big(\wp(x_n - x_{n-1}) - \wp(x_n + x_{n-1})\big)$$
$$= 2^{-2N} e^H, \tag{4.14}$$

where H is the Hamiltonian of system (1.1). Lemma 4.2 is proved. $\blacksquare$

For a generic point Q of the spectral curve Γ, that is, for a pair (w, z) that satisfies (4.12), there exists a unique solution $C_n = (c_n^i(t, Q))$ of the equations

$$C_{n+1}(t, Q) = L_n(t, z)C_n(t, Q), \qquad \partial_t C_n(t, Q) = M_n(t, z) \tag{4.15}$$

such that

$$C_{n+N}(t, Q) = wC_n(t, Q) \tag{4.16}$$

and the unique solution $C_n = (c_n^i(t, Q))$ is normalized by the condition

$$c_0^1(0, Q)\Phi\big(-x_0(0), z\big) + c_0^2(0, Q)\Phi\big(x_0(0), z\big) = 1. \tag{4.17}$$

Remark. Normalization (4.17) corresponds to a usual normalization $\Psi_0(0, 0, Q) = 1$ of the solution $\Psi_n(x, t, Q)$ of (3.5) defined by (3.8).

Theorem 4.1. The coordinates $c_n^i(t, Q)$ of the vector-valued function $C_n(t, Q)$ are meromorphic functions on Γ except at the preimages $P_\pm$ of $z = 0$. Their poles $\gamma_1, \ldots, \gamma_{N+1}$ do not depend on n and t. The projections $z(\gamma_s)$ of these poles on Γ_0 satisfy the constraint

$$\sum_{s=1}^{N+1} z(\gamma_s) = 0. \tag{4.18}$$

In the neighborhoods of $P_\pm$, the coordinates of $C_n(t, Q)$ have the form

$$c_n^1(t, Q) = z^{\mp n}\chi_{n,\pm}^1(t, z)e^{(\pm t + x_n(t))z^{-1}}, \tag{4.19}$$
$$c_n^2(t, Q) = z^{\mp n}\chi_{n,\pm}^2(t, z)e^{(\pm t - x_n(t))z^{-1}}, \tag{4.20}$$

where $\chi_{n,\pm}^i(t, z)$ are regular functions of z:

$$\chi_{n,+}^i(t, z) = z\chi_{n,+}^i(t) + O(z^2), \qquad \chi_{n,-}^i(t, z) = \chi_{n,-}^i(t) + z\chi_{n,-}^{i,1}(t) + O(z^2) \tag{4.21}$$

such that the leading coefficients of their expansions have the form

$$\chi^1_{n,+}(t) = c(t)(1 - \dot{x}_n), \qquad \chi^2_{n,+}(t) = c(t)(1 + \dot{x}_n), \quad c(0) = 1, \tag{4.22}$$

$$\chi^1_{n,-}(t) = s_n(t), \qquad \chi^2_{n,-}(t) = -s_n(t), \tag{4.23}$$

where functions s_n satisfy the relation

$$s_{n+1} = 2^{-2} \left(\dot{x}^2_{n+1} - 1 \right) \left[\wp(x_{n+1} - x_n) - \wp(x_{n+1} + x_n) \right] s_n. \tag{4.24}$$

$\square$

Proof. Vector-columns $S_n^{(1)}$ and $S_n^{(2)}$ of the matrix-function

$$S_0^{ij} = \delta_{ij}, \qquad S_n(t, z) = \prod_{m=0}^{n-1} L_m(t, z), \quad n > 0, \tag{4.25}$$

are holomorphic functions on Γ_0 except at $z = 0$. They satisfy the equation $S_{n+1}^{(i)} = L_n S_n^{(i)}$. Therefore, the Bloch solution C_n of (4.15) has the form

$$C_n(t, Q) = h_1(Q) S_n^{(1)}(t, z) + h_2(Q) S_n^{(2)}(t, z), \tag{4.26}$$

where $h_i(Q), Q = (w, z) \in \Gamma$, are the coordinates of the normalized eigenvector of the monodromy matrix $T(z)$, corresponding to the eigenvalue w. They are equal to

$$h_1(Q) = \frac{1}{r(Q)} T^{12}(z), \qquad h_2(Q) = \frac{1}{r(Q)} \left(w - T^{11}(z) \right), \tag{4.27}$$

where $T^{ij}(z)$ are entries of the monodromy matrix and the normalization constant $r(Q)$ equals

$$r(Q) = T^{12}(z) \Phi\left(-x_0(0), z \right) + \left(w - T^{11}(z) \right) \Phi\left(x_0(0), z \right). \tag{4.28}$$

The function $r(Q)$ has the pole of degree $N + 1$ at P_+ and the pole of degree N at P_-. Therefore, it has $2N + 1$ zeros.

Let us show that N of these zeros are situated over roots of the equation $T^{12}(z) = 0$ on one of the sheets of Γ. Indeed, if $T^{12}(z) = 0$, then eigenvalues $w(z)$ of the monodromy matrix are equal to $T^{11}(z)$ or $T^{12}(z)$. Therefore, $r = 0$ at the points $Q = (T^{11}(z), z)$. Equations (4.27) imply that C_n has no poles at these points. The poles γ_s of $C_n(t, Q)$ on Γ outside the punctures $P_\pm$ are the other zeros of $r(Q)$ and do not depend on n and t. Let us now prove that they satisfy (4.18).

The function $r^*(z) = r(Q)r(Q^\sigma)$ with $\sigma : Q \to Q^\sigma$ as a permutation of sheets of Γ, is a well-defined function on Γ_0 with the pole of degree $2N + 1$ at $z = 0$. As it was shown above, it is divisible by $T^{12}(z)$. Therefore, the ratio $r^*(z)/T^{12}(z)$ is an elliptic function with the pole of degree $N + 1$ at $z = 0$ and zeros at the points $z(\gamma_s)$. Divisors of zeros and poles of an elliptic function are equivalent. Therefore, (4.18) is proved.

From (4.7), it follows that the vector-function $\widetilde{C}_n = g_n^{-1} C_n$ is a Bloch solution of the equation $\widetilde{C}_{n+1} = \widetilde{L}_n \widetilde{C}_n$. Let us first consider the neighborhood of the puncture P_+, which corresponds to the branch $w = z^{-N}(1 + O(z))$ of the eigenvalue of the monodromy matrix.

The vector-function $X_n(t)$ with the coordinates given by (4.22) satisfies the equation $X_{n+1} = \widetilde{L}_n^0 X_n$, where $\widetilde{L}_n^0$ is defined in (4.8). That implies that, in the neighborhood of P_+, the vector-function $C_n(t, Q)$ has the form stated in the theorem up to a time-dependent factor $f_+(t, z)$. Substitution of (4.19), (4.20) into $\partial_t C_n = M_n C_n$ shows that $\partial_t f = O(z)$. Therefore, the analytical properties of C_n near P_+ are established.

Now we prove by induction that at P_- equalities (4.19), (4.20), and (4.23) hold. For $n = 0$, they are fulfilled by the normalization conditions. Let us first prove that if (4.19), (4.20), and (4.23) hold for $n' \leq n$, then

$$2\kappa_n = \big(\zeta(x_{n+1} + x_n) - \zeta(x_{n+1} - x_n)\big)s_n + \chi_{n,-}^{1,1} + \chi_{n,-}^{1,2} = 0. \tag{4.29}$$

Indeed, $C_{n+1} = L_n C_n$ implies that $\widetilde{C}_{n+1}$ at P_- has the form

$$\widetilde{C}_{n+1} = z^n \begin{pmatrix} (1 - \dot{x}_{n+1})\kappa_n \\ (1 + \dot{x}_{n+1})\kappa_n \end{pmatrix} + O(z^{n+1}). \tag{4.30}$$

Hence,

$$\widetilde{C}_N = \left(\prod_{m=n+1}^{N-1} L_m\right) \widetilde{C}_{n+1} = z^{2n-N-1} \begin{pmatrix} (1 - \dot{x}_0)\kappa_n \\ (1 + \dot{x}_0)\kappa_n \end{pmatrix} + O\left(z^{2n-N}\right). \tag{4.31}$$

If $\kappa_n \neq 0$, then the last equality contradicts the monodromy property $\widetilde{C}_N = w\widetilde{C}_0 = O(z^N)$. Therefore, $\kappa_n = 0$, and then (4.30) shows that $\widetilde{C}_{n+1}$ has zero of order $n+1$ at P_-. Therefore, a step of induction for equalities (4.19), (4.20) is proved. The same arguments show that if (4.23) does not hold, then the vector $\widetilde{C}_N$ has zero of order $(2n - N)$, which again contradicts the relation $\widetilde{C}_N = O(z^N)$.

Equalities (4.19), (4.20) near P_- are proved, possibly up to a time-dependent factor $f_-(t, z)$. Their substitution into $\partial_t C_n = M_n C_n$ shows that $\partial_t f_- = O(z)$ and completes a proof of (4.19), (4.20), and (4.23).

Let $\mathcal{C}_n(z)$ be a matrix formed by the vectors $C_n(t, Q_i(z))$, corresponding to two different sheets $Q_i(z) = (w_i(z), z)$ of Γ. This matrix is defined up to a permutation of sheets. From (4.19)–(4.23), it follows that in the neighborhood of $z = 0$,

$$
\begin{aligned}
\mathcal{C}_n(z) = {} & \begin{pmatrix} e^{x_n\zeta(z)} & 0 \\ 0 & e^{-x_n\zeta(z)} \end{pmatrix} \left[\begin{pmatrix} (1-\dot{x}_n)c & s_n \\ (1+\dot{x}_n)c & -s_n \end{pmatrix} + O(z) \right] \\
& \times \begin{pmatrix} z^{-n+1}e^{t\zeta(z)} & 0 \\ 0 & z^n e^{-t\zeta(z)} \end{pmatrix}.
\end{aligned}
\tag{4.32}
$$

Therefore,

$$
\det\mathcal{C}_n = -2cs_n z + O(z^2),
\tag{4.33}
$$

and from the definition of $\mathcal{C}_n$, we have $\mathcal{C}_{n+1} = L_n \mathcal{C}_n$. Hence,

$$
s_{n+1} = s_n \det L_n,
\tag{4.34}
$$

which coincides with (4.24). Theorem 4.1 is proved.   ∎

The correspondence that assigns a set of algebro-geometric data $\{\Gamma, D\}$ to each solution $x_n(t) = x_{n+N}(t)$ of (1.1), is a direct spectral transform. The following statement shows that the results of Section 2 can be seen as the inverse spectral transform.

Corollary 4.1. The solution

$$
\Psi_n(x, t, Q) = c_n^1(t, Q)\Phi(x - x_n(t), t) + c_n^2(t, Q)\Phi(x + x_n(t), t)
\tag{4.35}
$$

of (3.5) is equal to $\Psi_n(x, t, Q) = c(t)\psi_n(x, t, Q)$, where $\psi_n(x, t, Q)$ is the Baker-Akhiezer function corresponding to Γ and the divisor D of the poles of C_n; the factor $c(t)$ is defined in (4.22).

All the solutions $x_n(t)$ of (1.1) have the form $x_n = (1/2)(x_n^1 - x_n^2)$, where $x_n^i(t)$ are roots of

$$
\theta(nU + x_n^i(t)V + Wt + Z) = 0.
\tag{4.36}
$$

Here $\theta(z)$ is the Riemann theta-function corresponding to Γ; vectors U, V, W are defined by (2.8), (2.9); vector Z corresponds to the divisor D via the Abel transform.   □

As follows from the Theorem 4.1, the function Ψ_n defined by (4.35) has the same analytical properties on Γ as the function $c(t)\psi_n$. Therefore, they coincide. Equation (4.36) immediately follows from (2.10) for ψ_n.

5 Action-angle variables

Until now we have not used the Hamiltonian structure of (1.1). Moreover, a priori, it is not clear why a system that has arisen as a pole system of elliptic solutions of the 2-dimensional Toda lattice is Hamiltonian. The general algebro-geometric approach, which allows us to derive a Hamiltonian structure starting from the Lax representation, was proposed and developed in [28], [29], and [25].

The main goal of this section is to construct action-angle variables for (1.1). First of all, let us summarize the necessary results of the previous sections. A point (p_n, x_n) of the phase space $\mathcal{M}$ of the system defines a matrix function $L_n(z)$ with the help of the formulae

$$L_n^{ij} = 2^{-1} h_{n+1}^i \Phi\big(x_{n+1}^i - x_n^j, z\big), \tag{5.1}$$

$$x_n^1 = x_n, \qquad x_n^2 = -x_n, \qquad h_n^1 = h_n - 1, \qquad h_n^2 = -h_n - 1, \qquad h_n = \frac{1 + e^{p_n}}{1 - e^{p_n}}. \tag{5.2}$$

This function defines the spectral curve Γ (with the help of (4.12)) and the divisor D of poles $\gamma_1, \ldots, \gamma_{N+1}$ of the Baker-Akhiezer function $C_n(Q) = (c_n^1(Q), c_n^2(Q))$:

$$C_{n+1}(Q) = L_n(z)C_n(Q), \qquad C_N(Q) = wC_0(Q), \quad Q = (w, z) \in \Gamma, \tag{5.3}$$

normalized by the condition

$$c_0^1(0, Q)\Phi(-x_0, z) + c_0^2(Q)\Phi(x_0, z) = 1. \tag{5.4}$$

The divisor D satisfies (4.18), that is, it defines a point of an odd part of the Jacobian $J^{Pr}(\Gamma) \in J(\Gamma)$, which is defined as a fiber of the projection

$$\{\gamma_1, \ldots, \gamma_{N+1}\} \in J(\Gamma) \longmapsto 2\omega\phi_+ = \sum_{s=1} z(\gamma_s) \in \Gamma_0, \tag{5.5}$$

corresponding to $\phi_+ = 0$. All the fibers are equivalent and can be identified with the Prym variety of Γ. Note that a shift of ϕ_+ corresponds to the shift $x \to x + a$ for the solution (1.8) of a 2-dimensional Toda lattice.

The correspondence

$$(p_n, x_n) \in \mathcal{M} \longmapsto \{\Gamma, D \in J^{Pr}(\Gamma)\} \tag{5.6}$$

is an isomorphism. The coefficients (u_i, Λ) of (1.6) are integrals of the Hamiltonian system (1.1). Equations (1.1) on a fiber over Γ of the map (5.6) are linearized by the Abel transform (2.6).

The main goal of this section is to construct the action variables that are canonically conjugated to the coordinates $\phi_1, \ldots, \phi_{N-1}, \phi_-$:

$$\phi_k = \sum_{s=1}^{N+1} A_k(\gamma_s), \qquad \phi_- = \phi_N - \phi_{N+1}, \tag{5.7}$$

on the Prymmian $J^{Pr}(\Gamma)$. Note that $\phi_+ = \phi_N + \phi_{N+1}$.

Theorem 5.1. The transformation

$$(x_n, p_n) \longmapsto (\phi_1, \ldots, \phi_{N-1}, \phi_-; I_1, \ldots I_N), \tag{5.8}$$

where I_k are a-periods of the differential $dS = \ln(\Lambda^{-N} w)\, dz$:

$$I_k = \oint_{a_k} \ln\left(\Lambda^{-N} w\right) dz, \tag{5.9}$$

is a canonical transformation, that is,

$$\sum_{n=1}^{N} dp_n \wedge dx_n = \sum_{k=1}^{N-1} \left(\delta I_k \wedge \delta\phi_k\right) + \delta I_N \wedge \delta\phi_-. \tag{5.10}$$
$$\square$$

Proof. First of all, following the approach proposed in [28], we define a symplectic structure on $\mathcal{M}$ in terms of the Lax operator and its eigenfunctions. After that, we calculate it in two different ways, which immediately imply (5.10).

The external differential $\delta L_n(z)$ can be seen as an operator-valued 1-form on $\mathcal{M}$. Canonically normalized eigenfunction $C_n(Q)$ of $L_n(z)$ is the vector-valued function on $\mathcal{M}$. Hence, its differential is a vector-valued 1-form. Let us define a 2-form ω on $\mathcal{M}$ by the formula

$$\omega = \frac{1}{2}\left(\operatorname*{res}_{P_+} \Omega + \operatorname*{res}_{P_-} \Omega\right), \tag{5.11}$$

where

$$\Omega = \left\langle C_{n+1}^*(Q)\delta L_n(z) \wedge \delta C_n(Q) \right\rangle dz. \tag{5.12}$$

Here and below, $\langle \cdot \rangle$ stands for the sum over a period of a periodic in n function, that is,

$$\langle f_n \rangle = \sum_{n=0}^{N-1} f_n;$$

$C_n^*(Q)$ is the dual Baker-Akhiezer function, which is defined as a covector (row-vector) solution of the equation

$$C_{n+1}^*(Q)L_n(z) = C_n^*(Q), \qquad C_N^*(Q) = w^{-1}C_0^*(Q), \tag{5.13}$$

normalized by the condition

$$C_0^*(Q)C_0(Q) = 1. \tag{5.14}$$

The form ω can be rewritten as

$$\omega = \frac{1}{2} \operatorname*{res}_0 \operatorname{Tr} \left\langle \left(\mathcal{C}_{n+1}^{-1}(z)\delta L_n(z) \wedge \delta \mathcal{C}_n(z) \right) \right\rangle dz, \tag{5.15}$$

where $\mathcal{C}_n(z)$ is a matrix with the columns $C_n(Q_j(z))$, $Q_j(z) = (z, w_j)$ corresponding to different sheets of Γ.

Note that $C_n^*(Q)$ are rows of the matrix $\mathcal{C}_n^{-1}(z)$. This implies that $C_n^*(Q)$ as a function on the spectral curve is meromorphic outside the punctures, has poles at the branching points of the spectral curve, and has zeros at the poles γ_s of $C_n(Q)$. These analytical properties are used in the proof of the following lemma.

Lemma 5.1. The 2-form ω equals

$$\omega = \sum_{s=1}^{N+1} \delta z(\gamma_s) \wedge \delta \ln w(\gamma_s). \tag{5.16}$$

$\square$

The meaning of the right-hand side of this formula is as follows. The spectral curve by definition arises with the meromorphic function $w(Q)$ and the multi-valued holomorphic function $z(Q)$. Their evaluations $w(\gamma_s), z(\gamma_s)$ at the points γ_s define functions on $\mathcal{M}$, and the wedge product of their external differentials is a 2-form on $\mathcal{M}$.

Proof. The differential Ω, defined by (5.12), is a meromorphic differential on the spectral curve. (The essential singularities of the factors cancel each other at the punctures.) Therefore, the sum of its residues at the punctures is equal to the sum of other residues with negative sign. There are poles of two types.

First of all, Ω has poles at the poles γ_s of C_n. Note that δC_n has the pole of the second order at γ_s. Taking into account that C_n^* has zero at γ_s, we obtain

$$\operatorname*{res}_{\gamma_s} \Omega = \left\langle C_{n+1}^* \delta L_n C_n \right\rangle \wedge \delta z(\gamma_s). \tag{5.17}$$

From (5.3) and (5.13), it follows that

$$\left\langle C_{n+1}^* \delta L_n C_n \right\rangle = \left\langle C_N^* \left(\prod_{m=n+1}^{N-1} L_m \right) \delta L_n \left(\prod_{m=0}^{n-1} L_m \right) C_0 \right\rangle = \left(C_N^* \delta T C_0 \right), \qquad (5.18)$$

where T is the monodromy matrix. Using the standard formula for the variation of the eigenvalue of an operator $\delta w = C_0^* \delta T C_0$, we obtain that

$$\operatorname*{res}_{\gamma_s} \Omega = \delta \ln w(\gamma_s) \wedge \delta z(\gamma_s). \qquad (5.19)$$

The second set of poles of Ω is a set of branching points q_i of the cover. The pole of C_n^* at q_i cancels with the zero of the differential dz, $dz(q_i) = 0$, considered as differential on Γ. The vector-function C_n is holomorphic at q_i. If we take an expansion of C_n in the local coordinate $(z - z(q_i))^{1/2}$ (in general position when the branching point is simple) and consider its variation, we get

$$\delta C_n = -\frac{dC_n}{dz} \delta z(q_i) + O(1). \qquad (5.20)$$

Therefore, δC_n has a simple pole at q_i. In a similar way, we obtain

$$\delta w = -\frac{dw}{dz} \delta z(q_i). \qquad (5.21)$$

Equalities (5.20) and (5.21) imply that

$$\operatorname*{res}_{q_i} \Omega = \operatorname*{res}_{q_i} \left[\left\langle C_{n+1}^* \delta L_n dC_n \right\rangle \wedge \frac{\delta w \, dz}{dw} \right]. \qquad (5.22)$$

At q_i, we have $dL_n(q_i) = 0$. Therefore, in a way similar to (5.18), we get

$$\operatorname*{res}_{q_i} \Omega = \operatorname*{res}_{q_i} \left[\left(C_N^* \delta T dC_0 \right) \wedge \frac{\delta w \, dz}{dw} \right]. \qquad (5.23)$$

Due to skew-symmetry of the wedge product, we may replace δT in (5.23) by $(\delta T - \delta w)$. Then using identities $C_N^*(\delta T - \delta w) = \delta C_N^*(w - T)$ and $(w - T)dC_0 = (dT - dw)C_0$, we obtain

$$\operatorname*{res}_{q_i} \Omega = -\operatorname*{res}_{q_i} \left(\delta C_N^* C_0 \right) \wedge \delta w \, dz = \operatorname*{res}_{q_i} \left(C_N^* \delta C_0 \right) \wedge \delta w \, dz. \qquad (5.24)$$

Note that the dT does not contribute to the residue, because $dT(q_i) = 0$.

Expansions (4.19), (4.20) near the punctures imply that

$$\operatorname*{res}_{P_\pm} \left(C_N^* \delta C_0\right) \wedge \delta w \, dz = 0. \tag{5.25}$$

Therefore,

$$
\begin{aligned}
\sum_{q_i} \operatorname*{res}_{q_i} \left(C_N^* \delta C_0\right) \wedge \delta w \, dz &= -\sum_{s=1}^{N+1} \operatorname*{res}_{\gamma_s} \left(C_N^* \delta C_0\right) \wedge \delta w \, dz \\
&= \sum_{s=1}^{N+1} \delta \ln w(\gamma_s) \wedge \delta z(\gamma_s).
\end{aligned} \tag{5.26}
$$

The sum of (5.19) and (5.26) gives (5.16), because

$$2\omega = -\sum_{s=1}^{N+1} \operatorname*{res}_{\gamma_s} \Omega - \sum_{q_i} \operatorname*{res}_{q_i} \Omega. \tag{5.27}$$

∎

Our next goal is to prove the following statement.

Lemma 5.2. The symplectic form given by (5.11) coincides with the canonical symplectic structure

$$\omega = \sum_{n=0}^{N+1} \delta p_n \wedge \delta x_n. \tag{5.28}$$

□

Proof. Using the gauge transformation (4.7)

$$L_n = g_{n+1} \tilde{L}_n g_n^{-1}, \qquad C_n = g_n \tilde{C}, \quad g_n = \begin{pmatrix} e^{x_n \zeta(z)} & 0 \\ 0 & e^{-x_n \zeta(z)} \end{pmatrix},$$

we obtain

$$
\begin{aligned}
\omega = \frac{1}{2} \operatorname*{res}_0 \operatorname{Tr} \Big\langle &\tilde{C}_{n+1}^{-1}(z) \delta \tilde{L}_n(z) \wedge \delta \tilde{C}_n(z) + \tilde{C}_{n+1}^{-1} \delta \tilde{L}_n \wedge \delta f_n \tilde{C}_n \\
&- \tilde{C}_{n+1}^{-1} \left(\delta f_{n+1} \wedge \delta \tilde{L}_n + \delta f_{n+1} \wedge \tilde{L}_n \delta f_n\right) \tilde{C}_n \Big\rangle dz,
\end{aligned} \tag{5.29}
$$

where $\delta f_n = \delta g_n g_n^{-1}$. From (4.32), using the equality

$$
\begin{aligned}
\big(\zeta(x_{n+1} + x_n) - \zeta(x_{n+1} - x_n)\big) \delta s_n &+ \delta\big(\chi_{n,-}^{1,1} + \chi_{n,-}^{1,2}\big) \\
&= s_n \delta\big(\zeta(x_{n+1} - x_n) - \zeta(x_{n+1} + x_n)\big),
\end{aligned}
$$

which follows from (4.28), we obtain that the first term in (5.29)

$$J_1 = \operatorname*{res}_0 \left\langle \operatorname{Tr}\left(\tilde{\mathbb{C}}_{n+1}^{-1}(z)\delta\tilde{L}_n(z)\wedge\delta\tilde{\mathbb{C}}_n(z)\right)\right\rangle dz$$

is equal to

$$J_1 = \left\langle \frac{s_n}{2s_{n+1}}\delta h_{n+1}\wedge\delta\left(\zeta(x_n-x_{n+1})+\zeta(x_{n+1}+x_n)\right)\right\rangle.$$

Equation (4.34) implies that

$$J_1 = \left\langle \frac{2\delta h_{n+1}\wedge\delta x_{n+1}}{h_{n+1}^2-1} - \frac{2\delta h_{n+1}\wedge\delta x_n}{h_{n+1}^2-1}\left(\frac{\wp(x_n-x_{n+1})+\wp(x_n+x_{n+1})}{\wp(x_n-x_{n+1})-\wp(x_n+x_{n+1})}\right)\right\rangle. \quad (5.30)$$

The second term in (5.29) is equal to

$$J_2 = \left\langle \operatorname*{res}_0 \operatorname{Tr}\left(\tilde{\mathbb{C}}_{n+1}^{-1}\delta\tilde{L}_n\wedge\delta f_n\tilde{\mathbb{C}}_n\right)\right\rangle dz = \left\langle \operatorname*{res}_0 \operatorname{Tr}\left(\tilde{L}_n^{-1}\delta\tilde{L}_n\wedge\delta f_n\right)\right\rangle dz.$$

From definition (5.1) of L_n, by direct calculations, we obtain that

$$J_2 = \left\langle \frac{2\delta h_{n+1}\wedge\delta x_n}{h_{n+1}^2-1}\left(\frac{\wp(x_n-x_{n+1})+\wp(x_n+x_{n+1})}{\wp(x_n-x_{n+1})-\wp(x_n+x_{n+1})}\right)\right\rangle. \quad (5.31)$$

At last, the third term in (5.29) is equal to

$$\begin{aligned}J_3 &= -\left\langle \operatorname*{res}_0 \operatorname{Tr}\left(\tilde{\mathbb{C}}_{n+1}^{-1}\delta f_{n+1}\wedge\delta\tilde{L}_n\tilde{\mathbb{C}}_n\right)\right\rangle dz\\ &= \left\langle \operatorname*{res}_0 \operatorname{Tr}\left((\delta\tilde{L}_n)\tilde{L}_n^{-1}\wedge\delta f_{n+1}\right)\right\rangle dz,\end{aligned} \quad (5.32)$$

because

$$J_4 = \left\langle \operatorname*{res}_0 \operatorname{Tr}\left(\tilde{\mathbb{C}}_{n+1}^{-1}\delta f_{n+1}\wedge\tilde{L}_n\delta f_n\right)\tilde{\mathbb{C}}_n\right\rangle dz = 0. \quad (5.33)$$

In order to prove (5.33), let us note that at $z=0$,

$$f_n(z) = z^{-1}f_n^0 + O(z^3), \quad f_n^0 = \begin{pmatrix} x_n & 0 \\ 0 & -x_n \end{pmatrix}. \quad (5.34)$$

Therefore,

$$\begin{aligned}J_4 &= \left\langle \operatorname*{res}_0 \operatorname{Tr}\left(\tilde{\mathbb{C}}_{n+1}^{-1}\delta f_{n+1}^0\wedge\tilde{L}_n\delta f_n^0\right)\tilde{\mathbb{C}}_n\right\rangle\wp(z)\,dz\\ &= \left\langle \operatorname*{res}_0 \operatorname{Tr}\left(\mathbb{C}_{n+1}^{-1}\delta f_{n+1}^0\wedge L_n\delta f_n^0\right)\mathbb{C}_n\right\rangle\wp(z)\,dz.\end{aligned} \quad (5.35)$$

The last term in (5.35) is equal to the sum of residues at the punctures $P_\pm$ of the differential

$$\langle \mathcal{C}_{n+1}^* \delta f_{n+1}^0 \wedge L_n \delta f_n^0 \mathcal{C}_n \rangle \wp(z)\, dz,$$

which is holomorphic on Γ outside the punctures. Hence, $J_4 = 0$.

From (5.32), by direct calculations, we obtain that

$$J_3 = \left\langle \frac{2\delta h_{n+1} \wedge \delta x_{n+1}}{h_{n+1}^2 - 1} \right\rangle. \tag{5.36}$$

Equations (5.30), (5.31), and (5.36) imply (5.28). Lemma 5.2 is proved.   ∎

Now we are ready to complete the proof of Theorem 5.1. Equations (5.16) and (2.14) imply that

$$\omega = -\delta\alpha, \qquad \alpha = \sum_{s=1}^{N+1} \int_{P_+}^{\gamma_s} \delta \ln \left(\Lambda^{-N} w \right) dz. \tag{5.37}$$

Indeed, we have

$$\delta\alpha = \sum_{s=1}^{N+1} \delta \ln w(\gamma_s) \wedge \delta z(\gamma_s) - N\delta \ln \Lambda \wedge \delta \sum_{s=1}^{N+1} z(\gamma_s). \tag{5.38}$$

The last term in (5.38) equals zero on the fibers $\phi_+ = \text{const}$ of the map (5.5).

The differential $dS = \ln(\Lambda^{-N} w)\, dz$ is multivalued on Γ, but following the arguments of [28], we can show that its derivatives with respect to $I_k, k = 1, \ldots, N$ (which can be considered as coordinates on a space of curves given by (1.6)) are holomorphic differentials. The differential dS is odd with respect to the permutation of sheets of Γ. Therefore, $I_{N+1} = -I_N$ and the definition of I_k implies that

$$\frac{\partial}{\partial I_k} dS = d\Omega_k^h, \quad k = 1, \ldots, N - 1, \qquad \frac{\partial}{\partial I_N} dS = d\Omega_N^h - d\Omega_{N+1}^h. \tag{5.39}$$

Equations (5.7) and (2.6) imply that

$$\alpha = \sum_{k=1}^{N-1} (\phi_k \delta I_k) + \phi_- \delta I_N, \tag{5.40}$$

and using (5.37), we finally obtain (5.10). Theorem 5.1 is proved.   ∎

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