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Elliptic Families of Solutions of the Kadomtsev–Petviashvili Equation and the Field Elliptic Calogero–Moser System*

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Abstract. We present a Lax pair for the field elliptic Calogero–Moser system and establish a connection between this system and the Kadomtsev–Petviashvili equation. Namely, we consider elliptic families of solutions of the KP equation such that their poles satisfy a constraint of being *balanced*. We show that the dynamics of these poles is described by a reduction of the field elliptic CM system.

We construct a wide class of solutions to the field elliptic CM system by showing that any N-fold branched cover of an elliptic curve gives rise to an elliptic family of solutions of the KP equation with balanced poles.

Key words: KP equation, Calogero–Moser system, Lax pair.

1. Introduction

The main goal of this paper is to establish a connection between the field analog of the elliptic Calogero–Moser system (CM) introduced in [10] and the Kadomtsev–Petviashvili equation (KP). This connection is a next step along the line that goes back to the paper [1], where the dynamics of the poles of elliptic (rational or trigonometric) solutions of the Korteweg–de Vries equation (KdV) was described in terms of commuting flows of the elliptic (rational or trigonometric) CM system.

It was shown in the earlier paper [7] of one of the authors that the constrained correspondence between a theory of the elliptic CM system and a theory of the elliptic solutions of the KdV equation becomes an isomorphism for the case of the KP equation. It turns out that a function $u(x, y, t)$ that is an elliptic function with respect to the variable x satisfies the KP equation if and only if it has the form

$$u(x,y,t) = -2\sum_{i=1}^{N} \wp(x - q_i(y,t)) + c, \tag{1.1}$$

and its poles q_i as functions of y satisfy the equations of motion of the elliptic CM system. The latter is a system of N particles on an elliptic curve with pairwise interactions. Its Hamiltonian has the form

$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i\neq j}\wp(q_i - q_j),$$

where $\wp(q)$ is the Weierstrass $\wp$-function. The dynamics of the particles q_i with respect to t coincides with the commuting flow generated by the third Hamiltonian H_3 of the system. Recall that the elliptic CM system is a completely integrable system. It admits a Lax representation $\dot{L} = [L, M]$, where $L = L(z)$ and $M = M(z)$ are $(N \times N)$ matrices depending on a spectral parameter z [4]. The involutive integrals H_n are defined as $H_n = n^{-1}\operatorname{Tr} L^n$.

An explicit theta-functional formula for algebro-geometric solutions of the KP equation provides an *algebraic* solution of the Cauchy problem for the elliptic CM system [7]. Namely, the positions $q_i(y)$ of the particles at any time y are roots of the equation

$$\theta(\vec{U}q_i + \vec{V}y + \vec{Z} \mid B) = 0,$$

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where $\theta(z \mid B)$ is the Riemann theta function constructed with the help of the matrix of b-periods of the holomorphic differentials on a *time-independent* spectral curve Γ. The spectral curve is given by $R(k,z) = \det(kI - L(z)) = 0$, and the vectors $\vec{U}$, $\vec{V}$, $\vec{Z}$ are determined by the initial data.

The correspondence between finite-dimensional integrable systems and pole systems of various soliton equation has been extensively studied in [2, 8, 9, 11, 12]. A general scheme for constructing such systems using a specific inverse problem for linear equations with elliptic coefficients is presented in [8].

The problem we address in this paper is as follows. The KP equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x\right) \tag{1.2}$$

is the first equation in a hierarchy of commuting flows. A general solution of the entire hierarchy is known to be of the form

$$u(x,y,t,t_4,\dots) = 2\frac{\partial^2}{\partial x^2}\ln\tau(x,y,t,t_4,\dots), \qquad x = t_1,\ y = t_2,\ t = t_3,$$

where τ is the so-called KP *tau* function. We consider solutions u that are elliptic function with respect to some variable t_k or a linear combination $\lambda = \sum_k \alpha_k t_k$ of times.

It is instructive first to consider the algebraic-geometric solutions of the KP equation. According to [6], any smooth algebraic curve Γ with a puncture defines a solution of the KP hierarchy by the formula

$$u = 2\,\frac{\partial^2}{\partial x^2}\ln\theta\biggl(\sum_k \vec{U}_k t_k + \vec{Z} \;\Big|\; B\biggr), \qquad x = t_1, \tag{1.3}$$

where, as before, B is the matrix of b-periods of the normalized holomorphic differentials on Γ and $\vec{Z}$ is the vector of Riemann constants. The vectors $\vec{U}_k$ are the vectors of b-periods of certain meromorphic differentials on Γ. The algebraic-geometric solution is elliptic with respect to some direction if there is a vector $\vec{\Lambda}$ that spans an elliptic curve $\mathscr{E}$ embedded in the Jacobian $J(\Gamma)$. This is a nontrivial constraint, and the corresponding algebraic curves form a subspace of codimension $g-1$ in the moduli space of all curves. If the vector $\vec{\Lambda}$ does exist, then the theta-divisor intersects the shifted elliptic curve $\mathscr{E} + \sum_k \vec{U}_k t_k$ at finitely many points $\lambda_i(t_1, t_2, \dots)$.

A straightforward verification shows that if $u(x,y,t,\lambda)$ is an elliptic family of solutions of the KP equation, then it has the form

$$u = -2\sum_{i=1}^{N}[\lambda_{i\,x}^2\wp(\lambda-\lambda_i) - \lambda_{i\,xx}\zeta(\lambda-\lambda_i)] + c(x,y,t), \qquad \lambda_i = \lambda_i(x,y,t). \tag{1.4}$$

Since u is an elliptic function, it follows that the sum of its residues is zero, and therefore, $\sum_i \lambda_{ixx} = 0$. We shall consider only solutions u with poles λ_i satisfying an additional constraint. Namely, we say that the poles λ_i, $i = 1,\dots,N$, are *balanced* if they can be represented in the form

$$\lambda_i(x,y,t) = q_i(x,y,t) - hx, \qquad \sum_{i=1}^{N} q_i(x,y,t) = \text{const}, \tag{1.5}$$

where h is an arbitrary nonzero constant. We prove that if the poles of u are balanced, then the functions $q_i(x,y)$ satisfy the equations

$$\begin{aligned} q_{i\,yy} = &-\left\{\frac{q_{i\,y}^2}{h - q_{i\,x}}\right\}_x + \frac{1}{Nh}(h - q_{i\,x})\sum_{k=1}^{N}\left\{\frac{q_{k\,y}^2}{h - q_{k\,x}}\right\}_x \\ &+ 2(h - q_{i\,x})\frac{\delta U(q)}{\delta q_i} - \frac{2}{Nh}(h - q_{i\,x})\sum_{k=1}^{N}(h - q_{k\,x})\frac{\delta U(q)}{\delta q_k}, \qquad 1 \leqslant i \leqslant N, \end{aligned} \tag{1.6}$$

where

$$U(q)=\sum_{i=1}^{N}\frac{q_{i\,xx}^2}{4(h-q_{i\,x})}-\frac{1}{2}\sum_{j\neq i}[(h-q_{j\,x})q_{i\,xx}-(h-q_{i\,x})q_{j\,xx}]\zeta(q_i-q_j)$$
$$+\frac{1}{2}\sum_{j\neq i}[(h-q_{j\,x})^2(h-q_{i\,x})+(h-q_{j\,x})(h-q_{i\,x})^2]\wp(q_i-q_j). \tag{1.7}$$

Here $\delta/\delta q_i$ is the variational derivative. Since $U(q)$ depends only on q_i and their first and second derivatives, we have

$$\frac{\delta U(q)}{\delta q_i}=\frac{\partial U(q)}{\partial q_i}-\frac{d}{dx}\frac{\partial U(q)}{\partial q_{i\,x}}+\frac{d^2}{dx^2}\frac{\partial U(q)}{\partial q_{i\,xx}},\qquad 1\leqslant i\leqslant N,$$

Equations (1.6) can be identified with a reduction of a special case of the Hamiltonian system introduced in [10]. We refer to the latter system as a field analog of the elliptic Calogero–Moser system. The phase space of this system is the space of functions $q_1(x),\dots,q_N(x)$, $p_1(x),\dots,p_N(x)$, the Poisson brackets are given by

$$\{q_i(x),p_j(\tilde{x})\}=\delta_{ij}\delta(x-\tilde{x}),$$

and the Hamiltonian is equal to

$$\widehat{H}=\int H(x)\,dx,\qquad H=\sum_{i=1}^{N}p_i^2(h-q_{i\,x})-\frac{1}{Nh}\biggl(\sum_{i=1}^{N}p_i(h-q_{i\,x})\biggr)^2-\widetilde{U}(q), \tag{1.8}$$

where

$$\widetilde{U}(q)=U(q)+\frac{\partial}{\partial x}\biggl(\frac{h}{2}\sum_{i\neq j}(q_{i\,x}-q_{j\,x})\zeta(q_i-q_j)\biggr).$$

The corresponding equations of motion are presented in Section 3 (see (3.1)). Note that if the q_i are independent of x, then (1.8) is reduced to the Hamiltonian of the elliptic CM system.

In particular, for $N=2$ the Hamiltonian reduction of this system corresponding to the constraint $\sum_i q_i=0$ is a Hamiltonian system on the space of two functions $q(x)$, $p(x)$, where we set

$$q=q_1=-q_2,\qquad \frac{1}{h}p(h^2-q_x^2)=p_1(h-q_x)=-p_2(h-q_x).$$

The Poisson brackets are canonical, i.e., $\{q(x),p(\tilde{x})\}=\delta(x-\tilde{x})$, while the Hamiltonian density H in the coordinates $\{p,q\}$ can be rewritten as

$$H=\frac{2}{h}p^2(h^2-q_x^2)-h\,\frac{q_{xx}^2}{2(h^2-q_x^2)}-2h(h^2-3q_x^2)\wp(2q).$$

A. Shabat noticed that the equations of motion given by this Hamiltonian are equivalent to the Landau–Lifshitz equation. This case $N=2$ was independently studied in [13].

The paper is organized as follows. In Sections 2 and 3 we show that the field analog of the elliptic CM system describes a solution of the inverse Picard type problem for the linear equation

$$\biggl(\frac{\partial}{\partial y}-\mathscr{L}\biggr)\psi(x,y,\lambda)=0,\qquad \mathscr{L}=\frac{\partial^2}{\partial x^2}+u(x,y,\lambda), \tag{1.9}$$

which is one of the equations in the auxiliary linear problem for the KP equation. Namely, it turns out that if equation (1.9) with a family of potentials of the form (1.4) elliptic in λ has N linearly independent *double-Bloch* solutions meromorphic in λ, then the variables $q_i=\lambda_i+hx$ satisfy the equations of motion generated by the Hamiltonian (1.8). Just as in [7], this inverse problem provides a Lax representation for the Hamiltonian system (3.1).

In Section 4, we show that if $u(x,y,t,\lambda)$ is an elliptic family of solutions of the KP equation with balanced poles, then the corresponding family of operators $\partial/\partial y-\mathscr{L}$ has infinitely many double-Bloch solutions. Consequently, the dynamics of $q_i(x,y,t)$ with respect to y coincides with the equations of motion of the field elliptic CM system. We are quite sure that the dynamics of q_i with respect to all times of the KP hierarchy coincides with the hierarchy of commuting flows for the system (3.1), but this question is yet open. We plan to investigate it elsewhere.

In the last section we consider the finite-gap solutions of the KP hierarchy corresponding to an algebraic curve which is an N-fold branched cover of the elliptic curve. We show that they are elliptic with respect to a certain linear combination λ of the times t_k. Moreover, as a function of λ these solutions have precisely N poles. Therefore, they provide a wide class of exact algebraic solutions of the field elliptic CM system.

The definitions and properties of classical elliptic functions and the Riemann θ-function are gathered in the appendix.

2. The Generating Problem

Let us choose a pair of periods $2\omega_1, 2\omega_2 \in \mathbb{C}$, where $\operatorname{Im}(\omega_2/\omega_1) > 0$. A meromorphic function $f(\lambda)$ is said to be *double-Bloch* if it satisfies the following monodromy properties:

$$f(\lambda + 2\omega_a) = B_a f(\lambda), \qquad a = 1, 2.$$

The complex constants B_a are called *Bloch multipliers*. Equivalently, $f(\lambda)$ is a section of a linear bundle over the elliptic curve $\mathscr{E} = \mathbb{C}/\mathbb{Z}[2\omega_1, 2\omega_2]$.

We consider the nonstationary Schrödinger operator

$$\partial_y - \mathscr{L} = \partial_y - \partial_{xx}^2 - u(x, y, \lambda), \qquad \partial_x = \partial/\partial x,\ \partial_y = \partial/\partial y,$$

where the potential $u(x, y, \lambda)$ is a double-periodic function of the variable λ. We do not assume any special dependence on the other variables. Our goal is to find the potentials $u(x, y, \lambda)$ such that the equation

$$(\partial_y - \mathscr{L})\psi(x, y, \lambda) = 0 \tag{2.1}$$

has *sufficiently many* double-Bloch solutions. The existence of such solutions turns out to be a very restrictive condition (see the discussion in [8]).

A basis in the space of the double-Bloch functions can be written in terms of the fundamental function $\Phi(\lambda, z)$ defined by the formula

$$\Phi(\lambda, z) = \frac{\sigma(z - \lambda)}{\sigma(z)\sigma(\lambda)}\, e^{\zeta(z)\lambda}. \tag{2.2}$$

This function is a solution of the Lamè equation

$$\Phi''(\lambda, z) = \Phi(\lambda, z)[\wp(z) + 2\wp(\lambda)]. \tag{2.3}$$

It follows from the monodromy properties of the Weierstrass functions that $\Phi(\lambda, z)$ is double-periodic as a function of z, although it is not elliptic in the classical sense because of the essential singularity at $z = 0$ for $\lambda \neq 0$. It also follows that $\Phi(\lambda, z)$ is double-Bloch as a function of λ; namely

$$\Phi(\lambda + 2\omega_a, z) = T_a(z)\Phi(\lambda, z), \qquad T_a(z) = \exp[2\omega_a\zeta(z) - 2\eta_a z], \quad a = 1, 2.$$

In the fundamental domain of the lattice defined by the periods $2\omega_1$ and $2\omega_2$, the function $\Phi(\lambda, z)$ has a unique pole at the point $\lambda = 0$ with the following expansion in a neighborhood of this point:

$$\Phi(\lambda, z) = \lambda^{-1} + O(\lambda). \tag{2.4}$$

Let $f(\lambda)$ be a double-Bloch function with Bloch multipliers B_a. The gauge transformation

$$f(\lambda) \mapsto \widetilde{f}(\lambda) = f(\lambda)e^{k\lambda}$$

does not change the poles of f and produces a double-Bloch function $\widetilde{f}(\lambda)$ with Bloch multipliers $\widetilde{B}_a = B_a e^{2k\omega_a}$. The two pairs of Bloch multipliers B_a and $\widetilde{B}_a$ connected by such a relation are said to be equivalent. Note that for all equivalent pairs of Bloch multipliers the product $B_1^{\omega_2} B_2^{-\omega_1}$ is a constant depending only on the equivalence class. Further, note that any pair of Bloch multipliers can be represented in the form

$$B_a = T_a(z)e^{2\omega_a k}, \qquad a = 1, 2,$$

with an appropriate choice of the parameters z and k.

There is no differentiation with respect to the variable λ in equation (2.1). Thus, it suffices to study the double-Bloch solutions $\psi(x, t, \lambda)$ with Bloch multipliers B_a such that $B_a = T_a(z)$ for some z.

It follows from (2.4) that a double-Bloch function $f(\lambda)$ with simple poles λ_i in the fundamental domain and with Bloch multipliers $B_a = T_a(z)$ can be represented in the form

$$f(\lambda) = \sum_{i=1}^{N} s_i \Phi(\lambda - \lambda_i, z), \tag{2.5}$$

where s_i is the residue of the function $f(\lambda)$ at the pole λ_i. Indeed, the difference of the left- and right-hand sides in (2.5) is a double-Bloch function with the same Bloch multipliers as $f(\lambda)$. It is also holomorphic in the fundamental domain. Therefore, it is zero, since any nonzero double-Bloch function with at least one of the Bloch multipliers distinct from 1 has at least one pole in the fundamental domain.

Now we are in position to present the generating problem for equations (1.6).

Theorem 1. *Equation* (2.1)*, where the potential is given by*

$$u(x, y, \lambda) = -2 \sum_{i=1}^{N} [(\lambda_{i\,x})^2 \wp(\lambda - \lambda_i) + \lambda_{i\,xx}\, \zeta(\lambda - \lambda_i)] + c(x, y) \tag{2.6}$$

and has the balanced set of poles (1.5)*, has N linearly independent double-Bloch solutions with Bloch multipliers $T_a(z)$, that is, solutions of the form* (2.5)*, if and only if*

$$c(x, y) = \frac{2}{Nh} U(q) - \frac{1}{2Nh} \sum_{i=1}^{N} \frac{q_{i\,y}^2}{h - q_{i\,x}}, \tag{2.7}$$

and the functions $q_i(x, y)$ satisfy (1.6)*.*

If (2.1) *has N linearly independent solutions of the form* (2.5) *for some z, then they exist for all values of z.*

Proof. We begin with a remark. If $u(x, y, \lambda)$ is an elliptic function with a balanced set of poles, then it is necessarily of the form (2.6) provided that there exist N linearly independent double-Bloch solutions of (2.1) for all values of the parameter z in a neighborhood of $z = 0$.

Indeed, let us substitute (2.5) into (2.1). First, we conclude that the potential u has at most double poles at the points λ_i. Thus, the potential is of the form

$$u(\lambda, x, y) = \sum_{i=1}^{N} [a_i \wp(\lambda - \lambda_i) + b_i \zeta(\lambda - \lambda_i)] + c(x, y)$$

with some unknown coefficients $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$. Now the coefficients of the singular part of the right-hand side in (2.1) must be zero. The vanishing of the coefficient of $(\lambda - \lambda_i)^{-3}$ implies that $a_i = -2(\lambda_{i\,x})^2$. The vanishing of the coefficient of $(\lambda - \lambda_i)^{-2}$ gives the equations

$$2s_{i\,x}\lambda_{i\,x} = s_i(\lambda_{i\,y} - \lambda_{i\,xx} - b_i) - \sum_{j \neq i} s_j a_i \Phi(\lambda_i - \lambda_j, z). \tag{2.8}$$

Finally, the vanishing of the coefficient of $(\lambda-\lambda_i)^{-1}$ results in the equations

$$
\begin{aligned}
s_{i\,y}-s_{i\,xx}=s_i\Big(\lambda_{i\,x}^2\wp(z)+\sum_{j\neq i}[a_i\wp(\lambda_i-\lambda_j)+b_j\zeta(\lambda_i-\lambda_j)]+c\Big)\\
+\sum_{j\neq i}s_j(a_i\Phi'(\lambda_i-\lambda_j,z)+b_j\Phi(\lambda_i-\lambda_j,z)).
\end{aligned}
\tag{2.9}
$$

Equations (2.8) and (2.9) are linear equations on $s_i=s_i(x,y,z)$. If we introduce the vector $\vec{S}=(s_1,\dots,s_N)$ and the matrices $L=(L_{ij}),\ A=(A_{ij})$ with matrix elements

$$
L_{ij}=\delta_{ij}\xi_i+(1-\delta_{ij})\lambda_{i\,x}\Phi(\lambda_i-\lambda_j,z),\quad \text{where } \xi_i=\frac{\lambda_{i\,y}-\lambda_{i\,xx}-b_i}{2\lambda_{i\,x}},
\tag{2.10}
$$

and

$$
\begin{aligned}
A_{ij}=\delta_{ij}\Big(\lambda_{i\,x}^2\wp(z)+\sum_{j\neq i}[-2\lambda_{i\,x}^2\wp(\lambda_i-\lambda_j)+b_j\zeta(\lambda_i-\lambda_j)]+c\Big)\\
+(1-\delta_{ij})(-2\lambda_{i\,x}^2\Phi'(\lambda_i-\lambda_j,z)+b_j\Phi(\lambda_i-\lambda_j,z)).
\end{aligned}
$$

then equations (2.8) and (2.9) can be rewritten in the form

$$
\vec{S}_x=L\vec{S},\qquad \vec{S}_y=\vec{S}_{xx}+A\vec{S}=(L^2+L_x+A)\vec{S}.
\tag{2.11}
$$

Let $M=L^2+L_x+A$; then the compatibility of equations (2.11) is equivalent to the zero-curvature equation for L and M, i.e.

$$
L_y-M_x+[L,M]=0.
\tag{2.12}
$$

The entries of M can be computed with the help of the identities (A.2)

$$
\begin{aligned}
M_{ii}&=\lambda_{i\,x}\Big(\sum_{k=1}^N\lambda_{k\,x}\Big)\wp(z)+m_i^0,\\
M_{ij}&=-\lambda_{i\,x}\Big(\sum_{k=1}^N\lambda_{k\,x}\Big)\Phi'(\lambda_i-\lambda_j,z)+m_{ij}\Phi(\lambda_i-\lambda_j,z),\qquad i\neq j
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
m_i^0&=\xi_i^2+\xi_{i\,x}-\sum_{k\neq i}\lambda_{k\,x}(2\lambda_{k\,x}^2+\lambda_{i\,x})\wp(\lambda_i-\lambda_k)+\sum_{k\neq i}b_k\zeta(\lambda_i-\lambda_k)+c,\\
m_{ij}&=\lambda_{i\,x}(\xi_i+\xi_j)+\lambda_{i\,xx}+b_i+\sum_{k\neq i,j}\lambda_{i\,x}\lambda_{k\,x}\,\eta(\lambda_i,\lambda_k,\lambda_j).
\end{aligned}
$$

The coefficients b_i can be determined from the off-diagonal part of the zero curvature equation. The left-hand side of the equation corresponding to a pair of indexes $i\neq j$ is a double-periodic function of z. It is holomorphic except at $z=0$, where it has the form $O(z^{-3})\exp[(\lambda_i-\lambda_j)\zeta(z)]$. Such a function is zero if and only if the coefficients of z^{-3}, z^{-2}, and z^{-1} vanish. A straightforward computation shows that the coefficient of z^{-3} vanishes identically, while the coefficient at z^{-2} is equal to

$$
\Big(\sum_{k=1}^N\lambda_{k\,x}\Big)(b_i+2\lambda_{i\,xx}).
$$

Since our assumption prevents the first factor from vanishing, we conclude that $b_i=-2\lambda_{i\,xx}$. Given this, another straightforward computation shows that the coefficient of z^{-1} also vanishes identically.

The zero curvature equation (2.12) is not only a necessary but also a sufficient condition for (2.1) to have solutions of the form (2.5). Now the following lemma completes the proof of the theorem.

Lemma 1. *Let $L = (L_{ij}(x, y, z))$ and $M = (M_{ij}(x, y, z))$ be defined by formulas* (2.10) *and* (2.13)*, where $b_i = -2\lambda_{i\,xx}$ and the set of $\lambda_i(x, y)$, $i = 1, \dots, N$ is balanced. Then L and M satisfy equation* (2.12) *if and only if $c(x, y)$ is given by* (2.7) *and the functions $q_i(x, y)$ solve* (1.6)*.*

Proof. It was mentioned above that all off-diagonal equations in (2.12) become identities if $b_i = -2\lambda_{i\,xx}$. The diagonal part of the zero curvature equation (2.12) is simplified with the help of identities (A.2) and (A.3). Under the change of variables $\lambda_i = q_i - hx$, it acquires the form

$$\begin{aligned} q_{i\,yy} = -2(h - q_{i\,x})c_x &+ \left\{ \frac{q_{i\,xx}^2 - q_{i\,y}^2}{h - q_{i\,x}} + q_{i\,xxx} \right\}_x \\ &+ 4(h - q_{i\,x}) \sum_{j \neq i} [(h - q_{j\,x})^3 \wp'(q_i - q_j) \\ &\qquad - 3(h - q_{j\,x}) q_{j\,xx}\, \wp(q_i - q_j) + q_{j\,xxx}\, \zeta(q_i - q_j)]. \end{aligned} \tag{2.14}$$

Now consider the sum of equations (2.14) for all i from 1 to N. Since the poles are balanced, it follows that the left-hand side vanishes and the coefficient of c_x becomes $-2Nh$. The other terms on the right-hand side can be rewritten as

$$\frac{\partial}{\partial x} \left(- \sum_{i=1}^{N} \frac{q_{i\,y}^2}{h - q_{i\,x}} + 4U(q) \right).$$

Therefore, c is given by (2.7) up to an arbitrary function of y, which does not affect equations (2.14). Finally, substituting (2.7) into (2.14), we arrive at (1.6). $\square$

3. The Field Analog of the Elliptic Calogero–Moser System

In this section, we show that equations (1.6) can be obtained as a reduction of the field elliptic CM system.

In [15], the elliptic CM system was identified with a special case of the Hitchin system on an elliptic curve with a puncture. In [10], a Hamiltonian theory of zero curvature equations on algebraic curves was developed and identified with infinite-dimensional field analogs of the Hitchin system. In particular, it was shown that the zero curvature equation on an elliptic curve with a puncture can be viewed as a field generalization of the elliptic CM system.

The field elliptic CM system is a Hamiltonian system on the space of functions $\{q_i(x), p_i(x)\}_{i=1}^{N}$ equipped with the canonical Poisson brackets

$$\{q_i(x), q_j(\tilde{x})\} = \{p_i(x), p_j(\tilde{x})\} = 0, \quad \{q_i(x), p_j(\tilde{x})\} = \delta_{ij}\, \delta(x - \tilde{x}), \qquad 1 \leqslant i, j \leqslant N.$$

Its Hamiltonian is given by (1.8). Note that $\widetilde{U}(q)$ is an elliptic function in each of the variables q_i, $i = 1, \dots, N$. Substituting the definition of $\widetilde{U}(q)$ into (1.8), we obtain the following expression for the hamiltonian density:

$$\begin{aligned} H = \sum_{i=1}^{N} p_i^2 (h - q_{i\,x}) - \frac{1}{Nh} \left(\sum_{i=1}^{N} p_i (h - q_{i\,x}) \right)^2 - \sum_{i=1}^{N} \frac{q_{i\,xx}^2}{4(h - q_{i\,x})} - \frac{1}{2} \sum_{i \neq j} [q_{i\,x} q_{j\,xx} - q_{j\,x} q_{i\,xx}] \zeta(q_i - q_j) \\ + \frac{1}{2} \sum_{i \neq j} [(h - q_{i\,x})^2 (h - q_{j\,x}) + (h - q_{i\,x})(h - q_{j\,x})^2 - h(q_{i\,x} - q_{j\,x})^2] \wp(q_i - q_j). \end{aligned}$$

The equations of motion are

$$
\begin{aligned}
\dot q_i &= 2p_i(h-q_{i\,x}) - \frac{2}{Nh}\sum_{k=1}^{N} p_k(h-q_{k\,x})(h-q_{i\,x}),\\
\dot p_i &= -2p_ip_{i\,x} + \frac{2}{Nh}\left\{\sum_{k=1}^{N} p_ip_k(h-q_{k\,x})\right\}_x + \left\{\frac{q_{i\,xxx}}{2(h-q_{i\,x})} + \frac{q_{i\,xx}^2}{4(h-q_{i\,x})^2}\right\}_x\\
&\quad + 2\sum_{j\neq i}[q_{j\,xxx}\zeta(q_i-q_j) - 3(h-q_{j\,x})q_{j\,xx}\wp(q_i-q_j) + (h-q_{j\,x})^3\wp'(q_i-q_j)].
\end{aligned}
\tag{3.1}
$$

Let us make a remark on the notation. Throughout this section, dots stand for derivatives with respect to the variable y, which we treat as a time variable. In view of the connection with the KP equation,this time variable corresponds to the second time of the KP hierarchy, for which y is the standard notation.

One can readily verify that the subspace $\mathscr{N}$ defined by the constraint

$$
\sum_{i=1}^{N} q_i(x) = \text{const}, \tag{3.2}
$$

is invariant with respect to system (3.1). On that subspace, the first two terms of the Hamiltonian density H can be represented in the form

$$
H = \frac{1}{2Nh}\left(\sum_{i\neq j}(p_i-p_j)^2(h-q_{i\,x})(h-q_{j\,x})\right) - \widetilde{U}(q). \tag{3.3}
$$

Therefore, the Hamiltonian (1.8) restricted to $\mathscr{N}$ is invariant under the transformation

$$
p_i(x) \to p_i(x) + f(x), \tag{3.4}
$$

where $f(x)$ is an arbitrary function. The left-hand side of the constraint (3.2) is the first integral corresponding to this symmetry. The canonical symplectic form is also invariant with respect to ((3.4)). Therefore, the Hamiltonian system (3.1) restricted to $\mathscr{N}$ can be reduced to the quotient space. The reduction can be described as follows.

Consider the variables $\ell_i = p_i + \kappa$, $i = 1, \ldots, N$, where

$$
\kappa = -\frac{1}{Nh}\sum_{k=1}^{N} p_k(h-q_{k\,x}). \tag{3.5}
$$

They are invariant with respect to the symmetry (3.4) and satisfy the equation

$$
\sum_{k=1}^{N} \ell_k(h-q_{k\,x}) = 0. \tag{3.6}
$$

A straightforward substitution shows that equations (3.1) imply the system of equations

$$
\begin{aligned}
\dot q_i &= 2\ell_i(h-q_{ix}),\\
\dot \ell_i &= -2\ell_i\ell_{i\,x} + \frac{2}{Nh}\left\{\sum_{k=1}^{N} \ell_k^2(h-q_{kx}) - U(q)\right\}_x + \left\{\frac{q_{i\,xxx}}{2(h-q_{i\,x})} + \frac{q_{i\,xx}^2}{4(h-q_{i\,x})^2}\right\}_x\\
&\quad + 2\sum_{j\neq i}[(h-q_{j\,x})^3\wp'(q_i-q_j) - 3(h-q_{j\,x})q_{j\,xx}\,\wp(q_i-q_j) + q_{j\,xxx}\,\zeta(q_i-q_j)],
\end{aligned}
\tag{3.7}
$$

Theorem 2. *Equations* (1.6) *are equivalent to the restriction of system* (3.7) *to the subspace* $\mathscr{M}$ *defined by the constraints* (3.2) *and* (3.6).

Proof. Let us show that equations (1.6) imply (3.7). The first equations can be regarded as a definition of ℓ_i, $i=1,\dots,N$. Taking their derivative, we obtain

$$\ddot q_i = 2\dot\ell_i(h-q_{i\,x}) - 2\ell_i(2\ell_{i\,x}(h-q_{i\,x}) - 2\ell_i q_{i\,xx}). \tag{3.8}$$

Therefore,

$$\dot\ell_i = 2\ell_i\ell_{i\,x} - 2\ell_i^2\frac{q_{i\,xx}}{h-q_{i\,x}} + \frac{\ddot q_i}{2(h-q_{i\,x})}.$$

To obtain the second equation in (3.7) it suffices to substitute the right-hand side of (2.14) for $\ddot q_i$ and use formula (2.7).

Conversely, equation (3.8) can be used to derive (1.6) from (3.7). □

Note that a solution of (3.7) restricted to the subspace $\mathscr{M}$ defines a solution of (3.1) uniquely up to initial data. Namely, it can be verified directly that if $\kappa(x,y)$ is a solution of the equation

$$\dot\kappa = \left\{-\kappa^2 + \frac{2}{Nh}\sum_{k=1}^{N}\ell_i^2(h-q_{kx}) - \frac{2}{Nh}U(q)\right\}_x \tag{3.9}$$

and $\{\ell_i, q_i\}$ is a solution of (3.7) on $\mathscr{M}$, then $\{q_i, p_i=\ell_i-\kappa\}$ is a solution of (3.1).

Our final goal in this section is to present a Lax pair for the field elliptic CM system.

Theorem 3. *System* (3.1) *admits a zero curvature representation, i. e. it is equivalent to the matrix equation*

$$\widetilde L_y - \widetilde M_x + [\widetilde L, \widetilde M] = 0,$$

with Lax matrices $\widetilde L = (\widetilde L_{ij})$ *and* $\widetilde M = (\widetilde M_{ij})$ *of the form*

$$\begin{aligned}\widetilde L_{ij} &= -\delta_{ij}p_i + (1-\delta_{ij})\alpha_i\alpha_j\Phi(q_i-q_j,z),\\ \widetilde M_{ij} &= \delta_{ij}[-Nh\alpha_i^2\wp(z) + \widetilde m_i^0] + (1-\delta_{ij})\alpha_i\alpha_j[Nh\,\Phi'(q_i-q_j,z) - \widetilde m_{ij}\Phi(q_i-q_j,z)],\end{aligned} \tag{3.10}$$

where $\alpha_i^2 = q_{i\,x} - h$,

$$\widetilde m_i^0 = p_i^2 + \frac{\alpha_{ixx}}{\alpha_i} + 2\kappa p_i - \sum_{j\neq i}[\alpha_j^2(2\alpha_i^4+\alpha_j^2)\wp(q_i-q_j) + 4\alpha_i\alpha_{i\,x}\zeta(q_i-q_j)],$$

$$\widetilde m_{ij} = p_i + p_j + 2\kappa + \frac{\alpha_{i\,x}}{\alpha_i} - \frac{\alpha_{j\,x}}{\alpha_j} + \sum_{k\neq i,j}\alpha_k^2\eta(q_i,q_k,q_j),$$

and κ *is given by* (3.9).

Proof. If we subject the matrices L and M given by (2.10) and (2.13) to a gauge transformation

$$L \mapsto g_x g^{-1} + gLg^{-1}, \qquad M \mapsto g_y g^{-1} + gMg^{-1},$$

where $g=(g_{ij})$ is a diagonal matrix with $g_{ij} = \delta_{ij}(\lambda_{i\,x})^{-1/2}$, and then substitute $\lambda_i = q_i - hx$ and $\lambda_{i\,y}/2\lambda_{i\,x} = \ell_i$, then we obtain a Lax pair for system (3.7). To obtain (3.10), we apply another gauge transformation with $g = e^K I$ and substitute $\ell_i = p_i + \kappa$, $i=1,\dots,N$. Here $K = K(x,y) = \int^x \kappa(\widetilde x, y)\,d\widetilde x$. Note that $K_y = -\kappa^2 - c$ in view of (3.9) and (2.7). □

4. Elliptic Families of Solutions of the KP Equation

The KP equation (1.2) is equivalent to the commutation condition

$$[\partial_y - \mathscr{L}, \partial_t - \mathscr{A}] = 0, \qquad \partial_y = \partial/\partial y,\ \partial_t = \partial/\partial t, \tag{4.1}$$

for the auxiliary linear differential operators

$$\mathscr{L} = \partial_{xx}^2 + u(x,y,t), \qquad \mathscr{A} = \partial_{xxx}^3 + \frac{3}{2}u\partial_x + w(x,y,t), \qquad \partial_x = \frac{\partial}{\partial x}.$$

We use this representation to derive our main result.

Theorem 4. *Let $u(x,y,t,\lambda)$ be an elliptic family of solutions of the KP equation with balanced set of poles $\lambda_i(x,y,t) = q_i(x,y,t) - hx$, $i = 1,\dots,N$. Then $u(x,y,t,\lambda)$ has the form* (1.4), *and the dynamics of the functions $q_i(x,y,t)$ with respect to y is described by system* (1.6).

Proof. Substituting u into (1.2), we readily conclude that u may have poles in λ of at most second order. Moreover, matching the coefficients of the expansions of the left- and right-hand sides in (1.2) near the pole λ_i, we find that the principal part of the solution u coincides with the one given by (1.4).

The next step is to show that the operator equation (4.1) implies the existence of double-Bloch solutions for the equation $(\partial_y - \mathscr{L})\psi(x,y,t,\lambda) = 0$.

Let us define a matrix $S(x,y,t,z)$ as the solution of the linear differential equation $\partial_x S = LS$, where $L = (L_{ij})$ and

$$L_{ij} = \delta_{ij}\left(\frac{\lambda_{i\,y} + \lambda_{i\,xx}}{2\lambda_{i\,x}}\right) + (1-\delta_{ij})\lambda_{i\,x}\,\Phi(\lambda_i - \lambda_j, z),$$

with initial conditions $S(0,y,t,z) = S_0(y,t,z)$, where S_0 is a nonsingular matrix. By Φ we denote the row vector $(\Phi(\lambda-\lambda_1,z),\dots,\Phi(\lambda-\lambda_N,z))$. It follows readily that the vector $(\partial_y - \mathscr{L})\Phi S$ has at most simple poles at λ_i, $i = 1,\dots,N$. Therefore, it is equal to ΦD for some matrix D. The commutation relation (4.1) implies that $D_x = LD$. To show this, consider the vector

$$(\partial_t - \mathscr{A})\Phi D = (\partial_t - \mathscr{A})(\partial_t - \mathscr{L})\Phi S = (\partial_y - \mathscr{L})(\partial_t - \mathscr{A})\Phi S.$$

It has poles of at most third order, and therefore, the vector $(\partial_t - \mathscr{A})\Phi S$ has at most simple poles. In this case, however, the vector

$$(\partial_t - \mathscr{A})(\partial_t - \mathscr{L})\Phi S = (\partial_y - \mathscr{L})(\partial_t - \mathscr{A})\Phi S = (\partial_t - \mathscr{A})\Phi D$$

has poles of at most second order. The absence of third-order poles in the expression $(\partial_t - \mathscr{A})\Phi D$ is equivalent to the equation $D_x = LD$.

Since S and D are solutions of the same linear differential equation in x, it follows that they differ by a matrix independent of x: $D(x,y,t,z) = S(x,y,t,z)T(y,t,z)$. Let us define a matrix $F(y,t,z)$ from the equation $\partial_y F + TF = 0$ and the initial condition $F(0,t,z) = I$. Here I is the identity matrix. Let $\widetilde{S} = SF$; then

$$(\partial_y - \mathscr{L})\Phi\widetilde{S} = (\partial_y - \mathscr{L})\Phi SF = \Phi DF + \Phi SF_y = \Phi S\,(TF + F_y) = 0,$$

and the components of the vector $\Phi\widetilde{S}$ are independent double-Bloch solutions of (2.1).

To complete the proof, it suffices to apply Theorem 1. □

5. The Algebraic-Geometric Solutions

According to [6], a smooth genus g algebraic curve Γ with fixed local coordinate w at a puncture P_0 defines solutions of the entire KP hierarchy by the formula

$$u(t) = 2\frac{\partial^2}{\partial x^2}\ln\theta\left(\sum_k \vec{U}_k t_k + \vec{Z}\;\Big|\; B\right) + \text{const}\,.$$

Here $B = (B_{jk})$ is a matrix of b-periods of normalized holomorphic differentials ω_k^h, i.e.

$$\oint_{a_i}\omega_j^h = \delta_{ij}, \qquad B_{ij} = \oint_{b_i}\omega_j^h, \tag{5.1}$$

and the vectors $\vec{U}_k = (\vec{U}_k^j)$ are vectors of b-periods,

$$\vec{U}_k^j = \frac{1}{2\pi i}\oint_{b_j} d\Omega_k, \qquad\qquad \oint_{a_j} d\Omega_k = 0,$$

of the normalized meromorphic differentials $d\Omega_k$ of the second kind, defined by their expansions

$$d\Omega_k = dw^{-k} + O(1)dw \tag{5.2}$$

in a neighborhood of P_0.

Let Γ be an N-fold branched cover of an elliptic curve $\mathscr{E}$:

$$\rho\colon \Gamma \to \mathscr{E}.$$

Then the induced map of the Jacobians defines an embedding of $\mathscr{E}$ in $J(\Gamma)$, i.e. $\rho^*\mathscr{E} \subset J(\Gamma)$. Therefore, each N-fold cover of $\mathscr{E}$ defines an elliptic family of solutions of the KP equation. The following assertion shows that the corresponding solutions have exactly N poles. Moreover, if the local coordinate w at the puncture P_0 is $\rho^*(\lambda)$, then the poles are balanced. Here λ is a flat coordinate on $\mathscr{E}$.

Theorem 5. *Let Γ be a smooth N-fold branched cover of the elliptic curve $\mathscr{E}$, and let $P_0 \in \Gamma$ be a preimage of the point $\lambda = 0$ on $\mathscr{E}$. Let $d\Omega_k$ be a normalized meromorphic differential on Γ with the only pole at P_0 of the form* (5.2), *where $w = \rho^*(\lambda)$, and let $2\pi i\vec{U}$ and $2\pi i\vec{V}$ be the vectors of b-periods of the differentials $d\Omega_1$ and $d\Omega_2$, respectively. Then the equation*

$$\theta(\vec{\Lambda}\lambda + \vec{U}x + \vec{V}y \mid B) = 0 \tag{5.3}$$

has N balanced roots $\lambda_i(x,y) = q_i(x,y) - x/N$, $\sum_i q_i(x,y) = 0$, and the functions q_i satisfy system (1.6).

Proof. Let $2\omega_1$, $2\omega_2$ be the periods of $\mathscr{E}$ such that $\operatorname{Im}(\tau) = \operatorname{Im}(\omega_2/\omega_1) > 0$. The Jacobian $J(\Gamma)$ is the quotient of $\mathbb{C}^g$ by the lattice $\mathscr{B}$ spanned by the basis vectors $\vec{e}_i \in \mathbb{C}^g$, $i = 1, \ldots, g$ and the columns $\vec{B}_i = (B_{ij}) \in \mathbb{C}^g$, $i = 1, \ldots, g$, of the matrix B. Let $\vec{\Lambda}$ be a vector in $\mathbb{C}^g$ that spans $\rho^*\mathscr{E} \subset J(\Gamma)$. Note that not only $\vec{\Lambda} \in \mathscr{B}$, but also $\tau\vec{\Lambda} \in \mathscr{B}$.

The function $\theta(\sum_k \vec{U}_k t_k + \vec{\Lambda}\lambda + \vec{Z} \mid B)$ viewed as a function of λ has a finite number D of zeros. Its monodromy properties (A.4) imply that it can be rewritten in the form

$$\theta\left(\sum\nolimits_k \vec{U}_k t_k + \vec{\Lambda}\lambda + \vec{Z} \;\middle|\; B\right) = f(t)e^{c_1\lambda + c_2\lambda^2}\prod_{i=1}^{D}\sigma(\lambda - \lambda_i(t)),$$

where c_1 and c_2 are constants.

Note, that the λ_i are defined modulo the periods of $\mathscr{E}$. To count them, we integrate $d\ln\theta$ along the boundary of the fundamental domain of $\rho^*\mathscr{E}$ in $\mathbb{C}^g$.

The embedding of $\mathscr{E}$ in $J(\Gamma)$ is defined by the equivalence classes of the divisors $\rho^*(z) - \rho^*(0)$, where $\rho^*(z)$ is the divisor of preimages on Γ of a point $z \in \mathscr{E}$. Preimages on Γ of a- and b-cycles of $\mathscr{E}$ are some linear combination of the basis cycles on Γ, i.e.

$$\rho^* a = \sum_{k=1}^{g} n_k a_k + m_k b_k, \qquad \rho^* b = \sum_{k=1}^{g} n'_k a_k + m'_k b_k.$$

Therefore, the vector $\vec{\Lambda}$ is equal to

$$\vec{\Lambda} = \sum_{k=1}^{g} n_k \vec{e}_k + m_k \vec{B}_k, \qquad \tau\vec{\Lambda} = \sum_{k=1}^{g} n'_k \vec{e}_k + m'_k \vec{B}_k.$$

The usual residue argument implies that

$$2\pi i D = \oint_{\partial(\rho^*\mathscr{E})} d\ln\theta = \int_{\tau\vec{\Lambda}}\left(\int_{\vec{\Lambda}} d\ln\theta\right) - \int_{\vec{\Lambda}}\left(\int_{\tau\vec{\Lambda}} d\ln\theta\right)$$

The monodromy properties of the theta function imply that

$$D = \sum_{k=1}^{g}(n_k m'_k - n'_k m_k).$$

The right-hand side of the last formula is the intersection number of the cycles $\rho^* a$ and $\rho^* b$, i.e.,

$$D = (\rho^* a) \cap (\rho^* b) = N\,(a \cap b) = N,$$

and so the theta function has exactly N zeros λ_i, $i = 1, \dots, N$.

Now let us show that the set of λ_i is balanced. In a way similar to the residue argument above, we find that

$$-2\pi i \sum_{j=1}^{N} \frac{\partial \lambda_j}{\partial t_k} = \oint_{\partial(\rho^* \mathscr{E})} (\partial_{t_k} \ln \theta)\, d\lambda = \int_b d\lambda \biggl(\int_{\rho^* a} d\Omega_k \biggr) - \int_a d\lambda \biggl(\int_{\rho^* b} d\Omega_k \biggr). \tag{5.4}$$

Let $\operatorname{Tr} d\Omega = \rho_*(d\Omega_k)$ be the sum of $d\Omega_k$ on all sheets of Γ over a point $\lambda \in \mathscr{E}$. It is a meromorphic differential on $\mathscr{E}$. Since the local coordinate w near the puncture is defined by the projection ρ, we have

$$\operatorname{Tr} d\Omega_k = \frac{(-1)^k}{(k-1)!}\, \wp^{(k-1)}(\lambda)\, d\lambda + r_k\, d\lambda, \tag{5.5}$$

where r_k is a constant. The right-hand side of (5.4) can be rewritten as $2\pi i \operatorname{res}_{\lambda=0}(\operatorname{Tr} \Omega_k)\, d\lambda$. For $k > 1$, it is equal to zero, and for $k = 1$ we have

$$\operatorname*{res}_{\lambda=0} (\operatorname{Tr} \Omega_1)\, d\lambda = \operatorname*{res}_{\lambda=0} \zeta(\lambda)\, d\lambda = 1.$$

Therefore, we obtain

$$\sum_{i=1}^{N} \frac{\partial \lambda_j}{\partial x} = -1, \qquad \sum_{i=1}^{N} \frac{\partial \lambda_j}{\partial t_k} = 0, \quad k > 1, \tag{5.6}$$

and consequently, the λ_i, $i = 1, \dots, N$, satisfy (1.5). Note that our choice of a local coordinate near the puncture corresponds to $h = 1/N$. An arbitrary nonzero value of h can be obtained by setting $w = \rho^*(\lambda/Nh)$. The proof of Theorem 5 is complete. □

Remark. If the $q_i(x, y)$, $i = 1, \dots, N$ are periodic functions of x, the algebraic curve Γ can be identified with the spectral curve for the equation $(\partial_x - L)\vec{S} = 0$ (see [10]).

Appendix

A.1. Elliptic functions. Here we list the definitions and basic properties of the classical elliptic functions (see [3] for details). Let $2\omega_1, 2\omega_2 \in \mathbb{C}$ be a pair of periods, $\operatorname{Im}(\omega_2/\omega_1) > 0$. The Weierstrass sigma function is defined by the infinite product

$$\sigma(z) = z \prod\nolimits_{m^2+n^2 \neq 0} \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left\{\frac{z}{\omega_{mn}} + \frac{z^2}{2\omega_{mn}^2}\right\}, \qquad \omega_{mn} = 2m\omega_1 + 2n\omega_2.$$

The product converges for every z to an entire function with simple zeros at the points $z = \omega_{mn}$. The Weierstrass zeta function and $\wp$-function are then defined by

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \qquad \wp(z) = -\zeta'(z).$$

It follows directly from this definition that $\sigma(z)$ and $\zeta(z)$ are odd functions, while $\wp(z)$ is an even function. Under shifts of the periods, the Weierstrass functions are transformed as follows:

$$\sigma(z + 2\omega_a) = e^{2\eta_a(z+\omega_a)}\sigma(z), \quad \zeta(z + 2\omega_a) = \zeta(z) + 2\eta_a, \qquad a = 1, 2,$$

where $\eta_a = \zeta(\omega_a)$ and $\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2$. The $\wp$-function is double-periodic

$$\wp(z + 2\omega_1) = \wp(z + 2\omega_2) = \wp(z) = \wp(-z)$$

and can be regarded as a function on the elliptic curve $\Gamma = \mathbb{C}/\mathbb{Z}[2\omega_1, 2\omega_2]$, where it has the only (double) pole at $z = 0$. It is useful to write out the Laurent expansions of the Weierstrass functions in a neighborhood of $z = 0$:

$$\sigma(z) = z + O(z^5), \qquad \zeta(z) = \frac{1}{z} + O(z^3), \qquad \wp(z) = \frac{1}{z^2} + O(z^2).$$

A.2. Identities for the function $\boldsymbol{\Phi(\lambda, z)}$. Here we collect some useful identities involving the function $\Phi(\lambda, z)$ defined by (2.2).

The derivative of $\Phi(\lambda, z)$ with respect to the variable λ is equal to

$$\Phi'(\lambda, z) = \Phi(\lambda, z)[\zeta(z) - \zeta(\lambda) - \zeta(z - \lambda)] \tag{A.1}$$

We also have the following product identities:

$$\begin{aligned} \Phi(\lambda - \mu, z)\Phi(\mu - \lambda, z) &= \wp(z) - \wp(\lambda - \mu), \\ \Phi(\lambda - \nu, z)\Phi(\nu - \mu, z) &= -\Phi'(\lambda - \mu, z) + \Phi(\lambda - \mu, z)\eta(\lambda, \nu, \mu), \end{aligned} \tag{A.2}$$

where in the second equation we use the notation

$$\eta(\lambda, \nu, \mu) = \zeta(\lambda - \nu) + \zeta(\nu - \mu) - \zeta(\lambda - \mu).$$

Note that η is a completely antisymmetric function of its arguments. To complete the list of identities required for our computations, we differentiate formulas (A.2) and obtain

$$\begin{aligned} \Phi'(\lambda - \mu, z)\Phi(\mu - \lambda, z) - \Phi(\lambda - \mu, z)\Phi'(\mu - \lambda, z) &= -\wp'(\lambda - \mu), \\ \Phi'(\lambda - \nu, z)\Phi(\nu - \mu, z) - \Phi(\lambda - \nu, z)\Phi'(\nu - \mu, z) &= -\Phi(\lambda - \mu)[\wp(\lambda - \nu) - \wp(\nu - \mu)]. \end{aligned} \tag{A.3}$$

A.3. The Riemann $\boldsymbol{\theta}$-function. Let Γ be a genus g algebraic curve with given basis of cycles a_i, b_i, $i \leqslant 1 \leqslant g$, with intersections $a_i \circ b_j = \delta_{ij}$. Let B be the matrix of normalized holomorphic differentials ω_i^h, see (5.1). Then B is a Riemann matrix, i.e. a symmetric $g \times g$ matrix with positive definite imaginary part $\operatorname{Im} B > 0$.

The Riemann θ function associated with the curve Γ is the analytic function of g complex variables $\vec{z} = (z_1, \dots, z_g)$ defined by the Fourier expansion

$$\theta(\vec{z} \,|\, B) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{2\pi i(\vec{m}, \vec{z}) + \pi i(B\vec{m}, \vec{m})}.$$

The Riemann θ-function has the following monodromy properties with respect to the lattice $\mathscr{B}$ spanned by the basis vectors $\vec{e}_i \in \mathbb{C}^g$, $i = 1, \dots, g$, and the columns $B_i \in \mathbb{C}^g$ of the matrix B:

$$\begin{aligned} \theta(\vec{z} + \vec{n} \,|\, B) &= \theta(\vec{z} \,|\, B), \\ \theta(\vec{z} + B\vec{n} \,|\, B) &= \exp[-2\pi i(\vec{n}, \vec{z}) - \pi i(B\vec{n}, \vec{n})]\, \theta(\vec{z} \,|\, B). \end{aligned} \tag{A.4}$$

Here $\vec{n}$ is a vector with integer components.

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