Russian Math. Surveys **63**:6 1011–1022

Uspekhi Mat. Nauk **63**:6 19–30 DOI 10.1070/RM2008v063n06ABEH004576

Abelian solutions of the soliton equations and Riemann–Schottky problems

I. M. Krichever

Abstract. The present article is an exposition of the author's talk at the conference dedicated to the 70th birthday of S. P. Novikov. The talk contained the proof of Welters' conjecture which proposes a solution of the classical Riemann–Schottky problem of characterizing the Jacobians of smooth algebraic curves in terms of the existence of a trisecant of the associated Kummer variety, and a solution of another classical problem of algebraic geometry, that of characterizing the Prym varieties of unramified covers.

Contents

1. Introduction

The famous Novikov conjecture which asserts that the Jacobians of smooth algebraic curves are precisely those indecomposable principally polarized Abelian varieties whose theta-functions provide explicit solutions of the Kadomtsev–Petviashvili (KP) equation, fundamentally changed the relations between the classical algebraic geometry of Riemann surfaces and the theory of soliton equations. It turns out that the finite-gap, or algebro-geometric, theory of integration of non-linear equations developed in the mid-1970s can provide a powerful tool for approaching the fundamental problems of the geometry of Abelian varieties.

The basic tool of the general construction proposed by the author [1], [2] which establishes a correspondence between algebro-geometric data $\{\Gamma, P_\alpha, z_\alpha, S^{g+k-1}(\Gamma)\}$ and solutions of some soliton equation, is the notion of Baker–Akhiezer function. Here Γ is a smooth algebraic curve of genus g with marked points P_α, in whose neighborhoods we fix local coordinates z_α, and $S^{g+k-1}(\Gamma)$ is a symmetric product of the curve. The Baker–Akhiezer functions are determined by their analytic properties on the corresponding algebraic curve. These analytic properties are essentially an axiomatization of the analytic properties of the Bloch functions of

This work was partially supported by NSF grant no. DMS-04-05519.

AMS 2000 *Mathematics Subject Classification.* Primary 14H40, 14H42, 37K10.

finite-gap Schrödinger operators which were established in the initial period of the development of the theory of finite-gap integration of the Korteweg–de Vries equation [3]–[6].

A particular case of the general construction of algebro-geometric solutions of soliton equations is the following statement:

• *If a symmetric matrix B is the period matrix of a basis of normalized holomorphic differentials on some algebraic curve Γ, then the function $u(x,y,t)$ given by the formula*

$$u(x,y,t) = 2\partial_x^2 \log\theta(Ux + Vy + Wt + Z \mid B) \tag{1}$$

satisfies the KP equation

$$3u_{yy} = (4u_t - 6uu_x + u_{xxx})_x, \tag{2}$$

where U, V, W are the vectors of b-periods of normalized meromorphic differentials with pole at some point $P_0 \in \Gamma$ of orders 2, 3, *and* 4 *respectively.*

Here and below, for any symmetric matrix B with positive-definite imaginary part, $\theta(z \mid B)$ is the Riemann theta-function given by the formula

$$\theta(z) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \qquad (z,m) = m_1z_1 + \cdots + m_gz_g. \tag{3}$$

The formula (1) looks simple, but the definition of the matrix B in it as the period matrix of holomorphic differentials on some algebraic curve is rather indirect. This naturally raises a question about the possibility of a self-contained description of admissible parameters. A conjecture posed by Novikov in the framework of the general problem stated by him of an effective presentation of theta-functional formulae, is equivalent to the statement that the direct substitution of the formula (1) in the KP equation produces equations that characterize the period matrices of holomorphic differentials; that is, this should provide a solution of the famous Riemann–Schottky problem of characterizing the Jacobians of algebraic curves. The proof of Novikov's conjecture obtained in 1986 by Shiota [7] was the first effective solution of the above problem posed more than 120 years ago.

To explain the role of new ideas introduced by the Novikov conjecture in the solution of the Riemann–Schottky problem we give a brief description of the history of the question and of results obtained in the framework of classical algebro-geometric approaches to its solution (details can be found in [8]).

For $g = 2$ and 3, the dimensions of the moduli space $\mathscr{M}_g$ of smooth algebraic curves of genus g and the moduli space $\mathscr{A}_g$ of principally polarized g-dimensional Abelian varieties coincide. Therefore, as an immediate consequence of Torelli's theorem, we obtain that in this case a sufficiently general symmetric matrix B with positive-definite imaginary part is the period matrix of a Riemann surface. The only restriction is given by Martens' theorem which asserts the Jacobian variety of a Riemann surface is indecomposable, that is, is not isomorphic to a direct product of Abelian varieties of positive dimension. This condition can be effectively described in terms of theta-constants.

A non-trivial relation for the period matrix of a Riemann surface of genus 4 was obtained by Schottky in 1888. Since for $g = 4$ the space $\mathscr{M}_g$ is of codimension 1, the corresponding *Schottky relation* gives at least a local solution to the problem of characterizing the corresponding Jacobian varieties. The proof of the irreducibility of the variety defined by the Schottky relation was obtained only in 1981 by Igusa [9]. Generalizations of this relation to the case of curves of arbitrary genus were formulated as a conjecture in 1909 in a joint work of Schottky and Jung [10]. These generalizations were proved in the work of Farkas and Rauch [11]. Later van Geemen [12] proved that the Schottky–Jung relations give a local solution of the Riemann–Schottky problem. It is known that these relations obviously do not give a complete solution of the problem because the subvariety defined by these relations has extra components even for $g = 5$ (Donagi [13]).

A characterization of Jacobians in a geometrical form was proposed by Gunning [14], [15]. The basic tool of this characterization is Fay's trisecant identity [16].

Consider the map of a principally polarized Abelian variety X into a complex projective space $\mathbb{CP}^{2^g-1}$ given by a basic set of theta-functions of level two

$$\phi_2(z) = \Theta[\varepsilon_1, 0](z) : \cdots : \Theta[\varepsilon_{2^g}, 0](z). \tag{4}$$

These functions are even, so the map ϕ_2 may be written as a composition

$$X \xrightarrow{\pi} X/\sigma \xrightarrow{K} \mathbb{CP}^{2^g-1}, \tag{5}$$

where $\sigma(z) = -z$ is the involution of an Abelian variety and π is the projection onto the quotient space. The map K is called the Kummer map and its image $K(X)$ is called the Kummer variety. It is known that the Kummer map is an embedding of a variety with singularities. An N*-secant* of a Kummer variety is an $(N-2)$-dimensional plane in $\mathbb{CP}^{2^g-1}$ meeting $K(X)$ at N points. The existence of an N-secant passing through points $K(A_i)$, $i = 0, 1, \dots, N-1$, is equivalent to the condition of linear dependency of these points. Fay's trisecant identity immediately implies that if X is a Jacobian of an algebraic curve Γ, then any three distinct points of Γ determine *a one-parameter* family of trisecants. A slightly simplified form of Gunning's result asserts that the existence of such a one-parameter family of trisecants is not only necessary but also sufficient for a principally polarized Abelian variety to be the Jacobian of some algebraic curve.

The problem of formulating Gunning's geometrical criterion in terms of equations was far from trivial and its solution required some serious steps. The first of these were made in the works of Welters [17], [18], whose starting point was probably Mumford's remark [19] that the limiting case of Fay's trisecant identity gives the theta-functional formula (1) for algebro-geometrical solutions of the KP equation. An infinitesimal analogue of a trisecant is an inflection point of a Kummer variety, that is, a point A such that there is a line in $\mathbb{CP}^{2^g-1}$ containing the image of the formal 2-jet of some curve in X. According to [18] the condition of the existence of a *formal infinite* jet of inflection points is characteristic for Jacobians.

A fundamental fact of the theory of soliton equations is that each of these equations is related to a consistent system of equations, the so-called hierarchy of the equation. Algebro-geometric solutions of the KP hierarchy are given by the formula

$$u(t_1, t_2, \dots) = 2\partial_x^2 \log \theta\left(\sum_i U_i t_i + Z \mid B\right), \qquad t_1 = x, \quad t_2 = y, \quad t_3 = t. \tag{6}$$

The next step was the proof by Arbarello and De Concini [20] that Welters' characterization is equivalent to the following assertion: a matrix B is the period matrix of holomorphic differentials if and only if there are vectors U_i such that the function $u(t)$ given by the formula (6) satisfies the equations of the whole KP hierarchy.

Moreover, these authors were the first who proved that, for the characterization of Jacobians, the validity of only a finite number of these equations is sufficient. We should note that the estimate obtained in [20] for the number of equations of the KP hierarchy necessary for the characterization of Jacobians was certainly an overestimate. A trivial consequence of the theory of commuting ordinary differential operators is that it is sufficient to consider the first $N = g + 1$ equations of the hierarchy. The Novikov conjecture asserted that the number of equations *does not depend* on g and is $N = 1$.

The key step in the proof of the Novikov conjecture proposed by Shiota is the fact that if the function $u(x, y, t)$ given by formula (1) satisfies the KP equation, then there are vectors U_i such that the function $u(t)$ given by formula (6) is a solution of the KP hierarchy. The main difficulty encountered by Shiota was that the possibility of such an extension of a solution of the KP equation to that of the KP hierarchy is controlled by some, a priori non-trivial, homological obstruction. A sufficient condition for the triviality of this obstruction is the condition that the theta-divisor Θ does not contain a complex line parallel to the vector $U = U_1$. The proof of the last property was technically the most difficult part of Shiota's work. The significance of this part was clarified in [21].

2. Welters' trisecant conjecture

The interest in subjects related to the Riemann–Schottky problem did not weaken after the proof of Novikov's conjecture. First of all this is related to a series of other problems in the geometry of Abelian varieties. Among them we distinguish the problem of characterizing the principally polarized Abelian varieties which are the Prym varieties of double covers of algebraic curves, and also Welters' remarkable conjecture stating that for the characterization of Jacobians the existence of *one* trisecant is sufficient. Comparing Welters' conjecture with Gunning's theorem, which requires the existence of a one-parameter family of such secants, one can see how strong this last statement is.

We should note that there are three particular cases of Welters' conjecture corresponding to three possible configurations of intersection points (a, b, c) of the trisecant and the Kummer variety $K(X)$:

(i) all three points coincide ($a = b = c$);
(ii) two of them coincide ($a = b \neq c$);
(iii) all three points are distinct ($a \neq b \neq c \neq a$).

Of course if there is a *family* of secants the first two cases can be regarded as degenerations of the general case (iii). However, in the situation when there is only one secant, all three cases are independent and require separate treatment. The proof of the first case, (i), of Welters' conjecture was obtained by the author [22] by means of a new approach[1]. It turns out that for the solution of the Riemann–Schottky problem it is sufficient to consider not the whole KP equation but just one auxiliary linear equation. More precisely, we have the following theorem.

Theorem 1. *An indecomposable principally polarized Abelian variety* (X, θ) *is the Jacobian variety of a smooth algebraic curve of genus* g *if and only if there exist* g*-dimensional vectors* $U \neq 0$, V, A *and constants* p *and* E *satisfying one of the following three equivalent conditions*:

(A) *the following equality holds*

$$(\partial_y - \partial_x^2 + u)\psi = 0, \tag{7}$$

where

$$u = -2\partial_x^2 \log \theta(Ux + Vy + Z), \qquad \psi = \frac{\theta(A + Ux + Vy + Z)}{\theta(Ux + Vy + Z)}\, e^{px+Ey}, \tag{8}$$

and Z *is an arbitrary vector*;

(B) *for all theta characteristics* $\varepsilon \in \frac{1}{2}\mathbb{Z}_2^g$,

$$\big(\partial_V - \partial_U^2 - 2p\,\partial_U + (E - p^2)\big)\Theta[\varepsilon, 0](A/2) = 0 \tag{9}$$

(*here and below* ∂_U, ∂_V *are the derivatives along the directions given by the vectors* U *and* V, *respectively*);

(C) *on the theta-divisor* $\Theta = \{Z \in X \mid \theta(Z) = 0\}$,

$$\begin{aligned}[](\partial_V\theta)^2 - (\partial_U^2\theta)^2]\partial_U^2\theta + 2[\partial_U^2\theta\,\partial_U^3\theta - \partial_V\theta\,\partial_U\partial_V\theta]\partial_U\theta \\ + [\partial_V^2\theta - \partial_U^4\theta](\partial_U\theta)^2 = 0 \quad (\mathrm{mod}\ \theta).\end{aligned} \tag{10}$$

Equation (7) is one of the auxiliary linear equations for the KP equation. The direct substitution of the expressions (8) in this equation and the use of the addition formula for the Riemann theta-functions shows the equivalence of conditions (A) and (B) in the theorem. Equation (9) means that the image of the point $A/2$ under the Kummer map is an inflection point (case (i) of Welters' conjecture). Condition (C) is the relation that is really used in the proof of the theorem. Formally, it is weaker than the two other conditions because its derivation is only local and does not use an explicit form of the solution ψ of the equation (7), but only the condition of meromorphicity of the solution. More precisely, consider an entire function $\tau(x, y)$ of the complex variable x depending smoothly on a parameter y and assume that in a neighbourhood of a simple zero $\eta(y)$ of the function τ (that is, $\tau(\eta(y), y) = 0$) equation (7) with the potential $u = -2\partial_x^2 \log \tau$ has a meromorphic solution ψ. Then the equation

$$\ddot{\eta} = 2w \tag{11}$$

holds, where the 'dots' denote the derivatives with respect to the variable y, and w is the third coefficient of the Laurent expansion of the function u at the point η; that is, $u(x, y) = 2(x - \eta(y))^{-2} + v(y) + w(y)(x - \eta(y)) + \cdots$. Formally, if we represent τ in the form of an infinite product

$$\tau(x, y) = c(y) \prod_i (x - x_i(y)), \tag{12}$$

[1] Under different additional assumptions the corresponding statement was proved in the earlier works [23]–[25].

then equation (11) is equivalent to an infinite system of equations

$$\ddot{x}_i = -4 \sum_{j \neq i} \frac{1}{(x_i - x_j)^3}\,, \tag{13}$$

which, in the cases where τ is a rational, trigonometric, or elliptic polynomial, coincides with the equations of motion for the rational, trigonometric, or elliptic Calogero–Moser systems, respectively. Equation (11) for the zeros of the function $\tau = \theta(Ux + Vy + Z)$ was first derived in [25]. Expanding the function θ in a neighborhood of the points of its divisor $z \in \Theta$: $\theta(z) = 0$, it is easy to see that equation (11) is equivalent to equation (10).

The proof of Welters' conjecture was completed by the author in [26]. Although the proof of the conjecture in the differential-difference (ii) and completely discrete (iii) cases required a series of technical changes, the main ideas of the approach basically remain unchanged. For brevity we present the statement of the corresponding theorem only in the completely discrete case.

Theorem 2. *An indecomposable, principally polarized Abelian variety (X, θ) is the Jacobian variety of a smooth algebraic curve of genus g if and only if there exist g-dimensional vectors $U \neq V \neq A \neq U \pmod{\Lambda}$ such that one of the following three equivalent conditions holds*:

(A) *the difference equation*

$$\psi(m, n+1) = \psi(m+1, n) + u(m,n)\psi(m,n), \tag{14}$$

holds with

$$u(m,n) = \frac{\theta((m+1)U + (n+1)V + Z)\theta(mU + nV + Z)}{\theta\big(mU + (n+1)V + Z\big)\theta\big((m+1)U + nV + Z\big)} \tag{15}$$

and

$$\psi(m,n) = \frac{\theta(A + mU + nV + Z)}{\theta(mU + nV + Z)}\, e^{mp+nE}, \tag{16}$$

where p, E are constants and Z is an arbitrary vector;

(B) *the equations*

$$\Theta[\varepsilon, 0]\left(\frac{A - U - V}{2}\right) + e^{p}\Theta[\varepsilon, 0]\left(\frac{A + U - V}{2}\right) = e^{E}\Theta[\varepsilon, 0]\left(\frac{A + V - U}{2}\right) \tag{17}$$

are satisfied for all $\varepsilon \in \frac{1}{2}\mathbb{Z}_2^g$;

(C) *the equality*

$$\theta(Z+U)\theta(Z-V)\theta(Z-U+V) + \theta(Z-U)\theta(Z+V)\theta(Z+U-V) = 0 \pmod{\theta} \tag{18}$$

holds on the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$.

We should note that, in a certain sense, the replacement of the generating differential equation (7) in the statement of the previous theorem by the discrete equation (14) was predictable because the last equation is one of the auxiliary linear problems for the so-called discrete bilinear Hirota equation

$$\tau_n(l+1,m)\,\tau_n(l,m+1)-\tau_n(l,m)\,\tau_n(l+1,m+1)+\tau_{n+1}(l+1,m)\,\tau_{n-1}(l,m+1)=0, \tag{19}$$

which in the continuous limiting case gives the KP equation, and in an intermediate limiting case gives another fundamental equation, the two-dimensional Toda lattice equation.

As above, the equivalence of conditions (A) and (B) immediately follows from addition formulas. Condition (C), which is a discrete analog of the equality (10), is that which is really used in the proof of the theorem.

3. The problem of characterization of Prym varieties

An involution $\sigma\colon \Gamma \mapsto \Gamma$ on a smooth algebraic curve Γ naturally determines an involution $\sigma^*\colon J(\Gamma) \mapsto J(\Gamma)$ on its Jacobian. The odd subspace with respect to this involution is a sum of an Abelian variety of lower dimension, called the Prym variety, and a finite group. The restriction of the principal polarization of the Jacobian determines a polarization of the Prym variety which is principal if and only if the original involution of the curve has at most two fixed points.

The problem of characterizing the locus $\mathscr{P}_g$ of Prym varieties of dimension g in the space $\mathscr{A}_g$ of all principally polarized Abelian varieties is well known and during its more than 100-year-old history has attracted considerable interest. This problem is much more difficult than the Riemann–Schottky problem and until quite recently its solution in terms of a finite system of equations was completely open.

The problem of characterizing Prym varieties in the case of curves with an involution having two fixed points was solved in [27] in terms of the Schrödinger operators integrable with respect to one energy level. The theory of such operators was developed by Novikov and Veselov in [28], where the authors also introduced the corresponding non-linear equation, the so-called Novikov–Veselov equation.

Curves with an involution having a pair of fixed points can be regarded as a limit of unramified covers. A characterization of the Prym varieties in the latter case in terms of the existence of quadrisecants was obtained in the recent work [29] of the author and Grushevsky.

The existence of families of quadrisecants for curves with an involution having at most two fixed points was proved in [30], [32]. An analogue of Gunning's theorem asserting that the existence of a family of secants characterizes Prym varieties was proved by Debarre [32]. We note that the existence of one quadrisecant does not characterize Prym varieties. A counterexample to the naïve generalization of Welters' conjecture was constructed by Beauville and Debarre in the work [30].

It was proved in [29] that the existence of a *symmetric pair* of quadrisecants is a characteristic property for Prym varieties of unramified covers.

Theorem 3. (Geometric characterization of Prym varieties.) *An indecomposable principally polarized Abelian variety $(X,\theta)\in\mathscr{A}_g$ is in the closure of the locus of Prym varieties of smooth unramified double covers if and only if there exist four distinct points $p_1,p_2,p_3,p_4\in X$, none of them of order two, such that the images of* *the Kummer map of the eight points $p_1 \pm p_2 \pm p_3 \pm p_4$ lie on two quadrisecants* (*the corresponding quadruples of points are determined by the number of plus signs*).

We should note that the proof of this statement required constructing and developing the theory of a new integrable equation because before that, in contrast with all other cases, no non-linear equations whose algebro-geometric solutions are associated to unramified double covers were known.

The auxiliary linear equation of the corresponding analogue of the Novikov–Veselov equation is a discrete analogue of the potential Schrödinger equation and has the form

$$\psi(n+1,m+1) - u(n,m)\big(\psi(n+1,m) - \psi(n,m+1)\big) - \psi(n,m) = 0. \tag{20}$$

4. Abelian solutions of the soliton equations

Most recently, in joint works of the author and Shiota [33], [34], a general notion of Abelian solutions of soliton equations was introduced. This notion generalizes naturally classes of solutions expressed in the terms of the theta-functions of principally polarized Abelian varieties and the theory of elliptic solutions of the soliton equations.

The theory of elliptic solutions of the KP equation goes back to the remarkable work [35], where it was found that the dynamics of poles of the elliptic (rational or trigonometric) solutions of the Korteweg–de Vries equation can be described in terms of the elliptic (rational or trigonometric) Calogero–Moser (CM) system with certain constraints. It was observed in [24] that, when the constraints are removed, this restricted correspondence becomes an isomorphism when the elliptic solutions of the KP equation are considered. Recall that the elliptic CM system is a completely integrable Hamiltonian system with Hamiltonian

$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i\neq j} \wp(q_i - q_j),$$

where $\wp$ is the Weierstrass $\wp$-function. In [24], for the elliptic CM system a Lax representation $\dot{L} = [L, M]$ with spectral parameter was proposed, which made it possible to prove the algebraic integrability of the problem. It turns out that, for generic initial data, the positions of the particles $q = q_i(y)$ at any time are determined by the equation

$$\theta(Uq + Vy + Z) = 0,$$

where $\theta(Z)$ is the Riemann theta-function which corresponds to the spectral curve explicitly constructed from the initial data.

A correspondence between finite-dimensional integrable systems and the pole systems of various soliton equations was considered in [36]–[39]. A general scheme of constructing such systems is presented in [40]. In [41] it was generalized to the case of field analogues of CM type systems (see also [42]).

According to [33] the solution $u(x, y, t)$ of the KP equation is called *Abelian* if it has the form

$$u = -2\partial_x^2 \log \tau(Ux + z, y, t), \tag{21}$$

where $z \in \mathbb{C}^n$ and $0 \neq U \in \mathbb{C}^n$ are n-dimensional vectors, and where for all y, t the function $\tau(\,\cdot\,, y, t)$ is a holomorphic section of some line bundle $\mathscr{L} = \mathscr{L}(y,t)$ on the Abelian variety $X = \mathbb{C}^n/\Lambda$; that is, for all vectors $\lambda \in \Lambda$ it satisfies the following monodromy conditions:

$$\tau(z+\lambda, y, t) = e^{a_\lambda \cdot z + b_\lambda}\tau(z,y,t) \tag{22}$$

for some $a_\lambda \in \mathbb{C}^n$, $b_\lambda = b_\lambda(y,t) \in \mathbb{C}$.

In the case of sections of the canonical line bundle on a principally polarized Abelian variety the corresponding theta-function is unique up to normalization. Hence the ansatz (21) assumes the form $u = -2\partial_x^2 \log\theta(Ux + Z(y,t) + z)$. Since flows commute with each other, the dependence of the vector $Z(y,t)$ must be linear:

$$u = -2\partial_x^2 \log\theta(Ux + Vy + Wt + z). \tag{23}$$

Therefore, the problem of classification of such Abelian solutions is the same problem as posed by Novikov.

In the case of one-dimensional Abelian varieties the problem of classification of Abelian solutions is the problem of classification of those elliptic solutions which are distinguished amongst the general algebro-geometric solutions by the condition that the corresponding vector U generates an elliptic curve embedded into the Jacobian of the spectral curve.

Note that, for any vector U, the closure of the group $\{Ux \mid x \in \mathbb{C}\}$ is an Abelian subvariety $X \subset J(\Gamma)$. So when this closure does not coincide with the whole Jacobian, we get non-trivial examples of Abelian solutions. Briefly, the main result on the classification of Abelian solutions of KP obtained in [33] can be formulated as the statement that all the Abelian solutions are obtained in this manner.

To avoid some technical complications we give the formulation of the corresponding theorem in the situation of general position.

Theorem 4. *Suppose that $u(x,y,t)$ is an Abelian solution of the KP equation such that the subgroup $\{Ux \mid x \in \mathbb{C}\}$ is dense in X. Then there exists a unique algebraic curve Γ with marked smooth point $P \in \Gamma$, a holomorphic embedding $j_0\colon X \to J(\Gamma)$, and a torsion-free rank-1 sheaf $\mathscr{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ on Γ of degree $g-1$, where $g = g(\Gamma)$ is the arithmetic genus of Γ, such that, with the notation $j(z) = j_0(z) \otimes \mathscr{F}$,*

$$\tau(Ux + z, y, t) = \rho(z,y,t)\hat{\tau}(x, y, t, 0, \dots \mid \Gamma, P, j(z)), \tag{24}$$

where $\hat{\tau}(t_1, t_2, t_3, \dots \mid \Gamma, P, \mathscr{F})$ is a KP τ-function corresponding to the data $(\Gamma, P, \mathscr{F})$ and the function $\rho(z,y,t) \not\equiv 0$ satisfies the condition $\partial_U \rho = 0$.

Note that if Γ is smooth, then

$$\hat{\tau}(x, t_2, t_3, \dots \mid \Gamma, P, j(z)) = \theta\Big(Ux + \sum V_i t_i + j(z) \mid B(\Gamma)\Big) e^{Q(x, t_2, t_3, \dots)}, \tag{25}$$

where $V_i \in \mathbb{C}^n$, Q is a quadratic form, and $B(\Gamma)$ is the corresponding period matrix. A linearization on the Jacobian $J(\Gamma)$ of the non-linear (y,t)-dynamics for $\tau(z,y,t)$ indicates the possibility of the existence of integrable systems on spaces of theta-functions of higher level. A CM system is an example of such a system for $n = 1$.

Bibliography

[1] И. М. Кричевер, “Интегрирование нелинейных уравнений методами алгебраической геометрии”, *Функц. анализ и его прил.* **11**:1 (1977), 15–31; English transl., I. M. Krichever, “Integration of nonlinear equations by the methods of algebraic geometry”, *Funct. Anal. Appl.* **11**:1 (1977), 12–26.

[2] И. М. Кричевер, “Методы алгебраической геометрии в теории нелинейных уравнений”, *УМН* **32**:6 (1977), 183–208; English transl., I. M. Krichever, “Methods of algebraic geometry in the theory of non-linear equations”, *Russian Math. Surveys* **32**:6 (1977), 185–213.

[3] Б. А. Дубровин, В. Б. Матвеев, С. П. Новиков, “Нелинейные уравнения типа Кортевега–де Фриза, конечнозонные линейные операторы и абелевы многообразия”, *УМН* **31**:1 (1976), 55–136; English transl., B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, “Non-linear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties”, *Russian Math. Surveys* **31**:1 (1976), 59–146.

[4] А. Р. Итс, В. Б. Матвеев, “Об операторах Хилла с конечным числом лакун”, *Функц. анализ и его прил.* **9**:1 (1975), 69–70; English transl., A. R. Its and V. B. Matveev, “Hill’s operator with finitely many gaps”, *Funct. Anal. Appl.* **9**:1 (1975), 65–66.

[5] H. P. McKean and P. van Moerbeke, “The spectrum of Hill’s equation”, *Invent. Math.* **30**:3 (1975), 217–274.

[6] P. D. Lax, “Periodic solutions of the KdV equation”, *Comm. Pure Appl. Math.* **28** (1975), 141–188.

[7] T. Shiota, “Characterization of Jacobian varieties in terms of soliton equations”, *Invent. Math.* **83**:2 (1986), 333–382.

[8] И. А. Тайманов, “Секущие абелевых многообразий, тэта-функции и солитонные уравнения”, *УМН* **52**:1 (1997), 149–224; English transl., I. A. Taimanov, “Secants of Abelian varieties, theta functions, and soliton equations”, *Russian Math. Surveys* **52**:1 (1997), 147–218.

[9] Jun-ichi Igusa, “On the irreducibility of Schottky’s divisor”, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28**:3 (1981), 531–545.

[10] F. Schottky and H. Jung, “Neue Sätze über Symmetralfunktionen und die Abel’schen Funktionen der Riemann’schen Theorie”, *Sitzungsber. Akad. Berlin*, 1909, 282–297.

[11] H. M. Farkas and H. E. Rauch, “Period relations of Schottky type on Riemann surfaces”, *Ann. of Math.* (2) **92**:3 (1970), 434–461.

[12] B. van Geemen, “Siegel modular forms vanishing on the moduli space of curves”, *Invent. Math.* **78**:2 (1984), 329–349.

[13] R. Donagi, “Big Schottky”, *Invent. Math.* **89**:3 (1987), 569–599.

[14] R. C. Gunning, “Some curves in Abelian varieties”, *Invent. Math.* **66**:3 (1982), 377–389.

[15] R. C. Gunning, “Some identities for abelian integrals”, *Amer. J. Math.* **108**:1 (1986), 39–74.

[16] J. D. Fay, “Theta functions on Riemann surfaces”, *Lecture Notes in Math.*, vol. 352, Springer-Verlag, Berlin–New York 1973.

[17] G. E. Welters, “On flexes of the Kummer variety (note on a theorem of R. C. Gunning)”, *Indag. Math.* **45**:4 (1983), 501–520.

[18] G. E. Welters, “A criterion for Jacobi varieties”, *Ann. of Math.* (2) **120**:3 (1984), 497–504.

[19] D. Mumford, “An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg–de Vries equation and related nonlinear equations”, *Proceedings of the International Symposium on Algebraic Geometry* (Kyoto, 1977), Kinokuniya Book Store, Kyoto 1978, pp. 115–153.

[20] E. Arbarello and C. De Concini, “On a set of equations characterizing Riemann matrices”, *Ann. of Math.* (2) **120**:1 (1984), 119–140.

[21] E. Arbarello and C. De Concini, “Another proof of a conjecture of S. P. Novikov on periods of Abelian integrals on Riemann surfaces”, *Duke Math. J.* **54**:1 (1987), 163–178.

[22] I. M. Krichever, “Integrable linear equations and the Riemann–Schottky problem”, *Algebraic geometry and number theory*, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA 2006, pp. 497–514.

[23] G. Marini, "A geometrical proof of Shiota's theorem on a conjecture of S. P. Novikov", *Compos. Math.* **111**:3 (1998), 305–322.

[24] И. М. Кричевер, "Эллиптические решения уравнения Кадомцева–Петвиашвили и интегрируемые системы частиц", *Функц. анализ и его прил.* **14**:4 (1980), 45–54; English transl., I. M. Krichever, "Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles", *Funct. Anal. Appl.* **14**:4 (1980), 282–290.

[25] E. Arbarello, I. Krichever, and G. Marini, "Characterizing Jacobians via flexes of the Kummer variety", *Math. Res. Lett.* **13**:1 (2006), 109–123.

[26] I. Krichever, "Characterizing Jacobians via trisecants of the Kummer variety", *Ann. of Math.*, 2008 (to appear); `arXiv: math/0605625`, 2006.

[27] I. Krichever, "A characterization of Prym varieties", *Int. Math. Res. Not.*, 2006, Art. ID 81476, 36 pp.

[28] А. П. Веселов, С. П. Новиков, "Конечнозонные двумерные потенциальные операторы Шрёдингера. Явные формулы и эволюционные уравнения", *Докл. АН СССР* **279**:1 (1984), 20–24; English transl., S. P. Novikov and A. P. Veselov, "Finite-gap two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations", *Soviet Math. Dokl.* **30**:3 (1984), 588–591.

[29] S. Grushevsky and I. Krichever, *Integrable discrete Schrödinger equations and a characterization of Prym varieties by a pair of quadrisecants*; `arXiv: 0705.2829v1`, 2007.

[30] A. Beauville and O. Debarre, "Sur le problème de Schottky pour les variétés de Prym", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **14**:4 (1988), 613–623.

[31] J. Fay, "On the even-order vanishing of Jacobian theta functions", *Duke Math. J.* **51**:1 (1984), 109–132.

[32] O. Debarre, "Vers une stratification de l'espace des modules des variétés abéliennes principalement polarisées", *Complex algebraic varieties* (Bayreuth, 1990), Lecture Notes in Math., vol. 1507, Springer, Berlin 1992, pp. 71–86.

[33] I. Krichever and T. Shiota, "Abelian solutions of the KP equation,", Amer. Math. Soc. Transl. Ser. 2, vol. 224, Amer. Math. Soc., Providence, RI 2008, pp. 173–192; `arXiv: 0804.0274`, 2008.

[34] I. Krichever and T. Shiota, *Abelian solutions of the soliton equations and geometry of abelian varieties*, `arXiv: 0804.0794`, 2008.

[35] H. Airault, H. McKean, and J. Moser, "Rational and elliptic solutions of the Korteweg–de Vries equation and related many-body problem", *Comm. Pure Appl. Math.* **30**:1 (1977), 95–148.

[36] I. M. Krichever, O. Babelon, E. Billey, and M. Talon, "Spin generalisation of the Calogero–Moser system and the matrix KP equation", *Topics in topology and mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 170, Amer. Math. Soc., Providence, RI 1995, pp. 83–119.

[37] И. М. Кричевер, А. В. Забродин, "Спиновое обобщение модели Рейсенарса–Шнайдера, неабелева двумеризованная цепочка Тода и представления алгебры Склянина", *УМН* **50**:6 (1995), 3–56; English transl., I. Krichever and A. Zabrodin, "Spin generalization of the Ruijsenaars–Schneider model, the non-Abelian 2D Toda chain, and representations of the Sklyanin algebra", *Russian Math. Surveys* **50**:6 (1995), 1101–1150.

[38] I. Krichever, O. Lipan, P. Wiegmann, and A. Zabrodin, "Quantum integrable models and discrete classical Hirota equations", *Comm. Math. Phys.* **188**:2 (1997), 267–304.

[39] I. Krichever, "Elliptic analog of the Toda lattice", *Internat. Math. Res. Notices*, 2000, no. 8, 383–412.

[40] I. Krichever, "Elliptic solutions to difference nonlinear equations and nested Bethe ansatz equations", *Calogero–Moser–Sutherland models* (Montréal, QC, 1997), CRM Ser. Math. Phys., Springer, New York 2000, pp. 249–271.

[41] А. А. Ахметшин, Ю. С. Вольвовский, И. М. Кричевер, "Эллиптические семейства решений уравнения Кадомцева–Петвиашвили и полевой аналог эллиптической системы Калоджеро–Мозера", *Функц. анализ и его прил.* **36**:4 (2002), 1–17; English transl., A. A. Akhmetshin, I. M. Krichever, and Yu. S. Volvovskii, "Elliptic families of solutions of the Kadomtsev–Petviashvili equation and the field elliptic Calogero–Moser system", *Funct. Anal. Appl.* **36**:4 (2002), 253–266.

[42] A. Levin, M. Olshanetsky, and A. Zotov, *Hitchin systems – symplectic maps and two-dimensional version*, arXiv: nlin/0110045v3, 2001.

I. M. Krichever
Landau Institute for Theoretical Physics,
Russian Academy of Sciences;
Columbia University, New York, USA
E-mail: krichev@math.columbia.edu

Received 2/SEP/08
Translated by Yu. PROKHOROV