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Real Normalized Differentials and Arbarello's Conjecture*

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Abstract. Using meromorphic differentials with real periods, we prove Arbarello's conjecture that any compact complex cycle of dimension $g-n$ in the moduli space $\mathcal{M}_g$ of smooth algebraic curves of genus g must intersect the locus of curves having a Weierstrass point of order at most n.

Key words: moduli space of algebraic curves, integrable system, real normalized differential.

1. Introduction

In [1] Arbarello considered the natural filtration

$$\mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots \subset \mathcal{W}_{g-1} \subset \mathcal{W}_g = \mathcal{M}_g \tag{1}$$

of the moduli space $\mathcal{M}_g$ of smooth Riemann surfaces of genus g by subvarieties $\mathcal{W}_n$ of curves having a Weierstrass point of order at most n (i.e., by subvarieties of curves on which there exists a meromorphic function with one pole of order at most n). He conjectured that *any compact complex cycle in $\mathcal{M}_g$ of dimension at least $g-n$ must intersect $\mathcal{W}_n$*. Since $\mathcal{W}_2$ is the subvariety of hyperelliptic curves, which is affine, Arbarello's conjecture implies that *the dimension of any compact complex cycle in $\mathcal{M}_g$ does not exceed $g-2$*. This was proved later by Diaz in [4] with the help of a modification of the Arbarello filtration. Another modification of the Arbarello filtration was used by Looijenga, who proved that *the tautological classes of degree greater than $g-2$ vanish in the Chow ring of $\mathcal{M}_g$*. This implies Diaz' result (indeed, the Hodge class λ_1 is ample in $\mathcal{M}_g$, and thus for any complete d-dimensional subvariety $X \subset \mathcal{M}_g$ we have $\lambda_1^d \cdot X > 0$, while $\lambda_1^{g-1} = 0$, because the latter class is tautological).

The main goal of this paper is to prove Arbarello's conjecture, which has remained open until now in spite of the attention which it attracted during many years. (The highly nontrivial nature of this problem has found its explanation in the recent work [2], where it was shown that the strata of the Arbarello filtration are almost never affine.) Our proof uses certain constructions of the Whitham perturbation theory of integrable systems, which were proposed in [10] and [11] and further developed in [12] and [13]. These constructions have already found their applications in problems of topological quantum field theory (WDVV equations) and $N = 2$ supersymmetric gauge theory [5] (see also [3] and the references therein). The application of these constructions to the study of the geometry of the moduli spaces of curves was initiated in author's joint works with S. Grushevsky (see [7] and [8]).

In [7] we gave a new proof of Diaz' theorem, and in [8] we proved the triviality of certain tautological classes. Both results had been known, but their new proofs suggested that the further development of the Whitham theory constructions, which was the main goal of [7] and [8], might lead to the creation of new methods applicable to a wide spectrum of algebraic-geometric problems.

The notion of *real-normalized* meromorphic differentials is central in Whitham theory. By definition a real normalized differential is a meromorphic differential all of whose periods are real. The possibility of application of this notion is based on the fact that, on any algebraic curve and for any fixed set of "singular parts," there exists a unique real normalized differential having prescribed singularities at marked points.

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In [7] real normalized differentials of the third kind, i.e., differentials with two simple poles, were used. Arbarello's conjecture involves curves with one puncture, which has tempted the author to try to use real normalized differentials of the second kind in its proof. However, the naïve attempt to use an argument similar to that in [7] runs almost immediately into a serious obstacle related to the noncompactness of the space of the singular parts parameterizing such differentials. This obstacle is similar to those arising in all attempts to prove Arbarello's conjecture (see details in the book [6]).

In the next section we present the necessary extensions and partial compactification of the previously known constructions. Namely, we define a foliation structure on the space of real normalized differentials of the second kind with poles of order *at most* $n_\alpha + 1$ at a marked point p_α. In previous works the foliation structure was defined on the moduli space of real normalized differentials with *fixed* order of poles. As previously, each leaf of the foliation is a (locally) smooth complex subvariety. At the end of Section 2 we show that the foliation structure on the moduli space of real normalized differentials induces a foliation structure on the quotient space of these differentials modulo the action of the multiplicative group of *positive real numbers*.

In Section 3 we introduce an additional tool needed for the proof of the main theorem, namely, the notion of cycles "dual" to the zeros of real normalized differentials. We prove that the "dual cycles" generate the homology group $H_1(\Gamma, \mathbb{Z})$. It seems to the author that the construction of these cycles is of independent interest and deserves a separate study. The proof of Arbarello's conjecture is given in Section 4.

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2. Foliations Defined by Real Normalized Differentials

Let $\mathcal{M}_{g,k}^{(n)}$, $n = (n_1, \ldots, n_k)$, be the moduli space of smooth algebraic curves Γ of genus g with fixed *singular parts of poles of order* $n_\alpha + 1$ *without residue* in a neighborhood of marked distinct points $p_\alpha \in \Gamma$, $\alpha = 1, \ldots, k$. We recall that choosing a singular part of a pole of order $n_\alpha + 1$ at a point p_α means choosing an equivalence class of pairs (z_α, R_α), where (i) z_α is a local coordinate in a neighborhood of p_α such that $z_\alpha(p_\alpha) = 0$; (ii) R_α is a polynomial of the form $R_\alpha = \sum_{i=0}^{n_\alpha} r_{\alpha,i} z_\alpha^{-i-1}$. Pairs (z_α, R_α) and (w_α, R'_α) belong to the same equivalence class if $R'\, dw_\alpha = R_\alpha\, dz_\alpha + O(1)\, dz_\alpha$, $w_\alpha = w_\alpha(z_\alpha)$. The coefficient $r_{\alpha,0}$ in the polynomial is the residue of the singular part; i.e., for singular parts without residues, we have $r_{\alpha,0} = 0$.

The nondegeneracy of the imaginary part of the Riemann matrix of the b-periods of normalized holomorphic differentials on a smooth genus g algebraic curve Γ implies the following assertion.

Lemma 2.1. *For any fixed singular parts of poles with purely imaginary residues, there exists a unique meromorphic differential* Ψ *having prescribed singular part at marked points* p_α *and such that its periods over all cycles on* Γ *are real, i.e.*,

$$\operatorname{Im}\left(\oint_c \Psi\right) = 0 \quad \forall c \in H^1(\Gamma, \mathbb{Z}). \tag{2}$$

(For a detailed proof see Proposition 3.4 in [7].)

Remark 2.2. Although throughout the paper we consider only real normalized differentials of the second kind (i.e., having no residues at poles), most of the constructions can be easily extended to the case of meromorphic differentials with purely imaginary residues. For the first time, real normalization as a defining property of quasi-momentum differentials in the spectral theory of linear operators with quasi-periodic coefficients was introduced in [9] and [10] (where it was called absolute normalization).

Let $\mathcal{M}_{g,k}^{\langle n\rangle}$ be the moduli space of smooth algebraic curves Γ of genus g with fixed *nontrivial* real normalized meromorphic differential Ψ of the second kind having poles of order at most $n_\alpha + 1$ at marked points $p_\alpha \in \Gamma$. Lemma 2.1 identifies $\mathcal{M}_{g,k}^{\langle n\rangle}$ with the moduli space of curves with fixed nontrivial set of singular parts of poles of orders *at most* $n_\alpha + 1$ without residues at the marked points. The latter is the total space of a vector bundle of rank $|n| := \sum_\alpha n_a$ over $\mathcal{M}_{g,k}$ with the zero section removed. Therefore, the identification of a real normalized differential with its singular part defines a complex structure on $\mathcal{M}_{g,k}^{\langle n\rangle}$.

Our next goal is to introduce the structure of a foliation on $\mathcal{M}_{g,k}^{\langle n\rangle}$. The periods of the differential Ψ define a cohomology class $\Pi \in H^1(\Gamma, \mathbb{R})$. Let ∇ be the Gauss–Manin connection on the Hodge bundle over $\mathcal{M}_{g,k}$, whose fiber over (Γ, p_α) is $H^1(\Gamma, \mathbb{Z})$. Then the equation

$$\nabla_X \Pi = 0 \tag{3}$$

considered as an equation for the tangent vector $X \in T(\mathcal{M}_{g,k}^{\langle n\rangle})$ defines a subspace of the tangent space at every point of $\mathcal{M}_{g,k}^{\langle n\rangle}$. The distribution of these subspaces is integrable and defines a foliation on $\mathcal{M}_{g,k}^{\langle n\rangle}$. The rigorous definition is as follows.

Definition 2.3. A leaf $\mathcal{L}$ of the foliation on $\mathcal{M}_{g,k}^{\langle n\rangle}$ is defined as the locus along which the periods of the corresponding differentials remain (covariantly) constant.

Remark 2.4. In the previous works [11]–[13] (see also [7] and [8]) the structure of the foliation was defined on the moduli space $\mathcal{M}_{g,k}^{(n)}$ of real normalized differentials having poles of exact order $n_\alpha + 1$ at the marked points. The definition given above extends the construction to the moduli space $\mathcal{M}_{g,k}^{\langle n\rangle}$ corresponding to real normalized differentials with poles of order at most $n_\alpha + 1$.

The foliation defined above is real-analytic, but its leaves are complex (so the foliation is real-analytic only "in the transverse direction"). Indeed, locally, in a neighborhood U of any curve Γ_0, one can always choose a basis of cycles on every curve $\Gamma \in U$ which continuously varies with the variation of Γ. Therefore, locally a leaf $\mathcal{L}$ is defined by equations which mean the existence of a differential Ψ on Γ whose periods over the chosen basis of cycles $A_1, \dots, A_g, B_1, \dots, B_g$ take prescribed values $a_1, \dots, a_g, b_1, \dots, b_g$. These are holomorphic conditions, and thus each leaf is a locally complex subvariety $\mathcal{L} \subset \mathcal{M}_{g,k}^{\langle n\rangle}$. If a different basis of $H_1(\Gamma, \mathbb{Z})$ is chosen, the periods of Ψ over the new basis are still fixed along a leaf (although take different numerical values).

Remark 2.5. In [7] the leaves of the foliation on $\mathcal{M}_{g,k}^{(n)}$ were called big leaves, as opposed to small leaves defined by periods of two real normalized differentials. Big leaves can be regarded as a generalization of the Hurwitz spaces of covers of $\mathbb{P}^1$. More precisely, if $k > 1$ or $k = 1$ but $n_1 > 1$, then the corresponding foliation contains a special leaf $\mathcal{L}_0 \subset \mathcal{M}_{g,k}^{\langle n\rangle}$ on which all periods of Ψ vanish, i.e., Ψ is exact, $\Psi = dF$. On the curves corresponding to points of the open set $\mathcal{L}_0 \cap \mathcal{M}_{g,k}^{(n)}$ the meromorphic function F has poles of orders n_a at p_α, i.e., F defines a cover of $\mathbb{P}^1$ with prescribed types of branching over one point (infinity). Hence $\mathcal{L}_0 \cap \mathcal{M}_{g,k}^{(n)}$ can be identified with a C^*-bundle over the Hurwitz space $\mathcal{H}^{(n_1,\dots,n_k)}$.

Theorem 2.6. *A leaf $\mathcal{L}$ is a smooth (local) complex subvariety of real codimension $2g$ (i.e., of complex dimension $d_g^{(n)} = 2g - 3 + |n| + k$).*

Remark 2.7. Theorem 2.6 is a generalization to the case of $\mathcal{M}_{g,k}^{\langle n\rangle}$ of the following assertion proved in [12]: *The open part of $\mathcal{L}$, namely, $\mathcal{L} \cap \mathcal{M}_{g,k}^{(n)}$, is smooth.*

Proof. As mentioned above, locally, after a basis of cycles on the curves in a neighborhood of $\Gamma_0 \in \mathcal{M}_g$ is chosen, the leaf $\mathcal{L}$ passing through a point $(\Gamma_0, \Psi_0) \in \mathcal{M}_{g,k}^{\langle n\rangle}$ is defined by the equations

$$\oint_{A_j} \Psi = a_j, \qquad \oint_{B_j} \Psi = b_j, \tag{4}$$

which mean that on a curve Γ near Γ_0 there exists a differential which has the same periods as the differential on Γ_0 (the periods are automatically real). To prove the theorem, we must show that these equations are independent. It turns out that this is indeed the case; moreover, the set of periods considered as a set of local functions on $\mathcal{M}_{g,k}^{\langle n\rangle}$ can be explicitly completed to a local coordinate system near any point of $(\Gamma_0, \Psi_0) \in \mathcal{M}_{g,k}^{\langle n\rangle}$. □

The construction of such local coordinates on $\mathcal{M}_{g,k}^{(n)}$ is given in [12]. The set of holomorphic coordinates on $\mathcal{L} \cap \mathcal{M}_{g,k}^{(n)}$ is similar to that used in the theory of Hurwitz spaces. The coordinates are the *critical* values of the corresponding Abelian integral

$$F(p) = c + \int^p \Psi, \qquad p \in \Gamma, \tag{5}$$

which is a multivalued meromorphic function on Γ. On $\mathcal{M}_{g,k}^{(n)}$ the differential has poles of orders $n_\alpha + 1$ at p_α. Therefore, the zero divisor of Ψ is of degree $d_g^{(n)} + 1$. At a generic point of $\mathcal{L} \cap \mathcal{M}_{g,k}^{(n)}$, where the zeros q_s of Ψ are distinct, the coordinates on $\mathcal{L}$ are the values of F at these critical points, that is,

$$\varphi_s = F(q_s), \quad \Psi(q_s) = 0, \qquad s = 0, \dots, d_g^{(n)}, \tag{6}$$

normalized by the condition $\sum_s \varphi_s = 0$. Of course, these coordinates depend on the path of integration needed to define F in a neighborhood of q_s. The normalization is needed to define the additional constant c in (5). Near points of $\mathcal{L} \cap \mathcal{M}_{g,k}^{(n)}$ where the corresponding differential has a multiple zero $q_{s_1} = \dots = q_{s_r}$ the local coordinates are symmetric polynomials $\sigma_i(\varphi_{s_1}, \dots, \varphi_{s_r})$, $i = 1, \dots, r$ (it is assumed here that the paths of integrations determining the critical values φ_{s_k} are chosen consistently; for more details see [12]).

Remark 2.8. A direct corollary of the real normalization is the statement that the imaginary parts $f_s = \operatorname{Im} \varphi_s$ of the critical values are independent of the paths of integration and depend only on the numbering of critical points. These points can be arranged in decreasing order:

$$f_0 \geqslant f_1 \geqslant \dots \geqslant f_{d-1} \geqslant f_d, \qquad d = d_g^{(n)}. \tag{7}$$

Then each function f_j is a well-defined continuous function on $\mathcal{M}_{g,k}^{(n)}$ whose restriction to $\mathcal{L} \cap \mathcal{M}_{g,k}^{(n)}$ is a piecewise harmonic function. Moreover, as shown in [7], the first function f_0 restricted to $\mathcal{L} \cap \mathcal{M}_{g,k}^{(n)}$ is a *subharmonic function*, i.e, a function to which the maximum principle can be applied: *If f_0 has a local maximum on a complex subvariety of $\mathcal{L}$, then it is constant on this subvariety.*

Now we are going to introduce a set of local coordinates in a neighborhood of any point (Γ_0, Ψ_0) of an arbitrary stratum $\mathcal{M}_{g,k}^{(m)} \subset \mathcal{M}_{g,k}^{\langle n\rangle}$. Let $\mathcal{C}$ be the universal curve over $\mathcal{M}_{g,k}^{\langle n\rangle}$, i.e., the bundle whose fiber over $(\Gamma, \Psi) \in \mathcal{M}_{g,k}^{\langle n\rangle}$ is the curve Γ itself. The marked points determine sections s_α of $\mathcal{C}$. A choice of local coordinates z_α near p_α on each of the curve Γ in a neighborhood of Γ_0 is equivalent to a choice of a local trivialization of a neighborhood of s_α in $\mathcal{C}$. After such a trivialization is fixed, the coefficients $r_{\alpha,i}$ in the Laurent expansion

$$\Psi = \sum_{i=1}^{n_\alpha} r_{\alpha,i} z_\alpha^{-i-1}\, dz_\alpha + O(1)\, dz_\alpha \tag{8}$$

of Ψ near p_α can be regarded as local functions on $\mathcal{M}_{g,k}^{\langle n\rangle}$.

If $(\Gamma_0, \Psi_0) \in \mathcal{M}_{g,k}^{(m)}$, then one can choose a neighborhood U_ε of (Γ_0, Ψ_0) in $\mathcal{M}_{g,k}^{\langle n\rangle}$ so that in this neighborhood the following inequalities hold:

$$0 < r < |r_{\alpha,m_\alpha}|, \quad |r_{\alpha,i}| < \varepsilon, \qquad i = m_\alpha + 1, \dots, n_\alpha, \tag{9}$$

where r is a constant.

If ε is small enough, then the differential Ψ has $d_g^{(m)} + 1$ zeros q_s outside neighborhoods of the marked points p_a, which tend to the zeros of Ψ_0 on Γ_0 as $\varepsilon \to 0$. The remaining $\sum_\alpha (n_a - m_\alpha)$ zeros of Ψ tend to the marked points p_α. Recall that the critical values of F depend on the choice of the constant c in (5). Under the normalization $\sum_s \varphi_s = 0$ chosen above, which fixes this constant, all critical values may have no limit on $\mathcal{M}_{g,k}^{(m)}$. In what follows, we define "finite critical values" φ_s, $s = 0, \ldots, d_g^{(m)}$, of F locally in U_ε by the normalization $\sum_{s=0}^{d_g^{(m)}} \varphi_s = 0$.

Let us introduce the following set of functions $\{x_A\}$ in $U_\varepsilon \subset \mathcal{M}_{g,k}^{\langle n\rangle}$:

(i) the leading coefficients $r_{\alpha,i}$, $i = m_\alpha + 1, \ldots, n_\alpha$, in expansion (8);

(ii) the finite critical values φ_s, $s = 1, \ldots, d_g^{(m)}$, of F if Ψ_0 has simple zeros; if Ψ_0 has a multiple zero $q_{s_1} = \cdots = q_{s_r}$, then the corresponding subset of finite critical values of F should be replaced by symmetric polynomials.

Lemma 2.9. *Near each point* $(\Gamma_0, \Psi_0) \in \mathcal{M}_{g,k}^{(m)} \subset \mathcal{M}_{g,k}^{\langle n\rangle}$ *the periods* a_j *and* b_j, $j = 1, \ldots, g$, *of* Ψ *and the real and imaginary parts of the functions* x_A *(see* (i) *and* (ii) *above) have linearly independent differentials and thus define a real-analytic local coordinate system on* $\mathcal{M}_{g,k}^{\langle n\rangle}$.

The proof of Lemma 2.9 is similar to that of Theorem 1 in [12]. The key points are as follows. Suppose that the differentials of the functions under consideration are linearly dependent at (Γ_0, Ψ_0) (and thus the functions do not determine local coordinates near (Γ_0, Ψ_0)). Then there exists a one-parameter family of points $(\Gamma_t, \Psi_t) \in \mathcal{M}_{g,k}^{\langle n\rangle}$ with real parameter t such that the derivative of any of the above functions with respect to the parameter vanishes at $t = 0$.

Recall that locally on each of the curves Γ_t we have already fixed a basis A_j, B_j for cycles (needed for the definition of periods (4)). Let $\omega_j(t)$ be the basis of holomorphic differentials on Γ_t dual to A_j, and let $v_j(p,t) := \int_{q_1(t)}^p \omega_i(t)$ denote the corresponding Abelian integrals, which are multivalued functions of $p \in \Gamma_t$ depending on the choice of the path of integration. By $F_t(p) := \int_{q_1(t)}^p \Psi_t$ we denote the integral of the chosen meromorphic differential along the same path.

Consider how v_j varies in t. To attach meaning to a partial derivative with respect to the parameter, we must determine how the point p changes under the variation of Γ_t. For this purpose, we use F_t to determine a local coordinate on the universal cover of Γ. After this the variation of the point p under the variation of the parameter is determined from the implicit equation $F = F_t(p_t)$. In other words, the fixation of F allows us to define a connection on the space of Abelian integrals.

Our goal is to prove that, under the assumptions made above, the *partial derivatives with fixed* F defined by

$$\partial_t v_j(F) := \frac{\partial}{\partial t} v_j(F_t^{-1}(F), t)\bigg|_{t=0} \tag{10}$$

vanish. Consider the surface Γ_t cut along basis cycles so that the integration paths $\int_{q_1(t)}^p$ in the definition of all Abelian integrals do not intersect these cuts, i.e., are contained in the simply connected cut surface. Then expression (10) defines a meromorphic function on the cut surface Γ_0 with possible poles at the zeros of Ψ (where F^{-1} is singular) and discontinuities along the cuts. However, if the periods of Ψ_t do not change under the variation of t, then (10) has constant "jumps" along the cuts, and if the critical values φ_s do not change, then (10) has no poles at the zeros of Ψ. By the chain rule the partial derivatives with fixed F and partial derivatives with *fixed* z_a (determining a trivialization of a neighborhood of the marked point $p_{\alpha,t} \in \Gamma_t$) are related by the equation

$$\partial_t v_j(F) = \partial_t v_j(z_\alpha) - \frac{\omega_j}{\Psi_0}\, \partial_t F(z_\alpha)\bigg|_{t=0}. \tag{11}$$

By our assumption the first $(n_\alpha - m_\alpha)$ leading coefficients in expansion (8) do not change. Hence it follows from (11) that (10) has no pole at p_α. Thus, the differential of expression (10) is a holomorphic differential on Γ_0 with zero A-periods; therefore, it is identically zero and hence has zero B-periods. Since the B-periods of ω_j form the period matrix τ of Γ_0, we have

$$\frac{\partial}{\partial t} \tau_{ij}(t)\bigg|_{t=0} = 0.$$

The infinitesimal version of Torelli's theorem says that the differential of the period map $\tau\colon \mathcal{M}_g \to \mathcal{A}_g$ is nondegenerate outside the variety of hyperelliptic curves. Therefore, the above relation cannot hold unless Γ_0 is a hyperelliptic curve. For a hyperelliptic curve the kernel of the homomorphism $d\tau$ is one-dimensional and transverse to the tangent space of the subvariety of hyperelliptic curves. Therefore, to complete the proof, it suffices to show that if Γ_0 is a hyperelliptic curve, then the tangent vector to the family of curves Γ_t at $t = 0$ is tangent to the subvariety of hyperelliptic curves. The proof of this assertion coincides with the proof of its analogue in [12].

Corollary 2.10. *The set of functions x_A defines a system of local complex coordinates on the leaf $\mathcal{L}$ passing through the point* (Γ_0, Ψ_0).

For what follows we need a partial compactification of $\mathcal{M}_{g,k}^{\langle n\rangle}$. The space $\mathcal{M}_{g,k}^{\langle n\rangle}$ of real normalized differentials is invariant under multiplication by *real numbers*. Let $\mathcal{P}_{g,k}^{\langle n\rangle} = \mathcal{M}_{g,k}^{\langle n\rangle}/R_+$ be the quotient space by the action of the multiplicative group of positive real numbers. The fiber of the forgetful map $\mathcal{P}_{g,k}^{\langle n\rangle} \to \mathcal{M}_g$ is the space of nontrivial "normalized" singular parts. It is isomorphic to the sphere $S^{2|n|-1}$.

The canonical foliation of $\mathcal{M}_{g,k}^{\langle n\rangle}$ defined above induces a foliation structure on $\mathcal{P}_{g,k}^{\langle n\rangle}$.

Definition 2.11. A leaf $[\mathcal{L}]$ of the foliation on $\mathcal{P}_{g,k}^{\langle n\rangle}$ is defined to be the projection of the locus in $\mathcal{M}_{g,k}^{\langle n\rangle}$ along which the ratio of any two periods of the corresponding differentials remains constant (if two periods are zero, they both must remain zero).

Multiplication by real numbers acts "transversally" on all leaves $\mathcal{L}$ of the big foliation except on the Hurwitz leaf $\mathcal{L}_0$ corresponding to exact differentials. Therefore, a leaf $[\mathcal{L}]$ passing through a point $(\Gamma, [\Psi]) \in \mathcal{P}_{g,k}^{\langle n\rangle}$, which is not in the image of $\mathcal{L}_0$, is locally isomorphic to the leaf $\mathcal{L} \in \mathcal{M}_{g,k}^{\langle n\rangle}$ passing through a point $(\Gamma, \Psi) \in \mathcal{M}_{g,k}^{\langle n\rangle}$ in the preimage of $(\Gamma, [\Psi])$. Hence $[\mathcal{L}]$ has a natural complex structure. (Notice that we cannot treat $[\mathcal{L}]$ as a local complex subvariety anymore, because $\mathcal{P}_{g,k}^{\langle n\rangle}$ is not a complex variety.)

The leaf $[\mathcal{L}_0]$ is the only singular leaf of the foliation. It is not complex, and its real dimension is less by 1 than the dimension of all other leaves. It is isomorphic to a S^1-bundle over the Hurwitz space.

3. Dual Cycles and Periods

In this section we introduce yet another notion needed for the proof of Arbarello's conjecture given in the next section, namely, the notion of cycles dual to critical points of real normalized differentials.

For simplicity, we consider only the case where $(\Gamma, \Psi) \in \mathcal{M}_{g,1}^{(n)}$, that is, the case of real normalized differentials having poles only at one marked point ($k = 1$). By the definition of the real normalization, the imaginary part $\Phi = \operatorname{Im} F$ of an Abelian integral F of such a differential is a single-valued harmonic function on $\Gamma \setminus p_1$. The level curves $\Phi_h := \{p \in \Gamma : \Phi(p) = h\}$ of this function are cycles that are smooth everywhere except at p_1 and at q_s if $h = f_s = \Phi(q_s)$. For h large enough ($h > f_0$), the level curve is the union of n "loops" in a small neighborhood of p_1. The real part of $F(p)$ is multivalued. Nevertheless, everywhere except at the zeros of Ψ, the directions along which the real part remains (locally) constant is well defined. We refer to the integral lines of these directions as *imaginary* rays. It will be always assumed that they are oriented so that Φ increases in the direction of orientation. In a small neighborhood of p_1 one can always fix a single-valued branch of F. If the imaginary ray going from a point p does not pass through the zeros of Ψ, then the values of F at the points of the ray belonging to the neighborhood of p_1 uniquely determines the real part of $F(p)$. This allows one to define a single-valued holomorphic branch of $F(p)$ on $\Gamma \setminus \Sigma$, where Σ is the graph whose edges are imaginary rays that begin at p_1 and at zeros of Ψ and end at zeros of Ψ. By continuity F can be extended to Γ cut along the edges of Σ. The limits

$F^{\pm}(p)$, $p \in \Sigma$, on the two sides of each cut are generally distinct. The discontinuity of F, i.e., the "jump" function $j(p) := F^{+}(p) - F^{-}(p)$, is constant on each of the edges.

Let $\mathcal{D}$ be an open subset of $\mathcal{M}_{g,1}^{(n)}$ where the differentials have simple zeros with "distinct" real parts of critical values. The latter means that the imaginary rays emanating from any zero of the differential do not contain any other zero, i.e., they end at p_1. In this case the graph has $2g+n-1$ connected components Σ_s. Each component is the union of two imaginary rays, both beginning at p_1 and ending at one of the zeros q_s of Ψ. Along these rays the imaginary part Φ increases from $-\infty$ to f_s, i.e., each of the zeros q_s of Ψ is a "tip" of Σ_s. At the same time, for each zero q_s, there are two imaginary rays beginning at q_s and ending at p_1. Along these rays Φ increases from f_s to $+\infty$. Reversing the orientation of one of these rays, we can define a closed oriented cycle σ_s on the curve.

Although the differential Ψ is singular on σ_s, the period of Ψ corresponding to the homology class $[\sigma_s] \in H_1(\Gamma, \mathbb{Z})$ of σ_s is well defined (recall that Ψ has no residue at p_1). It equals

$$\pi_s := \oint_{[\sigma_s]} \Psi = r_s^1 - r_s^2, \tag{12}$$

where r_s^1 and r_s^2 are the values of the real part of $F(q_s)$ asymptotically defined along each of the rays. Notice that on the edges of the graph Σ the jump function equals $j(p) = \pm\pi_s$, $p \in \Sigma_s$.

Remark 3.1. In the simplest case of real normalized differentials having one pole of the second order ($n = 1$) the construction of dual cycles looks especially attractive. In this case, if Ψ has simple zeros with distinct real parts of critical values, then it defines precisely $2g$ dual cycles on the corresponding curve. It is easy to see that each of the dual cycles represents a nontrivial homology class. Indeed, for $n = 1$, the Abelian integral F of Ψ has a simple pole at p_1. Therefore, the period of Ψ over $[\sigma_s]$ never vanishes: $\pi_s \neq 0$. Hence the cycle is not homologous to zero: $[\sigma_s] \neq 0$. We show below that the classes $[\sigma_s]$ are linearly independent and thus determine a basis in $H_1(\Gamma, \mathbb{Z})$ (see Lemma 3.2 below).

At the points (Γ, Ψ) where Ψ has a multiple zero or the integral F has critical values with "coinciding" real parts the structure of the graph Σ may be combinatorially nontrivial. Still, for each zero q_s of Ψ, there are only *finitely* many (arbitrarily ordered) semi-infinite paths γ_s^i along imaginary directions which start at q_s and end at p_1. A pair of such paths determines an oriented cycle $\sigma_s^{ij} = \gamma_s^i \cup (-\gamma_s^j)$, $i < j$.

Lemma 3.2. *The homology classes $[\sigma_s^{ij}]$ of dual cycles generate the homology group $H_1(\Gamma, \mathbb{Z})$.*

The proof of the lemma is yet another exercise in the application of the argument used in [12] and, in a slightly generalized form, in the proof of Theorem 2.6. Indeed, suppose that the classes $[\sigma_s^{ij}]$ defined by the differential Ψ_0 on a curve Γ_0 do not generate $H_1(\Gamma_0, \mathbb{Z})$; then by Theorem 1 in [12] there is a one-parameter deformation $(\Gamma_t, \Psi_t) \in \mathcal{M}_{g,1}^{(n)}$ such that the derivatives of the critical values φ_s and the periods π_s^{ij} along this family vanish at $t = 0$.

Recall once again that locally it can always be assumed that on each of the curves Γ_t one has a fixed basis A_j, B_j for cycles. As above, let $\omega_j(t)$ be the basis of holomorphic differentials on Γ_t dual to A_j. Let us fix a branch of the corresponding Abelian integral v_j in the sectors of a neighborhood of p_1 where $\Phi \to +\infty$. Analytically continuing this branch along imaginary rays (with reverse orientation), we can define a single-valued branch of v_j on $\Gamma_t \setminus \Sigma_t$ and then extend it by continuity to both sides of each edge of Σ_t. The jumps of v_j on the edges of the graph are linear combinations of the periods of ω_j over dual cycles. As before, consider the partial derivative $\partial_t v_j(F, t)$ with fixed F at $t = 0$. If the derivatives of φ_s and the jumps of F on the edges of Σ vanish (as assumed), then the derivative $\partial_t v_j(F, t)|_{t=0}$ is holomorphic on $\Gamma_0 \setminus \Sigma_0$ and has constant jumps on Σ_0. Therefore, the differential of this expression is a holomorphic differential on Γ_0 with zero A-periods. Hence it is identically zero. It follows that the matrix of b-periods of ω_j does not change along the family at $t = 0$. The remaining steps of the proof are identical to those in [12].

Let r_s^i be the "real values" of $F(q_s)$ defined asymptotically by the paths γ_s^i. In the general case, the jumps $j(p)$ on the edges of Σ are linear combinations with integer coefficients of the periods of Ψ over the dual cycles, which are

$$\pi_s^{ij} := \oint_{[\sigma_s^{ij}]} \Psi = r_s^i - r_s^j. \tag{13}$$

4. Proof of Arbarello's Conjecture

To motivate the further steps in the proof of Arbarello's conjecture, we outline a new proof of Diaz' theorem using real-normalized differentials with one pole of the second order.

Diaz' theorem revisited. Let ϕ_s be a "weighted critical value" of F defined by the formula

$$\phi_s = \frac{\varphi_s}{|\pi_s|}, \qquad |\pi_s| := \min_{ij}\{|\pi_s^{ij}| \neq 0\}, \tag{14}$$

where the minimum is taken over a finite set of nonzero periods dual to the critical value. As emphasized in Remark 3.1, at least one of the periods dual to a critical point of a real normalized differential with one pole of the second order does not vanish. Therefore, ϕ_s is a well-defined local function on $\mathcal{M}_{g,1}^{(1)}$. The imaginary parts of the weighted critical values $g_s = \operatorname{Im}\phi_s$ are independent of the paths of integration and depend only on the numbering of the critical points. Let us arrange them in decreasing order:

$$g_0 \geqslant g_1 \geqslant \cdots \geqslant g_{2g-2} \geqslant g_{2g-1}, \tag{15}$$

Then each of the functions g_s can be seen as a well-defined function on $\mathcal{M}_{g,1}^{(1)}$ (cf. Remark 2.8). It is continuous on the open set $\mathcal{D} \subset \mathcal{M}_{g,1}^{(1)}$, where the corresponding differentials Ψ have simple zeros with distinct real parts of critical values. (Recall that the latter means that the imaginary rays emanating from a zero of the differential do not contain any other zeros.) Moreover, g_s restricted to $\mathcal{L} \cap \mathcal{D}$ is a piecewise harmonic function.

From (14) it is easy to see that the value of g_s at any point of $\mathcal{M}_{g,1}^{(1)}$ is equal to the maximum limit at this point of values of g_s on $\mathcal{D}$. Therefore, g_s is an *upper semicontinuous* function on $\mathcal{M}_{g,1}^{(1)}$.

Lemma 4.1. *The function g_0 restricted to any complex subvariety of $\mathcal{L}$ has a local maximum if and only if it is constant on this subvariety.*

Proof. By the definition of $\mathcal{L}$, the periods π_s are constant on $\mathcal{L} \cap \mathcal{D}$; therefore, on $\mathcal{L} \cap \mathcal{D}$ the function g_0, being the maximum of harmonic functions, has local maximum only if it is constant (subharmonic). In order to show that g_0 has no local maximum at the points of discontinuity, it is sufficient to notice that the "directions" of its discontinuity which correspond to a change of the real part $\operatorname{Re}\varphi_0$ are always transversal to the directions along which the imaginary part $\operatorname{Im}\varphi_0$ changes.

Let X be a compact complex cycle in $\mathcal{M}_g$. Its preimage Y under the forgetful map $\mathcal{M}_{g,1}^{(1)} \to \mathcal{M}_g$ is noncompact, but the quotient space $Z = Y/R_+$ is compact. The function g_0 is homogeneous with respect to multiplication of real normalized differentials by positive real numbers; thus, it determines an upper semicontinuous function g_0' on $\mathcal{P}_{g,1}^{\langle 1\rangle}$. Since Z is compact, there is a point $(\Gamma_0, [\Psi_0])$ of Z where the restriction of g_0' on Z attains its maximum. Let us fix a preimage $(\Gamma_0, \Psi_0) \in \mathcal{M}_{g,1}^{(1)}$ of this point. At this preimage the function g_0 attains its maximum on Y.

Let $\mathcal{L}$ be the leaf of the big foliation passing through $(\Gamma_0, \Psi_0) \in \mathcal{M}_{g,1}^{(1)}$. At this point the function g_0 restricted to $\mathcal{L} \cap Y$ has a local maximum. By Lemma 4.1 it must be constant on $\mathcal{L} \cap Y$.

Let $Y_0 \in Y$ be a preimage of the compact set $Z_0 \subset Z$ at which g_0' takes its maximum value. On this compact set the second function g_1' must attain its maximum. As shown above, Y_0 is foliated by the leaves $\mathcal{L} \cap Y$. On these leaves the second function g_1 is subharmonic, i.e., it must be a constant. Continuing by induction, we see that all functions g_s are constants on $\mathcal{L} \cap Y$. If the g_s are constants, then the critical values φ_s are (locally) constant on $\mathcal{L} \cap Y$ as well. The functions

φ_s are local coordinates on $\mathcal{L}$. Therefore, $\mathcal{L} \cap Y$ must be at most zero-dimensional. But if X is of dimension greater that $g-2$, then $\mathcal{L} \cap Y$ is at least one-dimensional. This contradiction completes the proof of Diaz' theorem. $\square$

The proof given above is almost a carbon copy of the proof of Diaz' theorem given in [7] with the only (but important) modification consisting in the replacement of critical values by weighted critical values. The proof of Arbarello's conjecture follows mainly the same line of reasoning but requires further modifications for the following reasons: (a) the periods over cycles dual to critical values of a real normalized differential having a pole of order $n+1>2$ may vanish; (b) only part of the critical values of such a differential have finite limits on smaller strata $\mathcal{M}_{g,1}^{(m)}$. Notice that these two issues are interrelated, at least in the neighborhood U_ε of $\mathcal{M}_{g,1}^{(m)}$, $m<n$. Indeed, for sufficiently small ε, periods over the cycles dual to the critical points belonging to a small neighborhood of p_1 are always zero, because the cycles themselves are homologically trivial.

For further use, let us introduce the corresponding stratification on $\mathcal{M}_{g,1}^{\langle n\rangle}$. First, for $(\Gamma,\Psi) \in \mathcal{M}_{g,1}^{\langle n\rangle}$, consider the subset S of critical points of Ψ such that at least one of the periods of Ψ over the dual cycles σ_s^{ij} is nonzero, i.e., $q_s \in S$ if there exists a $\pi_s^{ij} \neq 0$.

Definition 4.2. The stratum $\mathcal{S}_k$ of $\mathcal{M}_{g,1}^{\langle n\rangle}$ is defined to be the locus where Ψ has exactly k zeros with nonvanishing periods over the dual cycles, i.e., $|S|=k$.

At the points of $\mathcal{S}_k$ we normalize the critical values of the corresponding differential by the condition $\sum_{q_s\in S}\varphi_s = 0$. As before, the imaginary parts $\operatorname{Im}\phi_s$, $s \in S$, of the weighted critical values (14) can be arranged in decreasing order:

$$w_0^k \geqslant \cdots \geqslant w_{k-1}^k. \tag{16}$$

After that w_j^k can be seen as well-defined upper semicontinuous functions on $\mathcal{S}_k$. The functions w_j^k are continuous on $\mathcal{S}_k \cap \mathcal{D}$, and w_0^k is subharmonic on $\mathcal{S}_k \cap \mathcal{L}$.

On a leaf $\mathcal{L}$ the periods of Ψ over any continuously varying cycles are constants. Therefore, on $\mathcal{L}$, the locally constant function $|S|$ is discontinuous only at those points at which one of the zeros q_s of the corresponding differential "escapes" from S under a variation which makes the real part of its critical value different from the others. Notice that if q_s leaves S under a variation moving it to one side of the edge, then it remains in S under variations moving it on the other side of the edge. Therefore, the intersection $\mathcal{L} \cap \overline{\mathcal{S}_k}$ of the leaf with the closure of $\mathcal{S}_k$ is a complex domain with boundary. Notice also that the imaginary part of the vanishing critical value should be less than the imaginary part of one of the critical values corresponding to the top vertex of the graph edge, which the vanishing critical value intersects under the variation. At the same time, the set of nontrivial periods q_s (before the variation) is a subset of the periods dual to the top vertex of the edge. Hence the vanishing weighted critical value satisfies the inequality $\operatorname{Im}\phi_s \le w_0^k$. Thus, at the common points of the boundaries of $\mathcal{L}\cap\overline{\mathcal{S}_k}$ and $\mathcal{L}\cap\overline{\mathcal{S}_{k+1}}$ the functions w_0^k and w_0^{k+1} coincide, i.e., on each leaf $\mathcal{L}$ of the foliation there is a well-defined subharmonic function $W_0^{\mathcal{L}}$. It should be emphasized that the above argument is valid only when consideration is restricted to the leaves of the foliation. Under variations changing periods the vanishing critical value may tend to infinity, and there may exist no globally defined upper semicontinuous function W_0.

Now, we are ready to present the proof of the main theorem.

Theorem 4.3. *Any compact complex cycle in $\mathcal{M}_g$ of dimension at least $g-n$ must intersect the subvariety $\mathcal{W}_n \subset \mathcal{M}_g$ of smooth algebraic curves of genus g having a Weierstrass point of order at most n.*

Proof. Let X be a complex compact cycle in $\mathcal{M}_g$ that does not intersect $\mathcal{W}_n$. Then its preimage Y under the forgetful map $\mathcal{M}_{g,1}^{\langle n\rangle} \to \mathcal{M}_g$ does not intersect the leaf $\mathcal{L}_0 \subset \mathcal{M}_{g,1}^{\langle n\rangle}$ corresponding to exact differentials. Therefore, for each point $(\Gamma,\Psi)\in Y$, the corresponding set S of critical points is always nonempty. Let k_0 be the minimum value of $|S|$ on Y, i.e., the minimum number such that $Y^k = \mathcal{S}_k \cap Y$ is nonempty.

For arbitrary k, the set Y^k is nonclosed, partly because of the vanishing of some periods at limit points. Although k_0 is the minimal number of critical values with nonzero dual periods, the subset Y^{k_0} may be nonclosed under variations along leaves. The limit points of Y^{k_0} not belonging to Y^{k_0} are boundary points of domains $Y^k \cap \mathcal{L}$ with $k > k_0$ which contain "vanishing" critical values. As shown above, the function $w_0^{k_0}$ has a continuous extension to the closure $\overline{Y^{k_0}}$.

The quotient space $\overline{Z^{k_0}} = \overline{Y^{k_0}}/R_+$ is compact. The function w_0 induces an upper semicontinuous function w_0' on $\overline{Z^{k_0}}$, which must attain its maximum at some point. Let $\mathcal{L}$ be the leaf of the foliation passing through any preimage of this point in $\overline{Y^{k_0}}$. The intersection $\mathcal{L} \cap \overline{Y^{k_0}}$ is a complex domain with boundary. The argument used in the proof of Lemma 4.1 shows that $w_0^{k_0}$ is subharmonic both at the interior points of the domain and on its boundary. Therefore, w_0 must be constant on $\mathcal{L} \cap \overline{Y^{k_0}}$.

If it is a constant, then the next function $w_1^{k_0}$ on $\mathcal{L} \cap Y^{k_0}$ is subharmonic and must be a constant from the same considerations. Continuing by induction, we see that all functions w_s are constant on $\mathcal{L} \cap Y^{k_0}$. If the $w_s^{k_0}$ are constant, then the corresponding critical values φ_s, $q_s \in S$, are constant as well.

Consider the set $Y_0 \subset Y$ for which among the zeros of the corresponding differential there is a k_0-tuple of zeros of Ψ such that (a) the ordered set $w_0 \geqslant \cdots \geqslant w_{k_0}$ of the imaginary parts of the critical values remain constant along the corresponding leaf of the foliation and (b) the imaginary part of the first critical value is normalized by the condition $f_0 = 1$. Notice that this normalization fixes the lifting of $Y_0/R_+ \subset Z = Y/R_+$ into Y.

The set Y_0 is compact (and nonempty, as shown above). Therefore, the continuous function $\widetilde{f} = \max_{s \in S} \operatorname{Im} \phi_s$ restricted to Y_0 attains its maximum. (Recall that, by definition, the zeros q_s, $s \in S$, of the differential Ψ are bounded away from p_1, and therefore the function $\widetilde{f}$ is bounded on Y.) The function $\widetilde{f}$ restricted to any leaf of the foliation is subharmonic. Hence the same argument as above proves that on $Y_0 \cap \mathcal{L}$ all critical values having nonzero dual period are constant.

Suppose that at least one of the remaining critical values φ_s, $q_s \notin S$, is nonconstant on $\mathcal{L} \cap Y_0$. Then, "moving" along $\mathcal{L} \cap Y_0$, the corresponding zero $q_s \notin S$ must cross an "unmovable" edge of the graph Σ with a nontrivial jump on it. After such a crossing at least one period dual to q_s becomes nonzero. If ϕ_s becomes constant after crossing the edge of the graph, then it must be constant before the crossing. Thus, all critical values are constant on $\mathcal{L} \cap Y_0$. By the definition of Y_0 we have $\mathcal{L} \cap Y_0 = \mathcal{L} \cap Y$. If the φ_s are constants, then $\mathcal{L} \cap Y$ is at most zero-dimensional, which contradicts the assumption that X has dimension at least $g - n$. This completes the proof of the theorem. □

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