Holomorphic bundles and scalar difference operators: one-point constructions

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As in [1], we shall consider a non-singular algebraic curve Γ with a distinguished point $P_0 = \infty$ and a local coordinate $z = k^{-1}$, $z(P_0) = 0$. We denote by $A = A(\Gamma, P_0)$ the ring of algebraic functions with a single pole at P_0. We prescribe the 'inverse problem data' as a sequence of points $(\gamma_1, \dots, \gamma_{lg})$, where l is the 'rank' and g is the genus of Γ, parameters α_{sj}, $s = 1, \dots, lg$, $j = 1, \dots, l-1$ and an $l \times l$ matrix-valued function $\chi_n^{(0)}(k)$ $(n \in \mathbb{Z})$ with the only non-zero elements $\chi^{(0)p,p+1} = 1$, $p \le l-1$, $\chi^{(0)lq} = a_q$, $q = 1, \dots, l$, where a_q are polynomials in k depending on n.

Theorem 1. *For any vector η_0 and generic data there exists a unique vector-valued 'Baker-Akhiezer' function $\psi_n(P)$, $P \in \Gamma$, meromorphic on $\Gamma \setminus P_0$ with poles of order one at the points γ_s, where the residues are connected by $(\mathrm{res}_{\gamma_s} \psi^{q+1}) = \alpha_{sq}(\mathrm{res}_{\gamma_s} \psi^1)$, $s = 1, \dots, lg$, $q = 1, \dots, l-1$. In a neighbourhood of $\infty = P_0$ the vector-valued function ψ has the asymptotics $\psi = \bigl[\eta_0 + \sum_{s\ge1} \eta_{sn} k^{-s}\bigr]\Psi^{(0)}$, $\Psi_x^{(0)} = \chi^{(0)}\Psi^{(0)}$ or $\Psi_{n+1}^{(0)} = \chi_n^{(0)}\Psi_n^{(0)}$, $\Psi^{(0)}$ being an $l \times l$-matrix.*

Theorem 2. *Suppose that the matrix $\chi^{(0)}$ depends on k in such a way that only one of the functions $a_{jn}(k)$ has the form $a_j = k - v_{j,n+1}^{(0)}$ and all the remaining a_q are independent of k for $q \ne j$. Then for any function $f(P) \in A(\Gamma, P_0)$ with a pole of order τ there is a unique operator L_f of the form*

$$L_f = \sum_{-M}^{+N} u_{pn} T^p,$$

where $N = \tau(l-j+1)$, $M = \tau(j-1)$, $T\psi_n = \psi_{n+1}$ and $M + N = \tau l$, such that the vector-valued Baker-Akhiezer function ψ constructed in Theorem 1 with $\eta_0 = (\eta_0^q)$, $\eta_0^q = \delta^{qj}$ satisfies the equation

$$L_f \psi = f\psi.$$

Remark. For $j = 1$ this assertion is contained in [1] and [2] in the continuous case. Recall that (α, γ) are the Tyurin parameters characterizing the framed holomorphic stable bundle η, where $c_1(\det \eta) = lg$. All the earlier constructions of commutative difference operators (of rank 1) required no fewer that two 'infinite' points on Γ. We note that symmetric operators $M = N$ are possible only in the case of even rank $l = 2j - 2$.

Following the idea of [1], we consider the multiparameter vector-valued Baker-Akhiezer function. It is determined by the same data $(\Gamma, P_0, \gamma_s, \alpha_{sj}, z = k^{-1})$ as in Theorem 1, but in addition for each new variable t_p a matrix $M^{(0p)}$, $p = 1, 2, \dots$ is given. The 'input' matrix $\Psi_n^{(0)}$ is determined by the equations $(t = (t_1, t_2, \dots))$

$$\Psi_{n+1}^{(0)} = \chi_n^{(0)}\Psi_n^{(0)}, \qquad \Psi_{t_p}^{(0)} = M^{(0p)}\Psi^{(0)}, \quad p = 1, 2, \dots,$$

where the $\chi^{(0)}$ are chosen as in Theorem 2.

Theorem 3. *For any $l \ge 2$ one can choose matrices $M^{(0p)}$, $p = 1, 2, \dots$, in such a way that the vector-valued Baker-Akhiezer function ψ determines a two-dimensionalized Toda lattice hierarchy of solutions of the inverse problem* $(\Gamma, P_0, z = k^{-1}, \gamma_s, \alpha_{sq})$. (The solution determined by ψ will be called a *solution of rank l*.)

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Example. Let $g = 1$, $l = 2$, $a_1 = -c_{n+1}^{(0)}$, $a_2 = k - v_{n+1}^{(0)}$ and let the data $(\gamma_1, \gamma_2, \alpha_1, \alpha_2)$ and the function $f(P) = \lambda = k^2$ be given on Γ. We use the Baker-Akhiezer vector Ψ_n to construct a matrix $\widehat{\Psi}_n$ with rows ψ_n, ψ_{n+1}. We have $\widehat{\Psi}_{n+1} = \chi_n \widehat{\Psi}_n$, where $\chi_n = (-c_{n+1}^{0,1}, k - v_{n+1}) + O(k^{-1})$. The poles of χ_n are at the points γ_{sn}, where $\gamma_{s0} = \gamma_s$, $s = 1, 2$. The zeros of $(\det \chi_n)$ are at the points $\gamma_{s,n+1}$. Moreover, $\alpha_{sn} \operatorname{res}_{\gamma_{sn}} \chi^{i1} = \operatorname{res}_{\gamma_{sn}} \chi^{i2}$, $i = 1, 2$, $\alpha_{s,n+1} = -\chi^{22}(\gamma_{s,n+1})$. The quantity $\gamma_{1n} + \gamma_{2n} = c$ is independent of n. The operators L_f can be computed effectively. For $f = \lambda = -P(z)$ and $c = 0$ we have the fourth-order symmetrizable operator

$$\lambda\psi_n = L_\lambda \psi_n = [(L_2)^2 + u_n]\psi_n, \qquad L_2\psi_n = \psi_{n+1} + v_n\psi_n + c_n\psi_{n-1},$$
$$u_n = -[\wp(\gamma_{n-1}) + \wp(\gamma_{n-2})] + b_{n-1} + b_{n-2}, \qquad \wp(z) = -\zeta'(z),$$
$$b_n = 2\wp'(\gamma_n)[\wp(\gamma_{n+1} + \gamma_n) - \wp(\gamma_{n+1} - \gamma_n)][\wp'(\gamma_{n+1} + \gamma_n) - \wp'(\gamma_{n+1} - \gamma_n)]^{-1},$$
$$c_n = (\alpha_{1n} - \alpha_{2n})^{-1}[\zeta(\gamma_{n+1} - \gamma_n) - \zeta(\gamma_{n+1} + \gamma_n) + 2\zeta(\gamma_n)].$$

Here $\gamma_n = \gamma_{1n}$ and v_n are arbitrary functions, α_{1n} and α_{2n} being determined by

$$\alpha_{1,n+1} = -v_{n+1} + \zeta(\gamma_{n+1}) + \frac{\alpha_{1n}}{\alpha_{1n} - \alpha_{2n}}\zeta(\gamma_{n+1} - \gamma_n) + \frac{\alpha_{2n}}{\alpha_{1n} - \alpha_{2n}}\zeta(\gamma_{n+1} + \gamma_n),$$
$$\alpha_{2,n+1} = -v_{n+1} - \zeta(\gamma_{n+1}) - \frac{\alpha_{1n}}{\alpha_{1n} - \alpha_{2n}}\zeta(\gamma_{n+1} - \gamma_n) - \frac{\alpha_{2n}}{\alpha_{1n} - \alpha_{2n}}\zeta(\gamma_{n+1} + \gamma_n).$$

We consider the time dynamics with $t = t_1$, $M^{(01)} = \chi_n^{(0)} + \operatorname{diag}(v_n^{(0)}, v_{n+1}^{(0)})$. For the Baker-Akhiezer matrix $\widehat{\Psi}_n(t)$ we have $\widehat{\Psi}_{nt} = M_n \widehat{\Psi}_n$, where $M_n = \chi_n + \operatorname{diag}(v_n, v_{n+1}) + O(k^{-1})$. From the compatibility of the variables n and t we obtain a non-linear system for $(c_n(t), v_n(t))$:

$$\dot{c}_{n+1} = c_{n+1}(v_{n+1} - v_m), \qquad \dot{v}_{n+1} = c_{n+2} - c_{n+1} + \varkappa_{n+1} - \varkappa_n,$$
$$\chi_n^{22} = k - v_n + \varkappa_n k^{-1} + O(k^{-2}).$$

This system is a discretization of the so-called 'Krichever-Novikov equation' in [1]. The coefficient $\varkappa_n$ can be computed explicitly using the Tyurin parameter dynamics. In a forthcoming note we shall consider two- (and more) point constructions of rank $l - 1$, in which case a number of new phenomena will appear.

Bibliography

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