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Analytic theory of difference equations with rational and elliptic coefficients and the Riemann–Hilbert problem

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Abstract. A new approach to the construction of the analytic theory of difference equations with rational and elliptic coefficients is proposed, based on the construction of canonical meromorphic solutions which are analytic along 'thick' paths. The concept of these solutions leads to the definition of local monodromies of difference equations. It is shown that, in the continuous limit, these local monodromies converge to monodromy matrices of differential equations. In the elliptic case a new type of isomonodromy transformations changing the periods of elliptic curves is constructed.

Contents

§ 1. Introduction

As is well known, the correlation functions of diverse statistical models, as well as a series of the most important characteristics in random matrix theory, can be expressed in terms of solutions of Painlevé-type *differential* equations (see [1]–[5] and the references therein). Discrete analogues of Painlevé equations have recently

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attracted considerable interest ([6], [7]), due largely to their relations to discrete probabilistic models ([8], [9]). As shown in [10], most discrete analogues of Painlevé equations can be treated in a unified way in the framework of the theory of *isomonodromy transformations* of systems of linear difference equations with rational coefficients.

The analytic theory of matrix linear difference equations

$$\Psi(z+1) = A(z)\Psi(z) \tag{1.1}$$

with rational coefficients goes back to fundamental results of Birkhoff ([11], [12]) which were the starting point of many investigations (see the monograph [13] and the references therein).

A rough classification of the equations (1.1) is given in the following terms: *regular*, *regular singular*, *mild*, *and wild* equations (see [13] for details). The terminology reflects the asymptotic formal theory of difference equations at infinity. The equation (1.1) with coefficients of the form

$$A = A_0 + \sum_{m=1}^{n} \frac{A_m}{z - z_m} \tag{1.2}$$

is *regular singular* if $A_0 = 1$. It is said to be *regular* if, in addition, the residue of A at infinity is trivial, that is, if $\sum_{m=1}^{n} A_m = 0$. The equations for which the leading coefficient A_0 is invertible are said to be *mild*. In this paper we restrict ourselves to the case of mild difference equations with diagonalizable leading coefficient A_0, and we assume that the poles z_m are not congruent, that is, $z_l - z_m$ is not an integer, $z_l - z_m \notin \mathbb{Z}$.

The equation (1.1) is invariant under the transformations $\Psi' = \rho^z\Psi$, $A' = \rho A(z)$, where ρ is a scalar. It is also invariant under the gauge transformations $\Psi' = g\Psi$, $A' = gA(z)g^{-1}$, $g \in SL_r$. Therefore, if the matrix A_0 is diagonalizable, then we can assume without loss of generality that it is diagonal:

$$A_0^{ij} = \rho_i \delta^{ij}, \qquad \det A_0 = \prod_j \rho_j = 1. \tag{1.3}$$

In addition, it is assumed throughout the paper that the residue of the trace of A at infinity is trivial,

$$\operatorname{Tr}(\operatorname{res}_\infty A dz) = \operatorname{Tr}\left(\sum_{m=1}^{n} A_m\right) = 0. \tag{1.4}$$

If the eigenvalues of A_0 are pairwise distinct, $\rho_i \neq \rho_j$, then the equation (1.1) has a unique formal solution $Y(z)$ of the form

$$Y = \left(1 + \sum_{s=1}^{\infty} \chi_s z^{-s}\right) e^{z \log A_0 + K \log z}, \tag{1.5}$$

where $K^{ij} = k_i \delta^{ij}$ is a diagonal matrix.

Difference equations with *polynomial* coefficients $\widetilde{A}$ were considered in [11] and [12]. We note that the general case of rational coefficients $A(z)$ reduces to the case of polynomial coefficients by the transformation

$$\widetilde{A} = A(z)\prod_m (z - z_m), \qquad \widetilde{\Psi} = \Psi \prod_m \Gamma(z - z_m), \tag{1.6}$$

where $\Gamma(z)$ is the gamma function. Birkhoff proved that if the ratios of the eigenvalues ρ_i of the leading coefficient of $\widetilde{A}$ are not real, $\operatorname{Im}(\rho_i/\rho_j) \neq 0$, then the equation (1.1) with polynomial coefficients has two canonical meromorphic solutions $\widetilde{\Psi}_r(z)$ and $\widetilde{\Psi}_l(z)$ which are holomorphic and can be asymptotically represented by the formal solution $\widetilde{Y}(z)$ in the half-planes $\operatorname{Re} z \gg 0$ and $\operatorname{Re} z \ll 0$, respectively. Birkhoff also proved that the connection matrix for these solutions,

$$\widetilde{S}(z) = \widetilde{\Psi}_r^{-1}(z)\widetilde{\Psi}_l(z), \tag{1.7}$$

which must be periodic for obvious reasons, is in fact a rational function of the variable $\exp(2\pi i z)$. The number of parameters occurring in this function is equal to the number of parameters in the coefficient $\widetilde{A}$. According to other results of Birkhoff, two polynomial matrix functions $\widetilde{A}'(z)$ and $\widetilde{A}(z)$ have the same connection matrix $S(z)$ if and only if there is a rational matrix $R(z)$ such that

$$\widetilde{A}'(z) = R(z+1)\widetilde{A}(z)R^{-1}(z). \tag{1.8}$$

Families of commuting transformations of the form (1.8) were explicitly constructed in [10]. It was also proved that, in the continuous limit, the commutativity conditions for a certain subset of these transformations converge to the classical Schlesinger equations [14].

Until now the key ideas of Birkhoff's approach to the analytic theory of difference equations have remained intact. The construction of actual solutions of (1.1) having a prescribed asymptotic behaviour at infinity in various sectors resembles the Stokes theory of differential equations with irregular singularities rather than the classical analytic theory of differential equations with regular singularities. No explicit analogue of the monodromy representation of $\pi_1(\mathbb{C}\setminus\{z_1,\dots,z_n\})$ giving the integrals of the Schlesinger equations exists in the framework of the Birkhoff theory. The obvious continuous limit of the connection matrix $S(z)$ gives only monodromy information at infinity and carries no information about local monodromies of the differential equation around the poles z_m. (Possibly for this reason Birkhoff eliminated the positions of poles by the transformation (1.6) from the very beginning and restricted consideration to the case of polynomial coefficients.)

The main goal of this paper is to develop a new approach to the analytic theory of difference equations with rational coefficients and to extend the treatment to the case of equations with *elliptic* coefficients. This approach is based on the construction for difference equations of meromorphic solutions which are holomorphic along *thick paths.*

To outline the main ideas of our approach, it is instructive to present the case completely opposite to that treated by Birkhoff. We mean the case of *real exponentials* ρ_i. Let us fix a real number x such that $x \neq \operatorname{Re} z_i$ and consider a matrix solution $\Psi_x(z)$ of (1.1) which is non-singular and holomorphic interior to the strip $z \in \Pi_x$ defined by the condition $x \leqslant \operatorname{Re} z \leqslant x+1$, and continuous up to the boundary. It is also required that Ψ_x have at most *polynomial* growth in the strip as $|\operatorname{Im} z| \to \infty$. One can readily show that if such a solution exists, then it is unique up to transformations of the form $\Psi'_x = \Psi_x(z)g$, $g \in GL_r$. Moreover, we prove that if Ψ_x exists, then it has the asymptotic representation

$$\Psi_x = Y g_x^{\pm}, \qquad \operatorname{Im} z \to \pm\infty, \tag{1.9}$$

which is certainly not obvious. To a certain extent, the ratio

$$g_x = g_x^{+}(g_x^{-})^{-1} \tag{1.10}$$

can be regarded as a transfer matrix of the solution along the 'thick' path Π_x from $-i\infty$ to $i\infty$.

Furthermore, we show that the solution Ψ_x *always exists* for $x \gg 0$ and $x \ll 0$. In both cases the corresponding solutions do not depend on x. Therefore, we obtain two meromorphic solutions Ψ_r and Ψ_l of the equation (1.1) that are holomorphic in the half-planes $\operatorname{Re} z \gg 0$ and $\operatorname{Re} z \ll 0$, respectively. The corresponding transfer matrices $g_r = g_x$, $x \gg 0$, and $g_l = g_x$, $x \ll 0$, are 'quasi'-upper or 'quasi'-lower triangular matrices, that is,

$$g_{r(l)}^{ii} = 1, \quad g_r^{ij} = 0 \quad \text{if} \quad \rho_i < \rho_j, \qquad g_l^{ij} = 0 \quad \text{if} \quad \rho_i > \rho_j. \tag{1.11}$$

This result clarifies the well-known fact that there are no Birkhoff solutions of (1.1) with uniform asymptotic representation in the half-planes $\operatorname{Re} z \gg 0$ and $\operatorname{Re} z \ll 0$ if $\operatorname{Im}(\rho_i/\rho_j) = 0$.

The solutions Ψ_r and Ψ_l can be uniquely normalized by the condition $g_x^{-} = 1$. In this case their connection matrix becomes

$$S(z) = \Psi_r^{-1}(z)\Psi_l(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}\,, \tag{1.12}$$

where

$$S_\infty = 1 + \sum_{m=1}^{n} S_m = g_r^{-1} e^{2\pi i K} g_l, \tag{1.13}$$

and K is a diagonal matrix coinciding with that in (2.5) below. To the author's knowledge, the explicit form (1.12) of the connection matrix, including the relations (1.11) and (1.13), is a new result even for the case of regular singular equations in which $g_{r(l)} = 1$ (this should be compared with Theorem 10.8 in [13]).

The direct monodromy map

$$A(z) \to S(z) \tag{1.14}$$

for regular singular equations and for mild equations is constructed in §§ 2 and 3, respectively. In § 2.2 we introduce the notion of *local monodromy* of a difference equation. This notion is first defined for three classes of regular singular equations.

We first deal with the case of special equations whose coefficients $A \in \mathcal{A}_0$ are of the form (1.2) and satisfy the condition $\det A(z) \equiv 1$. Another example of equations for which one can introduce the notion of local monodromy is given by unitary equations whose coefficients $A \in \mathcal{A}^U$ satisfy the relation $A^+(\bar{z}) = A^{-1}(z)$. The most important case in what follows is the case of small coefficients, that is, the case of equations with $|A_m| < \varepsilon$.

The existence of canonical solutions Ψ_x is equivalent to solubility of an auxiliary system of linear singular equations. The index of the corresponding system is equal to

$$\operatorname{ind}_x A = \frac{1}{2\pi i} \int_L d \log \det A, \qquad z \in L\colon\ \operatorname{Re} z = x. \tag{1.15}$$

It follows from fundamental results of the theory of singular integral equations [15] that if $\operatorname{ind}_x A = 0$, then the canonical solution Ψ_x exists for generic coefficients A.

The index $\operatorname{ind}_x A$ vanishes identically if $\det A = 1$. Therefore, for generic $A \in \mathcal{A}_0$ the solution Ψ_x exists for any $x \neq \operatorname{Re} z_k$. It turns out that this solution is x-*independent* when x varies between the values $\operatorname{Re} z_k$. To be definite, suppose that $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_n$. In this case we obtain a set of $n+1$ meromorphic solutions $\Psi_k(z)$ of (1.1) that are holomorphic in the domains $\operatorname{Re} z_k < \operatorname{Re} z < \operatorname{Re} z_{k+1} + 1$ (here $k = 0, \dots, n$, and we formally set $z_0 = -\infty$ and $z_{n+1} = \infty$ for brevity).

The local connection matrices $M_k = \Psi_k^{-1} \Psi_{k-1}$ have the form

$$M_k = 1 - \frac{m_k}{e^{2\pi i (z - z_k)} - 1}\,. \tag{1.16}$$

The value of M_k at $z = i\infty$ can be expressed as follows in terms of the transfer matrices g_m along the strips Π_x, $\operatorname{Re} z_m < x < \operatorname{Re} z_{m+1}$:

$$\mu_k = 1 + m_k = g_k^{-1} g_{k-1}. \tag{1.17}$$

The matrix μ_k is a discrete analogue of the *monodromy matrix* of a differential equation corresponding to a closed path from $-i\infty$ going around the pole z_k.

The monodromy matrices μ_k uniquely determine the local connection matrices $M_k(z)$ and the global connection matrix (1.12), which is equal to the product of the local matrices:

$$S(z) = M_n(z) \cdots M_1(z). \tag{1.18}$$

We note that every generic unimodular matrix $S(z)$, $\det S = 1$, of the form (1.12) has a unique representation (1.18) in the form of a product whose factors M_k are of the form (1.16). Therefore, the correspondence $S(z) \leftrightarrow \{\mu_k\}$ is one-to-one on open sets of the corresponding spaces.

In the three cases of difference equations treated in § 2.2, we show that the direct monodromy map (1.14) is one-to-one on dense open sets. The solution of the inverse monodromy problem of recovering the coefficients $A(z)$ from the monodromy data reduces to a certain Riemann–Hilbert factorization problem for matrix functions defined on a set of vertical lines. This reduction is based on the existence of a whole family of solutions $\Psi_l, \Psi_1, \dots, \Psi_r$ whose domains of analyticity overlap and cover the whole complex plane.

The solution of the inverse monodromy problem in the generic case is given in § 3. The direct monodromy map (1.14), when restricted to the subspace $\mathcal{A}_D$ of coefficients having a *fixed* determinant,

$$A \in \mathcal{A}_D \subset \mathcal{A}\colon \quad \det A(z) = D(z) = \frac{\prod_{\alpha=1}^{N}(z-\zeta_\alpha)}{\prod_{m=1}^{n}(z-z_m)^{h_m}}\,, \quad N = \sum_m h_m, \tag{1.19}$$

is injective on dense open sets if the zeros ζ_α of D are not congruent to one another; this follows immediately from the definition of canonical solutions. The construction of isomonodromy transformations plays the most important role in the construction and description of single-valued branches of the map inverse to (1.14).

Two rational functions D and D' of the form (1.19) are said to be *equivalent* if their zeros and poles are pairwise congruent, that is, $\zeta_\alpha - \zeta'_\alpha \in \mathbb{Z}$ and $z_m - z'_m \in \mathbb{Z}$. We prove that for any pair of equivalent functions there is a birational isomorphism $T_D^{D'}\colon \mathcal{A}_D \mapsto \mathcal{A}'_D$ preserving the monodromy data. Therefore, to prove that on an open subset of the space $\mathcal{S}_{\widehat{D}}$ of connection matrices with fixed determinant $\widehat{D} = D(w)$, $w = e^{2\pi i z}$, there is a map

$$\mathcal{S}_{\widehat{D}} \mapsto \mathcal{A}_D \tag{1.20}$$

which is inverse to the restriction to $\mathcal{A}_D$ of (1.14), it suffices to construct a map (1.20) for at least one D in each equivalence class $[D]$.

In each equivalence class $[D]$ there is a representative D such that its zeros and poles belong to Π_x. In this case the canonical meromorphic solutions Ψ_l and Ψ_r are holomorphic in the overlapping domains $\operatorname{Re} z < x+1$ and $\operatorname{Re} z > x$, respectively. The problem of reconstructing $\Psi_{r(l)}$ then reduces to a standard Riemann–Hilbert factorization problem on the line $\operatorname{Re} z = x + 1/2$.

In § 4 we consider the continuous limit of our construction. We prove that for sufficiently small h the canonical meromorphic solutions Ψ_x of the difference equation

$$\Psi(z+h) = \left(1 + hA_0 + h\sum_{m=1}^{n} \frac{A_m}{z-z_m}\right)\Psi(z) \tag{1.21}$$

exist for any x such that $|x - \operatorname{Re} z_m| > Ch$. Moreover, it turns out that, in the limit as $h \to 0$, this solution converges in a neighbourhood of the path $\operatorname{Re} z = x$ to a solution of the differential system

$$\frac{d\widehat{\Psi}}{dz} = \left(A_0 + \sum_{m=1}^{n} \frac{A_m}{z-z_m}\right)\widehat{\Psi}(z). \tag{1.22}$$

Hence, the monodromy matrices μ_k of the difference equation converge to monodromy matrices of the differential equation. For difference equations with real exponentials the transfer matrices $g_{r(l)}$ converge to the Stokes matrices of the equation (1.22) at infinity, which is an irregular singularity of (1.22). A similar result is obtained for the case of imaginary exponentials.

In § 5 we consider the analytic theory of difference equations with 'elliptic' coefficients. More precisely, we consider the equations

$$\Psi(z+h) = A(z)\Psi(z), \tag{1.23}$$

where the coefficients $A(z)$ are meromorphic $r \times r$ matrix functions with simple poles and satisfy the following monodromy properties:

$$A(z+2\omega_\alpha) = B_\alpha A(z) B_\alpha^{-1}, \qquad B_\alpha \in SL_r, \quad \alpha = 1, 2. \tag{1.24}$$

The relations (1.24) mean that the matrix $A(z)$ can be regarded as a meromorphic section of the vector bundle $\mathrm{Hom}(\mathcal{V}, \mathcal{V})$, where $\mathcal{V}$ is a holomorphic vector bundle over the elliptic curve Γ with periods $2\omega_\alpha$ and is defined by a pair of commuting matrices B_α. If the matrices B_α are diagonalizable, then we can assume without loss of generality that they are diagonal. Moreover, by using the gauge transformations given by diagonal matrices of the form G^z, one can reduce the problem to the case in which B_1 is the identity matrix. Let us represent the matrix B_2 in this gauge in the form $B_2 = e^{\pi i \hat{q}/\omega_1}$, where $\hat{q}$ is a diagonal matrix.

Without loss of generality we can assume that $\mathrm{Im}(\omega_2/h) > 0$. As in the case of rational coefficients, we define the canonical meromorphic solutions Ψ_x of the equation (1.23). They satisfy the *Bloch* monodromy property

$$\Psi_x(z+2\omega_2) = e^{\pi i \hat{q}/\omega_1} \Psi_x(z) e^{-2\pi i \hat{s}/h}, \tag{1.25}$$

where $\hat{s}$ is a diagonal matrix, $\hat{s}^{\,ij} = s_i \delta^{ij}$. The connection matrix S_x of two such solutions $\Psi_x(z)$ and $\Psi_{x+1}(z) = \Psi_x(z - 2\omega_1)$, that is,

$$\Psi_x(z) = \Psi_x(z-2\omega_1) S_x(z), \tag{1.26}$$

satisfies the relations

$$S_x(z+h) = S_x(z), \qquad S_x(z+2\omega_2) = e^{2\pi i \hat{s}/h} S_x(z) e^{-2\pi i \hat{s}/h} \tag{1.27}$$

and can be regarded as a section of the bundle over the elliptic curve with periods $(h, 2\omega_2)$.

The correspondence $A(z) \to S_x(z)$ is the direct monodromy map in the elliptic case. As in the rational case, single-valued branches of the inverse monodromy map are defined for coefficients with a fixed determinant. Isomonodromy transformations which change the positions of poles and zeros of the determinant of A can be constructed in almost the same way as in the rational case. We show that in the elliptic case there is a fundamentally new type of isomonodromy transformation which *changes the periods* of the corresponding elliptic curve. These transformations are of the form

$$A'(z) = \mathcal{R}(z+h) A(z) \mathcal{R}^{-1}(z) \tag{1.28}$$

and are determined by meromorphic solutions $\mathcal{R}$ of the difference equation

$$\mathcal{R}(z + 2\omega_1 + h) A(z) = \mathcal{R}(z), \tag{1.29}$$

which has the Bloch monodromy property

$$\mathcal{R}(z+2\omega_2) = e^{2\pi i \hat{q}'/(h+2\omega_1)} \mathcal{R}(z) e^{-\pi i \hat{q}/\omega_1}. \tag{1.30}$$

The existence of transformations of this kind shows that in the elliptic case there is a certain symmetry between the periods $2\omega_\alpha$ of the elliptic curves and the step h of the difference equation. We note that this type of symmetry for the q-analogue of the elliptic Bernard–Knizhnik–Zamolodchikov equations was discovered in [16].

§ 2. Meromorphic solutions of difference equations and the Riemann–Hilbert problem

The matrix differential equation $\partial_z \Psi = A(z)\Psi$ with rational coefficients has multivalued holomorphic solutions on the complex plane with punctures $\mathbb{C} \setminus \{z_m\}$, where the z_m are the poles of $A(z)$. The initial condition $\Psi(z_0) = 1$, $z_0 \neq z_m$, uniquely determines Ψ in a neighbourhood of z_0. This simple but fundamental fact is the starting point of the analytic theory of differential equations with rational coefficients. Analytic continuation of Ψ along paths in $\mathbb{C} \setminus \{z_m\}$ determines the monodromy representation, $\mu \colon \pi_1(\mathbb{C} \setminus \{z_m\}) \mapsto GL_r$.

The construction of meromorphic solutions of difference equations is less obvious. It can be reduced to a solution of the following auxiliary Riemann–Hilbert-type problem.

Problem I. *Find a continuous matrix function $\Phi(z)$ on the strip $\Pi_x \colon x \leqslant \operatorname{Re} z \leqslant x+1$ such that $\Phi(z)$ is meromorphic interior to Π_x, and the boundary values of $\Phi(z)$ on the two sides of the strip satisfy the equation*

$$\Phi^+(\xi+1) = A(\xi)\Phi^-(\xi), \qquad \xi = x + iy. \tag{2.1}$$

Every solution Φ of this problem can be extended to a function Ψ defined on the whole complex plane by using the equation (1.1). *A priori*, Ψ is meromorphic outside the lines $\operatorname{Re} z = x + l$, $l \in \mathbb{Z}$. On these lines Ψ is continuous by (2.1). We recall the following well-known property of analytic functions: if f is a continuous function in some domain D of the plane and is holomorphic in the complement $D \setminus L$ of a smooth arc L, then f is holomorphic on D. Therefore, Ψ is meromorphic on the whole complex plane and can be regarded as a meromorphic solution of (1.1).

The function $t = \tan(\pi z)$ defines a one-to-one conformal map of the interior of the strip Π_x onto the complex plane of the variable t with a cut between the punctures $t = \pm 1$. The problem (2.1) is transformed by this map into the standard Riemann–Hilbert factorization problem on the cut. By fundamental results of the theory of singular integral equations, the problem (2.1) always has a solution. Moreover, if the index (1.15) of the corresponding system of singular integral equations vanishes, then for any generic matrix $A(z)$ this problem has a *sectionally holomorphic* non-singular solution. This condition means that there is a constant $\alpha < 1$ such that the function $(t \pm 1)^\alpha \Phi(t)$ is bounded on the edges of the cut. In terms of the variable z a sectionally holomorphic solution Φ_x of Problem I is a non-singular matrix function holomorphic interior to Π_x and such that the growth of Φ_x at infinity satisfies the condition

$$\exists\, 0 \leqslant \alpha < 1 : \quad |\Phi(z)| < e^{2\pi\alpha|\operatorname{Im} z|}, \quad |\operatorname{Im} z| \to \infty. \tag{2.2}$$

This solution is unique up to a normalization of the form $\Phi'(z) = \Phi(z)g$, $g \in SL_r$.

Almost all results of this section require no additional information. We provide some details needed to describe the asymptotic behaviour of Ψ_x.

2.1. Regular singular equations. We begin with the case of regular singular difference equations, that is, equations (1.1) with coefficient $A(z)$ of the form

$$A = 1 + \sum_{i=m}^{n} \frac{A_m}{z - z_m}\,. \tag{2.3}$$

The equation (1.1) is invariant under the gauge transformations $A' = gAg^{-1}$, $\Psi' = g\Psi$, $g \in SL_r$. Thus, if the residue of Adz at infinity is diagonalizable, we can assume without loss of generality that

$$K = \operatorname{res}_\infty Adz = \sum_{m=1}^{n} A_m = \operatorname{diag}(k_1, \dots, k_r). \tag{2.4}$$

If $k_i - k_j \notin \mathbb{Z}$, then the equation (1.1) has a unique formal solution of the form

$$Y = \left(1 + \sum_{s=1}^{\infty} \chi_s z^{-s}\right) z^K. \tag{2.5}$$

The coefficients χ_s are determined by the equations obtained by substituting the series (2.5) into (1.1). These equations express the sum $[K, \chi_s] + s\chi_s$ in terms of the A_i and of $\chi_1, \dots, \chi_{s-1}$, and enable one to find all the coefficients χ_s by recursion.

Let $\mathcal{P}_x$ be the space of continuous functions $\Phi(z)$ on the strip Π_x that are holomorphic interior to it and have at most polynomial growth at infinity, that is,

$$\Phi \in \mathcal{P}_x: \quad \exists\, N, \quad |\Phi| < |z|^N, \quad z \in \Pi_x. \tag{2.6}$$

Lemma 2.1. *Let x be a real number such that $x \neq \operatorname{Re} z_j$. Then the following assertions hold.*

(a) *If $|x| \gg 0$, then the Riemann–Hilbert problem* (2.1) *always has a solution Φ_x belonging to $\mathcal{P}_x$. This solution is unique up to normalization.*

(b) *For a generic matrix $A(z)$ a solution $\Phi_x \in \mathcal{P}_x$ exists and is unique up to normalization for any x such that* $\operatorname{ind}_x A = 0$.

(c) *The solution Φ_x of the Riemann–Hilbert problem is asymptotically equal to*

$$\Phi_x(z) = Y(z) g_x^{\pm}, \qquad \operatorname{Im} z \to \pm\infty. \tag{2.7}$$

Remark. The assertion (c) of the lemma means that if $Y_{m'} = \bigl(1 + \sum_{s=1}^{m'} \chi_s z^{-s}\bigr) z^K$ is a partial sum of the formal series (2.5), then

$$|\Phi_x (Y_{m'} g_x^{\pm})^{-1} - 1| \leqslant O(|z|^{-m'-1}), \qquad \operatorname{Im} z \to \pm\infty. \tag{2.8}$$

For any $\varepsilon > 0$ the estimate (2.8) is uniform in the domain $z \in \Pi_{x,\varepsilon}$: $x + \varepsilon \leqslant \operatorname{Re} z \leqslant x + 1 - \varepsilon$.

Proof. Let us first show that if Φ_x exists, then it is unique up to normalization. The determinant of Φ_x is a holomorphic function interior to Π_x. Its boundary values satisfy the relation $\log\det \Phi_x^+(\xi + 1) = \log\det \Phi_x^-(\xi) + \log\det A(\xi)$. If $\operatorname{ind}_x A = 0$, then the principal part of the integral of $(d \log\det \Phi_x)$ along the boundary of Π_x vanishes. Therefore, if Φ_x is non-singular at least at one point, then it is non-singular at all points of Π_x. Suppose now that there are two solutions of the factorization problem. Then $g = \Phi_x^{-1}\Phi_x'$ is an entire periodic matrix function.

It can be regarded as a function $g(w)$ of the variable $w = e^{2\pi i z}$ which is holomorphic outside the points $w = 0$ and $w = \infty$. It follows from (2.6) that

$$\lim_{w\to 0} w g(w) = 0, \qquad \lim_{w\to\infty} w^{-1} g(w) = 0. \tag{2.9}$$

Therefore, $g(w)$ has an extension which is holomorphic at the points $w = 0$ and $w = \infty$. Thus, $g(w)$ is a constant matrix.

The problem (2.1) can be reduced to a system of singular integral equations in the standard way. Let us fix a positive integer m and denote by Y_m a holomorphic function on Π_x which coincides with Y up to order m at $\pm i\infty$ in the strip. If $0 \notin \Pi_x$, then for Y_m one can take the mth partial sum of the series (2.5). If $0 \in \Pi_x$, then we choose a point $x_0 \notin \Pi_x$ and take Y_m in the form

$$Y_m = \left(1 + \sum_{s=1}^{m} \widetilde{\chi}_s (z - x_0)^{-s}\right)(z - x_0)^K, \tag{2.10}$$

where the coefficients $\widetilde{\chi}_s$ are uniquely determined by the congruence

$$\left(1 + \sum_{s=1}^{m} \widetilde{\chi}_s (z - x_0)^{-s}\right)\left(\frac{z - x_0}{z}\right)^K \left(1 + \sum_{s=1}^{m} \chi_s z^{-s}\right)^{-1} = 1 + O(z^{-m-1}). \tag{2.11}$$

Each sectionally holomorphic function on Π_x can be represented by a Cauchy-type integral. Let us consider the function Φ_x given by the formula

$$\Phi_x = Y_m \phi, \qquad \phi = 1 + \int_L \varphi(\xi) k(z, \xi)\, d\xi, \tag{2.12}$$

where the integral is taken over the line $L = L_x$ given by $\operatorname{Re}\xi = x$, and the kernel in the integrand is equal to

$$k(z, \xi) = \frac{e^{\pi i(z-x)} + e^{-\pi i(z-x)}}{\bigl(e^{\pi i(\xi-x)} + e^{-\pi i(\xi-x)}\bigr)\bigl(e^{\pi i(\xi-z)} - e^{-\pi i(\xi-z)}\bigr)}\,. \tag{2.13}$$

Let H be the space of Hölder-class functions on L_x such that

$$\varphi \in H\colon \quad \exists\, \alpha < 1, \quad |\varphi(\xi)| < O(e^{\pi\alpha|\operatorname{Im}\xi|}). \tag{2.14}$$

If $\varphi \in H$, then the integral in (2.12) converges and defines a function ϕ holomorphic interior to Π_x and continuous up to the boundary. The boundary values $\phi^{\pm}$ of ϕ are given by the Sokhotskii–Plemelj formulae

$$\phi^-(\xi) = 1 + I_\varphi(\xi) - \frac{\varphi(\xi)}{2}\,, \qquad \phi^+(\xi + 1) = 1 + I_\varphi(\xi) + \frac{\varphi(\xi)}{2}\,, \tag{2.15}$$

where $I_\varphi(\xi)$ stands for the principle value of the integral:

$$I_\varphi(\xi) = \text{p.v.} \int_L \varphi(\xi') k(\xi, \xi')\, d\xi'. \tag{2.16}$$

The equation (2.1) is equivalent to the non-homogeneous singular integral equation

$$(\widetilde{A}+1)\,\varphi-2(\widetilde{A}-1)I_{\varphi}=2(\widetilde{A}-1), \tag{2.17}$$

where

$$\widetilde{A}=Y_m(\xi+1)^{-1}A(\xi)Y_m(\xi). \tag{2.18}$$

By the definition of Y_m, for large $|z|$ we have

$$|\widetilde{A}(\xi)-1|\leqslant O(|\xi|^{-m+\kappa}),\qquad \kappa=\max_{i,j}|k_i-k_j|. \tag{2.19}$$

For large $|x|$ the left-hand side of the inequality (2.19) is uniformly bounded by the expression $O(|x|^{-m+\kappa})$, and the equation (2.17) can be solved by iterations.

Let us consider the sequence of functions φ_n defined recursively by the equalities

$$(\widetilde{A}+1)\,\varphi_n-2(\widetilde{A}-1)I_{\varphi_{n-1}}=2(\widetilde{A}-1), \tag{2.20}$$

where we set $\varphi_0=0$. For $n>0$ the equation (2.20) implies that

$$(\widetilde{A}+1)(\varphi_{n+1}-\varphi_n)=2(\widetilde{A}-1)I_{(\varphi_n-\varphi_{n-1})}. \tag{2.21}$$

It follows from (2.21) that if the norm of $(\widetilde{A}-1)$ is small enough, then $|\varphi_{n+1}-\varphi_n|<c\varepsilon^n$, $\varepsilon<1$. In this case the sequence φ_n obviously converges to a continuous function φ, and this function satisfies the equation (2.17). Standard arguments of the theory of boundary-value problems (see [15] for details) show that φ is a Hölder-class function, which proves the first assertion of the lemma.

The left-hand side of (2.17) is a singular integral operator $K\colon H\to H$ for any x. It has a Fredholm regularization. The non-homogeneous equation (2.17) is soluble provided that the adjoint homogeneous equation

$$f(\xi)(\widetilde{A}(\xi)+1)-2\biggl(\text{p.v.}\int_L f(\xi')k(\xi',\xi)\,d\xi'\biggr)(\widetilde{A}(\xi)-1)=0 \tag{2.22}$$

for the row vector $f\in H_0$ has no solutions (see § 53 in [15]). Here H_0 stands for the space of Hölder-class functions *integrable* on L_x. Each solution of the equation (2.22) determines the row vector

$$F(z)=\cos^2(\pi(z-x))\biggl(\int_L f(\xi)k(\xi,z)\,d\xi\biggr)Y_m^{-1}(z), \tag{2.23}$$

which is a solution of the dual Riemann–Hilbert problem in Π_x,

$$F(\xi+1)A(\xi)=F(\xi),\qquad \xi\in L. \tag{2.24}$$

The Cauchy kernel $k(\xi,z)$ has a simple pole at $x'=x+1/2$. Hence, the function F is holomorphic interior to Π_x and vanishes at the point x', $F(x')=0$. This function is bounded as $|\operatorname{Im}z|\to\infty$. The non-existence of solutions F of this kind is an open condition. This implies the second statement of the lemma.

It follows from (2.17) and (2.19) that I_φ is bounded at infinity. Moreover, $|\varphi(\xi)| < O(|\xi|^{-m+\kappa})$. Let us show that for $z \in \Pi_{x,\varepsilon}$

$$\phi(z) = g^{\pm} + O(|z|^{-m+\kappa+1}), \qquad \operatorname{Im} z \to \pm\infty, \tag{2.25}$$

where

$$g^{\pm} = 1 - \frac{1}{2}\int_L (\tan(\pi i(\xi - x)) \pm 1)\varphi(\xi)\, d\xi. \tag{2.26}$$

We consider the case $\operatorname{Im} z \to \infty$. The integral in (2.12) can be represented as the sum of two integrals, I_1 and I_2. The first integral corresponds to integration over the interval L_1: $(x - i\infty, \xi_0)$, and the second to integration over the interval L_2: $(\xi_0, x + i\infty)$, where $\xi_0 = x + i \operatorname{Im} z/2$. The Cauchy kernel is uniformly bounded, $k(z,\xi) < C$, in the domain $\Pi_{x,\varepsilon}$. Therefore,

$$|I_2| < C \int_{L_2} |\varphi(\xi)|\, d\xi < O(|z|^{-m+\kappa+1}). \tag{2.27}$$

For $\xi \in L_1$ we have $|\xi - z| > \operatorname{Im} z/2$, which implies that

$$k(z,\xi) = k_+(\xi)(1 + O(e^{-\pi|z|})), \quad k_+(\xi) = (1 - \tan(\pi i(\xi - x))), \quad \xi \in L_1. \tag{2.28}$$

Hence,

$$|I_2 + 1 - g^+| < \left|\int_{L_2} k_+(\xi)\varphi(\xi)\, d\xi\right| + O(e^{-\pi|z|}) \int_L k_+(\xi)\, |\varphi(\xi)|\, d\xi < O(|z|^{-m+\kappa+1}). \tag{2.29}$$

One can prove the formula (2.25) similarly in the case $\operatorname{Im} z \to -\infty$.

As shown above, if a solution of the factorization problem exists, then it is unique. Therefore, the left-hand side of (2.12) does not depend on m. It follows from (2.25) that (2.8) holds for $m' < m - 2\kappa$. We now see that as $m \to \infty$ the formula (2.8) is valid for any m', which completes the proof of the lemma.

Theorem 2.1. *If $A_0 = 1$ and $k_i - k_j \notin \mathbb{Z}$, then*:

(A) *there are unique meromorphic solutions Ψ_l and Ψ_r of the equation* (1.1) *which are non-singular, holomorphic, and asymptotically equal to $Y(z)$ in the domains* $\operatorname{Re} z \ll 0$ *and* $\operatorname{Re} z \gg 0$, *respectively*;[1]

(B) *the matrix $S = \Psi_l^{-1}\Psi_r$ has the form*

$$S(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z - z_m)} - 1}, \qquad S_\infty = 1 + \sum_{m=1}^{n} S_m = e^{2\pi i K}. \tag{2.30}$$

The first statement of the theorem and the form of the connection matrix $S(z)$ are known (see Theorem 10.8 in [13]). The author could not found any explicit form of the matrix S_∞ in the literature. Birkhoff proved that $S_\infty = 1$ for regular equations (for which $K = 0$). In [13] it is stated only that the matrix S_∞ is non-singular.

Proof. As was already noted above, every solution Φ_x of the factorization problem (if such a solution exists) determines a meromorphic solution Ψ_x of the difference equation (1.1).

[1] In the asymptotic equalities $\Psi_{r(l)} = Y$ we assume the choice of a single-valued branch of $\log z$ on $\mathbb{C}$ with a cut along the ray $\arg z = \pi/2$.

Lemma 2.2. *Suppose that the factorization problem* (2.1) *is soluble for a pair of real numbers* x, y *with* $x < y$*. Then the connection matrix* $M_{x,y} = \Psi_y^{-1}\Psi_x$ *of the corresponding two solutions of the difference equation has the form*

$$M_{x,y} = 1 - \sum_{k \in J_{x,y}} \frac{m_{k,(x,y)}}{e^{2\pi i(z-z_k)} - 1}, \tag{2.31}$$

where the sum is taken over the subset $J_{x,y}$ *of indices corresponding to the poles with* $x < \operatorname{Re} z_k < y$*.*

Proof. By definition, Ψ_x is holomorphic on Π_x. By construction, in the domain $\operatorname{Re} z > x+1$ this function has poles at the points $z_k + l$, $l = 1, 2, \dots$, for $\operatorname{Re} z_k > x$. Hence, $M_{x,y}$ has poles in the strip Π_y at the points congruent to the poles z_k, $k \in J_{xy}$. The function $M_{x,y}$ is periodic with respect to z. Arguments similar to those used above in proving that Φ_x is unique show that $M_{x,y}(w)$, regarded as a function of the variable $w = e^{2\pi i z}$, admits holomorphic continuation to the points $w = 0$ and $w = \infty$. Hence, $M_{x,y}(w)$ is a rational function of the variable w. This function is equal to 1 at $w = 0$ and has poles at the points $w_k = e^{2\pi i z_k}$, $k \in J_{xy}$. Therefore, $M_{x,y}$ can be represented in the form (2.31).

Remark. The above proof of the lemma shows simultaneously that the existence of Φ_x for generic data A and x such that $\operatorname{ind}_x A = 0$ follows readily from the existence of Ψ_l. Indeed, let us consider the function M_x of the form (2.31), where the sum is taken over all z_k such that $\operatorname{Re} z_k < x$. The condition that the function $\Psi_x = \Psi_l M_x^{-1}$ be holomorphic on Π_x is equivalent to a system of algebraic equations for the residues of M_x. If $\operatorname{ind}_x A = 0$, then the number of equations is equal to the number of unknowns. Therefore, the canonical meromorphic solutions Ψ_x always exist for generic data.

It follows from the lemma that Ψ_x is locally constant with respect to the variable x (Ψ_x is 'x-independent'). In particular, Ψ_x is x-independent on the interval $x < \min_k\{\operatorname{Re} z_k\}$. The corresponding function Ψ_l is a unique meromorphic solution of (1.1) which is holomorphic in the domain $\operatorname{Re} z \ll 0$, where it is asymptotically equal to Y as $\operatorname{Im} z \to -\infty$ and asymptotically equal to $Y g_l$ as $\operatorname{Im} z \to \infty$. For large $|x|$ the coefficient $(\widetilde{A} - 1)$ in (2.17) is uniformly bounded. Therefore, the solution φ (which decays by the rule $|\varphi(\xi)| < O(|\xi|^{-m+k})$ on both ends of the line L) is uniformly bounded by a quantity of order $O(|x|^{-m+k})$. In this case it follows from the equation (2.26) that $g_x^{\pm} = 1 + O(|x|^{-m+k})$. The matrix $g_l = g_x^{+}(g_x^{-})^{-1}$ is x-independent. Hence, $g_l = 1$, and Ψ_l is asymptotically equal to Y on the whole half-plane $\operatorname{Re} z \ll 0$. One can prove similarly that Ψ_x can be identified with Ψ_r for $x \gg 0$. This completes the proof of the assertion (A) of the theorem.

The formula (2.30) is a particular case of the formula (2.31). To complete the proof of the assertion (B), it suffices to recall that in the definition of Y one must fix a single-valued branch of $\log z$. In the above treatment it was always assumed that a branch is fixed on the z-plane by choosing a cut along the positive part of the imaginary axis. In this case the value of S at $-i\infty$ is 1, and the value at $i\infty$ is equal to the ratio of the values of z^K on the two edges of the cut.

2.2. Local monodromies. For the existence of a solution Φ_x of the boundary-value problem (2.1) it is necessary that $\operatorname{ind}_x A = 0$. If this condition is satisfied for all values of x, then we can introduce the notion of local monodromies for the difference equation (1.1).

Special regular singular equations. A regular singular equation (1.1) is said to be *special* if the residues A_i of $A(z)$ are rank-one matrices,

$$A(z) = 1 + \sum_{k=1}^{n} \frac{p_k q_k^T}{z - z_k}, \tag{2.32}$$

and the determinant of A is identically equal to 1, $\det A(z) = 1$. Here p_k and q_k stand for r-dimensional vectors regarded modulo transformations of the form

$$p_k \to c_k p_k, \qquad q_k \to c_k^{-1} q_k, \tag{2.33}$$

where the symbols c_k stand for scalars. The dimension of the space $\mathcal{A}_0$ of these matrices is equal to $2N(r-2)$. An explicit parameterization of an open set in the space $\mathcal{A}_0$ can be obtained by ordering the poles and representing $A(z)$ in the multiplicative form

$$A(z) \in \mathcal{A}_0\colon \quad A(z) = \left(1 + \frac{a_n b_n^T}{z - z_n}\right) \cdots \left(1 + \frac{a_1 b_1^T}{z - z_1}\right), \tag{2.34}$$

where a_k and b_k are pairs of orthogonal vectors,

$$b_k^T a_k = 0, \tag{2.35}$$

which are regarded modulo transformations of the form (2.33). The equalities (2.35) imply that

$$\left(1 + \frac{a_k b_k^T}{z - z_k}\right)^{-1} = \left(1 - \frac{a_k b_k^T}{z - z_k}\right) \mapsto \det\left(1 + \frac{a_k b_k^T}{z - z_k}\right) = 1. \tag{2.36}$$

It follows from (2.34) and (2.35) that the parameters p_j and q_j in the additive representation (2.32) of the matrix function A satisfy the conditions

$$q_k^T l_k^{-1} p_k = 0, \qquad l_k = 1 + \sum_{m \neq k}^{N} \frac{p_m q_m^T}{z_k - z_m}. \tag{2.37}$$

For the matrices $A \in \mathcal{A}_0$ the gauge assumption (2.4) has the form

$$\sum_{m=1}^{n} p_m q_m^T = \sum_{m=1}^{n} a_m b_m^T = \mathrm{diag}(k_1, \ldots, k_r) = K. \tag{2.38}$$

It is assumed throughout the present subsection that the real parts $r_k = \mathrm{Re}\, z_k$ of the poles are distinct and ordered, $r_k < r_m$ for $k < m$. We introduce the notation $r_0 = -\infty$ and $r_{n+1} = \infty$ for brevity.

Theorem 2.2. (i) *For a generic matrix* $A \in \mathcal{A}_0$ *satisfying the condition* (2.38), *where* $k_i - k_j \notin \mathbb{Z}$, *the corresponding special regular singular equation* (1.1) *has a unique set of meromorphic solutions* Ψ_k, $k = 0, 1, \dots, n$, *which are holomorphic in the strips* $r_k < \operatorname{Re} z < r_{k+1} + 1$ *and are asymptotically equal in these strips to* $Y g_k^{\pm}$ *as* $\operatorname{Im} z \to \pm\infty$, *where* $g_k^- = 1$.

(ii) *The local connection matrices* $M_k = \Psi_k^{-1}\Psi_{k-1}$, $k = 1, \dots, n$, *are of the form*

$$M_k = 1 - \frac{\alpha_k \beta_k^T}{e^{2\pi i(z-z_k)} - 1}, \tag{2.39}$$

where (α_k, β_k) *are pairs of orthogonal vectors,*

$$\beta_k^T \alpha_k = 0, \tag{2.40}$$

which are regarded modulo transformations of the form (2.33) *and satisfy the condition*

$$(1 + \alpha_n \beta_n^T) \cdots (1 + \alpha_1 \beta_1^T) = e^{2\pi i K}. \tag{2.41}$$

(iii) *The map* $\{a_m, b_m\} \mapsto \{\alpha_k, \beta_k\}$ *of pairs of orthogonal vectors regarded modulo transformations of the form* (2.33) *is a one-to-one correspondence between open sets of the varieties defined by the conditions* (2.35), (2.38) *and* (2.40), (2.41), *respectively.*

Proof. As shown above, a solution $\Phi_x \in \mathcal{P}_x$ of the factorization problem (2.1) exists if the homogeneous singular integral equation (2.22) has no solutions. This condition is of open type, and therefore for a fixed x and for generic A the corresponding meromorphic solution Ψ_x of the equation (1.1) exists. If $r_k < x < r_{k+1}$, then it follows from the equation (1.1) that Ψ_x has poles at the points $z_m + l$, $m = 1, 2, \dots$, for $k < m$, and at the points $z_m - l$, $l = 0, 1, 2, \dots$, for $m \leqslant k$. Therefore, Ψ_x is holomorphic in the strip $r_k < \operatorname{Re} z < r_{k+1} + 1$ and can be regarded as one of the desired solutions Ψ_k. The solutions Ψ_k exist for any k if A belongs to the intersection of finitely many open sets. This is still a condition of open type, and therefore a full set of solutions Ψ_k exists for a generic matrix A. These solutions are unique and have the asymptotic representation described in the assertion (i) of the theorem.

The residues of $A(z)$ are rank-one matrices. Therefore, the residue of M_k at z_k is also a rank-one matrix and can be represented in the form $\alpha_k \beta_k^T$. The formula (2.39) follows from (2.31). The condition $\det A = 1$ and the normalization $g_k^- = 1$ imply the equality $\det \Psi_k = 1$. Hence, $\det M_k = 1$, which is equivalent to (2.40). The global connection matrix is the product of the local matrices, $S = M_n \cdots M_1$. Therefore, it follows from (2.30) that the formula (2.41) holds. This completes the proof of (ii).

Let us now show that the map $\{a_m, b_m\} \mapsto \{\alpha_k, \beta_k\}$ defined on the open set $\mathcal{A}_0$ is injective. Indeed, suppose that there are two special regular singular equations having the same set of local connection matrices. The canonical solutions Ψ_k and Ψ_k' of these equations are holomorphic on the strips $\operatorname{Re} z \in (r_k, r_{k+1} + 1)$ and are asymptotically equal to $O(1) z^K g_k^{\pm}$, where $g_k^- = 1$, as $\operatorname{Im} z \to \pm\infty$. We note that the transfer matrices g_k^+ are the same for Ψ_k and Ψ_k', because they are equal to the products of the *monodromy matrices* $\mu_k = 1 + \alpha_k \beta_k^T$,

$$g_0^+ = 1, \quad g_k^+ = \mu_{k-1} \cdots \mu_1, \quad k > 1. \tag{2.42}$$

The matrix function which is equal to $\Psi_k' \Psi_k^{-1}$ in each of the strips is continuous on the boundaries of the strips. Hence, this is an entire function which is bounded at infinity. It tends to 1 as $\operatorname{Im} z \to -\infty$. Therefore, it is identically equal to 1.

The surjectivity of the map $\{a_m, b_m\} \mapsto \{\alpha_k, \beta_k\}$ onto an open set of connection matrices can be established by reducing the proof to a Riemann–Hilbert-type factorization problem. Let us fix a sufficiently small real number ε. The lines L_m: $\operatorname{Re} \xi = \operatorname{Re} z_m + \varepsilon$ divide the complex plane into $n+1$ domains $\mathcal{D}_k$, $k = 0, 1, \dots, n$.

Problem II. *Let a set of matrix functions $\mathcal{M}_j(\xi)$ on the lines L_j be given. Find matrix functions $\mathcal{X}_k(z)$ which are holomorphic on the domains $\mathcal{D}_k$ and continuous up to the boundaries and whose boundary values satisfy the equations*

$$\mathcal{X}_{k-1}^+(\xi) = \mathcal{X}_k^-(\xi)\mathcal{M}_k(\xi), \qquad \xi \in L_k. \tag{2.43}$$

We consider an arbitrary set of matrices M_k of the form (2.39) satisfying the conditions (2.40) and (2.41). Plemelj studied Problem II for the piecewise constant matrices

$$\mathcal{M}_k^0(\xi) = 1, \quad \operatorname{Im} \xi \geqslant 0, \qquad \mathcal{M}_k^0(\xi) = \mu_k, \quad \operatorname{Im} \xi < 0. \tag{2.44}$$

He showed that a solution of this problem exists if at least one of the monodromy matrices is diagonalizable [17]. Let $\mathcal{F}_k$ be solutions of this auxiliary problem. Using these solutions, we define a new set of functions $\mathcal{M}_j(\xi)$ by the formula

$$\mathcal{M}_k = \mathcal{F}_j^+ M_k (\mathcal{F}_k^-)^{-1}. \tag{2.45}$$

The function M_k tends to μ_k exponentially as $\operatorname{Im} z \to \infty$. Hence, $\mathcal{M}_k \to 1$ on both ends of L_k. In that case we can represent a solution of the corresponding factorization problem (2.43) in the form of a Cauchy-type integral,

$$\mathcal{X}(z) = 1 + \sum_k \int_{L_k} \frac{\chi_k(\xi)\, d\xi}{\xi - z}\,. \tag{2.46}$$

The formula (2.46) defines a holomorphic function $\mathcal{X}_k$ interior to each of the domains $\mathcal{D}_k$. Using the Sokhotskii–Plemelj formulae for their boundary values, we obtain a system of singular integral equations for χ_k,

$$\frac{1}{2}\chi_k(\xi)(\mathcal{M}_k(\xi) + 1) - \frac{1}{2\pi i} I_\chi(\xi)(\mathcal{M}_k(\xi) - 1) = (\mathcal{M}_k(\xi) - 1), \tag{2.47}$$

where $I_\chi(\xi)$ stands for the principle value of the integral,

$$I_\chi(\xi) = \text{p.v.} \sum_k \int_{L_k} \frac{\chi_k(\xi')\, d\xi'}{\xi' - \xi}\,. \tag{2.48}$$

The non-homogeneous term of the system vanishes at infinity. Therefore, for a generic set of matrices M_k the system of equations has a solution in the space of Hölder-class functions decaying at infinity. The corresponding functions $\mathcal{X}_k$ tend to the identity matrix at infinity. The functions $\mathcal{F}_k$ have the asymptotic behaviour

$O(1)z^K g_k^{\pm}$. Hence, the functions $\Psi_k = \mathcal{X}_k \mathcal{F}_k$ have the same asymptotic behaviour. Their boundary values satisfy the equation

$$\Psi_{k-1}^{+}(\xi) = \Psi_k^{-}(\xi) M_k(\xi), \qquad \xi \in L_k, \tag{2.49}$$

which can be used to continue the function Ψ_k meromorphically to the whole complex plane. The same equation implies that the function $A_k(z) = \Psi_k(z+1)\Psi_k^{-1}(z)$ is k-independent. In the domain $\mathcal{D}_k$ this function has a unique simple pole at z_k. Therefore, the function $A(z)$ is meromorphic with simple rank-one poles at the points z_k. It tends to the identity matrix at infinity and satisfies the condition $\det A = 1$, that is, $A \in \mathcal{A}_0$, and this completes the proof of the theorem.

Unitary difference equations. As was repeatedly stressed above, for a given real number x the canonical meromorphic solution Ψ_x exists only for generic difference equations. In this subsection we present an example of a class of difference equations for which the canonical meromorphic solutions *always exist*.

A difference equation is said to be *unitary* if its coefficients satisfy the relation

$$A(z) \in \mathcal{A}^U: \quad A^{+}(\bar{z}) = A^{-1}(z), \tag{2.50}$$

where A^+ is the Hermitian conjugate of A. An open set of these matrices can be parameterized by sets of unit vectors a_k,

$$A(z) = \prod_{k=1}^{n} \left(1 + a_k a_k^{+} \frac{z_k - \bar{z}_k}{z - z_k}\right), \qquad a_k^{+} a_k = |a_k|^2 = 1. \tag{2.51}$$

The factors in the product (2.51) are ordered in such a way that the indices increase from right to left. We recall that in this section we assume that the residue of A at infinity is a diagonal matrix,

$$\sum_{k=1}^{n} (z_k - \bar{z}_k) a_k a_k^{+} = K, \qquad K^{ij} = k_i \delta^{ij}, \quad k_i - k_j \notin \mathbb{Z}. \tag{2.52}$$

The equation (2.50) implies that $\overline{\det A(\bar{z})} = \det A^{-1}(z)$. Hence, for any $x \neq \operatorname{Re} z_k$ the index of the factorization problem (2.1) vanishes, $\operatorname{ind}_x A = 0$.

Lemma 2.3. *Let $A(z)$ be the coefficient of a regular singular unitary equation. Then for each $x \neq \operatorname{Re} z_k$ the boundary-value problem* (2.1) *has a non-singular holomorphic solution $\widetilde{\Phi}_x \in \mathcal{P}_x$ such that*

$$\widetilde{\Phi}_x^{+}(\bar{z}) = \widetilde{\Phi}_x^{-1}(z). \tag{2.53}$$

This solution is unique up to a unitary transformation

$$\widetilde{\Phi}'_x(z) = \widetilde{\Phi}_x(z) u, \qquad u \in U(r). \tag{2.54}$$

Proof. As was shown above, the Riemann–Hilbert problem (2.1) has a solution $\Phi \in \mathcal{P}_x$ if the dual boundary problem (2.24) has no vector solution bounded in a neighbourhood of $-i\infty$ and tending to zero more rapidly than any negative power of $\operatorname{Im} z$ at the other end of the strip. Suppose that a vector solution F of this kind exists. Then the scalar function $F(z)F^+(\bar z)$ is holomorphic in Π_x and tends to zero on both ends of the strip. Therefore, the integral of this function over the boundary of the upper half Π_x^+ of the strip Π_x exists and vanishes,

$$\oint_{\partial\Pi_x^+} F(z)F^+(\bar z)\,dz = 0, \qquad z \in \Pi_x^+ \subset \Pi_x\colon \quad \operatorname{Im} z \geqslant 0. \tag{2.55}$$

On the other hand, it follows from (2.50) that this function is periodic, that is, its values at $\xi = x + iy$ and $\xi + 1$ are the same. Therefore, the integral (2.55) is equal to the integral over the lower boundary of Π_x^+,

$$\oint_{\partial\Pi_x^+} F(z)F^+(\bar z)\,dz = \int_x^{x+1} |F(x')|^2\,dx' > 0. \tag{2.56}$$

The contradiction between (2.55) and (2.56) proves that a solution Φ_x exists. As was proved above, Φ_x is unique up to normalization. Let us fix this normalization by the condition that Φ_x has the asymptotic behaviour of Y as $\operatorname{Im} z \to -\infty$. At the other end of the strip this solution has the asymptotic behaviour of Yg_x (in this subsection we avoid the notation $g_x^{\pm}$ to avoid confusion with the symbol of Hermitian conjugation.)

Our next goal is to show that the matrix g_x is Hermitian and positive definite. Indeed, it follows from (2.50) that if Φ_x is a solution of the boundary-value problem, then the matrix $(\Phi_x^+(\bar z))^{-1}$ is also a solution of the same problem. Thus,

$$(\Phi_x^+(\bar z))^{-1} = \Phi_x(z)h, \qquad h \in GL_r. \tag{2.57}$$

This equality at the two ends of the strip is equivalent to the equations $gh = 1$ and $g^+h = 1$. Hence, $g = g^+$. The matrix function $\Phi_x^+(\bar z)\Phi_x(z)$ is holomorphic in Π_x and periodic. Hence, for any constant vector v we have

$$\oint_{\partial\Pi_x^+} v^+\Phi_x^+(\bar z)\Phi_x(z)v\,dz = 0 \;\longmapsto\; v^+gv = \int_x^{x+1} v\Phi_x^+(x')\Phi_x(x')v\,dx' > 0. \tag{2.58}$$

Thus, g is positive definite, and hence there is a matrix g_1 such that $g = g_1^+g_1$. It follows from the equation (2.57) that the function $\widetilde\Phi_x = \Phi_x g_1^{-1}$ satisfies the relation (2.53).

Theorem 2.3. *Let $A(z)$ be a matrix of the form* (2.51). *Then the following assertions hold.*

(i) *The difference equation* (1.1) *has a unique set of meromorphic solutions $\widetilde\Psi_k$, such that* (a) *the function $\widetilde\Psi_k$ is holomorphic in the strip $r_k < \operatorname{Re} z < r_{k+1} + 1$ and has at most polynomial growth as* $\operatorname{Im} z \to \pm\infty$; (b) $\widetilde\Psi_0 = (1 + O(z^{-1}))z^K$ *as* $\operatorname{Im} z \to -\infty$; (c) $\widetilde\Psi_k$ *satisfies the relation*

$$\widetilde\Psi_k^+(\bar z) = \widetilde\Psi_k^{-1}(z); \tag{2.59}$$

(d) *the local connection matrices* $M_k = \widetilde{\Psi}_k^{-1}\widetilde{\Psi}_{k-1}$ *have the form*

$$\widetilde{M}_k(z) = 1 - f_k(z)\alpha_k\alpha_k^+, \tag{2.60}$$

where

$$f_k(z) = (1+|w_k|)\,\frac{ww_k^{-1} - |w_k|^{-1}}{ww_k^{-1} - 1}\,, \qquad w = e^{2\pi iz}, \quad w_k = w(z_k), \tag{2.61}$$

and the α_k *stand for unit vectors,* $\alpha_k^+\alpha_k = 1$, *satisfying the condition*

$$(1-\nu_n\alpha_n\alpha_n^+)\cdots(1-\nu_1\alpha_1\alpha_1^+) = e^{\pi iK}, \qquad \nu_k = 1+|w_k|. \tag{2.62}$$

(ii) *The monodromy map* $\{a_k\} \mapsto \{\alpha_k\}$ *is a one-to-one correspondence of the algebraic varieties defined by the equations* (2.52) *and* (2.62).

Proof. Lemma 2.3 implies that solutions $\widetilde{\Psi}'_k$ satisfying the conditions (a) and (c) exist and are unique up to unitary normalization. The corresponding connection matrix $\widetilde{M}'_k$ is a rational function in the variable w, satisfies the relation

$$\widetilde{M}'^{\,+}_k(\bar z) = \widetilde{M}'^{\,-1}_k(z), \tag{2.63}$$

and has only one pole at the corresponding point w_k. The residue of this function is a rank-one matrix. One can immediately see that every matrix of this kind has a unique representation in the form $\widetilde{M}'_k = u_k\widetilde{M}_k$, where $\widetilde{M}_k$ is given by the formula (2.60) and $u_k \in U(r)$. The condition (b) uniquely determines the function $\widetilde{\Psi}_0$. After this, one can reduce the local connection matrices to the form $\widetilde{M}_k$ by modifying the normalization, $\widetilde{\Psi}'_k = \widetilde{\Psi}_k u_k$, $u_k \in U(r)$.

Up to a z-independent factor, the global connection matrix $\widetilde{S} = \widetilde{M}_n\cdots\widetilde{M}_1$ is equal to the global connection matrix S corresponding to the canonically normalized solutions Ψ_k used above, that is, $\widetilde{S} = \widetilde{S}(-i\infty)S(z)$. Therefore, using (2.63), we see that $S(i\infty) = \widetilde{S}^{-1}(-i\infty)\widetilde{S}(i\infty) = \widetilde{S}^2(i\infty)$. The left-hand side of (2.62) is equal to $\widetilde{S}(i\infty)$. Therefore, the equality (2.30) implies the formula (2.62).

The proof of the part (ii) of the theorem is almost identical to that of the last statement of Theorem 2.2.

Case of small norm. We now present another case, which is important for further considerations and for which, again in the case of general position, the notion of local monodromies around the poles of $A(z)$ can be introduced.

For simplicity we assume that $\operatorname{Re} z_k < \operatorname{Re} z_m$, $k < m$. Let us fix a positive number $\varepsilon \ll \max_{k,m}|\operatorname{Re} z_k - \operatorname{Re} z_m|$ and consider the space of matrix functions $A(z)$ of the form (2.3) such that the Euclidean norm of the coefficients satisfies the inequality $|A_k| < \varepsilon/2$. If ε is sufficiently small, then $A(z)$ is invertible for $|z - z_k| > \varepsilon$. In this case the zeros of $\det A$ are localized in neighbourhoods of the poles. Let the zeros be denoted by z_{ks}^-,

$$\det A(z_{ks}^-) = 0, \quad |z_k - z_{ks}^-| < \varepsilon, \quad s = 1,\dots,h_k = \operatorname{rank} A_k. \tag{2.64}$$

If the number ε is sufficiently small, then a solution of the singular integral equation (2.17) for $x_k = (\operatorname{Re} z_k + \operatorname{Re} z_{k+1})/2$, $k = 1,\dots,n-1$, can be constructed by the same iterations (2.20) used above for $|x| \gg 0$. The corresponding canonical solution $\Psi_k = \Psi_{x_k}$ of the equation (1.1) has poles at the points $z_m + l$, $l = 1,2,\dots$, $k \leqslant m$, and at the points $z_{ms}^- - l$, $l = 0,1,2,\dots$, $m \leqslant k$. As in the proof of Theorem 2.2, we obtain the following assertion.

Theorem 2.4. *There is an ε such that if $|A_k| < \varepsilon$ and if the A_k satisfy the condition* (2.4), *then the corresponding special regular singular equation* (1.1) *has a unique set of canonical meromorphic solutions Ψ_k, $k = 0, 1, \dots, n$, which are holomorphic in the strips $r_k + \varepsilon < \operatorname{Re} z < r_{k+1} + 1$, have at most polynomial growth as $|\operatorname{Im} z| \to \infty$, and are normalized by the conditions $\lim_{\operatorname{Im} z \to -\infty} \Psi_k z^{-K} = 1$.*

(i) *The solutions Ψ_k can be asymptotically represented in the form $Y g_k^{\pm}$ as $\operatorname{Im} z \to \pm\infty$, where $g_k^{-} = 1$ and $g_0^{+} = g_n^{+} = 1$.*

(ii) *The local connection matrices $M_k = \Psi_k^{-1} \Psi_{k-1}$, $k = 1, \dots, n$, have the form*

$$M_k = 1 - \frac{m_k}{e^{2\pi i (z - z_k)} - 1}, \tag{2.65}$$

where the matrices m_k satisfy the condition

$$(1 + m_n) \cdots (1 + m_1) = e^{2\pi i K}. \tag{2.66}$$

(iii) *The map $\{A_m\} \mapsto \{m_k\}$ is a one-to-one correspondence between the space of matrices with $|A_m| < \varepsilon$ that satisfy* (2.4) *and the intersection of an open neighbourhood of the point $(m_k = 0)$ with the variety defined by equation* (2.66).

2.3. Mild equations. In this subsection we extend the above results to the case of mild difference equations (1.1) with diagonalizable leading coefficient

$$A = A_0 + \sum_{m=1}^{n} \frac{A_m}{z - z_m}, \qquad A_0^{ij} = \rho_i \delta^{ij}. \tag{2.67}$$

If $\rho_i \neq \rho_j$, then the equation (1.1) has a unique formal solution of the form (1.5). The substitution of (1.5) into (1.1) gives a system of equations for the unknowns χ_s. The first non-trivial equation

$$[A_0, \chi_1] = \sum_{m=1}^{n} A_m - K \tag{2.68}$$

determines the diagonal matrix

$$K^{ij} = k_i \delta^{ij}, \qquad k_i = \sum_{m=1}^{n} A_m^{ii}, \tag{2.69}$$

and the off-diagonal part of the matrix χ_1. At each step, the corresponding equation recursively determines the diagonal entries of χ_{s-1} and the off-diagonal part of χ_s.

Let us consider first the case of *real exponentials*.

Theorem 2.5. *Let A be a matrix of the form* (2.67) *with $\rho_i \neq \rho_j$, $\operatorname{Im} \rho_i = 0$. Then the following assertions hold.*

(A) *There are meromorphic solutions $\Psi_{l(r)}$ of the equation* (1.1) *that are holomorphic in the domains $\operatorname{Re} z \ll 0$ and $\operatorname{Re} z \gg 0$, respectively, in which these solutions are asymptotically equal to $Y g_{l(r)}^{\pm}$, $g_{l(r)}^{-} = 1$, as $\operatorname{Im} z \to \pm\infty$; the matrices $g_{r(l)} = g_{r(l)}^{+}$ satisfy the conditions* (1.11),

$$g_{r(l)}^{ii} = 1, \quad g_r^{ij} = 0 \quad \text{if} \quad \rho_i < \rho_j, \qquad g_l^{ij} = 0 \quad \text{if} \quad \rho_i > \rho_j, \tag{2.70}$$

and these solutions are unique.

(B) *The global connection matrix* $S = (\Psi_r)^{-1}\Psi_l$ *has the form*

$$S(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}, \qquad S_\infty = 1 + \sum_{m=1}^{n} S_m = g_r^{-1} e^{2\pi i K} g_l. \tag{2.71}$$

In the case of real exponentials, $\operatorname{Im}\rho_i = 0$, the growth of the matrix $e^{z\log A_0 + zK}$ is at most polynomial as $|\operatorname{Im} z| \to \infty$, and practically all results proved above for regular singular equations remain valid. Lemma 2.1 needs no modifications at all. It implies the existence of meromorphic canonical solutions Ψ_r and Ψ_l of the equation (1.1). These solutions are asymptotically equal to $Yg^{\pm}_{l(r)}$ as $\operatorname{Im} z \to \pm\infty$, and they can be normalized by the condition $g^{-}_{l(r)} = 1$ in a unique way. The only substantial difference between the mild equations with real exponentials and the regular singular equations is that the mild equations need not satisfy the equality $g_{l(r)} = 1$. The coefficient $(\widetilde{A} - 1)$ of the equation (2.17) has the form

$$\widetilde{A} - 1 = e^{-z\log A_0 - K\log z} O(z^{-m}) e^{z\log A_0 + K\log z}. \tag{2.72}$$

It follows from (2.17) that the matrix function φ has the quasi-triangular form asymptotically. Moreover, the equation (2.26) implies the formula (2.70). The proof of the formula (2.71) is similar to that of (2.30).

We consider now the *Birkhoff case* of exponentials ρ_i with distinct imaginary parts of $\log\rho_i$. Below we assume that the branch of $\log\rho_i$ is chosen in such a way that

$$-\pi < \nu_i = \operatorname{Im}(\log\rho_i) \leqslant \pi. \tag{2.73}$$

Theorem 2.6. *Let A be a matrix of the form* (2.67) *and let* $\nu_i \neq \nu_j \neq 0$. *Then*:

(A) *there are meromorphic solutions* $\Psi_{l(r)}$ *of the equation* (1.1) *that are holomorphic in the domains* $\operatorname{Re} z \ll 0$ *and* $\operatorname{Re} z \gg 0$, *respectively, in which these solutions are asymptotically equal to* Y, $\operatorname{Im} z \to \pm\infty$, *and are unique*;

(B) *the connection matrix* $S = (\Psi_r)^{-1}\Psi_l$ *of these solutions is of the form*

$$S(z) = S_0 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1}, \tag{2.74}$$

where the terms S_0 *and* $S_\infty = 1 + \sum_{m=1}^{n} S_m$ *satisfy the relations*

$$S_0^{jj} = 1,\ \ S_0^{ij} = 0 \ \text{ if } \nu_i > \nu_j, \quad S_\infty^{jj} = e^{2\pi i k_j},\ \ S_\infty^{ij} = 0 \ \text{ if } \nu_i < \nu_j. \tag{2.75}$$

The first assertion of the theorem is a fundamental result of Birkhoff. Nevertheless, we outline the proof based on the Riemann–Hilbert problem (2.1). This enables us to show both the similarity and the difference between the Birkhoff case and the case of real exponentials. The differences are mainly related to the following simple fact: if $\nu_i \neq \nu_j$, then the formal series Y and Yg are asymptotically equal to each other as $\operatorname{Im} z \to \pm\infty$ if g is a quasi-upper-triangular or a quasi-lower-triangular matrix, respectively, with the diagonal entries equal to 1. It follows that the notion of transfer matrix g_x along the thick path Π_x introduced above has no intrinsic meaning in the Birkhoff case. The transfer matrix is hidden in the normalization of $\Psi_{l(r)}$ and manifests itself only in the form of the connection matrix S.

As above, the construction of a sectionally holomorphic solution Φ_x of the Riemann–Hilbert factorization problem (2.1) reduces to a singular integral equation. Let Φ_x be given by

$$\Phi_x = Y_m\phi, \qquad \phi = g + \int_L \varphi(\xi)k(z,\xi)\,d\xi. \tag{2.76}$$

The function Φ_x is a solution of the Riemann–Hilbert problem if $\varphi \in H$ satisfies the equation

$$(\widetilde{A}+1)\varphi - 2(\widetilde{A}-1)I_\varphi = 2(\widetilde{A}-1)g, \tag{2.77}$$

where $\widetilde{A}$ is given by the formula (2.18). For regular singular equations, as well as in the case of mild equations with real exponentials, the choice of the constant term g in (2.76) was unessential. However, this choice becomes crucial in the Birkhoff case.

Our next objective is to show that there is a unique matrix g with unit diagonal entries, $g^{ii}=1$, for which the equation (2.77) has a solution $\varphi \in H$ such that

$$|\varphi^{ij}(\xi)| < O(|y|^{-m+\kappa})e^{y\nu_{ij}}, \qquad \nu_{ij} = \nu_i - \nu_j, \quad y = \operatorname{Im}\xi \to \pm\infty. \tag{2.78}$$

If a smooth function φ satisfies (2.78), then the corresponding Cauchy integral has the asymptotic behaviour

$$\pm\nu_{ij} > 0\colon \quad \begin{cases} |I_\varphi^{ij}| < O(|y|^{-m+\kappa})e^{y\nu_{ij}}, & y \to \pm\infty, \\ |I_\varphi^{ij} - f_\varphi^{ij}| < O(|y|^{-m+\kappa})e^{y\nu_{ij}}, & y \to \mp\infty, \end{cases} \tag{2.79}$$

where

$$\pm\nu_{ij} > 0\colon \quad f_\varphi^{ij} = -\frac{1}{2}\int_L (\tan(\pi y) \pm 1)\varphi^{ij}(\xi)\,d\xi. \tag{2.80}$$

The proof of the second inequality in (2.79) is almost identical to that of (2.25). The first inequality can be obtained by similar arguments.

It follows from self-consistency of the equation (2.77) and the conditions (2.79) that

$$g = 1 - f_\varphi, \tag{2.81}$$

where the matrix f_φ is off-diagonal; it is given by (2.80). The equations (2.77) and (2.81) form a system of equations for the unknowns $\varphi(\xi)$ and g. This system can be solved for large $|x|$ by iterations. To this end, we take $\varphi_0 = 0$ and define φ_n recursively by the equation

$$(\widetilde{A}+1)\,\varphi_{n+1} = 2(\widetilde{A}-1)(1 + I_{\varphi_n} - f_{\varphi_n}). \tag{2.82}$$

It follows from (2.79) that if φ_n satisfies (2.78), then so does φ_{n+1}. The sequences $g_n = 1 - f_{\varphi_n}$ and φ_n converge and determine g and a solution φ of the corresponding equation (2.77), and this solution satisfies (2.79).

It follows from (2.79) that if ϕ and g are solutions of (2.77) and (2.81), then the off-diagonal entries of the matrix function Φ given by (2.76) have the following asymptotic behaviour on both ends of $\Pi_{x,\varepsilon}$:

$$|\phi^{ij}(z)| < O(|z|^{-m+\kappa})|(\rho_j/\rho_i)^z|, \qquad \operatorname{Im} z \to \pm\infty. \tag{2.83}$$

The asymptotic behaviour of the diagonal elements of ϕ coincides with that established above in the case of regular singular equations, that is,

$$|\phi^{jj}(z) - v_j^{\pm}| < O(|z|^{-m+\kappa+1}), \qquad \operatorname{Im} z \to \pm\infty, \tag{2.84}$$

where

$$v_j^{\pm} = 1 - \frac{1}{2}\int_L (\tan(\pi i(\xi - x)) \pm 1)\varphi^{jj}(\xi)\,d\xi. \tag{2.85}$$

As above, one can prove that if there is a sectionally holomorphic solution Φ_x of the Riemann–Hilbert problem (2.1), then this solution is unique. Therefore, (2.83) and (2.84) imply the following assertion.

Lemma 2.4. *For a generic matrix A such that* $\operatorname{ind}_x A = 0$ *there is a unique sectionally holomorphic solution Φ_x of the Riemann–Hilbert problem* (2.1) *for which Φ_x is asymptotically equal to Y as* $\operatorname{Im} z \to -\infty$ *and to Yv_x as* $\operatorname{Im} z \to \infty$, *where v_x is a diagonal matrix.*

For large $|x|$ the corresponding solutions Ψ_x of the equation (1.1) are x-independent. For $x \gg 0$ and $x \ll 0$ these solutions can be identified with the Birkhoff solutions Ψ_r and Ψ_l, respectively. We note that the functions φ^{ii} are uniformly bounded for large $|x|$ by $O(|x|^{-m+\kappa})$. Therefore, it follows from (2.84) that $v_{l(r)} = 1$. This proves the first assertion of the theorem.

It follows from (2.73) that the connection matrix S, when regarded as a function of the variable $w = e^{2\pi iz}$, has a holomorphic continuation to the points $w = 0$ and $w = \infty$. Therefore, this is a rational function of w having poles at the points $w_m = w(z_m)$. Hence, it has the form (2.71). It follows from the above arguments that the values of this function at the points $w = 0$ and $w = \infty$ are quasi-triangular matrices. This completes the proof of the theorem.

Local monodromies for mild equations can be introduced in the cases similar to those treated above in § 2.2. Namely, this is possible in the cases of *special* coefficients, *unitary* coefficients, and coefficients of *small norm*. The form of the local monodromy matrices in the case of mild equations with real exponentials was described in the Introduction. The other results of Section 2.2 also admit a straightforward generalization to the case of mild equations. For example, let us consider the special mild equations with imaginary exponentials.

Theorem 2.7. (i) *For a generic matrix A of the form*

$$A(z) = A_0\left(1 + \frac{a_n b_n^T}{z - z_n}\right)\cdots\left(1 + \frac{a_1 b_1^T}{z - z_1}\right), \tag{2.86}$$

where

$$\text{(a)}\quad A_0^{ij} = \rho_i\delta^{ij}, \quad \nu_i \ne \nu_j, \quad \nu_i = \operatorname{Im}(\log\rho_i),$$
$$\text{(b)}\quad b_k^T a_k = 0, \qquad \text{(c)}\quad \operatorname{Re} z_k < \operatorname{Re} z_m, \quad k < m,$$

the equation (1.1) *has a unique set of meromorphic solutions* Ψ_k, $k = 0, 1, \dots, n$, *which are holomorphic in the strips* $r_k < \operatorname{Re} z < r_{k+1} + 1$ *and are asymptotically equal to* $Yv_k^{\pm}$ *as* $\operatorname{Im} z \to \pm\infty$, *where* $v_k^- = 1$ *and the matrices* v_k^+ *are diagonal.*

(ii) *The local connection matrices* $M_k = \Psi_k^{-1}\Psi_{k-1}$, $k = 1, \dots, n$, *have the form*

$$M_k = m_{k0} - \frac{\alpha_k \beta_k^T}{e^{2\pi i(z-z_k)} - 1}\,, \tag{2.87}$$

where (α_k, β_k) *are pairs of orthogonal vectors considered modulo transformations of the form* (2.33), *and* m_{k0} *stands for a quasi-lower-triangular matrix such that* $M_k(i\infty)$ *is a quasi-upper-triangular matrix, that is,*

$$m_{k0}^{jj} = 1, \quad m_{k0}^{ij} = 0 \quad \text{if} \quad \nu_i > \nu_j, \qquad m_{k0}^{ij} = -\alpha_k^i \beta_k^j \quad \text{if} \quad \nu_i < \nu_j. \tag{2.88}$$

(iii) *The map of pairs of orthogonal vectors* $\{a_m, b_m\} \mapsto \{\alpha_k, \beta_k\}$ *considered modulo transformations of the form* (2.33) *is a one-to-one correspondence of open sets.*

In the case of small norms the structure of the local connection matrix M_k can be described in a similar way. Namely, it has the form

$$M_k = m_{k0} - \frac{m_{k1}}{e^{2\pi i(z-z_k)} - 1}\,, \tag{2.89}$$

where m_{k0} is a quasi-lower-triangular matrix and $m_{k0} + m_{k1}$ is a quasi-upper-triangular matrix. The discrete analogue of the local monodromy matrix is defined as the ratio

$$\mu_k = 1 + m_{k1} m_{k0}^{-1}. \tag{2.90}$$

We note that a generic matrix μ_k admits a unique factorization into a product of lower- and upper-triangular matrices. Therefore, in general position the matrix μ_k uniquely determines the matrices m_{k0} and m_{k1}, and, consequently, the local and global connection matrices.

§ 3. Isomonodromy transformations and the inverse monodromy problem

In this section we consider the map inverse to the direct monodromy map,

$$\{z_m, A_m\} \mapsto \{w_m, S_m\}, \qquad w_m = w(z_m) = e^{2\pi i z_m}. \tag{3.1}$$

For any fixed diagonalizable matrix A_0 the characterization of the equations (1.1) having the same monodromy data is identical to that given by Birkhoff in the case of imaginary exponentials.

Lemma 3.1. *Rational functions* $A(z)$ *and* $A'(z)$ *of the form* (1.2) *correspond under the map* (3.1) *to the same connection matrix* $S(z)$ *if and only if there is a rational matrix function* $R(z)$ *such that*

$$A'(z) = R(z+1)A(z)R^{-1}(z), \qquad R(\infty) = 1. \tag{3.2}$$

Proof. Let $\Psi_{l(r)}$ and $\Psi'_{l(r)}$ be canonical meromorphic solutions of the equation (1.1) with the coefficients $A(z)$ and $A'(z)$, respectively. If $\Psi_r^{-1}\Psi_l = (\Psi'_r)^{-1}\Psi'_l$, then

$$R = \Psi'_l \Psi_l^{-1} = \Psi'_r \Psi_r^{-1}. \tag{3.3}$$

By the definition of canonical solutions, the matrix function R is holomorphic for large $|\operatorname{Re} z|$. Moreover, if $A_0 = A'_0$ and $K = K'$, then $R \to 1$ as $|z| \to \infty$. Hence, R has only finitely many poles, and therefore R is a rational function in the variable z.

We denote by $\mathcal{A}_D$ the subspace of the space $\mathcal{A}$ of matrix functions having a fixed determinant:

$$A \in \mathcal{A}_D \subset \mathcal{A}: \qquad \det A(z) = D(z) = \frac{\prod_{\alpha=1}^{N}(z-\zeta_\alpha)}{\prod_{m=1}^{n}(z-z_m)^{h_m}}\,, \quad h_m = \operatorname{rk} A_m. \tag{3.4}$$

We note that the condition (1.4) is equivalent to the condition

$$\operatorname{tr} K = 0 \;\longleftrightarrow\; \sum_\alpha \zeta_\alpha = \sum_m h_m z_m. \tag{3.5}$$

Lemma 3.2. *If the zeros ζ_α are not congruent, that is, if $\zeta_\alpha - \zeta_\beta \notin \mathbb{Z}$, then the monodromy correspondence* (3.1) *restricted to $\mathcal{A}_D$ is injective.*

Proof. Let us consider a matrix function $A \in \mathcal{A}_D$ whose poles and zeros (of the determinant) are pairwise non-congruent. Suppose that there is a rational matrix function R which is equal to 1 at infinity, $R = 1 + O(z^{-1})$, and such that the matrix A' defined by (3.2) has the same determinant, that is, $A' \in \mathcal{A}_D$. Then the equality $R(z+1) = A'(z)R(z)A^{-1}(z)$ implies that R has poles of *constant* rank at the points $\zeta_\alpha + l$ and $z_m + l$, where $l \in \mathbb{Z}_+$. The matrix R is regular at infinity. Therefore, it must be regular everywhere. This implies that $R = 1$.

Rational functions D and D' are said to be *equivalent* if the sets of their poles z_m, z'_m and zeros ζ_α, ζ'_α are congruent to each other, that is, $z_m - z'_m \in \mathbb{Z}$, $\zeta_\alpha - \zeta'_\alpha \in \mathbb{Z}$, and satisfy the condition (3.5).

Lemma 3.3. *For each pair of equivalent rational functions D and D' there is a unique isomonodromy birational isomorphism*

$$T_D^{D'} : \mathcal{A}_D \mapsto \mathcal{A}_{D'}. \tag{3.6}$$

Proof. The construction of the transformations $T_D^{D'}$ is similar to the construction proposed in [10] in the case of polynomial coefficients $\widetilde{A}$. To begin with, we introduce two types of elementary transformations. They are birational and defined on open sets of the corresponding spaces. An elementary isomonodromy transformation of the first type is defined by a pair (z_k, ζ_α) and by an eigenvector of $A_k = \operatorname{res}_{z_k} A$ corresponding to a non-zero eigenvalue λ,

$$q^T A_k = \lambda q^T \neq 0. \tag{3.7}$$

Let us consider the matrix

$$R = 1 + \frac{pq^T}{z - z_k}\,, \tag{3.8}$$

where p is a null-vector of the matrix $A(\zeta_\alpha)$ normalized in such a way that

$$(q^T p) = z_k - \zeta_\alpha, \qquad A(\zeta_\alpha)p = 0. \tag{3.9}$$

Remark. If $z_k \neq \zeta_\alpha$, then the matrix R is defined only for an open subset of $\mathcal{A}_D$ for which the inner product $(q^T p)$ of the corresponding eigenvectors is non-zero.

It follows from the equations (3.9) that

$$R^{-1} = 1 - \frac{pq^T}{z - \zeta_\alpha}\,. \tag{3.10}$$

Furthermore, it follows from the second equation in (3.9) that the matrix A' given by (3.2) is regular at ζ_α. The matrix A' has a pole of rank one at $z_k - 1$. The rank of its residue at z_k is equal to the rank of the matrix $A_k R^{-1}(z_k)$. The left null-space of the last matrix contains both the null-space of A_m and the vector q^T. Hence, the residue of A' at z_k is of rank $h_k - 1$. In the same way, choosing another zero ζ_{α_2} of D and an eigenvector of $A'_k = \operatorname{res}_{z_k} A'$ corresponding to a non-zero eigenvalue, we construct a matrix function A'' with a pole at z_k of rank $h_k - 2$. Further iterations give a matrix $T_k^{\alpha_1,\dots,\alpha_{h_k}}(A)$ which is regular at z_k and has a pole of rank h_k at $z_k - 1$.

As follows from Lemma 3.2, the isomonodromy transformation $T_k^{\alpha_1,\dots,\alpha_{h_k}}$ is uniquely determined by the choice of a pole z_k and a subset of h_k zeros ζ_{α_s} of the function D. These transformations are analogues of the transformations introduced in [10] in the case of polynomial functions $A(z)$.

An elementary isomonodromy transformation of the second type is defined by a pair of zeros ζ_α and ζ_β of D. The corresponding matrix $R = R_{\alpha,\beta}$ is given by the formula

$$R_{\alpha,\beta} = 1 + \frac{p_\alpha q_\beta^T}{z - \zeta_\beta - 1}\,, \tag{3.11}$$

where the vectors p_α and q_β are determined by the relations

$$\text{(i)}\ \ A(\zeta_\alpha)p_\alpha = 0; \quad \text{(ii)}\ \ q_\beta^T A(\zeta_\beta) = 0; \quad \text{(iii)}\ \ (q_\beta^T p_\alpha) = \zeta_\beta - \zeta_\alpha + 1. \tag{3.12}$$

It follows from (3.12) (iii) that $R_{\alpha,\beta}^{-1} = 1 - p_\alpha q_\beta^T/(z - \zeta_\alpha)$. Then the equations (3.12) (i), (ii) imply that the matrix

$$T^{\alpha|\beta}(A) = R_{\alpha,\beta}^{-1}(z+1)A(z)R_{\alpha,\beta}^{-1}(z) = \left(1 + \frac{p_\alpha q_\beta^T}{z - \alpha_\beta}\right)A(z)\left(1 + \frac{p_\alpha q_\beta^T}{z - \alpha_\alpha}\right) \tag{3.13}$$

is regular and non-singular at the points ζ_α and ζ_β. Its set of poles coincides with that of A. The zeros of the determinant of this matrix are the points $\zeta_\alpha - 1$, $\zeta_\beta + 1$, and ζ_γ, $\gamma \neq \alpha, \beta$.

The transformation $T_D^{D'}$ can be obtained as a composition of elementary transformations. Indeed, if D and D' are equivalent, then the poles of D can be shifted to the poles of D' by transformations (or their inverses) of the first type. After that, $N - 1$ zeros can be shifted to $N - 1$ zeros of D' by transformations of the second type. In this case the equation (3.5) uniquely determines the position of the last zero. This proves the lemma.

The main result of this section is presented in the next theorem.

Theorem 3.1. *Let $A_0^{ij} = \rho_i \delta^{ij}$ and K be diagonal matrices and suppose that the rational matrix $S(w)$ has one of the following forms*: (a) (2.30) *if* $A_0 = 1$; (b) (2.70), (2.71) *if* $\operatorname{Im}\rho_i = 0$; (c) (2.74), (2.75) *if* $\operatorname{Im}(\log\rho_i) \neq \operatorname{Im}(\log\rho_j) \neq 0$. *Then in general position for each S and for each set of branches z_k, ζ_α of the logarithms of the poles and zeros of* $\det S(w)$ *there is a unique rational matrix function $A(z)$ of the form* (1.2) *such that $S(z)$ is the connection matrix of the corresponding difference equation* (1.1). *In this case*, $\det A(\zeta_\alpha) = 0$.

Proof. It was already proved that if $A(z)$ exists for some set of branches z_k, ζ_α, then in general position it exists and is unique for any equivalent set. Therefore, to prove the theorem, it suffices to construct an equation of the form (1.1) for which the given function S is the connection matrix of canonical solutions.

Let us fix a real number x such that the matrix $S(z)$ is regular and invertible on the line L: $\operatorname{Re} z = x$. We denote the half-planes $\operatorname{Re} z < x$ and $\operatorname{Re} z > x$ by $\mathcal{D}_l$ and $\mathcal{D}_r$, respectively, and we consider the following factorization problem.

Problem III. *Let S be given. Find invertible matrix functions $\mathcal{X}_l(z)$ and $\mathcal{X}_r(z)$ which are holomorphic and bounded interior to the domains $\mathcal{D}_l$ and $\mathcal{D}_r$, respectively, are continuous up to the boundaries, and for which the boundary values of the functions $\Psi_{l(r)} = \mathcal{X}_{l(r)} e^{z\log A_0 + K\log z}$ satisfy the equation*

$$\Psi_l(\xi) = \Psi_r(\xi) S(\xi), \qquad \xi \in L. \tag{3.14}$$

Lemma 3.4. *Problem* III *has a solution for any generic matrix S, and this solution is unique up to the normalization $\mathcal{X}'_{l(r)} = g\mathcal{X}_{l(r)}$.*

Proof. Let us consider the functions $\mathcal{X}_{l(r)}$ defined in each of the corresponding half-planes by the Cauchy integral

$$\mathcal{X}(z) = 1 + \frac{1}{2\pi i}\int_L \frac{\chi(\xi)\,d\xi}{\xi - z}. \tag{3.15}$$

The equation (3.14) is equivalent to the equation

$$\frac{1}{2}\chi(\xi)(\mathcal{M}(\xi)+1) - \frac{1}{2\pi i} I_\chi(\xi)(\mathcal{M}(\xi)-1) = (\mathcal{M}(\xi)-1), \tag{3.16}$$

where $\mathcal{M} = Y_0 S Y_0^{-1}$ and $Y_0 = e^{z\log A_0 + K\log z}$. If S is of the form (a) or (c), then $\mathcal{M}$ tends to 1 exponentially at infinity, and the equation (3.16) has a unique solution for any generic matrix S. In the case (b) (of mild equations with real exponentials), the coefficient $\mathcal{M}$ has no limit at infinity. The following slight modification of Problem III enables one to prove the lemma for the case (b) as well. Let us consider the functions $\mathcal{X}'_{l(r)}$ given by the Cauchy integral (3.15) along the line $\xi \in L'$: $\operatorname{Arg}(\xi - x) = \pi/2 + \varepsilon$, $\varepsilon > 0$. If $\chi(\xi)$, $\xi \in L'$, is a solution of the equation (3.16) on L' with the coefficient $\mathcal{M}' = Y_0 g_r S Y_0^{-1}$, then the boundary values of the functions $\Psi' = \mathcal{X}'_l Y_0$ and $\mathcal{X}'_r Y_0$ on L' satisfy the equation

$$\Psi'_l(\xi) = \Psi'_r(\xi) g_r S(\xi), \qquad \xi \in L'. \tag{3.17}$$

It follows from (2.70) that $\mathcal{M}'$ tends exponentially to the identity matrix 1 along L'. Therefore, in general position the solution χ of the corresponding equation (3.16) on L' exists and is unique. This solution determines a unique solution of the factorization problem (3.17). The equation (3.17) can be used for meromorphic continuation of the functions $\Psi'_{l(r)}$, which are originally defined in the half-planes separated by L'. If $\varepsilon > 0$ is small enough, then S is regular and invertible in the sectors between L and L'. Hence, the continuations of the functions Ψ'_l and Ψ'_r are holomorphic in the domains $\mathcal{D}_l$ and $\mathcal{D}_r$, respectively. Therefore, the functions $\Psi_l = \Psi'_l$ and $\Psi_r = \Psi'_r g_r$ are solutions of the factorization problem (3.14). This proves the lemma.

Let the functions $\Psi_{l(r)}$ form a solution of the factorization problem (3.14). Then the function

$$A(z) = \Psi_l(z+1)\Psi_l^{-1}(z) = \Psi_r(z+1)\Psi_r^{-1}(z) \tag{3.18}$$

is holomorphic in the domains $\operatorname{Re} z < x-1$ and $\operatorname{Re} z > x$. It tends to A_0 as $z \to \infty$. Interior to the strip Π_{x-1} the poles of A and A^{-1} coincide with the poles of S and S^{-1}, respectively. Hence, $A(z)$ is of the form (1.2), where $x - 1 < \operatorname{Re} z_m < x$. This proves the theorem.

§ 4. Continuous limit

Our next objective is to show that the canonical meromorphic solutions Ψ_x of the difference equation (1.21) converge to solutions of the differential equation (1.22) in the limit as $h \to 0$.

The construction of the meromorphic solutions Ψ_x of (1.21) that are holomorphic in the strip Π_x^h: $x < \operatorname{Re} z < x + h$ requires only slight modifications in the known formulae. As above, a sectionally holomorphic solution Φ_x of the factorization problem

$$\Phi_x^+(\xi + h) = (1 + hA(\xi))\Phi_x^-(\xi), \qquad \xi = x + iy, \tag{4.1}$$

can be represented by the Cauchy-type integral

$$\Phi_x = Y_0\phi, \quad \phi = 1 + \int_L \varphi(\xi) k_h(z,\xi)\, d\xi, \quad k_h = k(h^{-1}z, h^{-1}\xi), \tag{4.2}$$

where $k(z,\xi)$ is given by (2.13) and $Y_0 = e^{z\log(1+hA_0) + hK\log z}$. The residue of k_h at $z = \xi$ is equal to h, and therefore the boundary values of ϕ are

$$\phi^-(\xi) = -\frac{h\varphi(\xi)}{2} + 1 + I_\varphi(\xi), \qquad \phi^+(\xi+1) = \frac{h\varphi(\xi)}{2} + 1 + I_\varphi(\xi), \tag{4.3}$$

where I_φ stands for the principal value of the corresponding integral. The singular integral equation for φ, which is equivalent to (4.1), becomes

$$(2 + h\widetilde{A})\varphi - 2\widetilde{A} I_\varphi = 2\widetilde{A}, \tag{4.4}$$

where $\widetilde{A} = Y_0(\xi+1)^{-1}A(\xi)Y_0(\xi)$. If $|x - \operatorname{Re} z_k| > Ch$, then the equation (4.4) can be solved by iterations. The corresponding solution Ψ_x of the difference equation is x-independent on the intervals $\operatorname{Re} z_k + Ch < x < \operatorname{Re} z_{k+1} - Ch$. Thus, we conclude that for any $\varepsilon > 0$ and any rational function $A(z)$ of the form (1.2) there is an h_0 such that the equation (1.21) for $h < h_0$ has canonical meromorphic solutions Ψ_k that are holomorphic in the strips $z \in \mathcal{D}_k$: $\operatorname{Re} z_k + \varepsilon < \operatorname{Re} z < \operatorname{Re} z_{k+1} - \varepsilon$.

The existence of Ψ_k means that the local monodromy matrices μ_k are well defined for sufficiently small h for each A of the form (1.2). Hence, we can consider the continuous limit of these matrices.

Theorem 4.1. *The following assertions hold for the limit as $h \to 0$.*

(A) *The canonical meromorphic solutions Ψ_k of the difference equation* (1.21) *converge uniformly on $\mathcal{D}_k$ to solutions $\widehat{\Psi}_k$ of the differential equation* (1.22) *which are holomorphic on $\mathcal{D}_k$.*

(B) *The local monodromy matrix* (1.17) *of the difference equation converges to the monodromy of the corresponding solutions $\widehat{\Psi}_k$ along a closed path from $z = -i\infty$ going around the pole z_k.*

(C) *The upper- and lower-triangular matrices (g_r, g_l) and (S_0, S_∞) defined in* (2.70) *and* (2.75) *for the cases of real and imaginary exponents, respectively, converge to the Stokes matrices of the differential equation* (1.22).

The first assertion of the theorem follows from the fact that the singular integral equation for solutions of the Riemann–Hilbert factorization problem passes in the continuous limit to the differential equation (1.22). One can readily see that

$$k_h(z,\xi) = \begin{cases} 1 + O(h), & z - \xi > h\log h,\ \xi > h\log h, \\ O(h), & \xi - z > h\log h \text{ or } \xi < h\log h, \end{cases} \qquad z > h\log h. \tag{4.5}$$

Similar relations hold for $z < h\log h$. In both cases we have

$$I_\varphi(z) = \int_0^z \varphi(\xi)\, d\xi + O(h). \tag{4.6}$$

It follows from (4.4) and (4.6) that the function $\psi = 1 + I_\varphi$ satisfies the relation

$$\frac{d\psi}{dz} = A(z)\psi(z) + O(h). \tag{4.7}$$

On the line L_x: $\operatorname{Re} z = x$ the function Φ_x is equal to $\psi + 0(h)$. Hence, the function Φ_x converges to $\widehat{\Psi}_k$ on L_x. The convergence is uniform on $\mathcal{D}_k$ in the case of mild equations with real exponentials. In the Birkhoff case the convergence becomes uniform only for the special choice of the constant term g in the integral representation for Φ_x, which was chosen to be $g = 1$ in (4.2) (cf. (2.76)).

The second and third assertions of the theorem are direct corollaries of (A) and of the definitions of the local monodromy matrices μ_k and the matrices (g_r, g_l) and (S_0, S_∞).

§ 5. Difference equations on elliptic curves

In this section we construct direct and inverse monodromy maps for difference equations on an elliptic curve.

Let us consider the equation

$$\Psi(z+h)=A(z)\Psi(z), \tag{5.1}$$

where $A(z)$ is a meromorphic $r\times r$ matrix function with simple poles that satisfies the following monodromy properties:

$$A(z+2\omega_\alpha)=B_\alpha A(z)B_\alpha^{-1}, \qquad B_\alpha\in SL_r. \tag{5.2}$$

The matrix $A(z)$ can be regarded as a meromorphic section of the vector bundle $\operatorname{Hom}(\mathcal{V},\mathcal{V})$ over an elliptic curve Γ with periods $(2\omega_1,2\omega_2)$ satisfying the condition $\operatorname{Im}(\omega_2/\omega_1)>0$. Here $\mathcal{V}$ stands for a holomorphic vector bundle over Γ determined by a pair of commuting matrices B_α. We assume that the matrices B_α are diagonalizable. The equation (5.1) is invariant under the transformations $A'=GAG^{-1}$. Therefore, if the matrices B_α are diagonalizable, then we can assume without loss of generality that they are diagonal. Moreover, if G is a diagonal matrix, then the equation (5.1) is also invariant under the transformations

$$\Psi'=G^z\Psi, \quad A'=G^{z+h}A(z)G^{-z}, \quad G^{ij}=G^i\delta^{ij}. \tag{5.3}$$

The matrix A' has the following monodromy properties:

$$A'(z+2\omega_\alpha)=B'_\alpha A'(z)(B'_\alpha)^{-1}, \qquad B'_\alpha=G^{2\omega_\alpha}B_\alpha. \tag{5.4}$$

Therefore, if the matrices B_α are diagonalizable, then we can assume without loss of generality that

$$B_1^{lj}=\delta^{lj}, \qquad B_2^{lj}=e^{\pi i q_j/\omega_1}\delta^{ij}. \tag{5.5}$$

Below we assume that $q_i\ne q_j$. The entries of the matrix A can be expressed in terms of a standard Jacobi theta function, namely, $\theta_3(z)=\theta_3(z|\tau)$, where $\tau=\omega_2/\omega_1$. Let the function $\widetilde{\theta}$ be defined by the formula

$$\widetilde{\theta}(z)=\widetilde{\theta}(z|2\omega_1,2\omega_2)=\theta_3(z/2\omega_1|\omega_2/\omega_1). \tag{5.6}$$

The monodromy properties of θ_3 imply that

$$\widetilde{\theta}(z+2\omega_1)=\widetilde{\theta}(z), \qquad \widetilde{\theta}(z+2\omega_2)=-\widetilde{\theta}(z)e^{-\pi i z/\omega_1}. \tag{5.7}$$

The function $\widetilde{\theta}$ is an odd function: $\widetilde{\theta}(z)=-\widetilde{\theta}(-z)$. It follows from (5.7) that the entries of the matrix A satisfying (5.2) and (5.5) can be uniquely represented in the form

$$\begin{aligned} A^{ii}&=\rho_i+\sum_{m=1}^{n}A_m^i\widetilde{\zeta}(z-z_m), \quad \sum_m A_m^i=0, \\ A^{ij}&=\sum_{m=1}^{n}A_m^{ij}\frac{\widetilde{\theta}(z-q_i+q_j-z_m)}{\widetilde{\theta}(z-z_m)}, \quad i\ne j, \end{aligned} \tag{5.8}$$

where $\widetilde{\zeta} = \partial_z(\log \widetilde{\theta})$ and $z_m \in \mathbb{C}$ are the poles of $A(z)$ in the fundamental parallelogram of the quotient $\mathbb{C}/\Lambda$, $\Lambda = \{2n\omega_1, 2m\omega_2\}$, that is,

$$0 < r(z_m) < 1, \qquad 0 < u(z_m) < 1. \tag{5.9}$$

Here and below we use the notation $r(z)$ and $u(z)$ for the real coordinates of a point $z \in \mathbb{C}$ with respect to the basis $2\omega_\alpha$: $z = 2r\omega_1 + 2u\omega_2$, that is,

$$r(z) = \frac{z\bar{\omega}_2 - \bar{z}\omega_2}{2(\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2)}, \qquad u(z) = \frac{z\bar{\omega}_1 - \bar{z}\omega_1}{2(\omega_2\bar{\omega}_1 - \bar{\omega}_2\omega_1)}. \tag{5.10}$$

Throughout the section it is assumed that the poles of A are not congruent $(\operatorname{mod} h)$. In particular, $h^{-1}(z_m - z_k) \notin \mathbb{Z}$.

Our next objective is to construct canonical meromorphic solutions of the equation (5.1) with coefficients of the form (5.8). As above, this problem reduces to a suitable Riemann–Hilbert factorization problem. To be definite, we assume that the step h of the difference equation satisfies the condition

$$0 < r(h) < 1. \tag{5.11}$$

Let us fix a real number x and consider the following problem in the strip $z \in \Pi_x$: $x \leqslant r(z) \leqslant x + r(h)$.

Problem IV. *Find a continuous matrix function $\Phi(z)$ on Π_x holomorphic interior to Π_x for which the boundary values on the two sides of the strip satisfy the equation*

$$\Phi^+(\xi + h) = A(\xi)\Phi^-(\xi), \qquad r(\xi) = x. \tag{5.12}$$

The index of the problem is given by

$$\operatorname{ind}_x(A) = \int_{L_x} d\log\det A, \qquad \xi \in L_x\colon \quad r(\xi) = x. \tag{5.13}$$

Lemma 5.1. *For a generic matrix $A(z)$ such that $\operatorname{ind}_x(A) = 0$ there is a non-singular holomorphic solution Φ_x of* (5.12) *with the following monodromy properties*:

$$\Phi_x(z + 2\omega_2) = e^{\pi i\hat{q}/\omega_1}\Phi_x(z)e^{-2\pi i\hat{s}}, \tag{5.14}$$

where $\hat{q}$ is the diagonal matrix determining the monodromy properties (5.2), (5.5) *of the matrix A and $\hat{s}$ is a diagonal matrix, $\hat{s}^{ij} = s^i\delta^{ij}$. The solution Φ_x is unique up to a transformation of the form $\Phi'_x = \Phi_x F$, where F is diagonal.*

Proof. The lemma can readily be proved by using methods of algebraic geometry. Indeed, consider the following action of the lattice Λ_h generated by h and $2\omega_2$ on the linear space $(z, v) \in \mathbb{C} \times \mathbb{C}^r$:

$$(z, f) \to (z + h, A(z)f), \qquad (z, f) \to (z + 2\omega_2, B_2 f), \qquad B_2 = e^{\pi i\hat{q}/\omega_1}. \tag{5.15}$$

In this case the quotient space $\mathbb{C} \times \mathbb{C}^r/\Lambda_h$ is a vector bundle $\mathcal{V}$ on the elliptic curve Γ_h with periods $(h, 2\omega_2)$. It follows from (5.13) that the determinant bundle of $\mathcal{V}$ is of degree zero, $c_1(\mathcal{V}) = 0$. According to [18], any generic degree-zero vector bundle on an algebraic curve admits a flat holomorphic connection. Any basis of horizontal sections of this connection determines a holomorphic matrix function Φ' satisfying the relations $\Phi'(z+h) = A(z)\Phi'(z)V_1$, $\Phi'(z+2\omega_2) = B_2\Phi'(z)V_2$, where V_1, V_2 is a pair of commuting matrices. A change of the basis of horizontal sections corresponds to a transformation of the form $\Phi' \to \Phi g$, $V_i \to g^{-1}V_i g$. Therefore, in general position if the matrices V_i are diagonalizable, then we can assume without loss of generality that they are diagonal. A holomorphic solution of the problem (5.12) is given by the formula $\Phi_x = \Phi' V_1^{-z/h}$. It satisfies the relations (5.14), where $e^{-2\pi i\hat{s}/h} = V_2 V_1^{-2\omega_2/h}$. We refer to Φ_x as the *Bloch* solution of the factorization problem (5.12). We shall assume that $s_i \neq s_j$ in general position.

Suppose that there are two Bloch solutions Φ_x and Φ'_x of (5.12). It follows from the condition $\operatorname{ind}_x A = 0$ that Φ_x is non-singular on Π_x. Hence, the entries of the matrix function $F = \Phi_x^{-1}\Phi'_x$ are holomorphic matrix functions satisfying the relations

$$F^{lj}(z+h) = F(z), \qquad F^{lj}(z+2\omega_2) = F^{lj}(z)e^{2\pi i(s_l - s'_j)/h}. \tag{5.16}$$

Let us show that the equations (5.16) imply that $s_i = s'_i$ and $F^{ij} = 0$ for $i \neq j$ (we recall that $s_i \neq s_j$ by assumption). Indeed, consider the function

$$\widehat{F}^{ij} = F^{ij}\widetilde{\theta}_h(z + s_i - s'_j)/\widetilde{\theta}_h(z), \tag{5.17}$$

where $\widetilde{\theta}_h$ stands for the function defined by (5.7) for Γ_h, that is,

$$\widetilde{\theta}_h(z) = \widetilde{\theta}(z|h, 2\omega_2). \tag{5.18}$$

It follows from (5.16) that $\widehat{F}^{ij}$ is an elliptic function on Γ_h with a single simple pole at $z = 0$. There is no non-trivial function of this kind. Hence, $s_i = s'_i$ and $F^{ij} = 0$, $i \neq j$. This proves the lemma.

We are now ready to define the direct monodromy map for difference equations (5.1) with coefficients A of the form (5.8). As above, a holomorphic solution Φ_x of the factorization problem (5.12) determines a meromorphic solution $\Psi_x(z)$ of the equation (5.1). By (5.14), this solution satisfies the Bloch relation (1.25).

The matrix A has period $2\omega_1$. Thus,

$$\Phi_{x+1}(z + 2\omega_1) = \Phi_x(z), \qquad z \in \Pi_x. \tag{5.19}$$

Hence, $\Psi_x(z - 2\omega_1)$ is a Bloch solution of the equation (5.1), and this solution is analytic on the strip Π_{x+1}. We consider the connection matrix

$$S_x(z) = \Psi_x^{-1}(z - 2\omega_1)\Psi_x(z) \tag{5.20}$$

of two Bloch solutions.

For obvious reasons, the matrix S_x is h-periodic. It follows from the relation (1.25) that it also has the monodromy property

$$S(z+h) = S(z), \qquad S(z+2\omega_2) = e^{2\pi i\hat{s}/h}S(z)e^{-2\pi i\hat{s}/h}, \tag{5.21}$$

where $\hat{s}$ is the diagonal matrix in (5.14).

By definition, S_x depends on x. Let us fix x, by setting it to be zero for instance, and denote $S_{x=0}(z)$ by $S(z)$.

Theorem 5.1. *In general position the entries of the monodromy matrix $S(z)$ are of the form*

$$
\begin{aligned}
S^{ii} &= S_0^i + \sum_{m=1}^{n} S_m^i \zeta_h(z-z_m), \qquad \sum_{m=1}^{n} S_m^i = 0, \\
S^{ij} &= \sum_{m=1}^{n} S_m^{ij} \frac{\widetilde{\theta}_h(z-s_i+s_j-z_m)}{\widetilde{\theta}_h(z-z_m)}, \quad i \neq j,
\end{aligned}
\tag{5.22}
$$

where $\zeta_h = \partial_z(\log \widetilde{\theta}_h)$ and $\widetilde{\theta}_h$ is given by the formula (5.18).

We recall that the z_m are the poles of $A(z)$ in the fundamental domain of $\mathbb{C}/\Lambda$.

Proof. In the half-plane $r(z) > 0$ the function $\Psi_{x=0}$ has poles at the points $z_m + nh + 2m\omega_2$, $n = 1, 2, \dots$, $m \in \mathbb{Z}$. By definition, the function $\Psi_{x=1}$ is holomorphic on the strip $\Pi_{x=1}$. Therefore, the poles of the matrix S in the strip Π_1 are points congruent to z_m modulo the lattice Λ_h. Using (5.21), we obtain (5.22).

The correspondence constructed above,

$$
\{\rho_i, A_m^{ij}, q_i\} \mapsto \{S_0^i, S_m^{ij}, s_i\}
\tag{5.23}
$$

is called the *direct monodromy map.*

5.1. Local monodromies. The results proved above for difference equations with rational coefficients can be extended with minor technical modifications to the case of equations with elliptic coefficients. For example, an analogue of special regular singular equations with rational coefficients is given by the equations (5.1) with coefficients $A(z)$ whose residues A_m are rank-one matrices, whose determinant is identically equal to 1, $\det A(z) = 1$, and whose parameters q_i in (5.8) satisfy the relation

$$
\sum_{i=1}^{r} q_i = 0.
\tag{5.24}
$$

We denote by $\mathcal{A}_0(\Gamma)$ the set of all matrices of this kind. The dimension of $\mathcal{A}_0(\Gamma)$ is $\dim \mathcal{A}_0(\Gamma) = n(2r-1) - n + (r-1) = (2n+1)(r-1)$. The first term in this sum is equal to the dimension of the subspace of matrices of the form (5.8) that have rank-one residues. The second term in the sum is equal to the number of conditions equivalent to the equality $\det A = 1$. The last term is equal to the number of parameters q_i. Let us consider the quotient space $\mathcal{B}(\Gamma) = \mathcal{A}_0(\Gamma)/\mathbb{C}^{r-1}$ of the space $\mathcal{A}_0(\Gamma)$ by the action $A \to gAg^{-1}$ of the diagonal matrices g. The dimension of $\mathcal{B}(\Gamma)$ is $\dim \mathcal{B}(\Gamma) = 2n(r-1)$. An explicit parameterization of an open set of the space $\mathcal{B}(\Gamma)$ can be given as follows. We order the poles and consider the matrices $A(z)$ of the form

$$
A(z) = L_n(z) L_{n-1}(z) \cdots L_1(z),
\tag{5.25}
$$

where

$$
L_m^{ij} = f_m^i \frac{\widetilde{\theta}(z - q_{i,m+1} + q_{j,m} - z_m)}{\widetilde{\theta}(z-z_m)\widetilde{\theta}(q_{i,m+1} - q_{j,m})}
\tag{5.26}
$$

and the $q_{i,m}$ are complex numbers satisfying the condition (5.24) and the condition $q_{i,n+1} = q_{i,1}$.

The residue of L_m at z_m is of rank one. Hence, its determinant has at most a simple pole at z_m. It follows from the relation (5.24) for $q_{i,m}$ that the determinant $\det L_m$ is an elliptic function. Therefore, it is identically equal to some constant. The vector f_m can be normalized by the condition $\det L_m(z) = \det L(0) = 1$,

$$\prod_{i=1}^{r} f_i^{-1} = \det\left[\frac{\widetilde{\theta}(z_m + q_{i,m+1} - q_{j,m})}{\widetilde{\theta}(z_m)\widetilde{\theta}(q_{i,m+1} - q_{j,m})}\right]. \tag{5.27}$$

The number of parameters $(f_{i,m}, q_{i,m})$ in (5.25) satisfying the conditions (5.24) and (5.27) is equal to the dimension of $\mathcal{B}(\Gamma)$.

Let us assume that the first coordinates $r_m = r(z_m)$ of the poles of A in the basis $2\omega_\alpha$ are distinct, $r_l < r_m$ for $l < m$. For brevity, we use the notation $r_0 = 0$ and $r_{n+1} = 1$ below.

Theorem 5.2. *For a generic matrix $A \in \mathcal{A}_0(\Gamma)$ the equation* (5.1) *has a unique set of meromorphic solutions Ψ_k, $k = 0, 1, \dots, n$, that are holomorphic in the respective strips $r_k < r(z) < r_{k+1} + h$ and have the monodromy property*

$$\Psi_k(z + 2\omega_2) = e^{\pi i\hat{q}/\omega_1}\Psi_k(z)e^{-2\pi i\hat{s}_k/h}, \qquad \hat{s}_k^{ij} = s_{i,k}\delta^{ij}, \tag{5.28}$$

and for which the local connection matrices $M_k = \Psi_k^{-1}\Psi_{k-1}$, $k = 1, \dots, n$, are of the form

$$M_k = \alpha_{i,k}\frac{\widetilde{\theta}_h(z - s_{i,k} + s_{j,k-1} - z_k)}{\widetilde{\theta}_h(z - z_k)\widetilde{\theta}_h(s_{i,k} - s_{j,k-1})}, \tag{5.29}$$

where $s_{i,k}$ and $\alpha_{i,k}$ satisfy the relations

$$\sum_{i=1}^{r} s_{i,k} = 0, \qquad \prod_{i=1}^{r} \alpha_{i,k}^{-1} = \det\left[\frac{\widetilde{\theta}_h(z_k + s_{i,k} - s_{j,k-1})}{\widetilde{\theta}_h(z_k)\widetilde{\theta}_h(q_{i,k} - q_{j,k-1})}\right]. \tag{5.30}$$

The map $\{f_m^i, q_{i,m}\} \mapsto \{\alpha_k^i, s_{i,k}\}$ is a one-to-one correspondence of open subsets of the varieties given by the equations (5.24), (5.27) *and the equations* (5.30), *respectively.*

Proof. The existence of a meromorphic solution Ψ_k' which is holomorphic in the strip $r_k < r(z) < r_{k+1} + h$ and satisfies the relation (5.28) follows from Lemma 5.1. The matrix $M_k' = (\Psi_k')^{-1}\Psi_{k-1}'$ is h-periodic, that is, $M_k'(z+h) = M_k'(z)$. It follows from (5.28) that

$$M_k'(z + 2\omega_2) = e^{2\pi i\hat{s}_k/h}M_k'(z)e^{-2\pi i\hat{s}_{k-1}/h}.$$

In the strip Π_{r_k+h} this matrix has a simple pole at the point z_k, and the residue at this point is of rank one. Hence, *a priori* this matrix can be represented in the form

$$M_k' = \widetilde{\alpha}_{i,k}\beta_{j,k}\frac{\widetilde{\theta}_h(z - s_{i,k} + s_{j,k-1} - z_k)}{\widetilde{\theta}_h(z - z_k)\widetilde{\theta}_h(s_{i,k} - s_{j,k-1})}. \tag{5.31}$$

The solutions Ψ'_k are unique up to a transformation of the form $\Psi'_k = \Psi_k F_k$, where F_k is a diagonal matrix, $F_k^i \delta^{ij}$. If we set $F_{k-1}^j = \beta_{j,k}$, then the corresponding matrix $M_k = F_k^{-1} M'_k F_{k-1}$ has the form (5.29). The condition (5.30) is equivalent to the condition $\det M_k = 1$.

The proof of the last assertion of the theorem is standard in the framework of the present paper and reduces to a Riemann–Hilbert problem on the set of lines $r_1(z) = r_{1,m} + \varepsilon$. The solubility of this problem for a generic set of matrices M_k follows from the Riemann–Roch theorem.

Remark. An elliptic analogue of our unitary equations can be defined in the case of real elliptic curves. A generalization of the corresponding results obtained above for the rational case is straightforward.

5.2. Isomonodromy transformations. The characterization of the equations (5.1) on Γ having the same monodromy data is a straightforward generalization of the corresponding results in the rational case.

It follows from (5.2) that the determinant of $A \in \mathcal{A}(\Gamma)$ is an elliptic function,

$$\det A(z) = D(z) = c \frac{\prod_{\alpha=1}^{N} \widetilde{\theta}(z - \zeta_\alpha)}{\prod_{k=1}^{n} \widetilde{\theta}(z - z_k)^{h_k}}, \quad \sum_{\alpha=1}^{N} \zeta_\alpha = \sum_{k=1}^{n} h_k z_k, \quad N = \sum_j \nu_j. \tag{5.32}$$

The subspace of matrix functions having a fixed determinant $D(z)$ is denoted by $\mathcal{A}_D(\Gamma) \subset \mathcal{A}(\Gamma)$.

Lemma 5.2. (i) *Matrix functions $A(z)$ and $A'(z)$ of the form* (5.8) *correspond under the map* (5.23) *to the same connection matrix $S(z)$ if and only if they are related by the formula*

$$A'(z) = R(z+1) A(z) R^{-1}(z), \tag{5.33}$$

where the matrix R has the following monodromy properties:

$$R(z + 2\omega_1) = R(z), \qquad R(z + 2\omega_2) = e^{\pi i \hat{q}'/\omega_1} R(z)\, e^{-\pi i \hat{q}/\omega_1}. \tag{5.34}$$

(ii) *If the zeros ζ_α are not congruent, that is, if $(\zeta_\alpha - \zeta_\beta)h^{-1} \notin \mathbb{Z}$, then the monodromy correspondence* (5.23) *restricted to $\mathcal{A}_D(\Gamma)$ is injective.*

The proof of the lemma follows directly from the definition of $S(z)$ and from the monodromy properties of Bloch solutions of difference equations.

Elliptic functions D and D' are said to be *equivalent* if their poles z_i, z'_i and zeros $\zeta_\alpha, \zeta'_\alpha$ are pairwise congruent mod h, that is, if $(z_i - z'_i)h^{-1} \in \mathbb{Z}$ and $(\zeta_\alpha - \zeta'_\alpha)h^{-1} \in \mathbb{Z}$.

Theorem 5.3. *For each pair of equivalent elliptic functions D and D' there is a unique isomonodromy transformation*

$$T_D^{D'}(\Gamma) \colon \mathcal{A}_D(\Gamma) \mapsto \mathcal{A}_{D'}(\Gamma). \tag{5.35}$$

Proof. We consider a matrix $A(z) \in \mathcal{A}_D$ of the form (5.8). An elementary isomonodromy transformation of the first type is determined by a pair z_m, ζ_α and a left eigenvector v of $A_m = \operatorname{res}_{z_m} A$ corresponding to a non-zero eigenvalue λ (see (3.7)).

Let us consider a matrix $R(z)$ such that the entries of the inverse matrix are of the form

$$(R^{-1})^{ij} = p^i \frac{\widetilde{\theta}(z - q_i + q'_j - \zeta_\alpha)}{\widetilde{\theta}(z - \zeta_\alpha)}\,, \tag{5.36}$$

where the p^i are the coordinates of a null-vector of $A(\zeta_\alpha)$,

$$A(\zeta_\alpha)p = 0. \tag{5.37}$$

The residue of R^{-1} at ζ_α is of rank one. Therefore, the determinant of R^{-1} has a simple pole at ζ_α. If the parameters q'_i satisfy the condition

$$\sum_{i=1}^{r} q'_i = \zeta_\alpha - z_m + \sum_{i=1}^{m} q_i, \tag{5.38}$$

then $\det R^{-1}$ has a simple zero at z_m. In general position the parameters q'_j are uniquely determined by the equality (5.38) and by the equation

$$vR^{-1}(z_m) = 0. \tag{5.39}$$

It follows from (5.39) that R has the form

$$R^{ij} = v^j \frac{\widetilde{\theta}(z - q'_i + q_j - z_m)}{\widetilde{\theta}(z - z_m)}\,. \tag{5.40}$$

Let us now consider the matrix A' given by the formula (5.33). It follows from (5.37) that it is regular at ζ_α. The matrix A' has a first-order pole at $z_m - 1$. The rank of the residue of A' at z_m is equal to that of the matrix $A_m R^{-1}(z_m)$. The left null-space of this matrix contains both the null-space of A_m and the vector v. Hence, the residue of A' is of rank $h_m - 1$. As in the rational case, further iterations enable one to obtain a matrix $T_i^{\alpha_1,\ldots,\alpha_{h_i}}(A)$ that is regular at z_m and has a pole of order h_m at $z_m - h$.

It follows from Lemma 5.3 that the isomonodromy transformation $T_m^{\alpha_1,\ldots,\alpha_{h_m}}$ is uniquely determined by the choice of a pole z_m and a subset of h_m zeros ζ_{α_s} of the function D.

An elementary isomonodromy transformation of the second type is determined by a pair of zeros ζ_α and ζ_β of the function D. Let v_α and v_β be corresponding null-vectors, that is,

$$A(\zeta_\alpha)v_\alpha = 0, \qquad v_\beta^T A(\zeta_\beta) = 0. \tag{5.41}$$

Then, using the same arguments as above, we can see that there is a matrix function $R = R_{\alpha,\beta}$ of the form

$$R^{ij}_{\alpha,\beta} = v^j_\beta \frac{\widetilde{\theta}(z - q_i^{\alpha,\beta} + q_j - \zeta_\beta - h)}{\widetilde{\theta}(z - \zeta_\beta - h)} \tag{5.42}$$

such that

$$(R^{-1}_{\alpha,\beta})^{ij} = v^i_\alpha \frac{\widetilde{\theta}_1(z - q_i + q_j^{\alpha,\beta} - \zeta_\alpha)}{\widetilde{\theta}(z - \zeta_\alpha)}\,, \tag{5.43}$$

and this matrix function is unique up to a constant factor. It follows from the equations (5.41) that the matrix function

$$T^{\alpha|\beta}(A) = R^{-1}_{\alpha,\beta}(z+h)A(z)R^{-1}_{\alpha,\beta}(z)$$

is regular and invertible at the points ζ_α and ζ_β. Its poles coincide with the poles of A. The zeros of the determinant of this function are the points $\zeta_\alpha - h$, $\zeta_\beta + h$, and ζ_γ, $\gamma \neq \alpha, \beta$.

Every isomonodromy transformation $T_D^{D'}(\Gamma)$ can be obtained as a composition of elementary isomonodromy transformations. This completes the proof of the theorem.

Isomonodromy deformations changing elliptic curves. The isomonodromy transformations of the form $T_D^{D'}(\Gamma)$ are analogues of the isomonodromy transformations constructed in § 3 for difference equations with rational coefficients. In the elliptic case there is another type of isomonodromy transformations which has no analogue in the rational case for the following obvious reason: the corresponding transformations change the periods of the elliptic curves.

Our next goal is to define an elementary isomonodromy transformation of the third kind which preserves the poles of A and the zeros of its determinant.

Lemma 5.3. *For a generic matrix function $A(z)$ of the form* (5.8) *there is a meromorphic matrix function $\mathcal{R}(z)$ which is holomorphic in the strip Π_*: $0 < r(z) < 1 + r(h)$ and satisfies the conditions*

$$\mathcal{R}(z+2\omega_1+h)A(z) = \mathcal{R}(z), \qquad \mathcal{R}(z+2\omega_2) = e^{2\pi i\hat{q}'/(2\omega_1+h)}\mathcal{R}(z)e^{-\pi i\hat{q}/\omega_1}, \tag{5.44}$$

where $\hat{q}'$ is a diagonal matrix. The function $\mathcal{R}$ is unique up to multiplication by a diagonal matrix $F \in GL_r$, $\mathcal{R}' = F\mathcal{R}$.

A function $\mathcal{R}$ satisfying the relations (5.44) can be regarded as a canonical Bloch solution of the difference equation (5.44). The existence of a function of this kind for a generic matrix A follows from Lemma 5.1.

Let us now consider the matrix function $A' = \mathcal{R}(z+h)A(z)\mathcal{R}^{-1}(z)$. It follows from (5.44) that

$$A'(z+2\omega_1+h) = A'(z), \quad A'(z+2\omega_2) = e^{2\pi i\hat{q}'/(2\omega_1+h)}A'(z)e^{-2\pi i\hat{q}'/(2\omega_1+h)}. \tag{5.45}$$

Suppose that the matrix A is holomorphic and invertible in the strip $\Pi_{x=0}$. Then the poles of the matrix A' that belong to the fundamental parallelogram corresponding to the elliptic curve with periods $(2\omega_1+h, 2\omega_2)$ coincide with the corresponding poles z_m of A. The zeros of the determinant of A' in the same parallelogram coincide with the zeros ζ_α of the determinant $\det A$.

Remark. If the conditions $r(h) < r(z_m)$, $r(h) < r(\zeta_\alpha)$ fail to hold, then the extra pole or zero of the determinant of A' in the strip $\Pi_{x=1}$ is congruent $(\operatorname{mod} h)$ to a pole or zero of this determinant in Π_0.

Theorem 5.4. *If a matrix function A is regular and invertible on Π_0, then the transformation $A' = \mathcal{R}(z+h)A(z)\mathcal{R}^{-1}(z)$ is isomonodromic.*

To prove the theorem, it suffices to note that if the assumptions of the theorem hold, then the canonical Bloch solution Ψ_1 of (5.1) is holomorphic and invertible in the strip $\Pi_{1+r(h)}$. Therefore, the Bloch solutions of the equation (5.1) with coefficient A', which determine the connection matrix S', are given by

$$\Psi'_{x=0} = \mathcal{R}\Psi_0, \qquad \Psi'_{1+r(h)} = \mathcal{R}\Psi_1. \tag{5.46}$$

Hence, $S'(z) = S(z)$.

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