

# The periodic and open Toda lattice

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## Abstract

We develop algebro-geometrical approach for the open Toda lattice. For a finite Jacobi matrix we introduce a singular reducible Riemann surface and associated Baker–Akhiezer functions. We provide new explicit solution of inverse spectral problem for a finite Jacobi matrix. For the Toda lattice equations we obtain the explicit form of the equations of motion, the symplectic structure and Darboux coordinates. We develop similar approach for 2D open Toda. Explaining some the machinery we also make contact with the periodic case.

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# 1 Introduction

Until now the methods of integration for periodic and open Toda lattice were absolutely unrelated to each other. The periodic Toda, is a Hamiltonian system of  $N$ -particles with the Hamiltonian

$$H = \sum_{k=1}^N \frac{p_k^2}{2} + \sum_{k=1}^N e^{q_k - q_{k+1}}, \quad q_{N+1} = q_1 + I_0, \quad I_0 = \text{const.}$$

The solution of equations of motion was obtained by Krichever using general algebro-geometrical approach based on the concept of Baker-Akhiezer function [1]. The solution was expressed in terms of theta functions of the spectral curve, associated with the auxiliary spectral problem for an infinite periodic Jacobi matrix. For the open Toda with the Hamiltonian

$$H = \sum_{k=1}^N \frac{p_k^2}{2} + \sum_{k=1}^{N-1} e^{q_k - q_{k+1}}$$

the solution of equations of motion was obtained by Moser [2]. Moser reduced the original problem of integration to the inverse spectral problem for a finite Jacobi matrix. The latter was solved by Stieltjes [3] more than a hundred years ago using continuous fractions.

Though the Hamiltonian of open Toda lattice can be obtained from the periodic one as the limit  $I_0 \rightarrow \infty$ , the methods of solution are different. This is due to the fact that auxiliary spectral problem associated with a finite Jacobi matrix, unlike the spectral problem for the periodic infinite Jacobi matrix, does not determine a natural spectral curve. The first attempt to singularize smooth spectral curve of the periodic Toda to obtain a spectral curve for the open Toda chain goes back to seventies [4]. Recently, the interest in this problem was revived [5, 6] due to connections of the Toda lattice with Seiberg-Witten theory of supersymmetric  $SU(N)$  gauge theory [7]. (A relatively complete list of references on new insight upon the role of integrable systems in Seiberg-Witten theories [8]-[12] can be found in [13]-[17] and books [5, 18].)

The spectral curve proposed in [5, 6] for the open Toda lattice is determined by the equation

$$w_0 = P(E) = \prod_{k=1}^N (E - E_k) \tag{1.1}$$

and can be considered as the limit when  $I_0 \rightarrow \infty$  of the hyper-elliptic spectral curve

$$w_0 + \frac{\Lambda^2}{w_0} = P(E), \quad \Lambda^2 = e^{-I_0} \tag{1.2}$$

for the periodic case.

From algebro-geometrical viewpoint the limit  $\Lambda \rightarrow 0$  of hyper-elliptic curve leads to a singular curve, which is two copies of rational curve, corresponding to two sheets of hyper-elliptic curve, glued together at  $N$  points  $E_k$ . It is well known that Baker-Akhiezer functions introduced originally for smooth algebraic curves, are also an important tool of integration in the limiting case of singular curves. Multi-soliton and rational solutions of integrable

equations can be obtained within this approach (see in [19]). The main goal of the present paper is to demonstrate that algebro-geometrical approach based on the concept of Baker-Akhiezer function can be used in the case of *reducible* singular algebraic curves. As an outcome, we provide a solution to the inverse spectral problem for a finite Jacoby matrix which is different from the classical Stieltjes' solution.

In the recent papers by Krichever and Phong, [20, 21], the new approach using Baker-Akhiezer functions for construction of Hamiltonian theory of soliton type equations was developed. It provides a universal scheme for construction of angle-action type variables for these equations. In Section 4 we illustrate these ideas, and show that, the case of the open Toda lattice with singular curve can be treated similarly.

Finally, in section 5 we indicate how the ideas introduced in this paper can be applied to integration of open two-dimensional Toda lattice.

## 2 Periodic Toda lattice.

To begin with, let us first briefly recall algebro-geometric solution of the periodic Toda lattice. That will help us later to clarify algebro-geometric origin of our approach to the open Toda lattice.

Most of the material is standard and details can be found in [22, 23]. It is convenient to consider the periodic Toda lattice system as a subsystem of an infinite lattice. As it was found in [24, 25], the equations of motion

$$\dot{q}_k = p_k, \quad \dot{p}_k = -e^{q_k - q_{k+1}} + e^{q_{k-1} - q_k}. \quad (2.1)$$

$k = \dots, -1, 0, 1, \dots$  are equivalent to the Lax equation  $\dot{L} = [A, L]$  for auxiliary linear operators, where  $L$  and  $A$  are difference operators

$$(L\psi)_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} \quad (2.2)$$

$$(A\psi)_n = \frac{c_n}{2}\psi_{n+1} - \frac{c_{n-1}}{2}\psi_{n-1} \quad (2.3)$$

with coefficients

$$c_k = e^{(q_k - q_{k+1})/2}, \quad v_k = -p_k. \quad (2.4)$$

Let  $q_n$  satisfy the constraint  $q_n + I_0 = q_{n+N}$  which is invariant with respect to (2.1). The corresponding  $L$  and  $A$  operators are periodic ones:  $c_n = c_{n+N}$ ,  $v_n = v_{n+N}$ . The Floquet-Bloch solution is the solution of the periodic Schrödinger equation

$$(L\psi)_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} = E\psi_n \quad (2.5)$$

such that

$$\psi_{n+N} = w\psi_n. \quad (2.6)$$

Let  $T(E)$  be a restriction of the monodromy operator  $(Tf)_n = f_{n+N}$  on the invariant space of solutions to the equation  $(L\psi)_n = E\psi_n$ . The solutions  $\varphi$  and  $\theta$  of the Schrödinger equation,

normalized such that  $\varphi_0 = 1$ ,  $\varphi_1 = 0$  and  $\theta_0 = 0$ ,  $\theta_1 = 0$  define a basis in which the monodromy operator have the form:

$$T(E) = \begin{bmatrix} \varphi_N & \theta_N \\ \varphi_{N+1} & \theta_{N+1} \end{bmatrix}. \quad (2.7)$$

Then pairs of complex numbers  $(w, E)$  for which there is a common solution of equations (2.5,2.6) are defined by characteristic equation

$$R(w, E) = \det(w - T(E)) = w^2 - 2\Delta(E)w + 1 = 0. \quad (2.8)$$

The hyperelliptic Riemann surface  $\Gamma$  defined by equation (2.8) is called a *spectral curve* of the periodic Schrödinger operator. A point on the curve we denote by  $Q = (w, E)$ .

To make contact with the open Toda case it is useful to introduce a different representation of  $\Gamma$ . Consider the operator  $L(w)$  which is defined as a restriction of the infinite-dimensional operator  $L$  on  $N$ -dimensional space of functions that satisfy equation (2.6). The corresponding matrix has the form

$$L(w) = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & w^{-1}c_{N-1} \\ c_0 & v_1 & c_1 & \cdots & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ wc_{N-1} & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix} \quad (2.9)$$

From (2.9) it follows that

$$\det(E - L(w)) = P(E) - \Lambda w - \Lambda w^{-1} = 0, \quad \Lambda = (-1)^N \prod_{n=1}^N c_n = (-1)^N e^{-I_0/2}. \quad (2.10)$$

Coefficients of the monic polynomial  $P(E)$

$$P(E) = E^N + \sum_{i=0}^{N-1} u_i E^i \quad (2.11)$$

are polynomial functions of the coefficients  $c_n, v_n$  of  $L$ . For example,

$$u_{N-1} = -\sum_{n=1}^N v_n, \quad u_{N-2} = \left( \sum_{0 < i < j}^N v_i v_j - \sum_{i=1}^N c_i^2 \right). \quad (2.12)$$

Note, that  $P = 2\Lambda\Delta(E)$ . Change of the variable  $w_0 = \Lambda w$  transforms (2.10) into the desired formula (1.2).

The equation  $\partial_w R(w, E) = 0$  for branch points of  $\Gamma$  leads to  $\Delta^2(E) = 1$ . It has  $2N$  real roots which are points of periodic/anti-periodic spectrum for  $L$ , according to the sign of  $\Delta(E) = \pm 1$  at these points. For a generic configuration of particles these roots are distinct  $E_1 < E_2 < \cdots < E_{2N}$  and the curve  $\Gamma$  is smooth and has genus  $N - 1$ .

To make a model of the curve  $\Gamma$  we take two copies of the complex plane and make  $N$  cuts along the bands  $[E_1, E_2], \dots, [E_{2N-1}, E_{2N}]$ . Then glue two cut-planes together.

The multiplier is holomorphic in the affine part of the curve with pole/zero of degree  $N$  at the infinity  $P_+/P_-$ . The multivalued function of quasimomentum  $p(Q)$  is introduced by the formula  $w = e^{Np}$ . It has the asymptotic expansion

$$\pm p(E) = \log E - \sum_{k=0}^{\infty} \frac{H_k}{E^k}, \quad E = E(Q), \quad Q \rightarrow (P_{\pm}).$$

The coefficients are standard integrals of the periodic Toda

$$H_0 = -\frac{I_0}{2N}, \quad H_1 = \frac{1}{N} \sum_{k=1}^N v_k, \quad H_2 = \frac{1}{N} \sum_{k=1}^N \frac{v_k^2}{2} + \sum_{k=1}^N c_k^2.$$

The explicit formula for the Floquet-Bloch solution

$$\psi_n(Q) = \varphi_n(E) + \frac{w(Q) - \varphi_N(E)}{\theta_N(E)} \theta_n(E). \quad (2.13)$$

implies that

**Lemma 1** (i) *The Floquet-Bloch solution, normalized by the condition  $\psi_0 = 1$  becomes single-valued on the Riemann surface  $\Gamma$ . It has a single pole  $\gamma_k$  on each real oval  $a_k$ ,  $k = 1, \dots, N-1$ , which is a preimage of the  $[E_{2k}, E_{2k+1}]$ .*

(ii) *In the vicinity of infinities the Floquet solution has the asymptotics*

$$\psi_n(E) = E^{\pm n} e^{\pm(q_n - q_0)/2} \left( 1 + \sum_{s=1}^{\infty} \chi_s^{\pm}(n) E^{-s} \right),$$

where  $E = E(Q)$ ,  $Q \rightarrow (P_{\pm})$ .

Equation (2.13) implies that projections  $E(\gamma_s)$  of the poles of  $\psi$  are zeros of the polynomial  $\theta_N(E) : \theta_N(E(\gamma_s)) = 0$ . Therefore,  $\theta_n(E(\gamma_s))$  satisfies zero boundary conditions, and  $E(\gamma_s)$  are points of the Dirichlet spectrum for the Schrödinger operator.

The map which associates for  $N$ -periodic operator  $L$  with coefficients  $\{c_n, v_n\}$  a spectral curve  $\Gamma$ , and the divisor  $D = \{\gamma_1, \dots, \gamma_{N-1}\}$  of poles of the Floquet-Bloch solution.

$$\{c_n, v_n\} \rightarrow \{\Gamma, D\}, \quad (2.14)$$

is referred as *direct spectral transform*.

The spectral curve is defined by the variable  $\Lambda$ , and  $N$  coefficients  $u_i$  of the polynomial  $P(E)$ . Therefore, a space of the spectral data has the same dimension  $2N$  as the space of  $N$ -periodic operators. It is a fundamental fact in the theory that the map (2.14) is a bijective correspondence of generic points. We shall refer to the reverse construction

$$\{\Gamma, D\} \longmapsto \{c_n, v_n\}, \quad (2.15)$$

which recaptures the dynamical variables  $\{c_n, v_n\}$  from the geometric data  $\{\Gamma, D\}$ , as *the inverse problem*. As usual in the algebro-geometric theory of solitons, it will be based on the construction of a Baker-Akhiezer function.

**Lemma 2** [1] *Let  $\Gamma$  be a smooth hyperelliptic curve defined by equation (2.10). Then for a generic set of  $(N - 1)$  points  $\gamma_s \in \Gamma$  there exists a unique (up to a sign) function  $\Psi_n(t, Q)$  such that:*

(i) *it is meromorphic function on  $\Gamma$  outside infinities  $P_{\pm}$  and has at most simple poles at the points  $\gamma_s$ .*

(ii) *In the vicinity of infinities the Floquet solution has the asymptotics*

$$\Psi_n(E) = E^{\pm n} e^{\pm Et/2} b_n^{\pm 1}(t) \left( 1 + \sum_{s=1}^{\infty} \chi_s^{\pm}(n, t) E^{-s} \right),$$

where  $E = E(Q)$ ,  $Q \rightarrow (P_{\pm})$ .

Uniqueness of  $\Psi$  immediately implies the following theorem.

**Theorem 1** *The Baker-Akhiezer function  $\Psi_n(t, Q)$  associated with the spectral data  $\{\Gamma, D\}$  satisfies equations*

$$(L\Psi)_n(t, Q) = E\Psi_n(t, Q), \quad (\partial_t - A)\Psi_n(t, Q) = 0. \quad (2.16)$$

where the operators  $L$  and  $A$  have the form (2.2) and (2.3) with coefficients defined by the formulae

$$c_n(t) = b_n(t)b_{n+1}^{-1}(t), \quad v_n = \chi_1^+(n, t) - \chi_1^+(n + 1, t).$$

These coefficients are  $N$ -periodic functions of the variable  $n$ .

From the Lax equation, which is a compatibility condition for (2.16) we obtain that

$$\tilde{q}_n(t) = \ln b_n(t) \quad (2.17)$$

is a solution of the periodic Toda lattice normalized by the condition  $\tilde{q}_0(0) = 0$ . Note, that for  $t = 0$  the Baker-Akhiezer function has the same analytical properties as the Floquet-Bloch solutions. Therefore,  $\Psi_n(0, Q) = \psi_n(Q)$ , and using the spectral data  $\{\Gamma, D\}$  associated with  $\{c_n, v_n\}$  we solve the Cauchy problem for the periodic Toda lattice.

This solution can be written explicitly in terms of the Riemann  $\theta$ -function associated with matrix of  $b$ -periods of a normalized holomorphic differentials [22].

$$q_n(t) = q_0(0) + \frac{n}{N} I_0 + \ln \left( \frac{\theta((n-1)U + Vt + Z)\theta(Z)}{\theta(nU + Vt + Z)\theta(Z-U)} \right) \quad (2.18)$$

Here  $U$  and  $V$  are  $(N - 1)$ -dimensional vectors that are periods of certain meromorphic differentials on  $\Gamma$ , and  $Z$  is Abel transform of the divisor  $D$ .

We finish this brief account of solution of the periodic Toda lattice by the following remark. The Lax equation implies that if the dynamical variables  $\{c_n(t), v_n(t)\}$  evolve according to the Toda lattice equations then the spectral curve of the Schrödinger operator is *time-independent*. At the same time the poles, of the Floquet-Bloch solution normalized

by the condition  $\psi_0 = 1$  do depend on  $t$ . From (2.16) it follows that  $\psi_n(t, Q)$  and the Baker-Akhiezer function are proportional to each other. Therefore,

$$\psi_n(t, Q) = \Psi_n(t, Q)\Psi_0^{-1}(t, Q)$$

Hence, the divisor  $D(t)$  can be identified with the divisor of zeros of the function  $\Psi_0(t, Q)$ . An image of this divisor under the Abel transform evolves linearly,  $Z(t) = Z + Vt$ .

### 3 Open Toda lattice

Equations of motion for the open N-particle Toda lattice

$$\begin{aligned} \dot{q}_k &= p_k, & k &= 0, \dots, N-1, \\ \dot{p}_k &= -e^{q_k - q_{k+1}} + e^{q_{k-1} - q_k} & k &= 1, \dots, N-2, \\ \dot{p}_0 &= -e^{q_0 - q_1}, & \dot{p}_{N-1} &= e^{q_{N-2} - q_{N-1}}, \end{aligned}$$

have Lax representation  $\partial_t L_0 = [A_0, L_0]$  where  $L_0$  and  $A_0$  are finite-dimensional Jacobi matrices

$$L_0 = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & 0 \\ c_0 & v_1 & c_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ 0 & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix}, \quad A_0 = \frac{1}{2} \begin{bmatrix} 0 & c_0 & 0 & \cdots & 0 \\ -c_0 & 0 & c_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & -c_{N-3} & 0 & c_{N-2} \\ 0 & \cdots & 0 & -c_{N-2} & 0 \end{bmatrix} \quad (3.1)$$

where the variables  $c_n, v_n$  are defined by formulae (2.4), as in the periodic case.

Equations of motion are invariant with respect to the shift  $v_n \rightarrow v_n + \text{const}$ . Therefore, without loss of generality we will assume from now on that  $v_n$  are normalized by the constraint

$$\sum_{n=0}^{N-1} v_n = 0 \quad (3.2)$$

As it was mentioned in the introduction, the spectral problem for  $L$  does not contain spectral parameter. Let us introduced another auxiliary spectral problem, which does contain spectral parameter. The very same equations of motion are equivalent to the Lax equation for the following finite-dimensional operators:

$$L(w) = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & 0 \\ c_0 & v_1 & c_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ wc_{N-1} & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix}, \quad (3.3)$$

$$A(w) = \frac{1}{2} \begin{bmatrix} 0 & c_0 & 0 & \cdots & 0 \\ -c_0 & 0 & c_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & -c_{N-3} & 0 & c_{N-2} \\ wc_{N-1} & \cdots & 0 & -c_{N-2} & 0 \end{bmatrix}, \quad (3.4)$$

where

$$c_{N-1} = e^{(q_{N-1}-q_0)/2} = \prod_{n=0}^{N-2} c_n^{-1}. \quad (3.5)$$

Coordinates  $\psi_n(E)$ ,  $n = 0, \dots, N-2$ , of an eigenvector  $\psi$ ,  $L(w)\psi(E) = E\psi(E)$  can be found recurrently from equations

$$\begin{aligned} E\psi_0 &= c_0\psi_1 + v_0\psi_0, \\ E\psi_n &= c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1}, \quad n = 1, \dots, N-2. \end{aligned} \quad (3.6)$$

If we take  $\psi_0 = 1$ , then (3.6) implies that  $\psi_n(E)$  is a polynomial of degree  $n$

$$\psi_n(E) = \sum_{i=0}^n b_i(n)E^i, \quad b_n(n) = e^{(q_n-q_0)/2}. \quad (3.7)$$

The last equation

$$E\psi_{N-1} = c_{N-1}w\psi_0 + v_{N-1}\psi_{N-1} + c_{N-2}\psi_{N-2} \quad (3.8)$$

defines  $w = w(E)$ . It is a polynomial of degree  $N$ . From (3.5) it follows that  $w(E)$  is a monic polynomial. It can be also found from the characteristic equation

$$R_0(w, E) = \det(E - L(w)) = P(E) - w = 0, \quad P(E) = E^N + \sum_{i=0}^{N-2} u_i E^i. \quad (3.9)$$

which determines the components of the spectral curve. For  $w = 0$  the matrix  $L(w)$  coincides with  $L_0$ . Therefore, zeros  $E_k$  of the polynomial  $w(E)$

$$w(E) = \prod_{k=1}^N (E - E_k) \quad (3.10)$$

are eigenvalues of the matrix  $L_0$ . The spectral curve  $\Gamma_0$  is singular algebraic curve which is obtained by gluing to each other two copies of the  $E$ -plane along the set of  $N$  points  $E_k$ .

Let us introduce now a solution  $\varphi = \{\varphi_n(E)\}$  of the adjoint equation  $L^T(w)\varphi = E\varphi$ . If we normalize  $\varphi$  by the condition  $\varphi_{N-1} = 1$ , then the equations

$$\begin{aligned} E\varphi_{N-1} &= c_{N-2}\varphi_{N-2} + v_{N-1}\varphi_{N-1}, \\ E\varphi_n &= c_n\varphi_{n+1} + v_n\varphi_n + c_{n-1}\varphi_{n-1}, \quad n = N-2, \dots, 1. \end{aligned}$$

recurrently define  $\varphi_n(E)$ ,  $n = 0, \dots, N-2$ . It is a polynomial of degree  $N-1-n$  with the leading coefficient

$$\varphi_n(E) = e^{(q_{N-1}-q_n)/2} E^{N-1-n} + O(E^{N-2-n}).$$



The last equation

$$E\varphi_0 = c_0\varphi_1 + v_0\varphi_0 + wc_{N-1}\varphi_{N-1},$$

which determines  $w$  leads to the same formula (3.10).

Let  $\gamma_s$  be roots of the equation  $\varphi_0(E) = 0$ . As we will show, they interlace  $N$  roots of the determinant of  $L_0$ , i.e.

$$E_1 < \gamma_1 < E_2 < \dots < E_{N-1} < \gamma_{N-1} < E_N. \quad (3.11)$$

Coordinates of an eigenvector  $\psi^\sigma(E) = \psi_n^\sigma(E)$  for the matrix  $L^T(w)$ , normalized by the condition  $\psi_0^\sigma = 1$ , are equal to  $\psi_n^\sigma = \varphi_0^{-1}(E)\varphi_n(E)$ , and therefore, have the form

$$\psi_n^\sigma = \frac{\sum_{j=0}^{N-1-n} b_j^\sigma(n)E^j}{\prod_{s=1}^{N-1}(E - \gamma_s)}, \quad b_{N-1-n}^\sigma(n) = e^{(q_0 - q_n)/2}. \quad (3.12)$$

At  $w = 0$  matrices  $L(w)$  and  $L^T(w)$  coincides. Therefore, their eigenvectors are proportional to each other. Due to our choice of normalization, we conclude that

$$\psi_n(E_k) = \psi_n^\sigma(E_k). \quad (3.13)$$

Due to (3.13), a pair of functions  $\psi_n(E), \psi_n^\sigma(E)$  can be considered as a single-valued function on  $\Gamma_0$ . This is the Baker-Akhiezer function for the spectral curve  $\Gamma_0$ .

To prove (3.11) we represent the solution  $\varphi$  using truncated matrices

$$L_k = \begin{bmatrix} v_k & c_k & 0 & \dots & 0 \\ c_k & v_{k+1} & c_{k+1} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & c_{N-3} & v_{N-2} & c_{N-2} \\ 0 & \dots & 0 & c_{N-2} & v_{N-1} \end{bmatrix}, \quad (3.14)$$

We define  $\det(E - L_{N+1}) = 0$ ,  $\det(E - L_N) = 1$ . Using the recurrence relations

$$\det(E - L_k) = (E - v_k) \det(E - L_{k+1}) - c_k^2 \det(E - L_{k+2}),$$

it is easy to check, that

$$\varphi_n(E) = \frac{\det(E - L_{n+1})}{c_n \dots c_{N-2}}.$$

From this we see that  $\gamma_s$  are the spectrum of the matrix  $L_1$  which is obtained from  $L_0$  by canceling the first row and the first column. Now classical theorem of Sturm [26] implies (3.11). By analogy with the periodic case we call the points  $\gamma_s$  the divisor  $D_0$ . The points of  $D_0$  move between  $E_k$  under the Toda flow, but never change a sheet, contrary to the periodic case.

As in the periodic case, a map which associates to dynamical variables a set of spectral data  $\Gamma_0, D_0$  of the same dimension is a bijective correspondence

$$\{c_0, \dots, c_{N-2}, v_0, \dots, v_{N-1}\} \longleftrightarrow \{\Gamma_0, D_0\}. \quad (3.15)$$

In fact, the gluing conditions (3.13), and the normalization

$$b_n(n)b_{N-1-n}^\sigma(n) = 1, \quad (3.16)$$

which follows from (3.7) and (3.12), allow to reconstruct  $\psi_n$  and  $\psi_n^\sigma$ .

For each  $n$  the gluing conditions are equivalent to  $N$  linear equations for  $N + 1$  unknown coefficients  $b_i, b_j^\sigma$ . If we define the vector with  $N + 1$  components  $B_j = b_j, j = 0, \dots, n; B_j = b_{j-n-1}^\sigma, j = n + 1, \dots, N$  then the matrix of the corresponding linear system has the form

$$\begin{aligned} M_{ij}(n) &= r_i E_i^j, & j = 0, \dots, n \\ M_{ij}(n) &= -E_i^{j-n-1}, & j = n + 1, \dots, N, \end{aligned} \quad (3.17)$$

where  $r_i = \prod_{s=1}^{N-1} (E_i - \gamma_s), i = 1, \dots, N$ . Let  $\Delta_j(n)$  be the  $j$ -th minor obtained by canceling the  $j$ -th column. The coefficients  $B$ 's which satisfy the normalization condition are

$$B_j(n) = \frac{\Delta_j(n)}{\sqrt{\Delta_n(n) \Delta_N(n)}}, \quad j = 0, \dots, N. \quad (3.18)$$

Since  $B_n(n) = e^{(q_n - q_0)/2}$ , then

$$q_n - q_0 = 2 \ln B_n(n) = \ln \frac{\Delta_n(n)}{\Delta_N(n)}, \quad n = 0, \dots, N - 1. \quad (3.19)$$

Of course, there are sets of complex numbers  $\{E_k, \gamma_s\}$ , such that the right hand side of (3.19) is singular.

**Lemma 3** *Let the spectral data  $\{E_k, \gamma_s\}$  be real and satisfy conditions (3.11), then*

$$\Delta_n(n)\Delta_N(n) \neq 0. \quad (3.20)$$

Indeed, let  $\tilde{\psi}_n$  and  $\tilde{\psi}_n^\sigma$  be the unnormalized solution of the gluing conditions with the coefficients  $b_i, b_j^\sigma$ , corresponding to  $\tilde{B}_j(n) = \Delta_j(n)$ . The product  $\tilde{\psi}_n \tilde{\psi}_n^\sigma$  is a rational function of the form

$$\tilde{\psi}_n \tilde{\psi}_n^\sigma = \frac{F(E)}{\prod_{s=1}^{N-1} (E - \gamma_s)}, \quad F = \Delta_n(n)\Delta_N(n)E^{N-1} + O(E^{N-2}).$$

If the spectral data are real, then  $\tilde{\psi}_n(E)$  and  $\tilde{\psi}_n^\sigma(E)$  are real for real values of  $E$ . Due to the gluing conditions the product of these functions at  $E = E_k$  is positive. Therefore, the product  $\tilde{\psi}_n \tilde{\psi}_n^\sigma$  has at least one zero on each interval  $[E_k, E_{k+1}]$ , because it has a pole there. Hence, degree of the polynomial  $F$  can not be less than  $N - 1$ . That implies (3.20).

**Theorem 2** *Let  $\psi_n(E), \psi_n^\sigma(E)$  be functions associated with spectral data  $E_k, \gamma_s$ . Then they satisfy equations  $L(w)\psi = E\psi$  and  $L^T(w)\psi^\sigma = E\psi^\sigma$ , where*

$$c_n^2 = \frac{\Delta_n(n)\Delta_N(n+1)}{\Delta_{n+1}(n+1)\Delta_N(n)}, \quad n = 0, \dots, N - 2, \quad (3.21)$$

and

$$\begin{aligned} v_n &= \frac{\Delta_{n-1}(n)}{\Delta_n(n)} - \frac{\Delta_n(n+1)}{\Delta_{n+1}(n+1)}, \quad n = 1, \dots, N-2; \\ v_0 &= -\frac{\Delta_0(1)}{\Delta_1(1)}, \quad v_{N-1} = \frac{\Delta_{N-2}(N-1)}{\Delta_{N-1}(N-1)}. \end{aligned}$$

*Proof.* Functions  $\psi_n, \psi_n^\sigma$  at  $E \rightarrow \infty$  have the form

$$\psi_n(E) = E^n e^{(q_n - q_0)/2} \left( 1 + \sum_{s=1}^{\infty} \chi_s(n) E^{-s} \right), \quad (3.22)$$

$$\psi_n^\sigma(E) = E^{-n} e^{(q_0 - q_n)/2} \left( 1 + \sum_{s=1}^{\infty} \chi_s^\sigma(n) E^{-s} \right). \quad (3.23)$$

Note, that

$$\chi_1(n) = \frac{b_{n-1}(n)}{b_n(n)}, \quad \chi_1^\sigma(n) = \frac{b_{N-2-n}^\sigma(n)}{b_{N-1-n}^\sigma(n)} + \sum_{s=1}^{N-1} \gamma_s. \quad (3.24)$$

Here for  $n = 0$  and  $n = N-1$  we formally put  $b_{-1}(0) = b_{-1}^\sigma(N-1) = 0$ .

Let  $\phi_n(E)$  and  $\phi_n^\sigma(E)$  be coordinates of the vectors

$$\phi(E) = (L(w) - E)\psi(E), \quad \phi^\sigma(E) = (L^T(w) - E)\psi^\sigma(E). \quad (3.25)$$

If  $c_n$  are defined by (2.4), and  $v_n$  are defined by the equations

$$v_n = \chi_1(n) - \chi_1(n+1), \quad n = 0, \dots, N-2; \quad v_{N-1} = \chi_1(N-1), \quad (3.26)$$

then (3.22) implies that  $\phi$  near infinity has the form  $\phi_n = O(E^{n-1})$ . Therefore, we conclude that  $\phi_n$  is a polynomial of degree at most  $(n-1)$ . The leading term of expansion for  $\phi_n^\sigma$  near infinity is  $O(E^{-n})$ . Hence,  $\phi_n^\sigma$  has the same form as  $\psi_n^\sigma$ . Functions  $\phi_n$  and  $\phi_n^\sigma$  satisfy the same gluing conditions (3.13). They are equivalent to  $N$  linear equations for  $N$  unknown coefficients for corresponding polynomials. The matrix of this system is obtained from the matrix  $M$  (3.17) by canceling the  $n$ -th column. If this matrix is non degenerate,  $\Delta_n(n) \neq 0$ , we conclude that  $\phi = \phi^\sigma = 0$ . Formulae (3.18,3.24) allow to complete proof of the theorem.

Note, that the equation  $(L^T - E)\psi^\sigma = 0$  implies that

$$v_0 = -\chi_1^\sigma(N-1), \quad v_n = \chi_1^\sigma(n) - \chi_1^\sigma(n-1), \quad n = 1, \dots, N-1, \quad (3.27)$$

Therefore, we obtain the identities

$$\chi_1(n+1) + \chi_1^\sigma(n) = -v_0, \quad (3.28)$$

which will be used in the next section.

Solution of the Cauchy problem for the open Toda lattice can be obtained in a way which is almost identical to the periodic case. First of all, we defined time-dependent Baker-Akhiezer functions  $\psi_n(t, E)$ ,  $\psi_n^\sigma(t, E)$ , as functions of the form

$$\psi_n(t, E) = e^{Et/2} \left( \sum_{i=0}^n b_i(t, n) E^i \right), \quad (3.29)$$

$$\psi_n^\sigma(t, E) = e^{-Et/2} \left( \frac{\sum_{j=0}^{N-1-n} b_j^\sigma(t, n) E^j}{\prod_{s=1}^{N-1} (E - \gamma_s)} \right), \quad (3.30)$$

that satisfy gluing (3.13) and normalization conditions (3.16).

After that, without any change of arguments we obtain that they satisfy the equations  $(L(t, w) - E)\psi(t, E) = 0$ ,  $(L^T(t, w) - E)\psi^\sigma(t, E) = 0$ . In the similar way we get that these functions satisfy equations  $(\partial_t - A(t, w))\psi(t, E) = 0$ ,  $(\partial_t - A^T(t, w))\psi^\sigma(t, E) = 0$ . As a result of compatibility condition we get that functions  $q_n(t) = q_0(0) + \ln b_n(t, n)$  are solutions of the open Toda lattice.

The matrix of the linear system for  $b_i(t, n)$  and  $b_j^\sigma(t, n)$  has the same form as in (3.17) with the only difference that constants  $r_i$  should be replaced by  $r_i(t) = r_i e^{E_k t}$ . Therefore, we come to the following theorem.

**Theorem 3** *Let  $\Delta_j(t, n)$  be the  $j$ -th minor obtained by canceling the  $j$ -th column of the  $N \times (N + 1)$  matrix*

$$\begin{aligned} M_{ij}(t, n) &= r_i E_i^j e^{E_i t}, & j = 0, \dots, n \\ M_{ij}(t, n) &= -E_i^{j-n-1}, & j = n + 1, \dots, N, \end{aligned}$$

where  $r_i = \prod_{s=1}^{N-1} (E_i - \gamma_s)$ . Then the formula

$$q_n(t) = q_0(0) + \ln \frac{\Delta_n(t, n)}{\Delta_N(t, n)}$$

gives a solution of the Cauchy problem for the open Toda lattice equations.

## 4 Action-angle type variables

The main goal of this section is to show that Lax representation which depends on the spectral parameter allows to apply for Hamiltonian theory of open Toda lattice an algebro-geometric approach proposed in [20, 21]. The main idea of this approach is to introduce in a universal way two-form on a space of auxiliary linear operators, written in terms of the operator itself and its eigenfunctions. This form defines on proper subspaces a symplectic structure with respect to which Lax equation is Hamiltonian. Moreover, the way to define the symplectic structure leads directly to construction of Darboux coordinates.

In the case of consideration, we define such a form by the formula

$$\omega = \text{res}_\infty \left[ \langle \psi^+ \delta L \wedge \delta \psi \rangle + \langle \delta \psi^+ \wedge \delta L \psi \rangle \right] \frac{d \ln w}{\langle \psi^+ \psi \rangle}. \quad (4.1)$$

Here  $\delta L$  is an operator-valued one-form on a space of coefficients of the operator, i.e. in our case it is just a matrix with entries, which are one-forms

$$\delta L(w) = \begin{bmatrix} \delta v_0 & \delta c_0 & 0 & \cdots & 0 \\ \delta c_0 & \delta v_1 & \delta c_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \delta c_{N-3} & \delta v_{N-2} & \delta c_{N-2} \\ w\delta c_{N-1} & \cdots & 0 & \delta c_{N-2} & \delta v_{N-1} \end{bmatrix}. \quad (4.2)$$

An eigen-vector  $\psi$  of  $L$  can be seen as a vector-valued function on the space of the coefficients of  $L$ , and therefore, its external differential  $\delta\psi$  is a vector-valued one-form. The co(row)-vector  $\psi^+$  is an eigen-vector of  $L$  in the dual space,  $\psi^+L = E\psi^+$ , i.e.  $\psi^+ = (\psi^\sigma)^T$ . At last,  $\langle f^+g \rangle$  stands for pairing of vector and co-vector

$$\langle f^+g \rangle = \sum_{n=0}^{N-1} f_n^+ g_n.$$

Now, it is necessary to add an important remark, which clarifies our understanding of  $\delta\psi$ . In (4.2) variation of  $L$  was defined such that the spectral parameter  $w$  is fixed. An eigenvalue of  $L$  can be seen locally as a function  $E = E(w)$  defined implicitly by (3.9). Therefore, though  $\psi(E)$  is a polynomial in  $E$ , locally, we consider it as a function of  $w$ ,  $\psi(w) = \psi(E(w))$ , and partial derivatives in

$$\delta = \sum \left( \delta c_n \frac{\partial}{\partial c_n} + \delta v_n \frac{\partial}{\partial v_n} \right)$$

are taken for fixed values of  $w$ . A connection between  $\delta$  and variation  $\delta_0$ , which is defined if one keeps  $E$  fixed, is just a chain rule. Let  $f(E) = f(E, c_n, v_n)$  be a function of the variable  $E$  depending on  $\{c_n, v_n\}$  as on external parameters. Then

$$\delta f(w) = \delta f(E(w)) = \delta_0 f(E) + \frac{df}{dE} \delta E(w). \quad (4.3)$$

In particular, if we take  $f = w$ , then from (4.3) it follows that

$$0 = \delta_0 w(E) + \frac{dw}{dE} \delta E(w).$$

Therefore, (4.3) may be rewritten in the form

$$\delta f(w) = \delta_0 f(E) - \frac{df}{dw} \delta_0 w(E). \quad (4.4)$$

This formula shows that variation  $\delta$  leads to an appearance of poles at zeros of  $dw$ , which are critical points of the polynomial  $w(E)$ , where we, even locally, can not invert  $w$  and introduce  $E = E(w)$ .

Our immediate goal is to find an explicit formula for  $\omega$  in term of dynamical variables.

**Lemma 4** *Let  $\omega$  be given (4.1). Then on symplectic leaves of the space of dynamical variables  $\{c_n, v_n\}$ , defined by the constraint  $\sum_n v_n = \text{const}$ , the form  $\omega$  is equal to*

$$\omega = 2 \sum_{n=0}^{N-1} \delta q_n \wedge \delta v_n. \quad (4.5)$$

*Proof.* In order to simplify slightly formulae, we consider only the leaf  $\sum_n v_n = 0$ . For all the other leaves a proof is going in the same way.

At infinity the functions  $\psi_n$  and  $\psi^\sigma$  can be expanded as Laurent series in the variable

$$k = w^{1/N}(E) = E + O(E^{-1}).$$

From (3.22,3.23) we get

$$\psi_n(k) = k^n e^{(q_n - q_0)/2} \left( 1 + \sum_{s=1}^{\infty} \xi_s(n) k^{-s} \right), \quad (4.6)$$

$$\psi_n^\sigma(k) = k^{-n} e^{(q_0 - q_n)/2} \left( 1 + \sum_{s=1}^{\infty} \xi_s^\sigma(n) k^{-s} \right). \quad (4.7)$$

Note, that in this expansion  $\xi_1 = \chi_1$ ,  $\xi_1^\sigma = \chi_1^\sigma$ . Therefore, as before,

$$v_n = \xi_1(n) - \xi_1(n+1), \quad \xi_1(n+1) + \xi_1^\sigma(n) = -v_0 \quad n = 0, \dots, N-2, \quad (4.8)$$

$$\xi_1(N-1) = v_{N-1}. \quad (4.9)$$

The variable  $k$  is a constant with respect to our definition of  $\delta$ , and therefore, an expansion of  $\delta\psi$  can be obtained by taking variation of coefficients in (4.6,4.7).

Definition of  $\delta L$  implies

$$\langle \psi_n^+ \delta L \wedge \delta \psi \rangle = \sum_{n=0}^{N-1} \left( \delta v_n \wedge \psi_n^+ \delta \psi_n \right) + \sum_{n=0}^{N-2} \left( \delta c_n \wedge \left( \psi_n^+ \delta \psi_{n+1} + \psi_{n+1}^+ \delta \psi_n \right) \right)$$

From (4.8,4.9) it follows that at infinity

$$\frac{d \ln w}{\langle \psi^+ \psi \rangle} = \left( 1 + v_0 k^{-1} + O(k^{-2}) \right) \frac{dk}{k}. \quad (4.10)$$

Using expansions (4.6,4.7) and equation (4.8) we obtain

$$\omega = -2 \left[ \sum_{n=0}^{N-1} \delta v_n \wedge \delta(q_n - q_0) + \sum_{n=0}^{N-2} \delta(q_n - q_{n+1}) \wedge \delta \xi_1(n+1) \right]. \quad (4.11)$$

This equation implies (4.5), due to the relation  $\sum_n \delta v_n = 0$  and equations (4.8). The lemma is proved.

Representation of symplectic forms as a residue at infinity of the spectral curve of meromorphic differential  $d\Omega$  in the right hand side of (4.1) allows to find Darboux coordinates. These Darboux coordinates are connected with residues of the differential at finite points.

**Lemma 5** Let  $R = \prod_s (E - \gamma_s)$  be the monic polynomial with zeros at the poles of the Baker-Akhiezer function  $\psi^\sigma$ . Then the following identity holds

$$\frac{dw}{\langle \psi^+ \psi \rangle} = RdE, \quad (4.12)$$

*Proof.* The equation  $(L - E)\psi = 0$  implies that  $(dL - dE)\psi = (E - L)d\psi$ . Therefore,

$$\langle \psi^+ (dL - dE)\psi \rangle = \langle \psi^+ (E - L)d\psi \rangle = 0,$$

and we obtain the equality

$$dE \langle \psi^+ \psi \rangle = \langle \psi^+ dL\psi \rangle = c_{N-1} dw \left( \psi_{N-1}^+ \psi_0 \right).$$

From (3.12) we have that  $\psi_{N-1}^+ = c_{N-1}^{-1} R^{-1}$ . Lemma is proved.

This result implies that the differential  $d\Omega$ , besides  $E = \infty$ , may have poles at spectral points  $E_k$ , at poles  $\gamma_s$  of  $\psi^+$ , and at zeros  $z_i$  of  $dw$ . (The last poles are due to (4.4).)

First of all, let us show that  $d\Omega$  has no poles at  $E_k$ . At these points  $L = L_0$ , and therefore, is symmetric. Using gluing conditions we obtain

$$\langle \psi^+ \delta L \wedge \delta \psi \rangle + \langle \delta \psi^+ \wedge \delta L \psi \rangle |_{w=0} = \langle \psi^T \delta L_0 \wedge \delta \psi \rangle + \langle \delta \psi^T \wedge \delta L_0 \psi \rangle = 0,$$

due to skew-symmetry of the wedge product.

At  $E = \gamma_s$  the first term of  $d\Omega$  does not contribute to residue because poles of  $\psi^+$  cancel with zero of  $R$ . Nontrivial residue is due to second order pole of  $\delta \psi^+$ . We have

$$\begin{aligned} \text{res}_{\gamma_s} \frac{\langle \delta \psi^+ \wedge \delta L \psi \rangle}{\langle \psi^+ \psi \rangle} d \ln w &= \delta \ln w(\gamma_s) \wedge \left[ \frac{\langle \psi^+ \delta L \psi \rangle}{\langle \psi^+ \psi \rangle} \right]_{E=\gamma_s} \\ &= \delta \ln w(\gamma_s) \wedge \delta E(\gamma_s). \end{aligned} \quad (4.13)$$

In the last equality we use the well-known formula for variation of eigenvalue

$$\langle \psi^+ \delta L \psi \rangle = \delta E \langle \psi^+ \psi \rangle,$$

which immediately follows from the equation

$$\langle \psi^+ (\delta L - \delta E)\psi \rangle = \langle \psi^+ (L - E)\delta \psi \rangle = 0.$$

Consider now critical points  $z_i$  of the polynomial  $w(E) : dw(z_i) = 0$ . Formula (4.3) for  $\delta \psi$  implies

$$\text{res}_{z_i} \frac{\langle \psi^+ \delta L \wedge \delta \psi \rangle}{\langle \psi^+ \psi \rangle} d \ln w = \text{res}_{z_i} \frac{[\langle \psi^+ \delta L d\psi \rangle \wedge \delta E(w)] d \ln w}{dE \langle \psi^+ \psi \rangle} \quad (4.14)$$

Due to skew-symmetry of the wedge product,  $\delta L$  in the last formula may be replaced by  $(\delta L - \delta E)$ . The equations  $\psi^+ (\delta L - \delta E) = -\delta \psi^+ (L - E)$  and  $(L - E)d\psi = -(\delta L - \delta E)\psi$  imply

$$\langle \psi^+ (\delta L - \delta E)d\psi \rangle = \langle \delta \psi^+ (dL - dE)\psi \rangle .$$

Note that  $dL(z_i) = 0$ . Hence,

$$\operatorname{res}_{z_i} \frac{\langle \psi^+ \delta L \wedge \delta \psi \rangle}{\langle \psi^+ \psi \rangle} d \ln w = -\operatorname{res}_{z_i} \frac{(\langle \delta \psi^+ \psi \rangle \wedge \delta E) d \ln w}{\langle \psi^+ \psi \rangle}.$$

In the same way we obtain

$$\operatorname{res}_{z_i} \frac{\langle \delta \psi^+ \wedge \delta L \psi \rangle}{\langle \psi^+ \psi \rangle} d \ln w = -\operatorname{res}_{z_i} \frac{(\delta E \wedge \langle \psi^+ \delta \psi \rangle) d \ln w}{\langle \psi^+ \psi \rangle}.$$

Taking a sum of these equalities we get

$$\operatorname{res}_{z_i} d\Omega = \operatorname{res}_{z_i} \frac{(\langle \psi^+ \delta \psi - \delta \psi^+ \psi \rangle \wedge \delta E) d \ln w}{\langle \psi^+ \psi \rangle}.$$

Besides poles at the points  $z_i$ , the differential at the right hand side of the last equation has poles at the points  $\gamma_s$ , only. Indeed, it has no pole at the infinity, because due to constraint  $\sum_n \delta v_n = 0$  we have  $\delta E = O(k^{-1})$ . There is no pole of the differential at zeros  $E_k$  of  $w(E)$  due to gluing conditions.

Hence, we get

$$\sum_i \operatorname{res}_{z_i} d\Omega = -\sum_s \operatorname{res}_{\gamma_s} \frac{(\langle \psi^+ \delta \psi - \delta \psi^+ \psi \rangle \wedge \delta E) d \ln w}{\langle \psi^+ \psi \rangle} = \sum_s \delta \ln w(\gamma_s) \wedge \delta E(\gamma_s) \quad (4.15)$$

Sum of all the residues of  $d\Omega$  equals to zero. Therefore, using (4.13, 4.15) we obtain

**Lemma 6** *Let  $\omega$  be a two-form given by the formula (4.1). Then*

$$\omega = 2 \sum_{s=1}^{N-1} \delta E(\gamma_s) \wedge \delta \ln w(\gamma_s). \quad (4.16)$$

Finally, combining results of Lemma 4 and 6 we obtain

**Theorem 4** *Poles  $\gamma_s = E(\gamma_s)$  of the Baker-Akhiezer function and evaluation  $\ln w(\gamma_s)$  are Darboux coordinates for reduction of the canonical symplectic form on the leaf  $\mathcal{M}_I$ , defined by the constraint  $\sum_n v_n = I$ :*

$$\left( \sum_n \delta v_n \wedge \delta q_n \right)_{\mathcal{M}_I} = \sum_{s=1}^{N-1} \delta \ln w(\gamma_s) \wedge \delta \gamma_s.$$

From (3.9) we have  $w(\gamma_s) = \prod_k (\gamma_s - E_k)$ . Therefore,

$$\sum_{s=1}^{N-1} \delta \ln w(\gamma_s) \wedge \delta \gamma_s = \sum_{s=1}^{N-1} \sum_{k=1}^N \frac{\delta E_k \wedge \delta \gamma_s}{E_k - \gamma_s} = \sum_{k=1}^N \delta \ln r_k \wedge \delta E_k. \quad (4.17)$$

where  $r_k = R(E_k) = \prod_s (E_k - \gamma_s)$  are variables used in the previous section. From the formulae  $\varphi_0(E_k) \psi_{N-1}(E_k) = 1$  and  $\varphi_0(E_k) = c_{N-1} r_k$ ,  $k = 1, \dots, N$ ; we obtain

$$\sum_{k=1}^N \delta \ln r_k \wedge \delta E_k = \sum_{k=1}^N \delta \ln \varphi_0(E_k) \wedge \delta E_k = -\sum_{k=1}^N \delta \ln \psi_{N-1}(E_k) \wedge \delta E_k$$



Note that in all formulae the variables  $E_k$  are subject to the constraint  $\sum_k E_k = \sum_n v_n = I$ . The last representation for the symplectic structure in terms of first/last components of normalized eigenfunctions and eigenvalues of  $L_0$  is well-known, see [2].

We would like to emphasize that the form in (4.17) can be seen as a limit  $\Lambda \rightarrow 0$  of the formula for action-angle variables for the periodic Toda lattice. To see this, let us choose  $E_1, \dots, E_{N-1}$ , as independent variables, then

$$\sum_{k=1}^N \delta \ln r_k \wedge \delta E_k = \sum_{k=1}^{N-1} \delta (\ln r_k - \ln r_N) \wedge \delta E_k. \quad (4.18)$$

If we choose first  $(N-1)$  cuts of the hyperelliptic curve as a basis of  $a$ -cycles, then  $E_k$  are identified with limit of the action variables for the periodic Toda lattice:

$$E_k = \lim_{\Lambda \rightarrow 0} \frac{1}{2\pi i} \oint_{a_k} E d \ln w_0 ,$$

where  $w_0 = \Lambda w$  from (1.2). Limit of the normalized holomorphic differential  $d\Omega_k$  on the spectral curve of the periodic Toda lattice is equal to

$$\lim_{\Lambda \rightarrow 0} d\Omega_k = \frac{dE}{(E - E_k)} - \frac{dE}{(E - E_N)} .$$

Therefore, the variables  $(\ln r_k - \ln r_N)$  are identified with a limit of the angle variables  $\phi_k$  in the periodic case, which are result of the Abel transform of poles of the Floquet-Bloch solution

$$\ln r_k - \ln r_N = \lim_{\Lambda \rightarrow 0} \sum_s \int^{\gamma_s} d\Omega_k = \lim_{\Lambda \rightarrow 0} \phi_k .$$

## 5 2D open Toda lattice

Equations of motion of 2D open Toda lattice are equations for  $N$  unknown functions  $q_n(x, t)$ . In light-cone coordinates  $\xi = x + t$ ,  $\eta = x - t$  they have the form

$$\begin{aligned} \partial_{\xi\eta}^2 q_n &= e^{q_{n-1} - q_n} - e^{q_n - q_{n+1}}, \quad n = 1, \dots, N-2, \\ \partial_{\xi\eta}^2 q_0 &= -e^{q_0 - q_1}, \quad \partial_{\xi\eta}^2 q_{N-1} = e^{q_{N-2} - q_{N-1}} . \end{aligned} \quad (5.1)$$

We consider the Goursat initial value problem with initial data given on characteristics  $\partial_\xi q_n(\xi, 0)$  and  $q_n(0, \eta)$ .

In two-dimensional case an analog of the Lax representation is the zero-curvature equation  $[\partial_\xi - U_0, \partial_\eta - V_0] = 0$ , where  $U_0$  and  $V_0$  are finite-dimensional matrices

$$U_0 = \begin{bmatrix} v_0 & 1 & 0 & \cdots & 0 \\ 0 & v_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & v_{N-2} & 1 \\ 0 & \cdots & 0 & 0 & v_{N-1} \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ c_0^2 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & c_{N-3}^2 & 0 & 0 \\ 0 & \cdots & 0 & c_{N-2}^2 & 0 \end{bmatrix} \quad (5.2)$$

where, as before  $c_n^2 = e^{q_n - q_{n+1}}$  and  $v_n = -\partial_\xi q_n$ .

Note, that for  $N = 2$  equations (5.1) imply for  $\varphi = q_0 - q_1$  the Liouville equation

$$\partial_{\xi\eta}^2 \varphi = -2e^\varphi.$$

Explicit solution for the Liouville equation which contains two arbitrary was constructed by Liouville himself [27]. There are many approaches that lead to generalization of this solution for the case of arbitrary  $N$  (see, for example, [28, 29]).

The main goal of this section is to show that the general solution of the open 2D Toda lattice can be obtained as a degenerate case of the construction proposed in [30] for solving of periodic 2D Toda lattice. In [30] it was shown that any local solution of periodic 2D Toda lattice has an analog of the d'Alembert representation as a composition of functions depending on  $\xi$  (or  $\eta$ ), only. Nonlinear superposition of such right- (or left-) moving waves is achieved with the help of auxiliary *linear* Riemann-Hilbert problem. The corresponding matrix Baker-Akhiezer function which solves the Riemann-Hilbert problem has essential singularities at two points on complex plane of the spectral parameter  $w$ . If we represent this plane as two-fold cover with the help of the equation  $w + \Lambda^2 w^{-1} = E$  and take  $\Lambda \rightarrow 0$ , then the limit of the two-fold cover becomes a singular curve, which can be seen as two copies of the plane attached to each other at one point. In this limit, as we show the Riemann-Hilbert problem reduces to a system of linear equations and provides the general solution for 2D open Toda lattice.

Now let us provide details. Initial data which define the general solution are arbitrary functions of one variable  $a_0(\xi), \dots, a_{N-1}(\xi), b_0(\eta), \dots, b_{N-2}(\eta)$  which can be identified with initial data for  $q_n$  on the characteristics  $a_n(\xi) = v_n(\xi, 0), b_n(\eta) = c_n^2(0, \eta)$ .

Let  $\Phi(\xi, w)$  and  $\Phi^\sigma(\eta, w)$  matrix solutions of ordinary differential equations

$$(\partial_\xi - A(\xi, w)) \Phi(\xi, w) = 0, \quad (\partial_\eta - B(\eta, w)) \Phi^\sigma(\eta, w) = 0. \quad (5.3)$$

normalized by the initial condition

$$\Phi(0, w) = \Phi^\sigma(0, w) = 1, \quad (5.4)$$

Here  $A$  and  $B$  are the matrices

$$A(\xi, w) = \begin{bmatrix} a_0 & 1 & 0 & \cdots & 0 \\ 0 & a_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & a_{N-2} & 1 \\ w & \cdots & 0 & 0 & a_{N-1} \end{bmatrix}, \quad B(\eta, w) = \begin{bmatrix} 0 & 0 & 0 & \cdots & b_{N-1}w \\ b_0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & b_{N-3} & 0 & 0 \\ 0 & \cdots & 0 & b_{N-2} & 0 \end{bmatrix} \quad (5.5)$$

As before, we define  $b_{N-1}$  by the formula  $b_{N-1} = \prod_{n=0}^{N-2} b_n^{-1}$ .

Let  $R(\xi, \eta)$  and  $R^\sigma(\xi, \eta)$  be lower- and upper- triangular matrices

$$R_{ii} = 1, \quad R_{ij} = 0, i < j; \quad R_{ij}^\sigma = 0, i \geq j.$$

They are uniquely defined by the *gluing* condition at the point  $w = 0$

$$\Psi(\xi, \eta, 0) = \Psi^\sigma(\xi, \eta, 0) \quad (5.6)$$

for matrices

$$\Psi(\xi, \eta, w) = R(\xi, \eta)\Phi(\xi, w), \quad \Psi^\sigma(\xi, \eta, w) = R^\sigma(\xi, \eta)\Phi^\sigma(\eta, w), \quad (5.7)$$

If diagonal matrix  $H = H_i\delta_{ij}$ , upper- and lower- diagonal matrices  $B_+$ ,  $B_-$  with units on diagonals, are defined by the Borel decomposition of the matrix

$$\Phi(\xi, 0)\Phi^\sigma(\eta, 0)^{-1} = B_-HB_+, \quad (5.8)$$

then (5.6) implies that  $R = B_-^{-1}$ ,  $R^\sigma = HB_+$ . In particular we have  $R_{ii}^\sigma = H_i$ .

**Theorem 5** *The functions*

$$q_n(\xi, \eta) = -\ln R_{nn}^\sigma(\xi, \eta)$$

*solve the open 2D Toda lattice equations. They satisfy the initial conditions*

$$\partial_\xi q_n(\xi, 0) = -a_n(\xi), \quad q_n(0, \eta) - q_{n+1}(0, \eta) = \ln b_n(\eta).$$

*Proof.* Let us show first, that the matrix functions  $\Psi$  and  $\Psi^\sigma$ , satisfy linear equations

$$(\partial_\xi - U)\Psi = 0, \quad (\partial_\xi - U_0)\Psi^\sigma = 0, \quad (5.9)$$

where  $U_0 = U_0(\xi, \eta, w)$  has the form (5.2) with  $v_n = -\partial_\xi q_n$  and  $U = U_0 + we^-$ . Here  $e^-$  is the matrix with the only non-vanishing entry at the left lower corner:  $e_{ij}^- = \delta_{i, N-1}\delta_{0, j}$ .

The matrix  $\Psi$  is non-degenerate. Therefore, its logarithmic derivative  $U = \partial_\xi \Psi \Psi^{-1}$  is holomorphic function of  $w$ . Moreover, from (5.7) and condition that  $R$  is a lower-triangle matrix it follows that

$$U = R_\xi R^{-1} + RAR^{-1} \quad (5.10)$$

have the form  $U = we^- + \tilde{U}$ , where  $\tilde{U}$  does not depend on  $w$  and have vanishing entries over the first diagonal over the main diagonal

$$\tilde{U}_{i, i+1} = 1, \quad U_{ij} = 0, \quad j > i + 1. \quad (5.11)$$

On the other hand, from (5.7) we have that

$$U_0 = \partial_\xi \Psi^\sigma (\Psi^\sigma)^{-1} = R^\sigma (R^\sigma)^{-1} \quad (5.12)$$

is *upper-triangular* matrix with diagonal elements equal  $-\partial_\xi q_n$ . Gluing conditions (5.6) imply

$$U(\xi, \eta, 0) = \tilde{U}(\xi, \eta) = U_0(\xi, \eta) \quad (5.13)$$

Therefore, combining the conditions (5.11) and the fact that  $U_0$  is upper-triangular we conclude that  $U_0$  has the form (5.2).

In the similar way one proves that  $\Psi$  and  $\Psi^\sigma$ , satisfy linear equations

$$(\partial_\eta - V_0)\Psi = 0, \quad (\partial_\xi - V)\Psi^\sigma = 0, \quad (5.14)$$

where  $V = V_0 + w(e^-)^T$ . Compatibility conditions of (5.9) and (5.14) imply that  $q_n$  solves 2D Toda lattice equations.

At  $\eta = 0$  due to (5.4) we have that matrices  $R(\xi, 0), R^\sigma(\xi, 0)$  are defined by the Borel decomposition of the matrix

$$\Phi(\xi, 0) = R^{-1}(\xi, 0)R^\sigma(\xi, 0)$$

But this matrix is upper-triangular. Therefore,  $R(\xi, 0) = 1$  and

$$U_0(\xi, 0) = A(\xi).$$

In the same way we prove that  $V_0(0, \eta) = B(\eta)$ . The last two equalities gives initial conditions on the characteristics for  $q_n$ . The theorem is proved.

We would like to emphasize that the matrix-functions  $\Phi(\xi, 0)$  and  $\Phi^\sigma(\eta, 0)$ , which define through the Borel decomposition solutions of 2D Toda lattice, can be written explicitly. From (5.5) it follows that  $\Phi^\sigma(\eta, 0)$  is a lower-triangular matrix

$$\Phi_{ij}^\sigma(\eta) = 0, \quad i < j, \quad \Phi_{jj}^\sigma(\eta, 0) = 1, \quad (5.15)$$

Using (5.15) as initial condition for integration of equation (5.3),  $\partial_\eta \Phi_{kj}^\sigma = b_k \Phi_{k-1,j}^\sigma$ , we obtain that for  $i > j$  entries of  $\Phi^\sigma$  are given by multiple integrals

$$\Phi_{i,j}^\sigma(\eta) = \int_0^\eta \int_0^{\eta_{i-1}} \dots \int_0^{\eta_j} \prod_{k=1}^{i-j} b_{j+k}(\eta_{j+k-1}) d\eta_{j+k-1}. \quad (5.16)$$

In the same way we may right explicit formulae for the matrix  $\Phi(\xi)$ . It is upper triangular and if we introduce functions

$$\tilde{q}_n(\xi) = \exp\left(-\int_0^\xi v_n(\xi') d\xi'\right), \quad \tilde{b}_n = e^{\tilde{q}_n - \tilde{q}_{n+1}}$$

then

$$\Phi_{ij}(\xi) = 0, \quad i > j, \quad \Phi_{ii}(\xi) = e^{-\tilde{q}_i(\xi)}, \quad (5.17)$$

and for  $i < j$  its entries are given by multiple integrals

$$\Phi_{i,j}(\xi) = e^{-\tilde{q}_i(\xi)} \int_0^\xi \int_0^{\xi_{i+1}} \dots \int_0^{\xi_j} \prod_{k=1}^{j-i} \tilde{b}_{j-k}(\eta_{j-k+1}) d\eta_{j-k+1}. \quad (5.18)$$

Let  $M^{(n)}(\xi, \eta)$  be  $(n+1) \times (n+1)$  matrices which are upper left blocks

$$M_{ij}^{(n)} = M_{ij}, \quad 0 \leq i, j \leq n, \quad (5.19)$$

of the matrix

$$M(\xi, \eta) = \Phi(\xi, 0)\Phi^\sigma(\eta, 0)^{-1}. \quad (5.20)$$

Then well-known formula for diagonal matrix of the Borel decomposition (5.8)

$$H_n = \frac{D_n}{D_{n-1}}, \quad D_n = \det M^{(n)}, \quad D_{-1} = 1$$

together with Theorem 5 provide an explicit formula for the solution

$$q_n(\xi, \eta) = -\ln \frac{D_n}{D_{n-1}}$$

normalized by the condition

$$q_0(0, \eta) = 0$$

Note that equations of 2D Toda lattice are invariant with respect to the transformation  $q_n \rightarrow q_n + f(\eta)$ , where  $f(\eta)$  is an arbitrary function. This transformation does not change initial data on characteristics.

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