Algebraic versus Liouville integrability of the soliton systems

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This is an updated version of the talk given at ICMP 2003, Lisbon on the algebraic and Hamiltonian integrability of the soliton systems. New integrable models arising form the discrete zero-curvature equations with the spectral parameter on algebraic curves are discussed.

1. Introduction

Since the beginning of the 1970s the complete Liouville integrability has been considered as common characteristic features of all the soliton systems constructed in the framework of the inverse spectral transform method. Bi-Hamiltonian formalism, classical r-matrix approach or representation of these systems, as a result of the Hamiltonian reduction of "free" systems, are often considered as the defining starting points of the Hamiltonian theory of the soliton equations. Although each of these approaches has its own advantages, none of them is applicable to *all* known integrable systems.

More universal is the "definition" of the soliton equations as non-linear (ordinary or partial) differential equations that are equivalent to compatibility conditions of over-determined systems of linear equations. In this approach the direct and the inverse spectral transforms "solve" the non-linear equations with no use of the Hamiltonian theory. In particular, for finite-dimensional systems that admit the Lax representation $\partial_t L = [M, L]$, where $L(t,z)$ and $M(t,z)$ are *rational* matrix functions of the *spectral* parameter z, the algebro-geometric scheme based on the concept of the Baker-Akhiezer functions, gives an explicit solution of the equations in terms of the Riemann theta-functions. This scheme identifies the phase space of the systems with the Jacobian bundle over the space of the spectral curves. The Abel transform linearizes the motion along the fibers of the bundle. Although in this description the corresponding soliton system clearly exhibits dynamics of the completely integrable system, until the end of the 1990s there was no universal answer to the question: why the Lax equations are Hamiltonian.

The new approach to the Hamiltonian theory of the soliton equations developed in the works of D. H. Phong and the author [1, 2] is based on the discovery of some universal two-form defined on a space of meromorphic matrix-functions. Essentially there are two such universal forms. They can be traced back to the fact that there are two basic algebraic structures on a space of *operators.* The first one is the Lie algebra structure defined by the commutator of operators. The second one is the Lie group structure. In the framework of the r-matrix approach to the Hamiltonian theory of the integrable systems the first one usually is referred to as linear brackets and the second one as quadratic or Sklyanin brackets (see the book [3], survey [4] and references therein).

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A direct and simple corollary of the definition of the universal form is that its contraction by a vector-field defined by the Lax or the zero-curvature equations is an *exact* one-form. Therefore, whenever the form is non-degenerate the corresponding system is the Hamiltonian system.

In [5] this approach was extended to the case of the Lax and the zero-curvature equations with the spectral parameter on an algebraic genus $g > 0$ curve. The corresponding zero-curvature equations can be seen as a hierarchy of commuting flows on the space of admissible matrix-valued meromorphic functions $L(x, z)$, $z \in \Gamma$. The admissible meromorphic matrix-functions on a smooth genus g algebraic curve were identified with x-connections on x-parametric families $\mathcal{V}(x)$ of stable rank r and degree rg holomorphic vector bundles on Γ. In the stationary case, the factor-space $\mathcal{L}^{\mathcal{K}}/SL_r$ of x-independent connections $L(z)$ with the divisor of poles equivalent to the canonical divisor $\mathcal{K}$ is isomorphic to the phase space of the Hitchin system. The latter is the cotangent space $T^*(\mathcal{M})$ of the moduli space $\mathcal{M}$ of stable rank r and degree rg holomorphic vector-bundles on Γ. The non-stationary systems can be regarded as infinite-dimensional field analogs of the famous Hitchin system [7].

A discrete analog of x-parametric family of vector bundles is a sequence of vector bundles $\mathcal{V}_n \in \mathcal{M}$. The discrete analog of a meromorphic x-connection with the pole divisor D_+ is a chain $L_n(z)$ of meromorphic homomorphisms $L_n \in H^0(\mathrm{Hom}(\mathcal{V}_{n+1}, \mathcal{V}_n(D_+)))$. It is assumed that L_n is almost everywhere invertible and the inverse homomorphism has a fixed divisor of poles D_-, i.e. $L_n^{-1} \in H^0(\mathrm{Hom}(\mathcal{V}_n, \mathcal{V}_{n+1}(D_-)))$. In [6] it was shown that the space $\mathcal{L}_N(D_+, D_-)$ of periodic chains, considered modulo gauge transformations $L'_n = g_{n+1}L_n g_n$, $g_n \in GL_r$ is algebraically integrable. Namely, it was shown that an open set of the factor-space $\mathcal{L}_N(D_+, D_-)/GL_r^N$ is isomorphic to an open set of Jacobian bundle over the space $\mathcal{S}^{D_+,D_-}$ of the spectral curves $\hat{\Gamma}$. The spectral curves are defined by the characteristic equation for the monodromy operator $T = L_N L_{N-1} \cdots L_1$, which represents them as an r-sheet cover of the base curve $\pi : \hat{\Gamma} \longmapsto \Gamma$. The spectral transform identifies an open set of the *restricted* chains, corresponding to the sequences of bundles $\mathcal{V}_n$ with fixed determinant, with an open set of the bundle over the family of the spectral curves $\mathcal{S}^{D_+,D_-}$ with fibers $J_C(\hat{\Gamma})$, which are preimages in $J(\hat{\Gamma})$ of some point of $J(\Gamma)$. The spectral transform linearizes discrete analogs of the zero-curvature equations, which can be explicitly solved in terms of the Riemann theta-function of the spectral curve.

Although the algebraic integrability of the periodic chains on higher genus base curves goes almost identically to the genus zero case, their Hamiltonian theory exhibits new very unusual features. It turns out that in the case of rational and elliptic base curves the group version of the universal form ω restricted to the proper symplectic leaves $\mathcal{P}_* \subset \mathcal{L}_N(D_+, D_-) \,/\, GL_r^N$ is induced by the Sklyanin symplectic form on the space of the monodromy matrices. For $g > 1$ the form ω is degenerate on $\mathcal{P}_*$. That does not allow us to treat the corresponding systems within the framework of conventional Hamiltonian theory. At the same time the space $\mathcal{P}_*$ is equipped by g-parametric *family* of two-forms ω_{dz}, parameterized by the holomorphic differentials dz on Γ. For all of them the fibers $J_C(\hat{\Gamma})$ of the spectral bundle are maximum isotropic subspaces. For each of the flows defined by the Lax equations on $\mathcal{P}_*$, the contraction $i_{\partial_t}\omega_{dz}$ is an exact form δH^{dz}. Although each of the forms ω_{dz} is degenerate on $\mathcal{P}_*$, their family is *non-degenerate*.

In a certain sense the state-of-art described above is dual to that in the theory of bi-Hamiltonian systems. Usually the bi-Hamiltonian structure is defined on the Poisson manifold equipped by a family of compatible brackets. The vector-fields that are Hamiltonian with respect to one bracket are Hamiltonian with respect to the other ones, but correspond to different Hamiltonians. The drastic difference between the bi-Hamiltonian systems and the systems $\mathcal{P}_*$ of restricted chains is in the nature of the symplectic leaves. For the bi-Hamiltonian systems usually they are *globally* defined as levels of single-valued action-type variables. For $\mathcal{P}_*$ the form ω_{dz} becomes non-degenerate on levels of *multi-valued* angle-type functions.

It is worth to understand if there exists the general Hamiltonian-type setting, in which these characteristic features of $\mathcal{P}_*$ for $g > 1$ provide the basis for something that might be the notion of *super-integrable* systems. It is also possible, that there is no need for the new setting. The results presented in the last section provide some evidence that the Lax chains on the fixed base curve Γ might be "extended" to the conventional completely integrable Hamiltonian system. Namely, we show that for the rank $r = 2$ the space of the periodic Lax chains with *variable base curve* Γ is the Poisson manifold with leaves $\hat{\mathcal{P}}_\Delta$, corresponding to the chains (modulo gauge equivalence) with fixed determinant Δ of the monodromy matrix T, and with the fixed regular eigenvalue w of T at the punctures $P_k^\pm$. The universal form defines the structure of a completely integrable system on $\hat{\mathcal{P}}_\Delta$. The Hamiltonians of the Lax equations on $\hat{\mathcal{P}}_\Delta$ are in involution. They are given by the formula

$$H_f = \sum_{q=\hat{P}_k^\pm} \operatorname{res}_q(f \ln w)\, d\ln\Delta,$$

where f is a meromorphic function on Γ with poles at the punctures $P_k^\pm$. The common level of all the integrals H_f is identified with the Prim variety of the corresponding spectral curve.

2. The zero-genus case

Almost all $(1+1)$-soliton equations admit zero-curvature representation ([8])

$$\partial_t L - \partial_x M + [L, M] = 0, \tag{1}$$

where $L(x,t,z)$ and $M(x,t,z)$ are *rational* matrix functions of the *spectral* parameter z. The discrete analog of (1) is the equation

$$\partial_t L_n = M_{n+1} L_n - L_n M_n, \tag{2}$$

where, as before, $L_n = L_n(t,z)$ and $M_n = M_n(t,z)$ are rational functions of the spectral parameter. In both the cases poles of L and M are *fixed.* The singular parts of L and M at the poles are dynamical variables. Their number equals the number of equations equivalent to (1) or (2), respectively.

It is instructive to consider first the stationary solutions of (1) or (2), described by the conventional Lax equation

$$\partial_t L = [M, L]\,. \tag{3}$$

In this case the algebraic-geometrical and Hamiltonian integrability perfectly match each other. Their key ideas are as follows.

Let $\mathcal{L}(D)$ be a space of meromorphic $(r \times r)$ matrix functions

$$L(z) = u_0 + \sum_{m=1}^{n} \sum_{l=1}^{h_m} \frac{u_{ml}}{(z - z_m)^l} \tag{4}$$

with a fixed divisor of poles $D = \sum_{m=1}^{n} h_m z_m$. Let us assume for simplicity only that the matrix M has poles at points p_k distinct from the punctures z_m. Then (3) implies that the commutator $[M, L]$ is regular at p_k. The corresponding equations are algebraic and can be solved for M in terms of L. For example, if M has one simple pole, then it can be chosen in the form

$$M_{n,p}(z, L) = \frac{L^n(p)}{z - p}, \quad p \neq z_m. \tag{5}$$

If we identify the space $\mathcal{L}(D)$ with its own tangent space, then $[M_{n,p}, L]$ can be regarded as a tangent vector field $\partial_{n,p}$ to $\mathcal{L}(D)$, and the corresponding flow on $\mathcal{L}(D)$

$$\partial_{t_{n,p}} L = [M_{n,p}, L] \tag{6}$$

is a well-defined dynamical system. Standard arguments from the theory of solitons show that all these flows *commute* with each other.

The spectral transform. The Lax equation (3) implies that the spectral curve Γ, defined by the characteristic equation

$$R(k, z) = \det(k - L(t, z)) = k^r + \sum_{i=1}^{r} s_i(z) k^{r-i} = 0, \tag{7}$$

is *time-independent.* A space $\mathcal{S}(D)$ of the spectral curves is parameterized by the coefficients of the rational functions $s_i(z)$ having poles at the punctures z_m of orders ih_m. The dimension of $\mathcal{S}(D)$ equals $Nr(r+1)/2$, $N = \sum_m h_m = \deg D$.

By definition points $Q = (k, z) \in \Gamma$ of the spectral curve parameterize the eigenvectors of $L(z) \in \mathcal{L}(D)$,

$$L(z)\psi(Q) = k\psi(Q). \tag{8}$$

Let us normalize ψ by the constraint $\sum_{j=1}^{r} \psi^j = 1$. Then the coordinates of ψ are rational expressions of z and k. Therefore, they are meromorphic functions on Γ with common divisor $\hat{\gamma}$ of the poles. For generic case, when the spectral curve is smooth, the degree of this divisor equals $g + r - 1$, where $g = Nr(r-1)/2$ is the genus of Γ.

The spectral curve is invariant under the gauge transformation $L \to gLg^{-1}$, $g \in SL_r$. It turns out that the equivalence class $[\hat{\gamma}]$ of the pole divisor $\hat{g}$ which is a point the Jacobian variety $J(\Gamma)$ of Γ is also gauge invariant. Basic facts of the algebraic-geometrical integration theory (see [10] and references therein) are:

— *the spectral map $L \longmapsto \{\Gamma, \hat{\gamma}\}$ descends to bijective correspondence of generic points*

$$\mathcal{L}(D)/SL_r \leftrightarrow \{\Gamma, [\hat{\gamma}] \in J(\Gamma)\}; \tag{9}$$

— *the Lax equations (3), which are also gauge invariant, become* linear *flows on the Jacobian varieties and can be explicitly solved in terms of the Riemann theta-function.*

We stress once again that the construction of the commuting flows on $\mathcal{L}(D)$, and their linearization via the spectral transform, do not depend on a Hamiltonian structure.

Hamiltonian approach. In spite of the diversity of integrable models their Lax representation looks the same. The initial goal of [1, 2] was an attempt to find intrinsic patterns of the Hamiltonian theory in the Lax equation. It turned out that the simplest two-form on a space of "operators", which can be written in terms of the operator and its eigenfunctions only, has all the desired properties.

The case of Lax equations with the rational spectral parameter provides a transparent and instructive example of our approach. Let us define a two-form ω on $\mathcal{L}(D)$ by the formula

$$\omega = \frac{1}{2}\sum_a \operatorname{Res}_{z_a} \Omega\, dz, \quad \Omega = \operatorname{Tr}\left(\Psi^{-1}(z)\delta L(z) \wedge \delta\Psi(z)\right). \tag{10}$$

The sum is taken over the set of all the poles of L together with the pole of dz at $z_0 = \infty$, i.e., $z_a = \{z_0, z_1, \ldots, z_n\}$. We shall assume for simplicity that the normalization point z_0 does not coincide with any of the other punctures z_m. The case when z_0 coincides with one of the punctures can be treated with only slight technical modifications. The various components of the above formula are as follows. The entries of matrices u_0, u_{ml} can be viewed as coordinates on $\mathcal{L}(D)$. If we denote the exterior differentiation on $\mathcal{L}(D)$ by δ, then $\delta L(z)$ can be regarded as a matrix valued one-form on $\mathcal{L}(D)$

$$\delta L(z) = \delta u_0 + \sum_m \sum_{l=1}^{h_m} \frac{\delta u_{ml}}{(z - z_m)^l} \tag{11}$$

Let $\Psi(z)$ be the matrix whose columns are *normalized* eigenvectors of $L(z)$, i.e.

$$L(z)\Psi(z) = \Psi(z)K(z), \quad e_0\Psi = e_0, \tag{12}$$

where K is a diagonal matrix $K^{ij} = k_i\delta^{ij}$, and k_i are the eigenvalues of $L(z)$. The co-vector e_0 defining the normalization of the eigenvectors is $e_0 = (1, 1, \ldots, 1)$. The external differential $\delta\Psi$ of Ψ can be viewed as a one-form on $\mathcal{L}(D)$, and the formula (10) defines a two-form on $\mathcal{L}(D)$.

This form does depend on the choice of the normalization of the eigenvectors. A change of normalization vector e_0 leads to a transformation $\Psi(z) \to \Psi(z)' = \Psi(z)h(z)$, where $h(z)$ is a diagonal matrix. Under such transformation ω gets changed to

$$\omega' = \omega + \frac{1}{2}\sum_a \operatorname{Res}_{z_a} \operatorname{Tr}\left(\delta K \wedge \delta h h^{-1}\right) dz. \tag{13}$$

The last equation implies that the form ω is independent of the normalization on the subspaces of $\mathcal{L}(D)$, on which the one-form $\delta K\, dz$ is holomorphic at the punctures.

Let us fix a set of diagonal matrices $C = (C_0, C_m)$

$$C_0(z) = C_{0,0} + C_{0,1}z^{-1}, \quad C_m(z) = \sum_{l=1}^{h_m} C_{m,l}(z - z_m)^{-l}, \quad m > 0, \tag{14}$$

and define a subspace $\mathcal{L}_C = \mathcal{L}_C(D)$ of $\mathcal{L}(D)$ by the constraints

$$\text{(A) } K(z) = C_0(z) + O(z^{-2}), \quad z \to z_0, \qquad \text{(B) } K(z) = C_m(z) + O(1), \quad z \to z_m. \tag{15}$$

The number of independent constraints is $(N+2)r-1$ because $\operatorname{Tr} K(z) = \operatorname{Tr} L(z)$ is a meromorphic function of z. Thus $\dim \mathcal{L}_C = (\deg D)r(r-1) - 2r + r^2 + 1$. The restriction of δK to $\mathcal{L}_C$ is regular at the poles of L and has a zero of order 2 at z_0. Therefore, the form ω restricted to $\mathcal{L}_C$ is independent of the choice of the normalization of the eigenvectors.

The space $\mathcal{L}_C$ is invariant under the adjoint action $L \to gLg^{-1}$ of SL_r. Let

$$\mathcal{P}_C = \mathcal{L}_C / SL_r \tag{16}$$

be the quotient space. Its dimension equals $\dim \mathcal{P}_C = (\deg D)r(r-1) - 2r + 2$. Then we have

Theorem 2.1. (a) *The two-form ω defined by (10) restricted to $\mathcal{L}_C$ is gauge invariant and descends to a symplectic form on $\mathcal{P}$.* (b) *The Lax equation (6) is Hamiltonian with respect to ω. The Hamiltonian is*

$$H_{n,p} = -\frac{1}{(n+1)} \operatorname{Tr} L^{n+1}(p). \tag{17}$$

(c) *All the Hamiltonians $H_{n,p}$ are in involution with respect to ω.*

The proof of the theorem is very general and does not rely on any specific form of L. With slight technical modifications it is applicable for $(1+1)$ and $(2+1)$-soliton equation, as well.

It is necessary to mention that in all the cases, when the Hamiltonian theory of a corresponding integrable system has been known, the universal symplectic form coincides with the standard symplectic structure. For example, for the Lax equations with the rational parameter, it coincides with the symplectic structure which is a direct sum of the Kostant-Kirillov forms on the orbits of adjoint action on the singular part of L at the punctures z_m. One of the main advantages of the definition of the universal form ω is that it provides a straightforward way to construct the angle-action variables.

By definition the form ω equals to the sum of the residues of meromorphic form $\Omega\, dz$ at the punctures. The sum of all the residues of a meromorphic form equals zero. Therefore, ω equals with a negative sign to the sum of residues of Ωdz at the poles outside the punctures. This simple observation leads to the following result.

Theorem 2.2. *Let γ_s be the poles on the spectral curve of the normalized eigenvector ψ of the matrix function $L \in \mathcal{L}_C$. Then the two-form ω defined by (10) is equal to*

$$\omega = \sum_{s=1}^{\widehat{g}+r-1} \delta k(\gamma_s) \wedge \delta z(\gamma_s). \tag{18}$$

The meaning of the right hand side of this formula is as follows. The spectral curve is equipped by definition with the meromorphic function $k(Q)$. The evaluations $k(\gamma_s)$, $z(\gamma_s)$ at the points γ_s define functions on the space $\mathcal{L}(D)$, and the wedge product of their external differentials is a two-form on $\mathcal{L}_C$.

The universal symplectic form: logarithmic version. The basic symplectic form introduced above is related to the Lie algebra structure on the space of operators. We present now a construction of another symplectic structure, related to the Lie group structure, defined on suitable leaves in $\mathcal{L}(D)$.

Consider the open subspace of $\mathcal{L}(D)$ consisting of meromorphic matrix functions which are invertible at a generic point z, i.e. the subspace of matrices $L(z) \in \mathcal{L}(D)$ such that $L^{-1}(z)$ is also a meromorphic function. We define subspaces of $\mathcal{L}(D)$ with *fixed* divisor for the poles of $L^{-1}(z)$ as follows. Fix a set D_- of $(\deg D)r$ distinct points z_s^- and define a subspace $\mathcal{L}(D, D_-) \subset \mathcal{L}(D)$ by the constraints

$$L(z) \in \mathcal{L}(D, D_-) : \det L(z) = c\,\frac{\prod_{s=1}^{Nr}(z - z_s^-)}{\prod_{m=1}^{n}(z - z_m)^r}, \quad c = \text{const} \neq 0. \tag{19}$$

If $C_0(z)$ is the same as in (14), a subspace $\mathcal{L}_{C_0}^{gr} \subset \mathcal{L}(D, D_-)$ can be defined by the first set (A) of the constraints (15). The following two-form on $\mathcal{L}^{gr}$ is obviously a group version of (10)

$$\omega^{gr} = \frac{1}{2}\sum \operatorname{Res}_{z_a} \operatorname{Tr}\left(\Psi^{-1}L^{-1}(z)\,\delta L(z) \wedge \delta\Psi(z)\right)\,dz. \tag{20}$$

Here the sum is taken over all the punctures $z_a = \{z_0, z_m, z_s^-\}$. The subspace $\mathcal{L}_{C_0}^{gr}$ is invariant under the flows defined by the same Lax equations (6), which are also gauge invariant and therefore define flows on the quotient space $\mathcal{P}_{C_0}^{gr} = \mathcal{L}_{C_0}^{gr}/SL_r$.

Theorem 2.3. *The two-form ω^{gr} restricted to $\mathcal{L}_{C_0}^{gr}$ is independent on the normalization of the eigenvectors. It is gauge invariant and descends to a symplectic form on $\mathcal{P}_{C_0}^{gr}$. The Lax equation (6) is Hamiltonian with respect to ω^{gr}. The Hamiltonian is*

$$H_{n-1,p} = -\frac{1}{n}\operatorname{Tr} L^n(p). \tag{21}$$

All the Hamiltonians $H_{n,p}$ are in involution with respect to ω^{gr}.

The action-angle variables for the second symplectic form are group version of the action-angle variables for the first symplectic structure.

Theorem 2.4. *Let γ_s be the poles on the spectral curve of the normalized eigenvector ψ of the matrix function $L \in \mathcal{L}_{C_0}^{gr}$. Then the two-form ω^{gr} defined by (20) is equal to*

$$\omega^{gr} = \sum_{s=1}^{\widehat{g}+r-1} \delta \ln k(\gamma_s) \wedge \delta z(\gamma_s). \tag{22}$$

Theorems 2.1 and 2.3 provide a framework for the existence of so-called bi-Hamiltonian structures. It was first observed by Magri that the KdV hierarchy possesses a bi-Hamiltonian structure, in the sense that all the flows of the hierarchy are Hamiltonian with respect to two different symplectic structures. If H_n is the Hamiltonian generating the n-th flow of the KdV hierarchy with respect to the first Gardner-Zakharov-Faddeev symplectic form, then the same flow is generated by the Hamiltonian H_{n-1} with respect to the second Lenard-Magri symplectic form.

Periodic chains. The two symplectic structures ω and ω^{gr} are equally good in the case of a single matrix function $L(z)$, but there is a marked difference between them when periodic chains of operators are considered (see details in [13]). Let $L_n(z) \in \mathcal{L}(D)$ be a periodic chain of matrix-valued functions with a pole divisor D, $L_n = L_{n+N}$. The total space of such chains is $\mathcal{L}(D)^{\otimes N}$. The monodromy matrix

$$T_n(z) = L_{n+N-1}(z)\, L_{n+N-2}(z) \cdots L_n(z) \tag{23}$$

is a meromorphic matrix function with poles of order Nh_m at the puncture z_m, i.e. $T_n(z) \in \mathcal{L}(ND)$. For different n they are conjugated to each other. Thus the map

$$\mathcal{L}(D)^{\otimes N} \longmapsto \mathcal{L}(ND)/SL_r \tag{24}$$

is well-defined. However, the natural attempt to obtain a symplectic structure on the space $\mathcal{L}(D)^{\otimes N}$ by pulling back the first symplectic form ω on $\mathcal{L}(ND)$ runs immediately into obstacles. The main obstacle is that the form ω is only well-defined on the symplectic leaves of $\mathcal{L}(ND)$ consisting of matrices with fixed singular parts for the eigenvalues at the punctures. These constraints are *non-local*, and cannot be described in terms of constraints for each matrix $L_n(z)$ separately.

On the other hand, the second symplectic form ω^{gr} has essentially the desired local property. Indeed, let L_n be a chain of matrices such that $L_n \in \mathcal{L}(D, D_-)$. Then the monodromy matrix defines a map

$$\widehat{T} : \mathcal{L}(D, D_-)^{\otimes N} \longmapsto \mathcal{L}(ND, ND_-)/SL_r. \tag{25}$$

The group SL_r^N of z-independent matrices $g_n \in SL_r, g_n = g_{n+N}$ acts on $\mathcal{L}(D, D_-)^{\otimes N}$ by the gauge transformation

$$L_n \to g_{n+1} L_n g_n^{-1} \tag{26}$$

which is compatible with the monodromy matrix map (25). Let the space $\mathcal{P}_{\text{chain}}$ be defined as the corresponding quotient space of a preimage under $\widehat{T}$ of a symplectic leaf $\mathcal{L}_{C_0}^{gr} \subset \mathcal{L}(ND, ND_-)/SL_r$

$$\mathcal{P}_{\text{chain}} = \left(\widehat{T}^{-1}(\mathcal{P}_1^{C_0})\right) / SL_r^N. \tag{27}$$

The dimension of this space is equal to $\dim \mathcal{P}_{\text{chain}} = N(\deg D) r(r-1) - 2r + 2$.

Theorem 2.5. *The pull-back by $\widehat{T}$ of the second symplectic form $\omega_{\text{chain}} = \widehat{T}^*(\omega^{gr})$, restricted to $\widehat{T}^{-1}\left(\mathcal{L}_{C_0}^{gr}\right)$, is gauge invariant and descends to a symplectic form on $\mathcal{P}_{\text{chain}}$. It can also be expressed by the local expression*

$$\omega_{\text{chain}} = \frac{1}{2} \sum \operatorname{Res}_{z_a} \sum_{n=1}^{N} \operatorname{Tr}\left(\Psi_{n+1}^{-1} \delta L_n(z) \wedge \delta \Psi_n(z)\right) dz, \tag{28}$$

where $\Psi_{n+1} = L_n \Psi_n$, $\Psi_{n+N} = \Psi_n K$, $K^{ij} = \operatorname{diag}(k_i)\delta^{ij}$. All the coefficients of the characteristic polynomial of $T(z)$ are in involution with respect to this symplectic form. The number of independent integrals equals $\dim \mathcal{P}_{\text{chain}}/2$.

3. Periodic chains on algebraic curves

The Riemann-Roch theorem implies that naive generalization of equations (1, 2) for matrix functions, which are meromorphic on an algebraic curve Γ of genus $g > 0$, leads to an overdetermined system of equations. Indeed, the dimension of $r \times r$ matrix functions of fixed degree d divisor of poles in general position is $r^2(d-g+1)$. If the divisors of L and M have degrees n and m, then the commutator $[L, M]$ is of degree $n+m$. Thus the number of equations $r^2(n+m-g+1)$ exceeds the number $r^2(n+m-2g+1)$ of unknown functions modulo gauge equivalence (see details in [5]).

There are two ways to overcome this difficulty in defining zero curvature equations on algebraic curves. The first way is based on a choice of L with essential singularity at some point or with entries as sections of some bundle over the curve. The second way, based on a theory of *high rank* solutions of the Kadomtsev-Petviashvili equation, was discovered in [9]. There it was shown that if in addition to fixed poles the matrix functions L and M have rg moving poles of a special form, then the Lax equation is a well-defined system on the space of singular parts of L and M at fixed poles.

In [9] it was found, that if the matrix functions L and M have moving poles with special dependence on x and t besides fixed poles, then equation (1) is a well-defined system on the space of singular parts of L and M at fixed poles. A theory of the corresponding systems was developed in [5]. In is instructive to present its discrete analog, that a theory of the discrete curvature equations (2) with the spectral parameter an a smooth algebraic curve.

We begin by describing a suitable space of such functions L_n. Let Γ be a smooth genus g algebraic curve. According to [11], a generic stable, rank r and degree rg holomorphic vector bundle $\mathcal{V}$ on Γ is parameterized by a set of rg distinct points γ_s on Γ, and a set of r-dimensional vectors $\alpha_s = (\alpha_s^i)$, considered modulo scalar factor $\alpha_s \to \lambda_s \alpha_s$ and a common gauge transformation $\alpha_s^i \to g_j^i \alpha_s^j$, $g \in GL_r$, i.e. by a point of the factor-space

$$\mathcal{M}_0 = S^{rg}\left(\Gamma \times CP^{r-1}\right) / GL_r.$$

In [9, 12] the data $(\gamma, \alpha) = (\gamma_s, \alpha_s)$, $s = 1, \ldots, rg$, $i = 1, \ldots, r$, were called the Tyurin parameters.

Let $D_\pm$ be two effective divisors on Γ of the same degree $\mathcal{D}$. Below, if it is not stated otherwise, it is assumed that all the points of the divisors $D_\pm = \sum_{k=1}^{\mathcal{D}} P_k^\pm$ have multiplicity 1, $P_k^\pm \neq P_m^\pm, k \neq m$. For any sequence of the Tyurin parameters $(\gamma(n), \alpha(n))$ we introduce the space $\mathcal{L}_{\gamma(n),\alpha(n)}(D_+, D_-)$ of meromorphic matrix functions $L_n(q), q \in \Gamma$, such that:

1°. L_n is holomorphic except at the points γ_s, and at the points P_i^+ of D_+, where it has at most simple poles;

2°. the singular coefficient $L_s(n)$ of the Laurent expansion of L_n at γ_s

$$L_n(z) = \frac{L_s(n)}{z - z_s} + O(1), \quad z_s = z(\gamma_s), \tag{29}$$

is a rank 1 matrix of the form

$$L_s(n) = \beta_s(n)\alpha_s^T(n) \longleftrightarrow L_s^{ij}(n) = \beta_s^i(n)\alpha_s^j(n), \tag{30}$$

where $\beta_s(n)$ is a vector, and z is a local coordinate in the neighborhood of γ_s;

3°. the vector $\alpha_s^T(n+1)$ is a left null-vector of the evaluation of L_n at $\gamma_s(n+1)$, i.e.

$$\alpha_s(n+1)L_n(\gamma_s(n+1)) = 0; \tag{31}$$

4°. the determinant of $L_n(q)$ has simple poles at the points $P_k^+, \gamma_s(n)$, and simple zeros at the points $P_k^-, \gamma_s(n+1)$.

The last condition implies the following constraint for the equivalence classes of the divisors

$$[D_+] - [D_-] = \sum_s [\gamma_s(n+1) - \gamma_s(n)] \in J(\Gamma), \tag{32}$$

where $J(\Gamma)$ is the Jacobian of Γ. If $2N > g(r+1)$, then the Riemann-Roch theorem and simple counting of the number of the constraints (29)–(31) imply that the functional dimension of $\mathcal{L}_{\gamma(n),\alpha(n)}(D_+, D_-)$ (its dimension as the space of functions of the discrete variable n) equals $2\mathcal{D}(r-1) - gr^2 + g + r^2$.

The geometric interpretation of $\mathcal{L}_{\gamma(n),\alpha(n)}(D_+, D_-)$ is as follows. In the neighborhood of γ_s the space of local sections of the vector bundle $V_{\gamma,\alpha}$, corresponding to (γ, α), is the space $\mathcal{F}_s$ of meromorphic functions having a simple pole at γ_s of the form

$$f(z) = \frac{\lambda_s \alpha_s^T}{z - z(\gamma_s)} + O(1), \quad \lambda_s \in C. \tag{33}$$

Therefore, if $\mathcal{V}_n$ is a sequence of the vector bundles on Γ, corresponding to the sequence of the Tyurin parameters $(\gamma(n), \alpha(n))$, then the equivalence class $[L_n]$ of L_n modulo gauge transformation (26) can be seen as a homomorphism of the vector bundle $\mathcal{V}_{n+1}$ to the vector bundle $\mathcal{V}_n(D_+)$, obtained from $\mathcal{V}_n$ with the help of simple Hecke modification at the punctures P_k^+, i.e.

$$[L_n] \in \mathrm{Hom}(\mathcal{V}_{n+1}, \mathcal{V}_n(D_+)). \tag{34}$$

These homomorphisms are invertible almost everywhere. The inverse matrix-functions define the homomorphisms of the vector bundles

$$[L_n^{-1}] \in \mathrm{Hom}(\mathcal{V}_n, \mathcal{V}_{n+1}(D_-)). \tag{35}$$

The total space $\mathcal{L}_N(D_+, D_-)1$ of the chains, corresponding to all the sequences of the Tyurin parameters, is a bundle over the space of sequences of holomorphic vector bundles

$$\mathcal{L}(D_+, D_-) \longmapsto \{\mathcal{V}_n\}. \tag{36}$$

The fibers of this bundle are just the spaces $\mathcal{L}_{\gamma(n),\alpha(n)}(D_+, D_-)$.

Our next goal is to show algebraic integrability of the total space $\mathcal{L}_N(D_+, D_-)$ of the N-*periodic* chains, $L_n = L_{n+N}$ (see details in [6]). Equation (32) implies that the periodicity of chains requires the following constraint on the equivalence classes of the divisors $D_\pm$:

$$N([D_+] - [D_-]) = 0 \in J(\Gamma), \tag{37}$$

which will be always assumed below. The dimension of $\mathcal{L}_N(D_+, D_-)$ equals

$$\dim \mathcal{L}_N(D_+, D_-) = 2N\mathcal{D}(r-1) + Nr^2 + g. \tag{38}$$

Example. Consider the case of 1-periodic chains, i.e. the stationary case $L_n = L$. Let $D_+ = \mathcal{K}$ be the zero-divisor of a holomorphic differential dz, and let $\mathcal{L}^{\mathcal{K}}$ be a union of the spaces $\mathcal{L}_1(\mathcal{K}, D_-)$. Then the factor-space $\mathcal{L}^{\mathcal{K}}/SL_r$ is isomorphic to a phase space of the Hitchin system that is the cotangent bundle $T^*(\mathcal{M})$ to the moduli space of rank r stable vector ([5]).

Let $L_n \in \mathcal{L}_N(D_+, D_-)$ be a periodic chain. Then the Floque-Bloch solutions of the equation

$$\psi_{n+1} = L_n \psi_n \tag{39}$$

are solutions that are eigenfunctions for the monodromy operator

$$T_n \psi_n = \psi_{n+N} = w\psi_n, \quad T_n = L_{n+N-1} \cdots L_{n+1} L_n. \tag{40}$$

The monodromy matrix $T_n(q)$ belongs to the space of the Lax matrices introduced in [5], $T_n \in \mathcal{L}^{ND_+}$, $T_n^{-1} \in \mathcal{L}^{ND_-}$. The Floque-Bloch solutions are parameterized by the points $Q = (w, q)$, $q \in \Gamma$, of the spectral curve $\hat{\Gamma}$ defined by the characteristic equation

$$R(w, q) = \det (w \cdot 1 - T_{n_0}(q)) = w^r + \sum_{i=0}^{r-1} r_i(q) w^i = 0. \tag{41}$$

The coefficients $r_i(q)$ of the characteristic equation are meromorphic functions on Γ with the poles at the punctures P_k^+. Equation (7) defines an affine part of the spectral curve. Let us consider its compactification over the punctures P_k^+. As shown in [6], in the neighborhood of P_k^+ one of the roots of the characteristic equation has the form

$$w = (z - z(P_k^+))^{-N} \left(c_k^+ + O(z - z(P_k^+))\right). \tag{42}$$

The corresponding compactification point of $\hat{\Gamma}$ is smooth, and will be denoted by $\hat{P}_k^+$. In the general position all the other branches of $w(z)$ are regular at P_k^+. The coefficients $r_i(z)$ are the elementary symmetric polynomials of the branches of $w(z)$. Hence, all of them have poles at P_k^+ of order N. Note that the coefficient $r_0(z) = \det T_{n_0}$ has zero of order N at P_k^-.

The same arguments applied to L_n^{-1} show that over the puncture P_k^- there is one point of $\hat{\Gamma}$ denoted by $\hat{P}_k^-$ in the neighborhood of which w has zero of order N, i.e.,

$$w = \left(z - z\left(P_k^-\right)\right)^N \left(c_k^- + O(z - z(P_k^-))\right). \tag{43}$$

Let us fix a normalization of the Floque-Bloch solution by the condition that the sum of coordinates ψ_0^i of the vector ψ_0 equals 1, $\sum_{i=1}^r \psi_0^i = 1$. Then, the corresponding Floque-Bloch solution $\psi_n(Q)$ is well-defined for each point Q of $\hat{\Gamma}$.

Theorem 3.1. *The vector-function $\psi_n(Q)$ is a meromorphic vector-function on $\hat{\Gamma}$, such that: (i) outside the punctures $\hat{P}_k^{\pm}$ (which are the points of $\hat{\Gamma}$ situated on marked sheets over $P_k^{\pm}$) the divisor $\hat{\gamma}$ of its poles $\hat{\gamma}_\sigma$ is* n-independent*; (ii) at the punctures $\hat{P}_k^+$ and $\hat{P}_k^-$ the vector-function $\psi_n(Q)$ has poles and zeros of the order n, respectively; (iii) in the general position, when $\hat{\Gamma}$ is smooth, the number of these poles equals $\hat{g} + r - 1$, where*

$$\widehat{g} = N\mathcal{D}(r-1) + r(g-1) + 1 \tag{44}$$

is the genus of $\hat{\Gamma}$.

Let S^{D_+, D_-} be the space of the spectral curves, which can be seen as a space of the meromorphic functions $r_i(z)$ on Γ with poles of order N at the punctures P_k^+, and such that

r_0 has zeros of order N at the punctures P_k^-. The Riemann-Roch theorem implies that S^{D_+,D_-} is of dimension

$$\dim S^{D_+,D_-} = N\mathcal{D}(r-1) - (g-1)(r-1) + 1. \tag{45}$$

The characteristic equation (41) defines a map $\mathcal{L}_N(D_+, D_-) \to S^{D_+,D_-}$. Usual arguments show that this map on an open set is surjective. These arguments are based on solution of *the inverse spectral problem*, which reconstruct L_n, modulo gauge equivalence (26) from a generic set of spectral data: a smooth curve $\widehat{\Gamma}$ defined by $\{r_i\} \in S^{D_+,D_-}$, and a point of the Jacobian $J(\widehat{\Gamma})$.

Theorem 3.2. *The map described above $L_n \to \{\Gamma, \hat{\gamma}\}$ descends to a bijective correspondence of open sets*

$$\mathcal{L}_N(D_+, D_-) \,/\, GL_r^N \longmapsto \{\widehat{\Gamma} \in \mathcal{S}^{D_+,D_-},\ [\widehat{\gamma}] \in J(\Gamma)\}. \tag{46}$$

Restricted chains. Let us introduce subspaces $\mathcal{L}_{N,C,\Delta}^{D_+,D_-} \subset \mathcal{L}_N(D_+, D_-)$ of the Lax chains with *fixed* equivalence classes of the divisors of Tyurin parameters

$$[\gamma(n)] = C + n\,([D_+] - [D_-]) \in J(\Gamma), \tag{47}$$

and with *fixed* determinant $\det T = \Delta = r_0(q)$. The subspace of the corresponding spectral curves will be denoted by $\mathcal{S}_\Delta \in \mathcal{S}^{D_+,D_-}$. The points of $\mathcal{S}_\Delta$ are sets of functions $r_i(q)$, $i = 1, \ldots, r-1$, with the poles of order N at P_k^+. For the restricted chains the equivalence class $[\hat{\gamma}] \in J(\hat{\Gamma})$ of the poles of the Floque-Bloch solutions belongs to the abelian subvariety

$$J_C(\hat{\Gamma}) = \pi_*^{-1}\left(C + N(r-1)[D_+]/2\right), \quad \pi_* : J(\hat{\Gamma}) \longmapsto J(\Gamma). \tag{48}$$

Corollary 3.1. *The correspondence*

$$\mathcal{L}_{N,C,\Delta}^{D_+,D_-} \,/\, GL_r^N \leftrightarrow \{\hat{\Gamma} \in \mathcal{S}_\Delta, [\hat{\gamma}] \in J_C(\hat{\Gamma})\} \tag{49}$$

is one-to-one on the open sets.

Lax equations. In order to treat the zero-curvature equations (2) as a dynamical system on the space of chains, it is necessary to solve first a part of the equations and define M_n in terms of L_m. Unlike the stationary case considered above that can not be done, if M_n has fixed poles outside the punctures $P_k^\pm$. The singular parts of suitable matrix functions M_n at these points can be constructed locally in a way identical to the theory of discrete zero-curvature equations with the rational spectral parameter.

Note that $\det L_n$ has simple pole at P_k^+. Therefore, the residue of L_n is a matrix of rank 1, and can be written in the form $h_k(n)p_k^T(n)$, where $h_k(n)$ and $p_k(n)$ are r-dimensional vectors

Lemma 3.1. *Let $\tilde{L}_n$ be a formal series of the form*

$$\tilde{L}_n = h(n)p(n)^T\lambda^{-1} + \sum_{i=0}^{\infty} \chi_i(n)\lambda^i \tag{50}$$

where $h(n), p(n)$ are vectors and $\chi_i(n)$ are matrices. Then the equations

$$\phi_{n+1} = \tilde{L}_n \phi_n, \quad \phi^*_{n+1} \tilde{L}_n = \phi^*_n, \tag{51}$$

where ϕ_n and ϕ^*_n are r-dimensional vectors and co-vectors over the field of the Laurent series in the variable λ, have $(r-1)$-dimensional spaces of solutions of the form

$$\Phi_n = \sum_{i=0} \tilde{\xi}_i(n)\lambda^i, \quad \Phi^*_n = \sum_{i=0} \tilde{\xi}^*_i(n)\lambda^i. \tag{52}$$

The equations (51) have unique formal solutions of the form

$$\phi_n = \lambda^{-n}\left(\sum_{i=0}^{\infty} \xi_i(n)\lambda^i\right), \quad \phi^*_n = \lambda^n \left(\sum_{i=0}^{\infty} \xi^*_i(n)\lambda^i\right), \tag{53}$$

such that

$$(\Phi^*_n \phi_n) = (\phi^*_n \Phi_n) = 0, \quad (\phi^*_n \phi_n) = 1, \tag{54}$$

and normalized by the conditions

$$\chi_0(0) = q(-1), \quad \sum_{j=1}^{i} \chi_i^j(0) = 0, \quad i > 0. \tag{55}$$

For the proof of the lemma it is enough to substitute the formal series (52) or (53) in (51) and use recurrent relations for the coefficients of the Laurent series.

Let us fix a point P_0 on Γ and local coordinates in the neighborhoods of the punctures P_k^+. Then the Laurent expansion of L_n at the punctures defines with the help of the previous Lemma the formal series $\phi_n^{(k)}, \phi_n^{(*,k)}$. The same arguments, applied to the inverse give us formal solution $\phi_n^{(-k)}, \phi_n^{(*,-k)}$ in the neighborhoods of the points P_k^-. They have the form

$$\phi_n^- = \lambda^n \left(\sum_{i=0}^{\infty} \xi_i^-(n)\lambda^i\right), \quad \phi_n^{*,-} = \lambda^{-n}\left(\sum_{i=0}^{\infty} \xi_i^{*,-}(n)\lambda^i\right). \tag{56}$$

From the Riemann-Roch theorem (see details in [5]) it follows that there is a unique matrix function $M_n^{(\pm k,l)}$ such that:

(i) at the points $\gamma_s(n)$ it has simple poles of the form:

$$M_n = \frac{\mu_s(n)\alpha_s^T(n)}{z - z_s(n)} + O(1), \quad z_s(n) = z(\gamma_s(n)), \tag{57}$$

where $\mu_s(n)$ is a vector;

(ii) outside of the divisor γ it has pole at the point $P_k^\pm$, only, where

$$M_n^{(\pm k,l)} = \left(z - z(P_k^\pm)\right)^{-l} \phi_n^{(\pm k)} \phi_n^{(*,\pm k)} + O(1); \tag{58}$$

(iii) $M_n^{(\pm k,l)}$ is normalized by the condition $M_n^{(\pm k,l)}(P_0) = 0$.

Note, that although $\phi_n^{(\pm k)}$ and $\phi_n^{(*,\pm k)}$ are formal series, the constraint (58) involves only a finite number of their coefficients, and therefore, is well-defined.

Theorem 3.3.

(i) The equations

$$\partial_a L_n = M^a_{n+1} L_n - L_n M^a_n, \quad \partial_a = \partial/\partial t_a, \quad a = (\pm k, l), \tag{59}$$

define a hierarchy of commuting flows on an open set of $\mathcal{L}_N(D_+, D_-)$, which descends to the commuting hierarchy on an open set of $\mathcal{L}_N(D_+, D_-) \,/\, GL^N_r$.

(ii) The flows (59) are linearized by the spectral transform and can be explicitly solved in terms of the Riemann theta functions.

In general the flows (59) do not preserve the leaves of the foliation $\mathcal{L}^{D_+,D_-}_{N,C,\Delta} \subset \mathcal{L}_N(D_+, D_-)$. The linear combinations of basic flows which preserve the subspaces of the restricted chains are constructed as follows. Let f be a meromorphic function on Γ with poles only at the punctures $P^\pm_k$. Then we define

$$M^f_n = \sum_a c^f_a M^a_n, \tag{60}$$

where c^f_a are the coefficients of the singular part of the Laurent expansion

$$f = \sum_{l>0} c^f_{(\pm k,l)} \left(z - z(P^\pm_k)\right)^{-l} + O(1). \tag{61}$$

Theorem 3.4. *The equations*

$$\partial_f L_n = M^f_{n+1} L_n - L_n M^f_n, \quad \partial_f = \partial/\partial t_f, \tag{62}$$

define a hierarchy of commuting flows on an open set of $\mathcal{L}^{D_+,D_-}_{N,C,\Delta}$, which descends to the commuting hierarchy on an open set of $\mathcal{L}^{D_+,D_-}_{N,C,\Delta} \,/\, GL^N_r$.

Hamiltonian approach. At first glance the construction of the Hamiltonian theory for the periodic chains goes equally well on an arbitrary genus algebraic curve. The two-form $\Omega(z)$ on $\mathcal{L}_N(D_+, D_-)$ with values in the space of meromorphic functions on Γ by the formula identical to that in the genus zero case.

$$\Omega(z) = \sum_{n=0}^{N-1} \operatorname{Tr}\left(\Psi^{-1}_{n+1} \delta L_n \wedge \delta\Psi_n\right). \tag{63}$$

Let us fix a meromorphic differential dz on Γ with poles at a set of points q_m. Then the formula

$$\omega = -\frac{1}{2} \sum_{q\in I} \operatorname{res}_q \Omega\, dz, \quad I = \{\gamma_s, P^\pm_k, q_m\}, \tag{64}$$

defines a scalar-valued two-form on $\mathcal{L}_N(D_+, D_-)$. This form depends on a choice of the normalization of Ψ_n. A change of the normalization corresponds to the transformation $\Psi'_n = \Psi_n V$, where $V = V(z)$ is a diagonal matrix, which might depend on a point z of Γ. The corresponding transformation of Ω has the form:

$$\Omega' = \Omega + \delta\left(\operatorname{Tr}\left(\ln W v\right)\right), \quad v = \delta V V^{-1}. \tag{65}$$

Let $\mathcal{X}^{D_+,D_-}$ be a subspace of the chains $\mathcal{L}_N(D_+, D_-)$ such, that the restriction of $\delta(\ln w)\, dz$ to $\mathcal{X}^{D_+,D_-}$ is a differential *holomorphic* at all the preimages on $\hat{\Gamma}$ of the punctures $P^\pm_k$.

Theorem 3.5.

(i) The two-form ω, defined by (64) and restricted to $\mathcal{X}^{D_+,D_-}$, is independent of the choice of normalization of the Floque-Bloch solutions, and is gauge invariant, i.e. it descends to a form on $\mathcal{P} = \mathcal{X}^{D_+,D_-} \ / \ GL_r^N$.

(ii) Let $\widehat{\gamma}_s$ be the poles of the normalized Floque-Bloch solution ψ_n. Then

$$\omega = \sum_{s=1}^{\widehat{g}+r-1} \delta \ln w(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s). \tag{66}$$

By definition, a vector field ∂_t on a symplectic manifold is Hamiltonian, if the contraction $i_{\partial_t}\omega(X) = \omega(\partial_t, X)$ of the symplectic form is an exact one-form $dH(X)$. The function H is the Hamiltonian corresponding to the vector field ∂_t. The proof of the following theorem is almost identical to the proof of Theorem 4.2 in [5].

Theorem 3.6. *Let ∂_a be the vector fields corresponding to the Lax equations (59). Then the contraction of ω, defined by (64) and restricted to $\mathcal{P}$, equals*

$$i_{\partial_a}\omega = \delta H_a, \tag{67}$$

where

$$H_{(\pm k,l)} = \operatorname{res}_{P_k^\pm} \left(z - z\left(P_k^\pm\right)\right)^{-l} (\ln w)\, dz. \tag{68}$$

The theorem implies that Lax equations (59) are Hamiltonian whenever the form ω is non-degenerate.

The spectral map (46) identifies an open set of $\mathcal{L}_{N,C,\Delta}^{D_+,D_-} \ / \ GL_r^N$ with an open set of the Jacobian bundle over $\mathcal{S}_\Delta \subset \mathcal{S}^{D_+,D_-}$, i.e.

$$\mathcal{L}_{N,C,\Delta}^{D_+,D_-} \ / \ GL_r^N \longmapsto \mathcal{S}_\Delta. \tag{69}$$

The fibers of this bundle are isotropic subspaces for ω. Therefore, the form ω can be non-degenerate only if the base and the fibers of the bundle (69), restricted to $\mathcal{P}$, have the same dimension.

Consider the case $g > 0$. Let dz be a holomorphic differential on Γ. Then, for each branch of $w = w_i(z)$ the differential $\delta(\ln w)\, dz$ is always holomorphic at $P_k^\pm$. Hence, $\mathcal{P} = \mathcal{L}_N(D_+, D_-)/GL_r^N$. Recall that

$$\dim \mathcal{S}_\Delta = (r-1)(N\mathcal{D} - g + 1), \quad \dim J(\hat{\Gamma}) = (r-1)(N\mathcal{D} + g - 1) + g. \tag{70}$$

Therefore, for $g > 0$ the form ω is degenerate on $\mathcal{P}$. For $g = 1$ the space $\mathcal{P}$ is a Poisson manifold with the symplectic leaves, which are factor-spaces

$$\mathcal{P}_* = \mathcal{L}_{N,C,\Delta}^{D_+,D_-} \ / \ GL_r^N \tag{71}$$

of the restricted chains. In that case $\dim S_\Delta = \dim J_C(\hat{\Gamma}) = N\mathcal{D}(r-1)$. As in the genus zero case, the arguments identical to that used at the end of Section 4 in [5] prove that the form ω is non-degenerate on $\mathcal{P}_*$.

Corollary 3.2. *For $g=0$ and $g=1$ the form ω defined by (64) descends to the symplectic form on $\mathcal{P}_*$, which coincides with the pull-back of the Sklyanin symplectic structure restricted to the space of the monodromy operators. The Lax equations* (62) *are Hamiltonian with the Hamiltonians*

$$H_f = \sum_{q=\hat{P}_k^{\pm}} \operatorname{res}_q(\ln w) f\, dz. \tag{72}$$

The Hamiltonians H_f are in involution $\{H_f, H_h\} = 0$.

Now we are in the position to discuss the case $g > 1$ mentioned in the Introduction. The space $\mathcal{P}_*$ is equipped by g-parametric *family* of two-forms ω_{dz}, parameterized by the holomorphic differentials dz on Γ. For all of them the fibers $J_C(\hat{\Gamma})$ of the spectral bundle are maximum isotropic subspaces. For each vector-field ∂_f defined by (62) the equation $i_{\partial_f}\omega_{dz} = \delta H_f^{dz}$ holds.

Equation (10) implies that each of the forms ω_{dz} is degenerate on $\mathcal{P}_*$. Let us describe the kernel of ω_{dz}. According to Theorem 3.2, the tangent vectors to $J_C(\hat{\Gamma})$ are parameterized by the space $\mathcal{A}(\Gamma, P_\pm)$ of meromorphic functions f on Γ with the poles at $P_k^{\pm}$ modulo the following equivalence relation. The function f is equivalent to f_1, if there is a meromorphic function $F \in \mathcal{A}(\hat{\Gamma}, \hat{P}_k^{\pm})$ on $\hat{\Gamma}$ with the poles at $\hat{P}_k^{\pm}$, such that in the neighborhoods of these punctures the function $\pi^*(f - f_1) - F$ is regular. Let $K_{dz} \subset \mathcal{A}(\Gamma, P_\pm)$ be the subspace of functions such that there is a meromorphic function $\tilde{F}$ on $\hat{\Gamma}$ with poles at $\hat{P}_k^{\pm}$ and at the preimages $\pi^*(q_s)$, $dz(q_s) = 0$ of the zero-divisor of dz, and such that $f - \tilde{F}$ is regular at $\hat{P}_k^{\pm}$. Then, from equations (67) and (72) it follows that: $f \in K_{dz} \longmapsto i_{\partial_f}\omega_{dz} = 0$. Let $\mathcal{K}_{dz}$ be the factor-space of K_{dz} modulo the equivalence relation. Then the Riemann gap theorem implies that in the general position $\mathcal{K}_{dz}$ is of dimension $2(g-1)(r-1)$, which equals the dimension of the kernel of ω_{dz}. Therefore, the kernel of ω_{dz} is isomorphic to $\mathcal{K}_{dz}$. Using this isomorphism, it is easy to show that the intersection of all the kernels of the forms ω_{dz} is empty, and thus the family of these forms is *non-degenerate.*

4. Variable base curves

Until now it has been always assumed that the base curve is *fixed.* Let $\mathcal{M}_\Delta$ be the space of smooth genus g algebraic curves Γ with the fixed meromorphic function Δ, having poles and zeros of order N at punctures $P_k^{\pm}$, $k = 1, \ldots, \mathcal{D}$. For simplicity, we will assume that the punctures $P_k^{\pm}$ are distinct. The space $\mathcal{M}_\Delta$ is of dimension $\dim \mathcal{M}_\Delta = 2(\mathcal{D} + g - 1)$. The total space $\hat{\mathcal{L}}_{N,\Delta}$ of all the restricted chains corresponding to these data and the trivial equivalence class $C = 0 \in J(\Gamma)$ can be regarded as the bundle over $\mathcal{M}_\Delta$ with the fibers $\mathcal{L}_{N,0,\Delta}^{D_+,D_-} = \mathcal{L}_{N,0,\Delta}^{D_+,D_-}(\Gamma)$. By definition, the curve Γ corresponding to a point $(\Gamma, \Delta) \in \mathcal{M}_\Delta$ is equipped by the meromorphic differential $dz = d\ln\Delta$. The function Δ defines local coordinate everywhere on Γ except at zeros of its differential. Let ω_Δ be the form defined by (64), where $dz = d\ln\Delta$ and the variations of L_n and Ψ_n are taken with fixed Δ, i.e.

$$\omega_\Delta = -\frac{1}{2}\sum_{q\in I}\operatorname{res}_q \sum_{n=0}^{N-1} \operatorname{Tr}\left(\Psi_{n+1}^{-1}(\Delta)\delta L_n(\Delta) \wedge \delta\Psi_n(\Delta)\right) d\ln\Delta, \quad I = \{\gamma_s, P_k^{\pm}\}. \tag{73}$$

Then ω_Δ is well-defined on leaves $\hat{\mathcal{X}}_\Delta$ of the foliation on $\hat{\mathcal{L}}_{N,\Delta}$ defined by the condition: the differential $\delta\ln w(\Delta) d\ln\Delta$ restricted to $\hat{\mathcal{X}}_\Delta$ is *holomorphic* at the punctures $P_k^{\pm}$. This condition is equivalent to the following constraints. In the neighborhood of $P_k^{\pm}$ there are $(r-1)$ regular branches $w_i^{\pm}$ of the multi-valued function w, defined by the characteristic equation (41):

$$w_i^{\pm} = c_i^{\pm} + O(\Delta^{\mp 1}), \quad i = 1, \ldots, r-1. \tag{74}$$

The leaves $\hat{\mathcal{X}}_\Delta$ are defined by $2\mathcal{D}(r-1)$ constraints:

$$\delta c_i^{\pm} = 0 \longmapsto c_i^{\pm} = \text{const}_i^{\pm}. \tag{75}$$

Note that the differential $\delta \ln w(\Delta) d \ln \Delta$ is regular at $P_k^{\pm}$ for the singular branches of w, because the coefficients $c^{\pm}$ of the expansions (42) and (43) are also fixed due to the equation $c^{\pm}\prod_{i=1}^{r-1} c_i^{\pm} = 1$.

The factor-space $\hat{\mathcal{P}}_\Delta = \hat{\mathcal{X}}_\Delta / GL_r^N$ is of dimension $\dim \hat{\mathcal{P}}_\Delta = 2N\mathcal{D}(r-1) - 2\mathcal{D}(r-1) + 2(\mathcal{D}+g-1)$. The space $\hat{\mathcal{S}}_\Delta^0 \subset \hat{\mathcal{S}}_\Delta$ of the corresponding spectral curves is of dimension $\dim \hat{\mathcal{S}}_\Delta^0 = (r-1)(N\mathcal{D} - g + 1) - 2\mathcal{D}(r-1) + 2(\mathcal{D}+g-1)$. The second and the third terms in the last formulae are equal to the number of the constraints (75) and the dimension of $\mathcal{M}_\Delta$, respectively.

For the case $r = 2$ the last formulae imply the match of the dimensions $\dim \hat{\mathcal{P}}_\Delta = 2\dim \hat{\mathcal{S}}_\Delta^0$. For $r = 2$ the spectral curves are two-sheet cover of the base curves, and the fiber of the spectral bundle is the Prim variety $J_0(\hat{\Gamma}) = J_{\text{Prim}}(\hat{\Gamma})$.

Theorem 4.1. *For $r = 2$ the form ω_Δ defined by (73), and restricted to $\hat{\mathcal{P}}_\Delta$ is non-degenerate. If $\widehat{\gamma}_s$ are the poles of the normalized Floque-Bloch solution ψ_n, then*

$$\omega_\Delta = \sum_{s=1}^{\widehat{g}+r-1} \delta \ln w(\widehat{\gamma}_s) \wedge \delta \ln \Delta(\widehat{\gamma}_s) = \sum_{s=1}^{\widehat{g}+r-1} \delta \ln w(\widehat{\gamma}_s) \wedge \delta \ln w(\widehat{\gamma}_s^{\sigma}), \tag{76}$$

where $\sigma : \hat{\Gamma} \to \hat{\Gamma}$ is the involution, which permutes the sheets of $\hat{\Gamma}$ over Γ.

For every function $f \in \mathcal{A}(\Gamma, P_k^{\pm})$ the Lax equations (62) are Hamiltonian with the Hamiltonians

$$H_f = \sum_{q=\hat{P}_k^{\pm}} \text{res}_q (f \ln w)\, d \ln \Delta.$$

The Hamiltonians H_f are in involution. Their common level sets are fibers $J_{\text{Prim}}(\hat{\Gamma})$ of the spectral map.

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