

On the scaling limit of a singular integral operator

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Abstract The scaling limit and Schauder bounds are derived for a singular integral operator arising from a difference equation approach to monodromy problems.

Keywords Singular integrals · Hilbert transforms · Global Schauder estimates

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1 Introduction

Recently an alternative approach to Birkhoff's theory of difference equations [1] has been proposed in [4]. This approach leads naturally to local monodromies of difference equations, which should converge in principle to monodromy matrices of differential equations, thus providing a missing link in the theory of isomonodromic transformations of systems of linear difference equations (see e.g. [2, 3, 7] and references therein).

The key to the convergence process in [4] is the scaling limit of a certain singular integral operator I , arising from a Riemann–Hilbert problem. The operator I acts on functions ϕ defined on the vertical line with fixed abscisse at a , and its kernel $k(z, \xi)$ is given explicitly by

$$k(z, \xi) = \frac{e^{\pi i(z-a)} + e^{-\pi i(z-a)}}{(e^{i\pi(\xi-a)} + e^{-\pi i(\xi-a)})(e^{i\pi(\xi-z)} + e^{-\pi i(\xi-z)})} \quad (1.1)$$

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If we set $\pi z = \pi a + iy, \pi \xi = \pi a + i\eta, y, \eta \in \mathbf{R}$, and view ϕ as a function of η , we can define the following re-scaled versions I_λ of I ,

$$I_\lambda(\phi)(y) = P.V. \int_{-\infty}^{\infty} \frac{1}{e^{\lambda(y-\eta)} - e^{-\lambda(y-\eta)}} \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda \eta} + e^{\lambda \eta}} \phi(\eta) d\eta, \tag{1.2}$$

where $P.V.$ denotes principal values. As noted in [4], an essential property of the operators I_λ is their formal limit,

$$I_\lambda(\phi)(y) \rightarrow \int_0^y \phi(\xi) d\xi, \quad \lambda \rightarrow +\infty. \tag{1.3}$$

The purpose of the present paper is to provide a detailed study of the boundedness properties of the operators I and I_λ in suitable spaces of Schauder type, and to establish a precise version of the formal limit (1.3). Near the diagonal, the singularities of the kernels of I_λ are the same as for the Hilbert transform, and the techniques for handling the local behavior of such kernels are well known. The main novel feature in our case is rather their global behavior near ∞ . This global behavior prevents their boundedness on scale-invariant spaces, and accounts for the existence of non-trivial limits such as (1.3).

2 Schauder estimates with exponential growth

We introduce the following norms of Schauder type for functions on \mathbf{R} . Fix $\kappa \in \mathbf{R}$, $m \in \mathbf{Z}$, $0 < \alpha < 1$, and let $\Lambda_{(m,\kappa)}^\alpha$ be the space of functions ϕ on \mathbf{R} satisfying the conditions

$$\begin{aligned} |\phi(x)| &\leq C(1 + |x|)^m e^{\kappa|x|}, \\ |\phi(x) - \phi(y)| &\leq C|x - y|^\alpha \left\{ (1 + |x|)^m e^{\kappa|x|} + (1 + |y|)^m e^{\kappa|y|} \right\}, \end{aligned} \tag{2.1}$$

for all $x, y \in \mathbf{R}$. We define $\|\phi\|_{\Lambda_{(m,\kappa)}^\alpha}$ to be the infimum of the constants C for which these inequalities hold. We also require the space $\Lambda_{(\log,\kappa)}^\alpha$ and the corresponding norm $\|\phi\|_{\Lambda_{(\log,\kappa)}^\alpha}$ defined by the conditions

$$\begin{aligned} |\phi(x)| &\leq C \log(1 + |x|) e^{\kappa|x|}, \\ |\phi(x) - \phi(y)| &\leq C|x - y|^\alpha \left\{ \log(1 + |x|) e^{\kappa|x|} + \log(1 + |y|) e^{\kappa|y|} \right\}, \end{aligned} \tag{2.2}$$

The singular integral operator I can be expressed as

$$I(\phi)(y) = (e^{-y} + e^y)H\left(\frac{1}{e^{-\cdot} + e^{\cdot}}\phi(\cdot)\right), \tag{2.3}$$

where H is the following exponentially decaying version of the classical Hilbert transform,

$$\begin{aligned} (H\psi)(y) &= P.V. \int_{-\infty}^{\infty} \frac{1}{e^{y-\eta} - e^{-(y-\eta)}} \psi(\eta) d\eta \\ &\equiv \lim_{\epsilon \rightarrow 0} \int_{|y-\eta|>\epsilon} \frac{1}{e^{y-\eta} - e^{-(y-\eta)}} \psi(\eta) d\eta. \end{aligned} \tag{2.4}$$

Set

$$K(z) = \frac{1}{e^z - e^{-z}}. \tag{2.5}$$

Then the kernel $K(z)$ is $C^\infty(\mathbf{R} \setminus 0)$, odd, and satisfies

$$|K(z)| \leq C \begin{cases} |z|^{-1}, & \text{if } |z| \leq 1; \\ e^{-|z|}, & \text{if } |z| > 1. \end{cases} \quad |\partial_z K(z)| \leq C \begin{cases} |z|^{-2}, & \text{if } |z| \leq 1; \\ e^{-|z|}, & \text{if } |z| > 1. \end{cases} \tag{2.6}$$

In particular, these are better estimates than for the standard Hilbert transform kernel $K_0(z) = z^{-1}$, and it follows at once that the operator H is bounded on the standard Schauder spaces (see e.g. [5,6]). To obtain estimates for the operator I , we need the boundedness of H on the above spaces $\Lambda_{(m,\kappa)}^\alpha$, and this is provided by the following theorem:

Theorem 1 Fix $0 < \alpha < 1$, $m \in \mathbf{Z}$. The operator H is bounded on the following Schauder spaces,

$$\|H\psi\|_{\Lambda_{(m,\kappa)}^\alpha} \leq C_{m,\alpha,\kappa} \|\psi\|_{\Lambda_{(m,\kappa)}^\alpha}, \quad -1 < \kappa < 1. \tag{2.7}$$

For $\kappa = -1$, we have the following bounds, for $m \in \mathbf{Z}$, $m \geq -1$,

$$\begin{aligned} \|H\psi\|_{\Lambda_{(m+1,-1)}^\alpha} &\leq C_{m,\alpha} \|\psi\|_{\Lambda_{(m,-1)}^\alpha}, \quad m \geq 0, \\ \|H\psi\|_{\Lambda_{(log,-1)}^\alpha} &\leq C_\alpha \|\psi\|_{\Lambda_{(m,-1)}^\alpha}, \quad m = -1. \end{aligned} \tag{2.8}$$

Proof The method of proof is the standard method for Schauder estimates for singular integral operators. The only new feature here is the control of $H\psi(x)$ for x large. In view of the fact that $K(z)$ is odd and exponentially decreasing, we can write

$$H\psi(x) = \int_{-\infty}^{\infty} K(x-y) (\psi(y) - \psi(x)) dy, \tag{2.9}$$

where the integrals on the right-hand side are now convergent for $\psi \in \Lambda_{(m,\kappa)}^\alpha$ with $0 < \alpha < 1$, $\kappa < 1$. In particular,

$$\begin{aligned} |H\psi(x)| &\leq \int_{|x-y|<1} |x-y|^{-1+\alpha} ((1+|x|)^m e^{\kappa|x|} + (1+|y|)^m e^{\kappa|y|}) dy \\ &\quad + \int_{|x-y|\geq 1} e^{-|x-y|} ((1+|x|)^m e^{\kappa|x|} + (1+|y|)^m e^{\kappa|y|}) dy. \end{aligned} \tag{2.10}$$

These are clearly bounded for $|x|$ bounded, so we may assume that $|x| \geq 3$. In this case, $\frac{1}{2|x|} \leq |x| - 1 \leq |y| \leq |x| + 1 \leq 2|x|$ in the integral over the region $|x-y| < 1$, and $(1+|y|)^m e^{\kappa|y|} \leq C_\kappa (1+|x|)^m e^{\kappa|x|}$. Thus the first integral is bounded by $C(1+|x|)^m e^{\kappa|x|}$. The same upper bound for the second integral follows from the following lemma:

Lemma 1 For any $-1 < \kappa < 1$, and any $m \in \mathbf{Z}$, we have for all $|x| > 3$

$$\int_{\mathbf{R}} e^{-|z|} (1+|x-z|)^m e^{\kappa|x-z|} dz \leq C_{m,\kappa} (1+|x|)^m e^{\kappa|x|}. \tag{2.11}$$

For $\kappa = -1$, we have for $m \in \mathbf{Z}$, $m \geq -1$,

$$\int_{\mathbf{R}} e^{-|z|} (1+|x-z|)^m e^{-|x-z|} dz \leq C_m \begin{cases} e^{-|x|} (1+|x|)^{m+1}, & \text{if } m \geq 0 \\ e^{-|x|} \log(1+|x|), & \text{if } m = -1. \end{cases} \tag{2.12}$$

Proof of Lemma 1 We consider separately the cases of $0 \leq \kappa < 1$, $-1 < \kappa < 0$, and $\kappa = -1$. When $0 \leq \kappa < 1$, we write $e^{\kappa|x-z|} \leq e^{\kappa|x|} e^{\kappa|z|}$, and hence the integral on the left-hand side of the above inequality can be bounded by

$$\begin{aligned}
 & e^{\kappa|x|} \int_{|x-z| \geq \frac{1}{2}|x|} e^{-(1-\kappa)|z|} (1 + |x - z|)^m dz \\
 & + e^{\kappa|x|} \int_{|x-z| < \frac{1}{2}|x|} e^{-(1-\kappa)|z|} (1 + |x - z|)^m dz.
 \end{aligned} \tag{2.13}$$

In the first integral we can write

$$(1 + |x - z|)^m \leq C_m (1 + |x|)^m (1 + |z|)^m. \tag{2.14}$$

This is certainly true with $C_m = 1$ if $m \geq 0$. If $m < 0$, then we use the condition $|x - z| \geq \frac{1}{2}|x|$ to write $(1 + |x - z|)^m \leq 2^{-m} (1 + |x|)^m$, and the inequality still holds. Since $\kappa < 1$, the desired bound follows for the first integral. Next, in the second integral, we have $\frac{1}{2}|x| < |z| < \frac{3}{2}|x|$, and we can write

$$\begin{aligned}
 \int_{|x-z| < \frac{1}{2}|x|} e^{-(1-\kappa)|z|} (1 + |x - z|)^m dz & \leq e^{-\frac{1-\kappa}{2}|x|} \int_{|z| \leq \frac{3}{2}|x|} (1 + |x|)^{|m|} (1 + |z|)^{|m|} dz \\
 & \leq C_N (1 + |x|)^{-N},
 \end{aligned} \tag{2.15}$$

for arbitrary N . This proves the lemma when $0 \leq \kappa < 1$. When $-1 < \kappa < 0$, we write instead

$$e^{-|z|} e^{\kappa|x-z|} = e^{-(1+\kappa)|z|} e^{\kappa(|z|+|x-z|)} \leq e^{-(1+\kappa)|z|} e^{\kappa|x|} \tag{2.16}$$

and bound the integral on the left-hand side of the lemma by

$$\begin{aligned}
 & e^{\kappa|x|} \int_{|x-z| \geq \frac{1}{2}|x|} e^{-(1+\kappa)|z|} (1 + |x - z|)^m dz \\
 & + e^{\kappa|x|} \int_{|x-z| < \frac{1}{2}|x|} e^{-(1+\kappa)|z|} (1 + |x - z|)^m dz.
 \end{aligned} \tag{2.17}$$

The bounds for these integrals are now the same as in the previous case. This establishes the estimate (2.11). Finally, consider the case $\kappa = -1$. In the region of integration $|x - z| > 4|x|$, we have the integrand can be crudely bounded by $e^{-2|x|} (1 + |x - z|)^m e^{-\frac{1}{2}|x-z|}$, and hence the contribution of this region is $O(e^{-2|x|})$, which is better than we actually need. Thus it suffices to consider the region $|x - z| \leq 4|x|$. We write then

$$\int_{|x-z| < 4|x|} e^{-|z|} (1 + |x - z|)^m e^{-|x-z|} dz \leq e^{-|x|} \int_{|x-z| < 4|x|} (1 + |x - z|)^m dz \tag{2.18}$$

from which the desired estimate follows at once. The proof of the lemma is complete. □

We return to the proof of the theorem. Let $x, x' \in \mathbf{R}$ and set $\delta = |x - x'|$. The next step is to estimate $H\psi(x) - H\psi(x')$, which can be expressed as

$$\begin{aligned}
 & \int_{|y-x| < 3\delta} K(x - y)(\psi(y) - \psi(x)) dy - \int_{|y-x'| < 3\delta} K(x' - y)(\psi(y) - \psi(x')) dy \\
 & + \int_{|y-x| \geq 3\delta} K(x - y)(\psi(y) - \psi(x)) dy - \int_{|y-x'| \geq 3\delta} K(x' - y)(\psi(y) - \psi(x')) dy.
 \end{aligned} \tag{2.19}$$

The first two integrals can be estimated as in the bounds for $|H\psi(x)|$. For example,

$$\begin{aligned} \left| \int_{|y-x|<3\delta} K(x-y)(\psi(y) - \psi(x))dy \right| &\leq \|\psi\|_{\Lambda_{(m,\kappa)}^\alpha} \int_{|x-y|<3\delta} |x-y|^{-1+\alpha} \left\{ (1+|x|)^m e^{\kappa|x|} \right. \\ &\quad \left. + (1+|y|)^m e^{\kappa|y|} \right\} \\ &\leq C \|\psi\|_{\Lambda_{(m,\kappa)}^\alpha} \delta^\alpha (1+|x|)^m e^{\kappa|x|} \end{aligned} \tag{2.20}$$

since $(1+|y|)^m e^{\kappa|y|} \leq C(1+|x|)^m e^{\kappa|x|}$ for $|x| \geq 3$ and $\delta \ll 1$. To estimate the remaining two integrals, write

$$\begin{aligned} \int_{|y-x'| \geq 3\delta} K(x'-y)(\psi(y) - \psi(x'))dy &= \int_{|y-x'| \geq 3\delta} K(x'-y)(\psi(y) - \psi(x))dy \\ &= \int_{|y-x| \geq 3\delta} + \int_{|y-x'| \geq 3\delta, |y-x| < 3\delta} \\ &\quad - \int_{|y-x'| < 3\delta, |y-x| > 3\delta} \end{aligned} \tag{2.21}$$

The last two integrals on the right-hand side satisfy the desired bounds, because in their ranges of integration, we have $|y-x| \sim |y-x'| \sim \delta$, and the same arguments above apply. The remaining integral can be combined with the third integral in (2.19) to give

$$\int_{|y-x| > 3\delta} (K(x-y) - K(x'-y))(\psi(y) - \psi(x))dy. \tag{2.22}$$

Since we have

$$|K(x-y) - K(x'-y)| \leq |x-y| \cdot |\partial_x K(z)| \tag{2.23}$$

for some z in the segment between $x-y$ and $x'-y$, and hence $|z| \sim |x-y|$ when $|y-x| > 3|x-x'|$, we can write, in view of the bounds for the $|\partial_z K(z)|$,

$$\begin{aligned} &\left| \int_{|y-x| > 3\delta} (K(x-y) - K(x'-y))(\psi(y) - \psi(x))dy \right| \\ &\leq \delta \|\psi\|_{\Lambda_{m,\kappa}^\alpha} \int_{3\delta < |x-y| < 1} |x-y|^{-2+\alpha} \left\{ (1+|x|)^m e^{\kappa|x|} + (1+|y|)^m e^{\kappa|y|} \right\} dy \\ &\quad + \delta \|\psi\|_{\Lambda_{m,\kappa}^\alpha} \int_{|x-y| > 1} e^{-|x-y|} \left\{ (1+|x|)^m e^{\kappa|x|} + (1+|y|)^m e^{\kappa|y|} \right\} dy. \end{aligned} \tag{2.24}$$

The first integral on the right-hand side is bounded by

$$\begin{aligned} &\delta \|\psi\|_{\Lambda_{m,\kappa}^\alpha} \int_{3\delta < |x-y| < 1} |x-y|^{-2+\alpha} \left\{ (1+|x|)^m e^{\kappa|x|} + (1+|y|)^m e^{\kappa|y|} \right\} dy \\ &\leq \delta \|\psi\|_{\Lambda_{m,\kappa}^\alpha} (1+|x|)^m e^{\kappa|x|} \int_{3\delta < |x-y| < 1} |x-y|^{-2+\alpha} \leq C \delta^\alpha (1+|x|)^m e^{\kappa|x|}. \end{aligned} \tag{2.25}$$

Applying Lemma (1), we obtain similar bounds for the second integral. Altogether, we have shown that

$$|H\psi(x) - H\psi(x')| \leq C \|\psi\|_{\Lambda_{(m,\kappa)}^\alpha} |x-x'|^\alpha (1+|x|)^m e^{\kappa|x|} \tag{2.26}$$

for $|x - x'|$ small, and the theorem is proved when $-1 < \kappa < 1$. The case $\kappa = -1$ is established exactly in the same way, using the corresponding estimates in Lemma 1 for $\kappa = -1$ and $m \geq -1$. The proof of Theorem 1 is complete. \square

Theorem 2 For $0 < \kappa < 2$, $m \in \mathbf{Z}$, and $0 < \alpha < 1$, the operator I is bounded on the following Schauder spaces,

$$\|I(\phi)\|_{\Lambda_{(m,\kappa)}^\alpha} \leq C_{m,\kappa,\alpha} \|\phi\|_{\Lambda_{(m,\kappa)}^\alpha}, \quad 0 < \kappa < 2. \tag{2.27}$$

For $\kappa = 0$, $m \in \mathbf{Z}$, $m \geq -1$, the operator I satisfies the following bounds,

$$\begin{aligned} \|\phi\|_{\Lambda_{(m+1,\kappa)}^\alpha} &\leq C_{m,\alpha} \|\phi\|_{\Lambda_{(m,\kappa)}^\alpha}, \quad m \geq 0 \\ \|I(\phi)\|_{\Lambda_{(\log,\kappa)}^\alpha} &\leq C_\alpha \|\phi\|_{\Lambda_{(m,\kappa)}^\alpha}, \quad m = -1. \end{aligned} \tag{2.28}$$

Proof This is an easy consequence of Theorem 1, the fact that the map $\phi \rightarrow \psi(y) = \frac{1}{e^y + e^{-y}} \phi(y)$ is a one-to-one and onto map M from $\Lambda_{(m,\kappa)}^\alpha \rightarrow \Lambda_{(m,\kappa-1)}^\alpha$, with equivalent norms

$$\|\psi\|_{\Lambda_{(m,\kappa-1)}^\alpha} \sim \|\phi\|_{\Lambda_{(m,\kappa)}^\alpha}. \tag{2.29}$$

and the relation $I(\phi) = M^{-1}HM\phi$. \square

We observe that these bounds always require some space which is not scale-invariant. Thus bounds for I_λ cannot be obtained by scaling the bounds for I , and this explains partly the possibility of the scaling limits discussed in the next section.

3 The scaling limit of I_λ

We come now to the operators I_λ . The estimates for I in the previous section show that I_λ cannot be treated by simple scaling arguments from I . Instead, we shall study the bounds and limits for I_λ as $\lambda \rightarrow +\infty$ directly. It is simplest to carry this out for functions ϕ satisfying conditions of the form,

$$|\partial^l \phi(x)| \leq C_k (1 + |x|)^m, \quad 0 \leq l \leq k, \tag{3.1}$$

for fixed $m \in \mathbf{N}$, $k \in \mathbf{N}$, and norms $\|\phi\|_{\Lambda_{(m)}^k}$ defined to be the best constant C_k for which the above condition holds. The following theorem describes the limit of I_λ in these spaces, although it should be clear from the proof and from the previous section that other more precise versions can be formulated as well:

Theorem 3 Fix $m \in \mathbf{N}$. Then we have the following bounds, uniform in λ and in $\phi \in \Lambda_{(m)}^1$,

$$\|I_\lambda(\phi)(y) - \int_0^y \phi(\xi) d\xi\|_{\Lambda_{(m+1)}^0} \leq C_m \lambda^{-\frac{1}{2}} \|\phi\|_{\Lambda_{(m)}^1}. \tag{3.2}$$

Proof Formally, if we write

$$I_\lambda \phi(y) = \int K_\lambda(y, \eta) \phi(\eta) d\eta \tag{3.3}$$

with

$$K_\lambda(y, \eta) = \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda \eta} + e^{\lambda \eta}} \frac{1}{e^{\lambda(y-\eta)} - e^{-\lambda(y-\eta)}} \tag{3.4}$$

then for, say, $y > 0$, we have the pointwise limit

$$K_\lambda(y, \eta) \rightarrow \begin{cases} 1, & \text{if } 0 < \eta < y \\ 0, & \text{if } \eta < 0 \text{ or } \eta > y. \end{cases} \tag{3.5}$$

Thus, formally, the left-hand side of the expression in the theorem tends to 0 as $\lambda \rightarrow +\infty$. However, none of the integrals involved is uniformly nor absolutely convergent, and we have to proceed with care. Fix $y > 0$ (the case of $y < 0$ being similar). The key to the estimates is the following break-up of the principal value integral defining $I_\lambda(\phi)$,

$$\begin{aligned} (I_\lambda(\phi))(y) &= \int_0^y \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda t} - e^{-\lambda t}} (\psi(y - t) - \psi(y + t)) dt + \int_{|t|>y} \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda t} - e^{-\lambda t}} \psi(y - t) dt \\ &\equiv (A) + (B) \end{aligned} \tag{3.6}$$

with

$$\psi(\eta) = \frac{1}{e^{\lambda \eta} + e^{-\lambda \eta}} \phi(\eta). \tag{3.7}$$

To estimate (A), we apply Taylor’s formula

$$\psi(y - t) - \psi(y + t) = t \int_{-1}^1 \psi'(y - \rho t) d\rho \tag{3.8}$$

which gives in this particular case,

$$\begin{aligned} \psi(y - t) - \psi(y + t) &= t \int_{-1}^1 d\rho \left(\frac{1}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}} \phi'(y - \rho t) \right. \\ &\quad \left. - \lambda \frac{e^{\lambda(y-\rho t)} - e^{-\lambda(y-\rho t)}}{(e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)})^2} \phi(y - \rho t) \right) \end{aligned} \tag{3.9}$$

Thus (A) can be rewritten as

$$\begin{aligned} (A) &= \frac{1}{\lambda} \int_0^y dt \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \phi'(y - \rho t) \\ &\quad - \int_0^y dt \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \frac{e^{\lambda(y-\rho t)} - e^{-\lambda(y-\rho t)}}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}} \phi(y - \rho t) \\ &\equiv A_1 + A_0 \end{aligned} \tag{3.10}$$

where the function $\chi_\lambda(\rho, t)$ is defined by

$$\chi_\lambda(\rho, t) = \frac{\lambda t}{e^{\lambda t} - e^{-\lambda t}} \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}}. \tag{3.11}$$

The following sharp estimates for $\chi_\lambda(\rho, t)$ play an essential role in the sequel:

Lemma 2 *For all $0 < t < y$, the functions $\chi_\lambda(\rho, t)$ satisfy the following properties*

(a)

$$\frac{1}{2} \frac{\lambda t}{1 - e^{-2\lambda t}} e^{-\lambda t(1-\rho)} < \chi_\lambda(\rho, t) \leq 2 \frac{\lambda t}{1 - e^{-2\lambda t}} e^{-\lambda t(1-\rho)}, \quad |\rho| < 1, 0 < t < y$$

(b)

$$\frac{1}{2} \leq \int_{-1}^1 \chi_\lambda(\rho, t) d\rho \leq 2. \tag{3.12}$$

Proof In the region $0 < t < y$, we have $y - \rho t > 0$ for all $|\rho| < 1$, and thus

$$\frac{1}{2} e^{\lambda \rho t} \leq \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}} \leq 2 e^{\lambda \rho t}. \tag{3.13}$$

The upper bound implies (a), while the lower bound implies (b), when combined with the following explicit formula

$$\int_{-1}^1 e^{\lambda \rho t} d\rho = \frac{1}{\lambda t} (e^{\lambda t} - e^{-\lambda t}). \tag{3.14}$$

The proof of Lemma 2 is complete.

We can now show that $A_1 \rightarrow 0$ with a precise rate:

Lemma 3 *The term involving ϕ' above tends to 0 at the following rate,*

$$|A_1| \leq C_m \frac{1}{\lambda} \|\phi\|_{\Lambda_{(m,0)}^1} (1+y)^{m+1}. \tag{3.15}$$

Proof It suffices to write

$$|A_1| \leq \frac{1}{\lambda} \|\phi\|_{\Lambda_{(m)}^1} \int_0^y dt (1 + |y|^m + |t|^m) \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \tag{3.16}$$

and the desired estimate follows from the statement (b) of Lemma 2. □

The estimates in Lemma 2 show that $\chi_\lambda(\rho, t)$ provide an approximation of the Dirac measure concentrated at $\rho = 1$,

$$\frac{1}{\int_{-1}^1 \chi_\lambda(\mu, t) d\mu} \chi_\lambda(\rho, t) \rightarrow \delta(\mu - 1) \tag{3.17}$$

A precise version of this statement with sharp estimates is given in the next lemma. Set

$$\begin{aligned} & \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \frac{e^{-\lambda(y-\rho t)} - e^{\lambda(y-\rho t)}}{e^{-\lambda(y-\rho t)} + e^{\lambda(y-\rho t)}} \phi(y - \rho t) - \phi(y - t) \\ &= \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \left(\frac{e^{-\lambda(y-\rho t)} - e^{\lambda(y-\rho t)}}{e^{-\lambda(y-\rho t)} + e^{\lambda(y-\rho t)}} - 1 \right) \phi(y - \rho t) \\ &+ \int_{-1}^1 d\rho \chi_\lambda(\rho, t) (\phi(y - \rho t) - \phi(y - t)) + \left(\int_{-1}^1 d\rho \chi_\lambda(\rho, t) - 1 \right) \phi(y - t). \end{aligned} \tag{3.18}$$

Lemma 4 *For all $0 < t < y$, and any $\delta > 0$ and small, we have the following estimates, with absolute constants,*

(a)

$$\left| \int_{-1}^1 d\rho \chi_\lambda(\rho, t) - 1 \right| \cdot |\phi(y - t)| \leq \|\phi\|_{\Lambda_{(m)}^0} (1+y)^m (e^{-\lambda y} + e^{-2\lambda(y-t)}) \tag{3.19}$$

(b)

$$\left| \int_{-1}^1 d\rho \chi_\lambda(\rho, t) (\phi(y - \rho t) - \phi(y - t)) \right| \leq \|\phi\|_{\Lambda_{(m)}^1} (1 + y)^m (\delta t + e^{-\lambda \delta t}) \quad (3.20)$$

(c)

$$\begin{aligned} \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \left| \frac{e^{\lambda(y-\rho t)} - e^{-\lambda(y-\rho t)}}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}} - 1 \right| |\phi(y - \rho t)| \\ \leq C e^{-2\lambda(y-t)} \|\phi\|_{\Lambda_{(m)}^0} (1 + y)^m. \end{aligned} \quad (3.21)$$

Proof To prove (a), we write

$$\begin{aligned} \left| \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y-\rho t)} - e^{\lambda(y-\rho t)}} - e^{\lambda \rho t} \right| = e^{\lambda \rho t} \left| \frac{e^{-2\lambda y} - e^{-2\lambda(y-\rho t)}}{1 + e^{-2\lambda(y-\rho t)}} \right| \\ \leq e^{\lambda \rho t} (e^{-2\lambda y} + e^{-2\lambda(y-\rho t)}) \leq e^{\lambda \rho t} (e^{-2\lambda y} + e^{-2\lambda(y-t)}). \end{aligned} \quad (3.22)$$

In particular,

$$\left| \int_{-1}^1 d\rho \chi_\lambda(\rho, t) - \frac{\lambda t}{e^{\lambda t} - e^{-\lambda t}} \int_{-1}^1 d\rho e^{\lambda \rho t} \right| \leq \frac{\lambda t}{e^{\lambda t} - e^{-\lambda t}} \int_{-1}^1 d\rho e^{\lambda \rho t} (e^{-2\lambda y} + e^{-2\lambda(y-t)}). \quad (3.23)$$

Since $\int_{|\rho| \leq 1} d\rho e^{\lambda \rho t} = (\lambda t)^{-1} (e^{\lambda t} - e^{-\lambda t})$, the statement (a) follows.

To establish the statement (c), we begin by noting that

$$e^{-2\lambda(y-\rho t)} \leq 1 - \frac{e^{\lambda(y-\rho t)} - e^{-\lambda(y-\rho t)}}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}} = 2 \frac{e^{-2\lambda(y-\rho t)}}{1 + e^{-2\lambda(y-\rho t)}} \leq 2 e^{-2\lambda(y-\rho t)}. \quad (3.24)$$

Using the estimate for χ_λ in Lemma 2 and carrying out explicitly the integral in ρ gives

$$\begin{aligned} \int_{-1}^1 d\rho \chi_\lambda(\rho, t) \left| \frac{e^{\lambda(y-\rho t)} - e^{-\lambda(y-\rho t)}}{e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)}} - 1 \right| &\leq \frac{\lambda t}{1 - e^{-2\lambda t}} e^{-\lambda(t+2y)} \int_{-1}^1 d\rho e^{3\lambda \rho t} \\ &= \frac{1}{3} e^{-2\lambda(y-t)} \frac{1 - e^{-6\lambda t}}{1 - e^{-2\lambda t}} \leq C e^{-2\lambda(y-t)}, \end{aligned}$$

which implies immediately (c).

To establish (b), let $\delta > 0$ be any number sufficiently small and to be chosen suitably later. Write

$$\int_{-1}^1 d\rho \chi_\lambda(\rho, t) (\phi(y - \rho t) - \phi(y - t)) = \int_{-1}^{1-\delta} + \int_{1-\delta}^1 \equiv I_\delta + II_\delta. \quad (3.25)$$

The second term on the right-hand side can be estimated by,

$$|II_\delta| \leq \delta t \|\phi\|_{\Lambda_{(m)}^1} (1 + y)^m \int_{1-\delta}^1 d\rho \chi_\lambda(\rho, t) \leq \delta t \|\phi\|_{\Lambda_{(m)}^1} (1 + y)^m, \quad (3.26)$$

while the first term can be estimated using Lemma 2,

$$\begin{aligned}
 |I_\delta| &\leq 2 \|\phi\|_{\Lambda^1_{(m)}} (1+y)^m \frac{\lambda t}{1 - e^{-2\lambda t}} \int_{-1}^\delta d\rho e^{-\lambda t(1-\rho)} \\
 &= 2 \|\phi\|_{\Lambda^1_{(m)}} (1+y)^m e^{-\lambda \delta t} \frac{1 - e^{-\lambda t(2-\delta)}}{1 - e^{-2\lambda t}} \leq C \sup_{[0,2y]} |\phi| e^{-\lambda \delta t}. \tag{3.27}
 \end{aligned}$$

The proof of Lemma 4 is complete. □

We can now carry out the integral in t . The precise estimates are given in the next lemma:

Lemma 5 *For any $0 < y$, we have the following estimates,*

$$\begin{aligned}
 \left| \int_0^y dt \int_{-1}^1 d\rho \frac{\lambda t}{e^{\lambda t} - e^{-\lambda t}} (e^{\lambda y} + e^{-\lambda y}) \frac{e^{\lambda(y-\rho t)} - e^{-\lambda(y-\rho t)}}{(e^{\lambda(y-\rho t)} + e^{-\lambda(y-\rho t)})^2} \phi(y - \rho t) \right. \\
 \left. - \int_0^y dt \phi(y - t) \right| \leq C \|\phi\|_{\Lambda^1_{(m)}} (1+y)^m \left(\frac{y}{1 + \lambda y} + \frac{y}{\lambda^{\frac{1}{2}}} \right). \tag{3.28}
 \end{aligned}$$

with a constant C independent of y and of λ .

Proof In view of the defining formula (3.11) for the function $\chi_\lambda(\rho, t)$ and the break up (3.18), the left-hand side of the desired inequality is bounded by the integral in t of the three inequalities in Lemma 4. This gives the following upper bound,

$$\|\phi\|_{\Lambda^1_{(m)}} (1+y)^m \int_0^y dt (e^{-\lambda y} + 2e^{-2\lambda(y-t)} + \delta t + e^{-\delta \lambda t}). \tag{3.29}$$

The integral can be evaluated explicitly, and we find

$$ye^{-\lambda y} + \frac{1}{\lambda}(1 - e^{-2\lambda y}) + \frac{1}{2}\delta y^2 + \frac{1}{\delta\lambda}(1 - e^{-\delta\lambda}). \tag{3.30}$$

We consider the sum of the first two terms: when $\lambda y < 1$, it is bounded by $C y$, where C is an absolute constant. When $\lambda y \geq 1$, it is bounded by $C \lambda^{-1}$. Thus we have

$$ye^{-\lambda y} + \frac{1}{\lambda}(1 - e^{-2\lambda y}) \sim \frac{y}{1 + \lambda y}. \tag{3.31}$$

Next, we consider the optimal choice of δ so as to minimize the size of the sum of the remaining two terms in the above integral. We note that we may assume that $\delta\lambda > 1$, since otherwise the term $(\delta\lambda)^{-1}(1 - e^{-\delta\lambda})$ is of size 1, and we do not even get convergence to 0. Thus we should take $\delta\lambda > 1$, in which case the sum of the two remaining terms is of size

$$\delta y^2 + \frac{1}{\delta\lambda} \tag{3.32}$$

which attains its lowest size $y\lambda^{-\frac{1}{2}}$ if we set $\delta = y^{-1}\lambda^{-\frac{1}{2}}$. This gives the estimate stated in the lemma. □

We return now to the estimate of the contribution to $I_\lambda(\phi)(y)$ of the integral in t from the region $|t| > y$.

Lemma 6 For any $0 < y$, we have the following estimate

$$\left| \int_{|t|>y} \frac{1}{e^{\lambda t} - e^{-\lambda t}} \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-t)} + e^{-\lambda(y-t)}} \phi(y-t) dt \right| \leq C_m \frac{1}{\lambda} \|\phi\|_{\Lambda^0(m)} \left(1 + \log \left(1 + \frac{1}{\lambda y} \right) \right)$$

Proof Consider first the contribution from the region $t > y$. In this region, we have

$$\frac{1}{2} e^{\lambda(2y-t)} \leq \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-t)} + e^{-\lambda(y-t)}} \leq 2 \frac{e^{\lambda y}}{e^{\lambda(t-y)}} = 2 e^{\lambda(2y-t)} \tag{3.33}$$

Thus the contribution from the region $t > y$ to the integral on the left-hand side of the desired inequality can be bounded by

$$\begin{aligned} \int_{t>y} \frac{e^{-2\lambda(t-y)}}{1 - e^{-2\lambda t}} |\phi(y-t)| dt &= \int_0^\infty \frac{e^{-2\lambda s}}{1 - e^{-2\lambda(s+y)}} |\phi(-s)| ds \\ &\leq \|\phi\|_{\Lambda^0(m)} \int_0^\infty \frac{e^{-2\lambda s}}{1 - e^{-2\lambda(s+y)}} (1 + |s|^m) ds. \end{aligned} \tag{3.34}$$

We claim that for all $m \in \mathbf{N}$, we have

$$\int_0^\infty \frac{e^{-2\lambda s}}{1 - e^{-2\lambda(s+y)}} (1 + |s|^m) ds \leq C_m \frac{1}{\lambda} \left(1 + \log \left(1 + \frac{1}{\lambda y} \right) \right). \tag{3.35}$$

In fact, setting $\mu = e^{-2\lambda y}$ and making the change of variables $s \rightarrow u$, $e^{-2\lambda u} = s$, this integral can be rewritten as

$$\frac{1}{2\lambda} \int_0^1 \frac{du}{1 - u\mu} \left(1 + \frac{1}{2\lambda} \log \frac{1}{u} \right)^m. \tag{3.36}$$

We break it into two regions of integration $0 < u < \frac{1}{2}$ and $\frac{1}{2} \leq u \leq 1$. In the first region, the integral is of size

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{du}{1 - u\mu} \left(1 + \frac{1}{2\lambda} \log \frac{1}{u} \right)^m &\leq 2 \int_0^{\frac{1}{2}} \left(1 + \frac{1}{2\lambda} \log \frac{1}{u} \right)^m \\ &\leq \frac{2}{\lambda^m} \int_0^{e^{-\lambda}} \left(\log \frac{1}{u} \right)^m du + 2^{m+1} \int_{e^{-\lambda}}^{\frac{1}{2}} du \\ &\leq C_m. \end{aligned} \tag{3.37}$$

In the second region, we have

$$\int_{\frac{1}{2}}^1 \frac{du}{1 - u\mu} \left(1 + \frac{1}{2\lambda} \log \frac{1}{u} \right)^m \leq C_m \int_{\frac{1}{2}}^1 \frac{du}{1 - u\mu} \tag{3.38}$$

This last integral can be evaluated explicitly, and we find that it is bounded by $(1 + \log(1 + \frac{1}{\lambda y}))$. This is the desired estimate.

Next, consider the contribution of the region $t < -y$. In this region, we have instead

$$\frac{1}{2} e^{\lambda t} \leq \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-t)} + e^{-\lambda(y-t)}} \leq 2 \frac{e^{\lambda y}}{e^{\lambda(y-t)}} \leq 2 e^{\lambda t} \tag{3.39}$$

The contribution from $t < -y$ to the left-hand side of the desired inequality can then be bounded by

$$\|\phi\|_{\Lambda^0(m)} (1+y)^m \int_{-\infty}^{-y} \frac{e^{2\lambda t}}{1-e^{2\lambda t}} (1+|t|)^m dt \leq C_m \frac{1}{\lambda} \left(1 + \log \left(1 + \frac{1}{\lambda y}\right)\right), \tag{3.40}$$

as was to be shown. □

The bound provided by Lemma 6 involves a $\log(\lambda y)^{-1}$ term, and is not adequate for y close to 0. This is because the integral is only a principal value integral when $|t - y|$ is small, and the estimates we have just derived for the contribution of the region $t > y$ do not take into account the cancellations inherent to principal value integrals. This issue is addressed in the next lemma:

Lemma 7 *Assume that $0 < \lambda y < 1$. Then*

$$\left| \int_{y < |t| < 1} \frac{1}{e^{\lambda t} - e^{-\lambda t}} \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-t)} + e^{-\lambda(y-t)}} \phi(y-t) dt \right| \leq C \|\phi\|_{C^0_{[0,2]}} \left(y + \frac{1}{\lambda}\right) \tag{3.41}$$

$$\left| \int_{|t| > 1} \frac{1}{e^{\lambda t} - e^{-\lambda t}} \frac{e^{\lambda y} + e^{-\lambda y}}{e^{\lambda(y-t)} + e^{-\lambda(y-t)}} \phi(y-t) dt \right| \leq C_m \|\phi\|_{\Lambda^0(m)} \frac{1}{\lambda}. \tag{3.42}$$

Proof Since we can assume that λ is large, the condition that $\lambda y < 1$ implies that $y < 1$, say. We can exploit the cancellation by writing the integral over the region $y < |t| < 1$ in the form,

$$\int_{y < |t| < 1} = \int_y^1 \frac{1}{e^{\lambda t} - e^{-\lambda t}} \left\{ \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y-t)} + e^{\lambda(y-t)}} \phi(y-t) - \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y+t)} + e^{\lambda(y+t)}} \phi(y+t) \right\} dt \tag{3.43}$$

Next, the expression within brackets is written as,

$$\begin{aligned} & \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y-t)} + e^{\lambda(y-t)}} \phi(y-t) - \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y+t)} + e^{\lambda(y+t)}} \\ &= \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y-t)} + e^{\lambda(y-t)}} (\phi(y-t) - \phi(y+t)) \\ & \quad + \phi(y+t) \left\{ \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y-t)} + e^{\lambda(y-t)}} - \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y+t)} + e^{\lambda(y+t)}} \right\} \end{aligned} \tag{3.44}$$

The contribution of the first term on the right-hand side can be estimated as follows,

$$\left| \int_y^1 \frac{1}{e^{\lambda t} - e^{-\lambda t}} \frac{e^{-\lambda y} + e^{\lambda y}}{e^{-\lambda(y-t)} + e^{\lambda(y-t)}} (\phi(y-t) - \phi(y+t)) \right| \leq \|\phi\|_{C^1_{[-2,2]}} \int_y^1 \frac{t}{e^{\lambda t} - e^{-\lambda t}} e^{\lambda(2y-t)} dt$$

Since $\lambda y < 1$, we can estimate this last term crudely by

$$\int_y^1 \frac{t}{e^{\lambda t} - e^{-\lambda t}} e^{\lambda(2y-t)} dt \leq \frac{e^2}{\lambda} \int_y^1 \frac{t\lambda}{e^{2\lambda t} - 1} dt \leq \frac{C}{\lambda}, \tag{3.45}$$

since the function $u(e^{2u} - 1)^{-1}$ is a smooth and bounded function for $u \geq 0$. Next, to estimate the other contribution, we also exhibit the cancellation more clearly,

$$\begin{aligned} \frac{1}{e^{-\lambda(y-t)} + e^{\lambda(y-t)}} - \frac{1}{e^{-\lambda(y+t)} + e^{\lambda(y+t)}} &= \frac{1}{1 + e^{-2\lambda(t+y)}} \left(\frac{1}{e^{\lambda(t-y)}} - \frac{1}{e^{\lambda(t+y)}} \right) \\ & \quad + \frac{1}{e^{\lambda(t+y)}} \left(\frac{1}{1 + e^{-2\lambda(t-y)}} - \frac{1}{1 + e^{-2\lambda(y+t)}} \right) \end{aligned}$$

The first resulting group of terms can be estimated by

$$\left| \frac{1}{1 + e^{-2\lambda(t+y)}} \left(\frac{1}{e^{\lambda(t-y)}} - \frac{1}{e^{\lambda(t+y)}} \right) \right| \leq e^{-\lambda t} (e^{\lambda y} - e^{-\lambda y}) \leq C \lambda y e^{-\lambda t}, \tag{3.46}$$

and the corresponding integral in turn by,

$$\begin{aligned} \int_y^1 dt \frac{1}{e^{\lambda t} - e^{-\lambda t}} |\phi(y + t)| \cdot \left| \frac{1}{1 + e^{-2\lambda(t+y)}} \left(\frac{1}{e^{\lambda(t-y)}} - \frac{1}{e^{\lambda(t+y)}} \right) \right| \\ \leq \|\phi\|_{C_{[0,2]}^0} \lambda y \int_y^1 \frac{dt}{e^{2\lambda t} - 1}. \end{aligned} \tag{3.47}$$

To determine the size of this expression, we break it up as follows,

$$\begin{aligned} \int_y^1 \frac{dt}{e^{2\lambda t} - 1} &= \int_y^{\frac{1}{\lambda}} \frac{dt}{e^{2\lambda t} - 1} + \int_{\frac{1}{\lambda}}^1 \frac{dt}{e^{2\lambda t} - 1} \leq C \left(\int_y^{\frac{1}{\lambda}} \frac{dt}{\lambda t} + \int_{\frac{1}{\lambda}}^1 \frac{dt}{e^{2\lambda t}} \right) \\ &\leq C \frac{1}{\lambda} \left(\log \frac{1}{\lambda y} + 1 \right), \end{aligned} \tag{3.48}$$

and hence, since $\lambda y < 1$,

$$\int_y^1 dt \frac{1}{e^{\lambda t} - e^{-\lambda t}} |\phi(y + t)| \cdot \left| \frac{1}{1 + e^{-2\lambda(t+y)}} \left(\frac{1}{e^{\lambda(t-y)}} - \frac{1}{e^{\lambda(t+y)}} \right) \right| \leq \|\phi\|_{C_{[0,2]}^0} \left(\frac{1}{\lambda} + y \right). \tag{3.49}$$

The remaining group of terms in (3.46) can be estimated in a similar way,

$$\frac{1}{e^{\lambda(t+y)}} \left| \frac{1}{1 + e^{-2\lambda(t-y)}} - \frac{1}{1 + e^{-2\lambda(y+t)}} \right| \leq (e^{\lambda y} - e^{-\lambda y}) e^{-\lambda(3t+y)} \leq C \lambda y e^{-3\lambda t}, \tag{3.50}$$

and hence

$$\begin{aligned} \int_y^1 dt \frac{1}{e^{\lambda t} - e^{-\lambda t}} \frac{1}{e^{\lambda(t+y)}} \left| \frac{1}{1 + e^{-2\lambda(t-y)}} - \frac{1}{1 + e^{-2\lambda(y+t)}} \right| \cdot |\phi(y + t)| \\ \leq \|\phi\|_{C_{[0,2]}^0} \lambda y \int_y^1 \frac{e^{-3\lambda t}}{e^{\lambda t} - e^{-\lambda t}} dt, \end{aligned} \tag{3.51}$$

which is even smaller than the previous integral. Finally, to estimate the integral from the region $|t| > 1$, we have the simple estimate, since $\lambda y < 1$, say for $t > 0$,

$$\begin{aligned} \frac{1}{e^{\lambda t} - e^{-\lambda t}} \frac{e^{\lambda y} + e^{-\lambda y}}{e^{-\lambda(t-y)} + e^{\lambda(t-y)}} |\phi(y - t)| &\leq C \frac{1}{e^{\lambda t}} \frac{1}{e^{\lambda(t-y)}} \|\phi\|_{\Lambda_{(m)}^0} (1 + |t|)^m \\ &\leq C \|\phi\|_{\Lambda_{(m)}^0} (1 + |t|)^m e^{-2\lambda t} \end{aligned} \tag{3.52}$$

which implies readily the desired inequality upon integration in t . The proof of the lemma is complete. \square

Proof of Theorem 2 It suffices to combine all estimates from Lemmas 4, 5, and 6: when $\lambda y \geq 1$, we apply Lemmas 4 and 5, and when $\lambda y < 1$, we apply Lemmas 4 and 6. \square

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