Abelian solutions of the KP equation

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January 3, 2008

Abstract

We introduce the notion of abelian solutions of KP equations and show that all of them are algebro-geometric.

1 Introduction

The Kadomtsev-Petviashvili equation (KP)

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\Big(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x\Big) \tag{1.1}$$

is one of the most fundamental integrable equation of the soliton theory. Various classes of its exact solutions have been constructed and studied over the years. The purpose of this paper is to introduce and characterize a new class of solutions of the KP equation. We call a solution $u(x, y, t)$ of the KP equation *abelian* if it is of the form

$$u = -2\partial_x^2 \ln \tau(Ux + z, y, t)\,, \tag{1.2}$$

where x, y, $t \in \mathbb{C}$ and $z \in \mathbb{C}^n$ are independent variables, $0 \neq U \in \mathbb{C}^n$, and for all y, t the function $\tau(\cdot, y, t)$ is a holomorphic section of a line bundle $\mathcal{L} = \mathcal{L}(y, t)$ on an abelian variety $X = \mathbb{C}^n/\Lambda$, i.e., for all $\lambda \in \Lambda$ it satisfies the monodromy relations

$$\tau(z + \lambda, y, t) = e^{a_\lambda \cdot z + b_\lambda}\tau(z, y, t), \quad \text{for some } a_\lambda \in \mathbb{C}^n,\ b_\lambda = b_\lambda(y, t) \in \mathbb{C}\,. \tag{1.3}$$

There are two particular cases in which a complete characterization of the abelian solutions has been known for years. The first one is the case $n = 1$ of elliptic solutions of the KP equations. Theory of elliptic solutions of the KP equation goes back to the work [1], where it was found that the dynamics of poles of the elliptic (resp. rational or trigonometric)

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solutions of the Korteweg-de Vries equation can be described in terms of the elliptic (resp. rational or trigonometric) Calogero-Moser (CM) system with certain constraints. In [11] it was shown that when the constraints are removed this correspondence becomes an isomorphism between the solutions of the elliptic (resp. rational etc.) CM system and the elliptic (resp. rational etc.) solutions of the KP equation. Recall that the elliptic CM system is a completely integrable system with Lax representation $\dot{L} = [L, M]$, where $L = L(z)$ and $M = M(z)$ are $(N \times N)$ matrices depending on a spectral parameter $z \in \mathbb{C}$. The involutive integrals H_n are defined as $H_n = n^{-1} \operatorname{Tr} L^n$. A function $u(x, y, t)$ which is an elliptic function in x satisfies the KP equation if and only if it has the form

$$u(x, y, t) = 2 \sum_{i=1}^{N} \wp(x - q_i(y, t)) + c\,, \tag{1.4}$$

where $\wp(q)$ is the Weierstrass $\wp$-function ([6]), and its poles q_i as functions of y (resp. t) satisfy the equations of motion of the elliptic CM system, corresponding to the second Hamiltonian

$$H_2 = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - 2 \sum_{i \neq j} \wp(q_i - q_j)$$

(resp. the third Hamiltonian H_3).

An explicit theta-functional formula for algebro-geometric solutions of the KP equation provides an *algebraic* solution of the Cauchy problem for the elliptic CM system [11]. Namely, for generic initial data the positions $q = q_i(y, t)$ of the particles at any time y, t are roots of the equation

$$\theta(Uq + Vy + Wt + Z) = 0\,,$$

where $\theta(Z)$ is the Riemann theta-function of the Jacobian of *time-independent* spectral curve Γ, given by $R(k, z) = \det(kI - L(z)) = 0$ (hence Γ as well as the vectors U, V, W and Z depend on the initial data).

The correspondence between finite-dimensional integrable systems and pole systems of various soliton equations has been extensively studied in [5, 12, 13, 17, 18]. A general scheme of constructing such systems using a specific inverse problem for linear equations with elliptic coefficients is presented in [12]. In [2] it was generalized for the case of field analog of the CM system (see also [19]).

The second case in which a complete characterization of abelian solutions is known is the case of indecomposable principally polarized abelian variety (ppav). The corresponding θ-function is unique up to normalization, so that Ansatz (1.2) takes the form $u = -2\partial_x^2 \ln \theta(Ux + Z(y, t) + z)$. Since the flows commute, $Z(y, t)$ must be linear in y and t:

$$u = -2\partial_x^2 \ln \theta(Ux + Vy + Wt + z)\,. \tag{1.5}$$

Novikov conjectured that an indecomposable ppav (X, θ) is the Jacobian of a smooth genus g algebraic curve if and only if there exist vectors U $(\neq 0)$, V and W such that u given by (1.5) satisfies the KP equation. Novikov's conjecture was proved in [21], and until recently has remained the most effective solution of the Riemann-Schottky problem.

Besides these two cases of abelian solutions with known characterization, another may be worth mentioning. Let Γ be a curve, $P \in \Gamma$ a smooth point, and $\pi\colon \Gamma \to \Gamma_0$ a ramified covering map such that the curve Γ_0 has arithmetic genus $g_0 > 0$ and P is a branch point of the covering. Let $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$ be the (generalized) Jacobian of Γ, let $\mathrm{Nm}\colon J(\Gamma) \to J(\Gamma_0)$ be the reduced norm map as in [20], and let

$$X = \ker(\mathrm{Nm})^0 \subset J(\Gamma)$$

be the identity component of the kernel of Nm. Suppose X is compact. By assumption we have

$$\dim J(\Gamma) - \dim X = \dim J(\Gamma_0) = g_0 > 0,$$

so that X is a proper subvariety of $J(\Gamma)$, and the polarization on X induced by that on $J(\Gamma)$ is *not* principal. Now take a local coordinate $\zeta \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ at P, and define the KP flows on $\overline{\mathrm{Pic}^{g-1}}(\Gamma)$ using the data (Γ, P, ζ).

Suppose first that π is given by $\zeta \mapsto \zeta^m$ near P, i.e., $\zeta^m \in \pi^*(\mathfrak{m}_{\pi(P)} \setminus \mathfrak{m}_{\pi(P)}^2)$. Then for any $r \in \mathbb{Z}_{>0}$ *not* divisible by m we have $\prod_{j=0}^{m-1} e^{t_r/(\varepsilon^j\zeta)^r} = 1$, where $\varepsilon = e^{2\pi i/m}$, so that, as seen from the definition of the map e in (1.12) below, we have $e(0, \dots, 0, t_r, 0, \dots) \in X$, so the r-th KP orbit of $\mathcal{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ is contained in $\mathcal{F} \otimes X := \{\mathcal{F} \otimes \mathcal{L} \mid \mathcal{L} \in X\} \subset \overline{\mathrm{Pic}^{g-1}}(\Gamma)$.

In general, since for any $r_0 \in \mathbb{Z}_{>0}$ the space $\sum_{r \le r_0} \mathbb{C}\partial/\partial t_r$ is independent of the choice of ζ, for any $\zeta \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ and $0 < r < m$ (so in particular for $r = 1$), the r-th KP orbit of $\mathcal{F}$ is contained in $\mathcal{F} \otimes X$, and so it gives an abelian solution. Let us call this the *Prym-like* case.

An important subcase of it is the quasiperiodic solutions of Novikov-Veselov (NV) or BKP hierarchies. In the Prym-like case, just as in the NV/BKP case we can put singularities to Γ and Γ_0 in such a way that X remains compact, so it is more general than the KP quasiperiodic solutions, for which $J(\Gamma)$ itself is compact. Recall that NV or BKP quasiperiodic solutions can be obtained from Prym varieties $\mathrm{Prym}(\Gamma, \iota)$ of curves Γ with involution ι having two fixed points. The Riemann theta function of $J(\Gamma)$ restricted to a suitable translate of $\mathrm{Prym}(\Gamma, \iota)$ becomes the square of another holomorphic function, which defines the principal polarization on $\mathrm{Prym}(\Gamma, \iota)$. The Prym theta function becomes NV or BKP tau function, whose square is a special KP tau function with all *even* times set to zero, so any KP time-translate of it

- gives an abelian solution of the KP hierarchy with $n = \dim X$ being one-half the genus $g(\Gamma)$ of Γ, and
- defines twice the principal polarization on X.

A natural question may be whether these conditions characterize the (time-translates of) NV or BKP quasiperiodic solutions.

Hurwitz' formula tells us that in the Prym-like case $n = \dim(X) \ge g(\Gamma)/2$, where the equality holds only in the NV/BKP case. At the moment we have no examples of abelian solutions with $1 < n < g(\Gamma)/2$.

Integrable linear equations The KP equation can be seen as the compatibility condition for a system of linear equations. In [14], it is shown that only one of the *auxiliary* linear equations

$$\left(\partial_y - \partial_x^2 + u\right)\psi = 0 \tag{1.6}$$

suffices to characterize the Jacobian locus. We shall call this an *intebrable linear equation* although here both u and ψ are unknown, and the equation should be regarded nonlinear.

This result is stronger than the one given in terms of the KP equation (see details in [4]). In terms of the Kummer map it is equivalent to the characterization of the Jacobians via flexes of the Kummer varieties, which is one out of the three particular cases of the trisecant conjecture, first formulated in [22]. Two remaining cases of the trisecant conjecture were proved in [16]. The characterization problem of the Prym varieties among indecomposable ppav was solved in [8, 15].

The notion of abelian solutions can be extended to equation (1.6). A solution (u, ψ) of equation (1.6) is *abelian* if

$$u = -2\partial_x^2 \ln \tau(Ux + z, y) \quad \text{and} \quad \psi = \frac{\tau_A(Ux + z, y)}{\tau(Ux + z, y)}\, e^{p\,x + E\,y} \tag{1.7}$$

for some p, $E \in \mathbb{C}$ and $0 \neq U \in \mathbb{C}^n$, such that $\tau_A(z, y)$ and $\tau(z, y)$ are holomorphic functions of $(z, y) \in \mathbb{C}^n \times D$, where D is a neighborhood of 0 in $\mathbb{C}$, satisfying the monodromy properties

$$\tau(z + \lambda, y) = e^{a_\lambda \cdot z + b_\lambda(y)}\tau(z, y)\,, \quad \tau_A(z + \lambda, y) = e^{a_\lambda \cdot z + c_\lambda(y)}\tau_A(z, y) \tag{1.8}$$

for all λ in the period lattice Λ of an abelian variety $X = \mathbb{C}^n/\Lambda$.

Main result Our examples of abelian solutions of KP equation (1.1) or the integrable linear equation (1.6) can be extended to rank one algebro-geometric solutions of the KP hierarchy, for which $X \subset J(\Gamma)$, with Γ being the spectral curve. In this paper we follow the lines of [14] to observe that abelian solutions of (1.1) or (1.6) are rank one algebro-geometric, and $X \subset J(\Gamma)$ holds if the group $\mathbb{C}U = \{Ux \in X \mid x \in \mathbb{C}\}$ is Zariski dense in X.

Without loss of generality it will be assumed throughout the paper that

(∗) Λ is a maximal lattice satisfying the respective monodromy property, i.e., any $\lambda \in \mathbb{C}^n$ which satisfies condition (1.3) in the case of KP equation (1.1), or condition (1.8) in the case of equation (1.6), must belong to Λ.

Theorem 1.1 *Suppose that one of the following two conditions* (A), (B) *holds:*

(A) *for any $z \in \mathbb{C}^n$, and y, t in a neighborhood of the origin in $\mathbb{C}^2$, the function u given by (1.2), with τ satisfying the monodromy condition (1.3), is an abelian solution of the KP equation (1.1);*

(B) *for any $z \in \mathbb{C}^n$, and y in a neighborhood of the origin in $\mathbb{C}$, the pair (u, ψ) given by (1.7), with τ and τ_A satisfying the monodromy condition (1.8), is an abelian solution of equation (1.6), such that the following condition holds:*

(†) *the divisors $\Theta := \{(z, y) \in X \times D \mid \tau(z, y) = 0\}$ and $\Theta_A := \{(z, y) \in X \times D \mid \tau_A(z, y) = 0\}$ have no common component.*

Suppose, moreover, that condition $(*)$ *holds. Then there exist a unique irreducible algebraic curve* Γ*, a smooth point* $P \in \Gamma$*, a subabelian variety* Y *of* X *containing* $\mathbb{C}U$*, where* u *is as in (1.2) or (1.7), an injective homomorphism* $i: Y \hookrightarrow J(\Gamma)$*, a* Y*-invariant blow-up* $\pi: \tilde{X} \to X$ *(i.e., the* Y*-action on* X *lifts to* $\tilde{X}$*) with the center contained in*

$$\Sigma := \bigcap_{x \in \mathbb{C}} (\Theta + Ux)\,, \tag{1.9}$$

and a holomorphic map j *of* $\tilde{X}$ *to the space* $\overline{\mathrm{Pic}^{g-1}}(\Gamma)$ *of torsion-free rank 1 sheaves on* Γ *of degree* $g-1$*, where* $g = g(\Gamma)$ *is the arithmetic genus of* Γ*, such that for any given* $\tilde{z} \in \tilde{X}$ *the diagram*

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{e_1} & Y & \ni z' & \longmapsto & z' + \tilde{z} & \in & \tilde{X} \\ & & \downarrow{\scriptstyle i} & & & & & \downarrow{\scriptstyle j} \\ & & J(\Gamma) & \ni \mathcal{L} & \longmapsto & \mathcal{L} \otimes j(\tilde{z}) & \in & \overline{\mathrm{Pic}^{g-1}}(\Gamma) \end{array} \tag{1.10}$$

commutes, where $e_1(x) = Ux \in Y$*, and such that, locally in* $\tilde{z} \in \tilde{X}$*,*

$$\tau(Ux + z, y, t) = \rho(\tilde{z}, y, t)\, \widehat{\tau}(x, y, t, 0, \ldots \mid \Gamma, P, j(\tilde{z})) \tag{1.11}$$

(here the t*-variable is absent in case (B)), where* $z = \pi(\tilde{z})$*,* $\widehat{\tau}(t_1, t_2, t_3, \ldots \mid \Gamma, P, \mathcal{F})$ *is the KP tau-function defined by the data* $(\Gamma, P, \mathcal{F})$*, and* $\rho(\tilde{z}, y, t) \not\equiv 0$ *is a function of* $(\tilde{z}, y, t)$ *which satisfies* $\partial_U \rho = 0$*.*

Here are some remarks:

- the locus Σ, defined in (1.9), is a unique maximal ∂_U-invariant subset of Θ, and it will be called the *singular locus*,
- the main assumptions in either case (A) or (B), i.e., (1.1) or (1.6), contain excessive information. All what is used for their proof is a certain equation valid on the τ-divisor derived in Lemmas 3.1 and 3.2 below.
- time evolutions of the KP hierarchy can be described by extending the map e_1 in (1.10):

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{e_1} & Y & \xhookrightarrow{\cdot + \tilde{z}} & \tilde{X} \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathbb{C}^\infty & \xrightarrow{e} & J(\Gamma) & \xrightarrow{\cdot \otimes j(\tilde{z})} & \overline{\mathrm{Pic}^{g-1}}(\Gamma)\,, \end{array} \tag{1.10$'$}$$

  where $j_1: x \mapsto (x, 0, 0, \ldots)$, and by taking a local coordinate $\zeta \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ at P, the homomorphism e is defined by

$$e(t_1, t_2, \ldots) = \begin{cases} \mathcal{O} \text{ near } P \text{ and on } \Gamma \setminus \{P\}, \\ \text{glued to itself around } P \text{ by } e^{\sum t_i/\zeta^i}\,. \end{cases} \tag{1.12}$$

- the factor ρ in (1.11) is needed since multiplying τ and τ_A by a quantity independent of x has no effect on (1.2) or (1.7).

Since i and e are homomorphisms, they are lifted to linear maps on the universal coverings, as readily seen for the latter in the formula for tau when Γ is smooth:

$$\widehat{\tau}(x, t_2, t_3, \dots \mid \Gamma, P, j(\tilde{z})) = \theta\Big(Ux + \sum V_i t_i + j(\tilde{z}) \;\Big|\; B(\Gamma)\Big)\, e^{Q(x,t_2,t_3,\dots)}, \tag{1.13}$$

where $V_i \in \mathbb{C}^n$, Q is a quadratic form, $B(\Gamma)$ is the matrix of B-periods of Γ, and θ is the Riemann theta function. This linearization of nonlinear t_i-dynamics provides some evidence that there might be underlying integrable systems on the spaces of higher level theta-functions on an abelian variety. The CM system is an example of such a system for $n = 1$.

Blow-up and $\mathbb{P}^1$-family of solutions The space of tau functions is the total space, say $\mathcal{B}$, of a $\mathbb{C}^*$-bundle over $\overline{\mathrm{Pic}^{g-1}}(\Gamma)$. However, given $z \in X$ our τ as a function of x, y, t (or $\widehat{\tau}$ as a function of t_1, t_2, …) might be identically zero. So this maps X to $\mathcal{B} \cup \{0\}$, a space to which the projection from $\mathcal{B}$ to $\overline{\mathrm{Pic}^{g-1}}(\Gamma)$ cannot be continuously extended. Thus we often do need to blow up X to define j in (1.10). However, we observe

Remark 1.1 *No blow-up is needed if Γ is smooth.*

Proof. After dividing τ by the trivial factors (see Section 2), we assume the locus Σ is of codimension ≥ 2 in X.

Suppose Γ is smooth. Then $\overline{\mathrm{Pic}^{g-1}}(\Gamma) = J(\Gamma)$, and it is an abelian variety. Then any holomorphic map from $\mathbb{P}^1$ to it must be constant. Indeed, since $\mathbb{P}^1$ is simply connected, any such map can be lifted to a map from $\mathbb{P}^1$ to the universal covering of $J(\Gamma)$, i.e., an affine space. Hence it must be constant.

Assuming $\Sigma \neq \emptyset$, take any point $p_0 \in \Sigma$, and take a 2-dimensional plane $\Pi \subset X$ such that locally near p_0, the loci Σ and Π meet only at p_0. Take a coordinate system (a, b) on Π such that p_0 is the origin $a = b = 0$, and restrict the range of z to Π to obtain a family of τ parametrized by (a, b). Taylor expanding τ in a, b:

$$\tau(x, y; a, b) = \sum_{m,n \geq 0} \tau_{m,n}(x, y) a^m b^n,$$

where we omit the t-variable in case (A) (or the sequence t_3, t_4, … if τ is a KP τ-function), let N be the set of indices (m, n) for which $\tau_{m,n} \not\equiv 0$. Since $\tau(x, y, 0, 0) \equiv 0$ and τ is not divisible by a or b, we have $(0, 0) \notin N$, and $(m, 0) \in N$, $(0, n) \in N$ for some m, $n > 0$. Hence there exist positive integers p, q and C such that $N \subset \{(m, n) \mid pm + qn \geq C\}$, and such that N meets the line $pm + qn = C$ at least at two points. Then as the "lowest order" part of τ,

$$\tilde{\tau}(x, y; a, b) := \sum_{pm+qn=C} \tau_{m,n}(x, y) a^m b^n$$

is a family of solutions, and it is a weighted homogeneous polynomial of a and b. Then $\tilde{\tau}(x, y; a^p, b^q)$ is an (unweighted) homogeneous polynomial, giving a $\mathbb{P}^1$-family of τ mod $\mathbb{C}^\times$. Then by the fact noted above, this must be a constant family, so all the $\tau_{m,n}$ on the line $pm+qn = C$ must be a constant multiple of the same τ. Observing this on every edge of the polygon N, we see that as $(a, b) \to (0, 0)$ we have a well-defined limit of the corresponding sheaf $\mathcal{F}$, which means no blow-up is necessary around p_0. Since the point $p_0 \in \Sigma$ and the plane $\Pi \ni p_0$ are arbitrary, no blow-up is needed at all, so the remark follows.

Having this in mind, let us start with a curve with a node, and constract a nontrivial family of τ-functions of the form

$$\tau(\mathbf{t}, z'; a, b) = a\tau_0(\mathbf{t}, z') + b\tau_1(\mathbf{t}, z'), \quad (a, b) \in \mathbb{C}^2 \setminus \{(0,0)\}, \tag{1.14}$$

where τ_0 and τ_1 (and hence the entire family) depend on the same parameters $z' \in \mathbb{C}^d$ in such a way that $\tau_i(\mathbf{t}, z') = \tau_i(z' + Ut_1, t_2, \ldots)$, and satisfy the same monodromy conditions for a lattice $\Lambda' \subset \mathbb{C}^d$.

Such a family of quasiperiodic τ-functions should yield an example of abelian solutions for which blow-up is really needed: take an abelian variety $Z = \mathbb{C}^n/\Lambda$, and two $\mathbb{C}$-linearly independent functions θ_0 and θ_1 on $\mathbb{C}^n$ which satisfy the same monodromy conditions with respect to Λ (so the ratio θ_0/θ_1 is a meromorphic function on Z), let $Y = \mathbb{C}^d/\Lambda'$, $X = Y \times Z$, denote $X \ni z = (z', z'')$, with $z' \in Y$ and $z'' \in Z$, and define $\tau(\mathbf{t}, z)$ by replacing a and b in (1.14) by $\theta_0(z'')$ and $\theta_1(z'')$, respectively, i.e.,

$$\tau(\mathbf{t}, z', z'') = \theta_0(z'')\tau_0(\mathbf{t}, z') + \theta_1(z'')\tau_1(\mathbf{t}, z').$$

We need to blow up X along the intersection of zero loci of θ_0 and θ_1 to define a map to $\overline{\mathrm{Pic}^{g-1}}(\Gamma)$, where g is the arithmetic genus of Γ. Note also that the KP hierarchy has no control over the 2nd factor Z.

Construction of family (1.14) goes as follows. First, consider a simple Backlund transform applied to any quasiperiodic τ-function $\tau_0(\mathbf{t})$, $\mathbf{t} = (t_1, t_2, \ldots)$. This yields a family of τ-functions of the form (1.14), where

$$\tau_1 := X(p, q)\tau_0 := \exp\Big(\sum t_i(p^i - q^i)\Big) \exp\Big(\sum \frac{q^{-i} - p^{-i}}{i} \frac{\partial}{\partial t_i}\Big)\tau_0 \tag{1.15}$$

using Date et al.'s notation for vertex operator [7]. It is more common to take $a = 1$, but formula (1.14) gives a tau function as long as $(a, b) \neq (0, 0)$. Let us try to make τ_0 and τ_1 satisfy the same monodromy conditions. If τ_0 is a quasiperiodic solution associated with a smooth curve $\tilde{\Gamma}$ and a point $P \in \tilde{\Gamma}$, the effect of $a + bX(p, q)$ on τ_0 is to identify the points p and q on $\tilde{\Gamma}$ to make a curve Γ with node, and the fibres of line bundle on $\tilde{\Gamma}$ at p and q to obtain a line bundle on Γ if the ratio b/a is not 0 or ∞, or a torsion-free rank 1 sheaf on Γ in general.

As we saw in the paragraph on Prym-like solutions, suitably chosen $\tilde{\Gamma}$, P, p and q make entire family (1.14) of τ-functions quasiperiodic in t_1: suppose there exists a ramified covering map $\tilde{\pi}$ of $\tilde{\Gamma}$ to another smooth curve $\tilde{\Gamma}_0$ of genus $g_0 \geq 0$, such that P, p and q are branch points, and such that $\tilde{\pi}^{-1}(\tilde{\pi}(p)) = \{p\}$ and $\tilde{\pi}^{-1}(\tilde{\pi}(q)) = \{q\}$ hold. Identify p and q on $\tilde{\Gamma}$, and $\tilde{\pi}(p)$ and $\tilde{\pi}(q)$ on $\tilde{\Gamma}_0$ to obtain curves with nodes Γ and Γ_0, respectively, with a covering map $\pi\colon \Gamma \to \Gamma_0$. Note that $p_a(\Gamma_0) = g_0 + 1 \geq 1$. Since $J(\Gamma)$ (resp. $J(\Gamma_0)$) is a $\mathbb{C}^\times$-extension of $J(\tilde{\Gamma})$ (resp. $J(\tilde{\Gamma}_0)$), and since the restriction of $\mathrm{Nm}\colon J(\Gamma) \to J(\Gamma_0)$ to the $\mathbb{C}^\times$ does not vanish, the identity component Y of $\ker(\mathrm{Nm})$ is an abelian variety isogenous to that of $\ker(\widetilde{\mathrm{Nm}}\colon J(\tilde{\Gamma}) \to J(\tilde{\Gamma}_0))$. Hence, as seen in our construction of "Prym-like" solutions, the t_1-evolution associated to (Γ, P) is contained in Y and hence quasiperiodic, and for any $(a,b) \neq (0,0)$ the solution τ in (1.14) is quasiperiodic in t_1.

Next, let us construct a more explicit example. Starting with an elliptic curve $\tilde{\Gamma}$, we can easily adjust the monodromy conditions of τ_0 and τ_1 so that, after doubling one of the fundamental periods (or replacing $J(\tilde{\Gamma}) \simeq \tilde{\Gamma}$ by a double cover, Y, of it), τ_0 and τ_1 satisfy the same monodromy conditions. In this example we take $\tilde{\Gamma}_0 = \mathbb{P}^1$, so that $\ker(\widetilde{\mathrm{Nm}}) = J(\tilde{\Gamma}) = \tilde{\Gamma}$. That Y is a double cover of it fits the general picture above.

For brevity we restrict ourselves to the first three time variables $(t_1, t_2, t_3) = (x, y, t)$, and consider the first KP equation (1.1) only. It is a simple exercise on elliptic functions to work out the formulae for the whole KP hierarchy.

For $2\omega_1, 2\omega_3 \in \mathbb{C}^\times$ with $\Im(\omega_3/\omega_1) > 0$, let $\Lambda_0 := 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_3$ and $\tilde{\Gamma} := \mathbb{C}/\Lambda_0$. Denote by $\sum'$ (resp. $\prod'$) the sum (resp. product) over all $\omega \in \Lambda_0 \setminus \{0\}$. Defining Weierstrass' σ-function by

$$\sigma(z) := z \prod{}'(1 - z/\omega)\exp(z/\omega + (z/\omega)^2/2)$$

and using a well-known differential equation for $\wp(z) = -(\ln\sigma(z))''$, we have

$$\frac{D_z^4 \sigma \cdot \sigma}{\sigma^2} \equiv 2\partial_z^4 \ln\sigma + 12(\partial_z^2 \ln\sigma)^2 = g_2 := 60 \sum{}' \omega^{-4}\,.$$

Hence $\tau_0(x,y,t) := \tau_0(x,y,t,z) := e^{\alpha x t + \beta y^2}\sigma(z+x)$ is a z-dependent solution to the KP equation iff

$$g_2 + 12\beta - 8\alpha = 0. \tag{1.16}$$

Moreover, setting $\zeta(z) = (\ln\sigma(z))'$, $\omega_2 := -\omega_1 - \omega_3$ and $\eta_\nu = \zeta(\omega_\nu)$, where $\nu = 1, 2, 3$, we have

$$\sigma(z + 2\omega_\nu) = -\sigma(z)\exp(2\eta_\nu(z + \omega_\nu))$$

and

$$\begin{vmatrix} \eta_1 & \eta_3 \\ \omega_1 & \omega_3 \end{vmatrix} = \begin{vmatrix} \eta_3 & \eta_2 \\ \omega_3 & \omega_2 \end{vmatrix} = \begin{vmatrix} \eta_2 & \eta_1 \\ \omega_2 & \omega_1 \end{vmatrix} = \frac{\pi i}{2},$$

so that Weierstrass' co-sigma functions $\sigma_\mu(z) := \exp(-\eta_\mu z)\sigma(z + \omega_\mu)/\sigma(\omega_\mu)$, $\mu = 1, 2, 3$, satisfy

$$\sigma_\mu(z + 2\omega_\nu) = (-1)^{\delta_{\mu,\nu}} \exp(2\eta_\nu(z + \omega_\nu))\sigma_\mu(z), \quad \nu = 1, 2, 3,$$

i.e., σ_μ satisfies the same monodromy conditions as σ for the periods $2\omega_\mu$ and $4\omega_\nu$ $(\nu \neq \mu)$. On the other hand, (1.15) implies

$$\frac{\tau_1}{\tau_0} = C_{y,t}\exp(Ax)\frac{\sigma(x - 1/p + 1/q)}{\sigma(x)}, \tag{1.17}$$

where $A := p - q + \alpha(-1/p^3 + 1/q^3)/3$, and $C_{y,t} \neq 0$ is independent of x. Therefore, if we choose p, q and α so that

$$-\frac{1}{p} + \frac{1}{q} = \omega_\mu \quad \text{and} \quad p - q + \frac{\alpha}{3}\Big(-\frac{1}{p^3} + \frac{1}{q^3}\Big) = 2\eta_\mu \tag{1.18}$$

hold for some $\mu \in \{1,2,3\}$, then the right-hand side of (1.17) becomes σ_μ/σ up to a factor independent of x, so that τ_0 and τ_1 satisfy the same monodromy conditions for the periods $2\omega_\mu$ and $4\omega_\nu$ ($\nu \neq \mu$). We thus constructed a $\mathbb{P}^1$-family of solutions quasiperiodic with respect to the lattice $2\mathbb{Z}\omega_\mu + 4\mathbb{Z}\omega_\nu$, where $\nu \in \{1,2,3\} \setminus \{\mu\}$ is arbitrary.

The paper is organized as follows. In Sect. 2 we show basic properties of the zero loci of τ and τ_A, and observe the nature of condition (†). In Sect. 3 we construct a formal wave function, which is used in Sect. 4 to obtain commuting differential operators. Almost till the very end the proof of Theorem 1.1 goes along the lines of [21] (in case (A)) or [14] (in case (B)). Constructing a wave function is easier in case (A) than in case (B), and the rest of the proof is the same for both cases, so in what follows we mainly consider case (B).

2 Zero loci of τ and τ_A

Before constructing the wave function, let us observe some properties of the zero loci of τ and τ_A.

For a constant coefficient polynomial $P(\xi, \eta, \ldots)$, Hirota's bilinear differential operator $P(D) = P(D_x, D_y, \ldots)$ is defined by

$$P(D) f \cdot g := P(\partial_{x'}, \partial_{y'}, \ldots) f(x + x', y + y', \ldots) g(x - x', y - y', \ldots)\Big|_{x'=y'=\cdots=0}.$$

Putting (1.7) into the left-hand side of (1.6) and using

$$\partial_y \frac{\tau_A}{\tau} = \frac{D_y \tau_A \cdot \tau}{\tau^2}, \quad \partial_x \frac{\tau_A}{\tau} = \frac{D_x \tau_A \cdot \tau}{\tau^2} \quad \text{and} \quad \partial_x^2 \frac{\tau_A}{\tau} = \frac{D_x^2 \tau_A \cdot \tau}{\tau^2} - 2 \frac{\tau_A}{\tau} \frac{D_x^2 \tau \cdot \tau}{\tau^2},$$

we have $e^{-px-Ey} \tau^2 (\partial_y - \partial_x^2 + u)\psi = ((D_y + E) - (D_x + p)^2) \tau_A \cdot \tau$, so (1.6) is equivalent to

$$((D_y + E) - (D_x + p)^2) \tau_A \cdot \tau = 0. \tag{2.1}$$

This readily shows the symmetry $(x, y) \leftrightarrow (-x, -y)$, $\tau \leftrightarrow \tau_A$ of equation (1.6), and it is also handy to find the possible forms of common factors of τ and τ_A:

Lemma 2.1 *If*

$$\tau(x, y) = (x - x(y))^b \varphi \quad \textit{and} \quad \tau_A(x, y) = (x - x(y))^a \varphi_A, \tag{2.2}$$

where φ, $\varphi_A \neq 0$ at $x = x(y)$, then

$$a = \frac{\nu(\nu + 1)}{2}, \qquad b = \frac{\nu(\nu - 1)}{2}, \tag{2.3}$$

for some ν $(= a - b) \in \mathbb{Z}$. Conversely, for any $\nu \in \mathbb{Z}$ there is a solution of (2.1) of the form (2.2) with a, b given by (2.3).

Indeed, putting (2.2) into the left-hand side of (2.1) yields

$$-D_x^2\tau_A\cdot\tau+\cdots=-C_{ab}(x-x(y))^{a+b-2}\varphi_A\varphi+O((x-x(y))^{a+b-1})$$

with $C_{ab}=(a-b)^2-(a+b)$, which vanishes iff (2.3) holds. Conversely, for any holomorphic solution of the heat equation

$$f_y=f_{xx}\,,$$

e.g., $f\in\exp(y\partial_x^2)\mathbb{C}[x]$, the pair

$$\tau_A=e^{-px-Ey}f(x,\nu y)^a\quad\text{and}\quad\tau=f(x,\nu y)^b$$

give a solution of (2.1) of the form (2.2). This proves the lemma.

We have $|\nu|\le 1$ if ψ is a KP wave function evaluated at a finite value of spectral parameter k, so a nonempty zero locus with higher $|\nu|$ is an obstruction for the extension problem to be discussed in Sect. 3. Rather than trying to see what the occurrence of ($|\nu|>1$)-locus means to quasiperiodic solutions, in this paper we will simply choose to exclude these cases by assuming condition (†) in p. 4, or any one of the following:

(†′) ψ generically has a simple pole along $\Theta\setminus\Sigma$;

(‡) Θ and Θ_A are reduced, i.e., the zeros of τ and τ_A are generically simple;

(‡′) Θ and Θ_A are irreducible;

(‡″) Θ or Θ_A is reduced and irreducible.

Indeed, we have

Lemma 2.2 *For a solution of (1.6), conditions* (†), (†′) *and that* $|\nu|\le 1$ *on all components of* $\Theta\setminus\Sigma$*, are all equivalent, and if the solution is quasiperiodic, then they follow from any one of* (‡), (‡′) *and* (‡″).

Proof. Since a and b are positive (and one of them is greater than 1) when $|\nu|>1$, the first assertion is almost obvious. To be precise, we have to see that ν is constant on each component of Θ or Θ_A, or at least that the $|\nu|>1$ case cannot deform into the $|\nu|\le 1$ case, i.e., there is no parameter-dependent solution $(\tau(x,y,\zeta),\tau_A(x,y,\zeta))$ of (2.1) which looks like (2.2) with a, $b>0$ when the parameter $\zeta=0$ but not when $\zeta\ne 0$. Indeed, such a deformation would imply that when ζ is close to 0 there are b simple zeros of τ (and a simple zeros of τ_A) staying arbitrarily close to each other in a fixed interval of y. Since a or b must be greater than 1, we see, even without calculations, that this is unlikely from the usual, CM-like particle system interpretation of the motion of zeros of τ when $|\nu|\le 1$.

That (‡) implies $|\nu|\le 1$ is also equally obvious (the two conditions are equivalent if Σ is of codimension ≥ 2 in X).

Next, if Θ and Θ_A are irreducible and $|\nu|>1$ somewhere, then we have the same ν (and the same, unequal and positive a and b) all over Θ and Θ_A which have the same underlying set. Then τ and τ_A cannot define the same polarization on X which contradicts (1.8). Hence condition (‡′) also excludes the possibility of having $|\nu|>1$.

Criterion (‡″) may be useful since it involves only τ (or τ_A). If, e.g., Θ is reduced, then $b\le 1$, and the only case with $|\nu|>1$ we can have is $\nu=2$ ($a=3$, $b=1$). If, moreover, Θ is irreducible, then as a divisor $\Theta_A \geq 3\Theta$, so again τ and τ_A cannot define the same polarization on X. This completes the proof of the lemma.

Thus, in what follows we assume $|\nu| \leq 1$, so τ and τ_A have no common factor depending on x, and ψ has a simple pole along $\Theta \setminus \Sigma$. The latter form of the condition will be used in Sect. 3.

A pair (τ, τ_A) can also have "trivial" common factors. If $(\tau_0(x,y), \tau_{A0}(x,y))$ solves equation (2.1), then so does

$$(\rho(y)\tau_0(x,y), \rho(y)\tau_{A0}(x,y)) \tag{2.4}$$

for any $\rho(y)$. Such a factor $\rho(y)$ itself is harmless, but it may not if, e.g., it deforms into an x-*dependent* factor in a solution with parameters. So let us introduce a parameter z', and prove that a family of solutions of (2.1) which is a deformation of the pair in (2.4) must be of the form

$$(\rho(y,z')\tau_0(x,y;z'), \rho(y,z')\tau_{A0}(x,y;z'))\,, \tag{2.5}$$

where $(\tau_0(x,y;z'), \tau_{A0}(x,y;z'))$ is a family of solutions of (2.1) and $\rho(y,z')$ is a function of (y,z') (independent of x) such that

$$(\tau_0(x,y;0), \tau_{A0}(x,y;0)) = (\tau_0(x,y), \tau_{A0}(x,y)) \quad \text{and} \quad \rho(y,0) = \rho(y)\,.$$

As in the KP case, such a factorization is not free, but it can be proved using quasiperiodicity:

Lemma 2.3 *Let D and D' be neighborhoods of* 0 *in $\mathbb{C}$, let $d \in \mathbb{Z}_{>0}$, let Λ be a lattice in $\mathbb{C}^d$, and let $U \in \mathbb{C}^d$ be such that $\mathbb{C}U \bmod \Lambda$ is Zariski dense in $Y := \mathbb{C}^d/\Lambda$. Let (τ, τ_A) be a pair of functions defined on $\mathbb{C}^d \times D \times D'$, such that*

i) as a pair of functions of $(x,y) \in \mathbb{C} \times D$, $(\tau(z+Ux,y,z'), \tau_A(z+Ux,y,z'))$ solves (2.1),

ii) τ and τ_A satisfy the same monodromy conditions in z: for all $\lambda \in \Lambda$, there exist $a_\lambda \in \mathbb{C}^d$, $b_\lambda(y,z') \in \mathbb{C}$ such that

$$\tau(z+\lambda,y,z') = e^{a_\lambda \cdot z + b_\lambda(y,z')}\tau(z,y,z')\,, \quad \tau_A(z+\lambda,y,z') = e^{a_\lambda \cdot z + b_\lambda(y,z')}\tau_A(z,y,z')\,,$$

iii) $\tau(z,y,0) = y^m\tau_0(z,y)$, $\tau_A(z,y,0) = y^m\tau_{A0}(z,y)$ for some functions τ_0, τ_{A0} and $m \in \mathbb{Z}_{>0}$.

Then the whole family (τ, τ_A) must be factored as in (2.5), i.e., there exist a pair of functions (τ_0, τ_{A0}) defined on $\mathbb{C}^d \times D \times D'$ and a function ρ defined on $D \times D'$ such that $\rho(y,0) = y^m$, $\tau_0(z,y,0) = \tau_0(z,y)$ and $\tau_{A0}(z,y,0) = \tau_{A0}(z,y)$, and such that the factoring in iii) extends to z' away from 0*:*

$$\tau(z,y,z') = \rho(y,z')\tau_0(z,y,z')\,, \quad \tau_A(z,y,z') = \rho(y,z')\tau_{A0}(z,y,z')\,.$$

This is how we get the ρ in (1.11). Our choice of the factor y^m in iii) is for notational simplicity only. One can replace it by a more general $\rho_0(y)$.

Dividing τ and τ_A by the trivial factors, we may assume that Σ and

$$\Sigma_A := \bigcap_{x \in \mathbb{C}} (\Theta_A + Ux) \tag{2.6}$$

are of codimension ≥ 2 in $X \times D$. Then we can prove that $\Sigma = \Sigma_A$, and that Σ is not only ∂_U-invariant but also ∂_y-invariant.

Note that the division by trivial factors may change the monodromy condition (1.8), but it will not affect our argument in the following sections since the trivial factors ρ, λ_1 and λ_2 must be constant in the directions of the Zariski closure Y_U of line $\mathbb{C}U$ in X.

3 Construction of the wave function

In the core of the proof of Theorem is the construction of quasiperiodic wave function as in (3.8) below. Having a spectral parameter k, it contains much more information than the function ψ in (1.7). Taking (1.6) as a starting point, we closely follow the argument from the beginning of Section 2 through Lemma 3.2 of [14]. The construction is presented in two steps, first locally to show that (1.6) guarantees the single-valuedness of the wave function around each *simple* zero in x of $\tau(x, y)$, and then globally to maintain quasiperiodicity.

Step 1 Let $\tau(x, y)$ be a holomorphic function of the variable x in some domain in $\mathbb{C}$, depending smoothly on a parameter y and having only simple zeros at $x = x_i(y)$:

$$\tau(x_i(y), y) = 0, \quad \tau_x(x_i(y), y) \neq 0. \tag{3.1}$$

Let v_i and w_i be the second and the third Laurent coefficients of $u(x, y) = -2\partial_x^2 \ln \tau(x, y)$ at $x = x_i$, i.e.,

$$u(x, y) = \frac{2}{(x - x_i(y))^2} + v_i(y) + w_i(y)(x - x_i(y)) + \cdots . \tag{3.2}$$

Lemma 3.1 ([4]) *If equation (1.6) with the potential $u = -2\partial_x^2 \ln \tau(x, y)$ has a meromorphic solution $\psi_0(x, y)$, then the equations*

$$\ddot{x}_i = 2w_i \tag{3.3}$$

hold, where the "dots" stand for y-derivatives.

Proof. Consider the Laurent expansion of ψ_0 at $x = x_i$:

$$\psi_0 = \frac{\alpha_i}{x - x_i} + \beta_i + \gamma_i(x - x_i) + \delta_i(x - x_i)^2 + O((x - x_i)^3) . \tag{3.4}$$

All coefficients in this expansion are smooth functions of the variable y, and $\alpha_i \not\equiv 0$ due to condition ($\dagger'$) in p. 10. Substituting (3.2) and (3.4) into (1.6) gives a system of equations. The first three of them are

$$\alpha_i \dot{x}_i + 2\beta_i = 0 , \tag{3.5}$$

$$\dot{\alpha}_i + \alpha_i v_i + 2\gamma_i = 0 , \tag{3.6}$$

$$\dot{\beta}_i + v_i \beta_i - \gamma_i \dot{x}_i + \alpha_i w_i = 0 . \tag{3.7}$$

Taking the y-derivative of the first equation and using the others, we get (3.3).

The equation (3.3) is all what we are going to use below. Let us show that it is valid for any meromorphic solution of the KP equation. Namely: let $\tau(x,y,t)$ be a a holomorphic function of the variable x in some domain in $\mathbb{C}$, depending smoothly on parameters y, t and having only simple zeros $x = x_i(y,t)$.

Lemma 3.2 *If the function $u = -2\partial_x^2 \ln \tau(x,y,t)$ is a solution of the KP equation then the equations (3.3) hold.*

Proof. For the proof of the theorem it is enough to substitute the Laurant expansion of u at $x(y,t)$ into the KP equation and consider the coefficient in front of $(x - x_i(y,t))^{-3}$.

Next, let us show that equations (3.3) are sufficient for the existence of meromorphic wave solutions, i.e., solutions of the form

$$\psi(x,y,k) = e^{kx+(k^2+b)y}\left(1 + \sum_{s=1}^{\infty} \xi_s(x,y)\, k^{-s}\right), \tag{3.8}$$

where b is a constant, ξ_s are meromorphic functions, and the series in parentheses is a formal power series in k^{-1}.

Lemma 3.3 *Suppose that equations (3.3) for the zeros of $\tau(x,y)$ hold. Then there exist meromorphic wave solutions of equation (1.6) that have simple poles at x_i and are holomorphic everywhere else.*

Proof. Substituting (3.8) into (1.6) gives a recurrent system of equations

$$2\xi'_{s+1} = \dot{\xi}_s + (u+b)\xi_s - \xi''_s. \tag{3.9}$$

Adding b to u does not change the coefficient w_i in the expansion (3.2), so the presense of e^{by} in (3.8) has no effect on the assertion of the lemma. We are going to prove by induction that this system has meromorphic solutions with simple poles at $x = x_i$.

Let us expand ξ_s at $x = x_i$:

$$\xi_s = \frac{r_s}{x - x_i} + r_{s0} + r_{s1}(x - x_i) + O((x-x_i)^2), \tag{3.10}$$

where we omit the index i in the notation for the coefficients of this expansion, since it suffices to look at a neighborhood of each x_i. Suppose that ξ_s are defined and equation (3.9) has a meromorphic solution. Then the right-hand side of (3.9) has no residue at $x = x_i$, i.e.,

$$\mathrm{res}_{x_i}\,(\partial_y \xi_s + u\xi_s - \xi''_s) = \dot{r}_s + v_i r_s + 2r_{s1} = 0 \tag{3.11}$$

We need to show that the residue of the next equation vanishes also. From (3.9) it follows that the coefficients of the Laurent expansion for ξ_{s+1} are equal to

$$r_{s+1} = -\dot{x}_i r_s - 2r_{s0}, \tag{3.12}$$

$$2r_{s+1,1} = \dot{r}_{s0} - r_{s1} + w_i r_s + v_i r_{s0}. \tag{3.13}$$

These equations imply

$$\dot{r}_{s+1} + v_i r_{s+1} + 2r_{s+1,1} = -r_s(\ddot{x}_i - 2w_i) - \dot{x}_i(\dot{r}_s - v_i r_s s + 2r_{s1}) = 0,$$

and the lemma is proved.

Step 2 Let us now reintroduce z-dependence to τ, so that it is a function of $z + Ux \in \mathbb{C}^n$ and y. Our goal is to fix a *translation-invariant* normalization of ξ_s to define wave functions uniquely up to an x-independent factor.

We assume that y runs over a small neighborhood D of $0 \in \mathbb{C}$, and let $\mathbb{C}^{n*} := \mathbb{C}^n \times D$. Identify $\mathbb{C}^n$ with $\mathbb{C}^n \times \{0\} \subset \mathbb{C}^{n*}$, and hence $U \in \mathbb{C}^n$ with $(U, 0) \in \mathbb{C}^n \times \mathbb{C}$ and Λ with $\Lambda \times \{0\}$. A Λ-invariant subset of $\mathbb{C}^{n*}$ will be regarded as a subset of $X^* := X \times D = \mathbb{C}^{n*}/\Lambda$. Let $\Theta := \{(z, y) \in X^* \mid \tau(z, y) = 0\}$, $\Theta_1 := \{(z, y) \in X^* \mid \tau(z, y) = \partial_U \tau(z, y) = 0\}$. The singular locus $\Sigma = \bigcap_{x \in \mathbb{C}} (\Theta + Ux) = \bigcap_{x \in \mathbb{C}} (\Theta_1 + Ux)$ is a unique maximal $\mathbb{C}U$-invariant subset of Θ_1. As we observed in Section 2, dividing τ and τ_A by suitable ∂_U-invariant functions, we assume that Σ and Σ_A are of codimension ≥ 2 in X. Then Θ_1 is also of codimension ≥ 2 in X. Let $Y_U = \langle \mathbb{C}U \rangle$ be the Zariski closure of the group $\mathbb{C}U$ in X. Since it is a minimal $\mathbb{C}U$-invariant closed subset of X^*, Σ and Σ_A are Y_U-invariant, so that for any $(z, y) \in X^*$ we have either $Y_U \cap (\Sigma - (z, y)) = \emptyset$ or $Y_U \subset \Sigma - (z, y)$. The former is true outside a set of (z, y) of codimension ≥ 2 in X.

Let $\pi\colon \mathbb{C}^{n*} \to X^*$ be the covering map, let $\mathbb{C}^d = \pi^{-1}(Y_U)^0$ be the connected component of $\pi^{-1}(Y_U)$ through the origin, and let $\Lambda_U := \Lambda \cap \mathbb{C}^d$. Since $Y_U = \mathbb{C}^d/\Lambda_U$ is compact, Λ_U is a lattice in $\mathbb{C}^d$. Taking another vector subspace H of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{C}^d \oplus H$, we can write any $z \in \mathbb{C}^n$ as $z = z' + z''$, where $z' \in \mathbb{C}^d$ and $z'' \in H$. Consider τ as a function of $z' \in \mathbb{C}^d$ and $y \in D$ depending on parameters $z'' \in H$. The function $u(z, y) = -2\partial_U^2 \ln \tau$ is periodic with respect to Λ_U and, for each (z'', y), has a double pole in z' along the divisor $\Theta^U(z'', y) := (\Theta - (z'', y)) \cap Y_U$.

Lemma 3.4 *Suppose the equation*

$$\operatorname{res}_x \left(\partial_y^2 \ln \tau + 2 \left(\partial_x^2 \ln \tau \right)^2 \right) = 0 \tag{3.14}$$

for $\tau(Ux + z, y)$ holds, and let $\lambda_1, \ldots, \lambda_d$ be $\mathbb{C}$-linearly independent vectors in Λ_U. Then

(i) equation (1.6) with the potential $u(Ux + z, y)$ has a wave solution of the form $\psi = e^{kx + k^2 y} \phi(Ux + z, y, k)$ such that the coefficients $\xi_s(z, y)$ of the formal series

$$\phi(z, y, k) = e^{by} \left(1 + \sum_{s=1}^{\infty} \xi_s(z, y)\, k^{-s} \right) \tag{3.15}$$

are $(\lambda_1, \ldots, \lambda_d)$-periodic meromorphic functions of $(z, y) \in \mathbb{C}^{n*}$ with a simple pole along the divisor Θ^U,

$$\xi_s(z + \lambda_i, y) = \xi_s(z, y) = \frac{\tau_s(z, y)}{\tau(z, y)}, \quad i = 1, \ldots, d\,; \tag{3.16}$$

(ii) $\phi(z, y, k)$ is unique up to a factor which is ∂_U-invariant and holomorphic in z, i.e., if ϕ and ϕ_1 are two solutions, then we have

$$\phi_1(z, y, k) = \phi(z, y, k) \rho(z'', k)\,. \tag{3.17}$$

Proof. Let us temporarily modify formula (3.15) for ϕ:

$$\phi(z, y, k) = e^{by + \sum_{s=1}^{\infty} b_s(y) k^{-s}} \left(1 + \sum_{s=1}^{\infty} \xi_s(z, y)\, k^{-s} \right), \tag{3.15$'$}$$

where $b_s(y) = b_s(z'', y)$ are functions of y and z'', i.e., they are independent of z', such that $b_s(0) = b_s(z'', 0) = 0$. The factor $e^{\sum_{s=1}^{\infty} b_s(y)k^{-s}}$ can later be absorbed into $1 + \sum \xi_s k^{-s}$, so it is redundant, and harmless.

Substituting $\psi = e^{kx+k^2y}\phi$ and (3.15′) into equation (1.6), we find the recursion formulas

$$2\partial_U \xi_{s+1} = (\partial_y - \partial_U^2 + (u+b))\xi_s + \sum_{i=1}^{s} b_i' \xi_{s-i}\,, \quad s = 0,\ 1, \ldots, \tag{3.18}$$

where we set $\xi_0 = 1$. The first equation $2\partial_U \xi_1 = u + b$ can be solved explicitly:

$$\xi_1 = -\partial_U \ln \tau + (l_1, z)\,, \tag{3.19}$$

for a linear form $(l_1, \cdot)$ on $\mathbb{C}^d$, and $b = 2(l_1, U)$. The periodicity condition (3.16) for $s = 1$ is satisfied if and only if

$$(l_1, \lambda_i) = \partial_U \ln \tau(z + \lambda_i, y) - \partial_U \ln \tau(z, y) = a_{\lambda_i} \cdot U\,, \quad i = 1, \ldots, d, \tag{3.20}$$

where the last equality follows from (1.8). Since $\lambda_1, \ldots, \lambda_d$ are linearly independent vectors in $\mathbb{C}^d$, this determines the linear form $(l_1, \cdot)$ uniquely. The form $(l_1, \cdot)$ is independent of z and y since the right-hand side of (3.20) is, so $b = (l_1, U)$ is a constant.

Suppose we have $(\lambda_1, \ldots, \lambda_d)$-periodic functions $\xi_1, \ldots, \xi_r$, constant b, and functions $b_1(y), \ldots, b_{r-1}(y)$ of y solving equations (3.18) for $s < r$, and consider the equation for $s = r$, with tentative choice of $b_r \equiv 0$.

Equation (3.14) implies equations (3.3), which, as seen in Lemma 3.3, are sufficient for the local solvability of (3.9), and hence (3.18), away from the locus Θ_1 of multiple zeros of τ as a function of x. This set does not contain a ∂_U-invariant line away from Σ which is of codimension ≥ 2. Therefore, the sheaf $\mathcal{V}_0$ of ∂_U-invariant meromorphic functions on $\mathbb{C}^{n*} \setminus \Theta_1^U$ with poles along the divisor Θ^U coincides with the sheaf of ∂_U-invariant holomorphic functions. This implies the vanishing of $H^1(\mathbb{C}^{n*} \setminus \Theta_1^U, \mathcal{V}_0)$ and the existence of global meromorphic solutions ξ_s of (3.18), with a simple pole along the divisor Θ^U (see details in [21, 3]).

Let ξ_{r+1}^0 be a solution, not necessarily periodic, of (3.18) for $s = r$ with $b_r \equiv 0$. Then (3.18) implies that $\xi_{r+1}^0(z + \lambda_i, y) - \xi_{r+1}^0(z, y)$, $i = 1, \ldots, d$, are constant in the U-direction, hence they are constant on each translate of Y_U. Let $(l_{r+1}, \cdot)$ be a linear form on $\mathbb{C}^d$, depending on y and z'', such that

$$(l_{r+1}, \lambda_i) = \xi_{r+1}^0(z + \lambda_i, y) - \xi_{r+1}^0(z, y)\,, \quad i = 1, \ldots, d\,,$$

hold. Then

$$\xi_{r+1}(z, y) = \xi_{r+1}^0(z, y) + (l_{r+1}, z) \quad \text{and} \quad b_r(y) = 2\int_0^y (l_{r+1}(y'), U)dy', \tag{3.21}$$

together with the previously chosen ξ_s $(s \leq r)$ and b_s $(s < r)$, give a $(\lambda_1, \ldots, \lambda_d)$-periodic solution of (3.18) for $s = r$. This completes the induction step, proving (i) except for the latter half of (3.16), which is obvious if we compare the poles of ξ_s and the zeros of τ.

In each step of this construction, equation (3.18) determines ξ_{s+1} uniquely up to an additive constant $c_{s+1}(z'')$ depending on z''. Indeed, the constant may depend on y, but the effect of this y-dependence will be cancelled by b_{s+1} to be chosen in the next step, so we assume the $c_{s+1}(z'')$ is independent of y. Adding $c_{s+1}(z'')$ to $\xi_{s+1}(z,y)$ will affect the later steps of construction, but in terms of ϕ all the necessary changes can be done just by multiplying it by $1+c_{s+1}(z'')k^{-s-1}$. This proves (ii), with $\rho(z'',k)=\prod_s(1+c_{s+1}(z'')k^{-s-1})$. The lemma is thus proven.

4 Commuting differential operators

In this section, using the wave function ψ we show the existence of sufficiently many commuting differential operators, to obtain the curve Γ. From the point of view of the KP hierarchy, this amounts to showing the finite dimensionality of the orbit. There can be several approaches. For instance, given our specific form of quasiperiodicity condition, i.e., that τ is of the form $\tau(Ux+z,y)$ and is periodic in z with periods $\lambda_1, \ldots, \lambda_d$, knowing its zero locus is enough to recover τ, and hence u. Denote by Θ_0 the space of ample divisors on X which belong to the given polarization. Then equations (3.3) may be seen as a dynamical system on a subset of the tangent bundle $T(\Theta_0)$ of Θ_0. This set is finite dimensional, and we can realize the whole KP flows as commuting flows on this space, so the whole KP orbit must be finite dimensional. Rather than following this argument, here we give a proof by showing the finite dimensionality of certain space, to which we map an infinite sequence of differential operators, thus showing that sufficiently many linear combinations of the operators belong to the kernel of the map. We then identify this kernel with a space of commuting differential operators.

First define a pseudo-differential operator

$$\mathcal{L}=\partial_x+\sum_{s=1}^{\infty} w_s(z,y)\partial_x^{-s} \tag{4.1}$$

by

$$\mathcal{L}(Ux+z,\partial_x)\,\psi=k\,\psi\,, \tag{4.2}$$

or equivalently

$$\mathcal{L}(Ux+z,\partial_x)=\Phi\partial_x\Phi^{-1}\,, \tag{4.3}$$

where

$$\Phi=1+\sum_{s=1}^{\infty}\xi_s(Ux+z,y)\,\partial_x^{-s} \tag{4.4}$$

if $\phi=e^{-(kx+k^2y)}\psi$ is given by (3.15). So $\mathcal{L}$ is determined uniquely by ψ, and the ambiguity (3.17) in defining ψ does not affect $\mathcal{L}$, so it is determined by $u=-2\partial_x\ln\tau$ and the choice of vectors $\lambda_1, \ldots, \lambda_d$. Since u is Λ-periodic, so is $\mathcal{L}$; the coefficients $w_s(Z,y)$ of $\mathcal{L}$ are meromorphic functions on X^* with poles along the divisor Θ.

Consider now the differential parts of the pseudo-differential operators $\mathcal{L}^m$, namely, let $\mathcal{L}^m_+$ be the differential operator such that $\mathcal{L}^m_- := \mathcal{L}^m-\mathcal{L}^m_+ = F_m\partial^{-1}+F^1_m\partial^{-2}+F^2_m\partial^{-3}+O(\partial^{-4})$.

Here we denote ∂_x by ∂ for simplicity. The leading coefficient F_m of $\mathcal{L}_-^m$ is the residue of $\mathcal{L}^m$,

$$F_m = \text{res}_\partial \, \mathcal{L}^m \,, \tag{4.5}$$

and

$$F_m^i = \text{res}_\partial (\mathcal{L}^m \partial^i) \,. \tag{4.6}$$

From the construction of $\mathcal{L}$ it follows that $[\partial_y - \partial_x^2 + u, \mathcal{L}^n] = 0$. Hence

$$[\partial_y - \partial_x^2 + u, \mathcal{L}_+^m] = -[\partial_y - \partial_x^2 + u, \mathcal{L}_-^m] = 2\partial_x F_m \,. \tag{4.7}$$

The vanishing of the coefficients of ∂^{-1} and ∂^{-2} in the middle member of this equality implies the equations

$$2\partial_x F_m^1 = -\partial_x^2 F_m + \partial_y F_m. \tag{4.8}$$

$$2\partial_x F_m^2 = F_m u_x - \partial_x^2 F_m^1 + \partial_y F_m^1. \tag{4.9}$$

The functions F_m, F_m^i are differential polynomials in the coefficients w_s of $\mathcal{L}$. Hence, they are meromorphic functions on X.

Lemma 4.1 *The abelian functions F_m have at most the second order pole on the divisor Θ.*

Proof. The mth KP flow on Φ is defined by

$$\partial_{t_m}(\Phi) = \mathcal{L}_+^m \Phi - \Phi \partial^m = -\mathcal{L}_-^m \Phi \,. \tag{4.10}$$

Comparing the coefficients of ∂^{-1} on both sides of (4.10) and using (3.19), we obtain

$$F_m = \partial_x \partial_{t_m} \ln \tau \,. \tag{4.11}$$

So if we admit that the higher KP flows preserves the regularity of τ, the assertion follows immediately from (4.11).

Alternatively, by constructing the adjoint wave function, we can see that the 0th order pseudodifferential operator Φ^{-1} can be written in the form

$$\Phi^{-1} = 1 + \sum_{s=1}^{\infty} \partial^{-s} \circ \xi_s^*(Ux + z, y) \tag{4.12}$$

for some meromorphic functions ξ_s^* having a simple pole along Θ. Using (4.3), (4.4) and (4.12) we have

$$\mathcal{L}^m = \Phi \partial^m \Phi^{-1} = \sum_{r,s=0}^{\infty} \xi_r \partial^{m-r-s} \circ \xi_s^* \,,$$

where we set $\xi_0 = \xi_0^* = 1$. Since $\xi_r \partial^{m-r-s} \circ \xi_s^*$ does not yield negative order terms in ∂ if $m - r - s \geq 0$, this implies

$$\mathcal{L}_-^m = \sum_{r+s>m} \xi_r \partial^{m-r-s} \circ \xi_s^* = \sum_{r+s=m+1} \xi_r \xi_s^* \partial^{-1} + O(\partial^{-2}) \,, \tag{4.13}$$

hence $F_m = \sum_{r+s=m+1} \xi_r \xi_s^*$, proving the lemma.

Important remark In [14] this statement was crucial for the proof of the existence of commuting differential operators associated with u. Namely, it implies that for all but a finite number of positive integers n there exist constants $c_{n,i}$ such that

$$F_n(z,y) + \sum_{i=0}^{n-1} c_{n,i} F_i(z,y) = 0\,, \tag{4.14}$$

hence (4.7) would imply that the corresponding linear combinations $L_n := \mathcal{L}_+^n + \sum c_{n,i}\mathcal{L}_+^i$ commutes with $P := \partial_y - \partial_x^2 + u$. Not so: since these constants $c_{n,i}$ might depend on y, we might not have $[P, L_n] = 0$, and we cannot immediately make the next step and claim the existence of commuting operators (!).

So our next goal is to show that these constants in fact are y-independent. For that let us consider the functions F_m^1. Equation (4.8) (or (4.13)) implies that they have at most the third order pole on the divisor Θ. Moreover, if we expand F_m^1 near Θ,

$$F_n^1 = \frac{f_n^3}{\tau^3} + \frac{f_n^2}{\tau^2} + \frac{f_n^1}{\tau} + O(1)\,, \tag{4.15}$$

and use (4.11) so that F_n is of the form

$$F_n = \partial_U\left(\frac{q_n}{\tau} + O(1)\right) = -\frac{q_n \partial_U \tau}{\tau^2} + \frac{\partial_U q_n}{\tau} + O(1)\,, \tag{4.16}$$

then (4.8) implies

$$f_n^3 = -q_n(\partial_U\tau)^2, \quad f_n^2 = 0\,. \tag{4.17}$$

Let $\{F_\alpha^1 \mid \alpha \in A\}$, for finite set A, be a basis of the space $\mathcal{F}(y)$ spanned by $\{F_m^1\}$. Then for all $n \notin A$ there exist constants $c_{n,\alpha}(y)$ such that

$$F_n^1(z,y) = \sum_{\alpha\in A} c_{n,\alpha}(y) F_\alpha^1(z,y)\,. \tag{4.18}$$

Due to (4.17) it is equivalent to the equations

$$q_n(z,y) = \sum_\alpha c_{n,\alpha}(y) q_\alpha(z,y)\,, \tag{4.19}$$

$$f_n^1(z,y) = \sum_\alpha c_{n,\alpha}(y) f_\alpha^1(z,y)\,. \tag{4.20}$$

From equation (4.8) we get

$$\sum_\alpha (\partial_y c_{n,\alpha}) q_\alpha(z,y) = 0\,. \tag{4.21}$$

Taking a linear combination of (4.9) we get

$$2\partial_x\left(F_n^2 - \sum_\alpha c_{n,\alpha} F_\alpha^2\right) = \sum_\alpha (\partial_y c_{n,\alpha}) \frac{f_\alpha^1}{\tau} + O(1)\,. \tag{4.22}$$

The left-hand side has no "residue" on Θ, and that implies the equation

$$\sum_\alpha (\partial_y c_{n,\alpha}) f^1_\alpha = 0\,. \tag{4.23}$$

Equations (4.19) and (4.23) are equivalent to

$$\sum_\alpha (\partial_y c_{n,\alpha}) F^1_\alpha = 0\,. \tag{4.24}$$

By definition the functions F^1_α are linearly independent. Therefore $c_{n,\alpha}$ are y-independent and we can proceed as in [14]. Let us sketch the rest of the proof.

Now we have sufficiently many ordinary differential operators L_n, one for each $n \gg 0$, satisfying

$$[P, L_n] = 0\,. \tag{4.25}$$

Although $P = \partial_y - \partial_x^2 + u$ is a partial differential operator, this suffices to conclude that L_n's commute with each other. Indeed, (4.25) implies that $\tilde\psi := L_n\psi$ satisfies $P\tilde\psi = 0$. Since L_n is a linear combination of $\mathcal{L}^k_+$'s, one observes that $\tilde\psi$ also satisfies the same periodicity conditions as ψ, so by part (ii) of Lemma 3.4 it is equal to ψ up to a ∂_U-independent factor. This implies that the L_n's commute with $\mathcal{L}$ and with each other.

The coefficients $c_{n,\alpha}$ of linear combinations give the directions of trivial KP time evolutions, and the Laurent coefficients of the polar part at P of the corresponding functions $f_n(\zeta)$ on the curve Γ. Now let us find *all* the Laurent coefficients, not just the polar part, of $f_n(\zeta)$. Since L_n commutes with $\mathcal{L}$ and the latter is a first order operator, we can write L_n as a constant coefficient Laurent series in $\mathcal{L}^{-1}$:

$$L_n = \sum_{j=-\infty}^{n} c_{n,j}\mathcal{L}^j,$$

then $f_n(\zeta) = \sum_{j=-n}^{\infty} c_{n,-j}\zeta^j \in \mathbb{C}((\zeta))$ is the desired Laurent series for $f_n(\zeta)$. The coefficients $c_{n,j}$ are constant in z since they are periodic holomorphic functions. Hence the curve Γ is constant in z. They are also constant in y since KP flows do not deform a spectral curve.

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