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Triangular Reductions of the $2D$ Toda Hierarchy*

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Abstract. New reductions of the 2D Toda equations associated with lower-triangular difference operators are proposed. Their explicit Hamiltonian description is obtained.

Key words: integrable systems, bi-Hamiltonian theory, Baker–Akhiezer function.

1. Introduction

A recent burst of interest in the theory of linear difference operators was motivated by the connection between these operators and the theory of discrete-time integrable systems of new type (the pentagram map and its higher-dimensional generalizations), which have turned out to be closely related to representation theory (the Coxeter friezes), and the theory of cluster algebras.

The pentagram map is defined for n-gons in $\mathbb{RP}^2$ as follows: each vertex v_i of an n-gon $(v_1, \dots, v_n)$ is mapped to the intersection point of the two diagonals (v_{i-1}, v_{i+1}) and (v_i, v_{i+2}). If n and $k+1$ are coprime, then, as shown in [12], the moduli space of n-gons in $\mathbb{RP}^k$ is isomorphic, as an algebraic variety, to the space $\mathcal{E}_{k+1,n}$ of n-periodic linear difference equations

$$V_i = a_i^{(1)} V_{i-1} - a_i^{(2)} V_{i-2} + \dots + (-1)^{k-1} a_i^{(k)} V_{i-k} + (-1)^k V_{i-k-1} \tag{1.1}$$

whose *all solutions* are (anti)periodic:

$$V_{i+n} = (-1)^k V_i. \tag{1.2}$$

In [5] such equations were called *superperiodic.*

More generally, Eqs. (1.1) without constraints (1.2) correspond to the so-called twisted n-gons in $\mathbb{RP}^k$, that is, sequences of $v_j \in \mathbb{RP}^k$, $j \in \mathbb{Z}$, for which there is a projective linear transformation M of $\mathbb{RP}^k$ such that $v_{j+n} = M v_j$.

In [12] it was shown that the pentagram map is a discrete integrable system, i.e., it preserves a certain natural structure of a Poisson manifold on the space of n-periodic lower-triangular operators (1.1) of order 3, and a complete set of integrals of motion in involution for the pentagram map was constructed. The algebraic-geometric integrability of the pentagram map was proved in [13].

In [11] an explicit construction of a duality between the spaces $\mathcal{E}_{k+1,n}$ and $\mathcal{E}_{n-k-1,n}$ was proposed, which is a generalization of the classical Gale duality for n-gons. In [5] this duality was connected with the theory of commuting difference operators, and a spectral theory of strictly lower triangular difference operators

$$L = T^{-k-1} + \sum_{j=1}^{k} a_i^{(j)} T^{-j}, \qquad a_i^{(j)} = a_{i+n}^{(j)}, \tag{1.3}$$

was developed. Here T is the shift operator: $T\psi_j = \psi_{j+1}$. Throughout the paper it is assumed that the leading coefficient of L is non-zero:

$$a_i^{(1)} \neq 0. \tag{1.4}$$

The spectral theory of triangular difference operators is of interest in its own right. Our point of departure in this paper is the simple observation that the spectral theory of triangular operators is naturally connected with a special reduction of the $2D$ Toda hierarchy.

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Remark 1.1. For definiteness, in this paper we consider only the case of lower-triangular reductions, since the involution $L \to L^*$, where

$$L^* = T^{k+1} + \sum_{j=1}^{k} T^j a_i^{(j)} = T^{k+1} + \sum_{j=1}^{k} a_{i+j}^{(j)} T^j \tag{1.5}$$

is the formal adjoint operator, establishes an equivalence of the cases of lower- and upper-triangular operators.

Recall that the $2D$ Toda equation

$$\partial^2_{\xi\eta}\varphi_i = e^{\varphi_i - \varphi_{i+1}} - e^{\varphi_{i-1} - \varphi_i} \tag{1.6}$$

is a consistency condition for the two linear problems

$$\begin{cases} \partial_\xi \Psi_i = v_i \Psi_i + \Psi_{i-1}, \\ \partial_\eta \Psi_i = c_i \Psi_{i+1}, \quad c_i = e^{\varphi_i - \varphi_{i+1}}. \end{cases} \tag{1.7}$$

The full $2D$ Toda hierarchy is an infinite system of equations for a function $\varphi_i = \varphi_i(t_1^+, t_1^-, t_2^+, t_2^-, \ldots)$ depending on one discrete variable i and two sets of continuous variables $t_m^\pm$, which are usually referred to as the *times* of the hierarchy. In what follows, the times t_1^+ and t_1^- are identified with ξ and η. The hierarchy equations are a consistency condition for the system of linear problems

$$\partial_{t_m^\pm} \Psi = L_m^\pm \Psi, \tag{1.8}$$

where the $L_m^\pm$ are difference operators of the form

$$L_m^\pm = \sum_{j=0}^{m} a_{i,m}^{(j,\pm)} T^{\pm j} \tag{1.9}$$

with leading coefficients

$$a_{i,m}^{(m,-)} = 1, \qquad a_{i,m}^{(m,+)} = e^{\varphi_i - \varphi_{i+m}}. \tag{1.10}$$

It is easy to check that the consistency of the second equation in (1.7) with (1.8) implies

$$a_{i,m}^{(0,-)} = \partial_{t_m^-} \varphi_i, \qquad a_{i,m}^{(0,+)} = 0. \tag{1.11}$$

Remark 1.2. Importantly, the hierarchy of any soliton equation regarded as a linear space of commuting vector fields is well defined. However, as a rule, there is no canonical choice of "times" (or, equivalently, of a canonical basis of commuting vector fields). The condition that the operators $L_m^\pm$ are upper- (lower-)triangular operators of order m fixes this ambiguity only partially. This constraint determines times up to linear triangular transformations $\tilde{t}_m^\pm = t_m^\pm + \sum_{\mu<m} c_\mu^\pm t_\mu^\pm$. We consider this issue in more detail in Sections 2 and 3 below.

Let us fix one of the times of the hierarchy, t_{k+1}^- (or, more generally, a linear combination of the first $k+1$ times), and consider the solutions of the hierarchy that *do not* depend on it, i.e., such that

$$\partial_{t_{k+1}^-} \varphi_i = 0. \tag{1.12}$$

The space of such solutions can be identified with the space of auxiliary operators L_{k+1}^-. Note that from (1.11) it follows that under constraint (1.12) the operator $L = L_{k+1}^-$ becomes strictly lower-triangular, i.e., takes the form (1.3).

The restriction of the hierarchy flow associated with a time $t_m^\pm$ to the space of solutions stationary with respect to t_{k+1}^- can be seen as a finite-dimensional system admitting the Lax representation

$$\partial_{t_m^\pm} L = [L_m^\pm, L]. \tag{1.13}$$

For $\xi = t_1^+$, the auxiliary operator has the form $L_1^- = v_i + T^{-1}$ with $v_i = \partial_\xi \varphi_i$, and (1.13) is equivalent to the following system of equations for $a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}}$ and $a_i^{(j)}$, $j = 2, \dots, k$:

$$\begin{cases} \partial_\xi a_i^{(j)} = a_{i-1}^{(j-1)} - a_i^{(j-1)} + a_i^{(j)}(v_i - v_{i-j}), \quad j = 2, \dots, k, \\ 0 = a_{i-1}^{(k)} - a_i^{(k)} + (v_i - v_{i-k-1}), \quad v_i = \partial_\xi \varphi_i. \end{cases} \tag{1.14}$$

Similarly, for $\eta = t_1^+$, we obtain the system

$$\partial_\eta a_i^{(j)} = c_i a_{i+1}^{(j+1)} - c_{i-j-1} a_i^{(j+1)}, \qquad j = 1, \dots, k, \tag{1.15}$$

where $a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}}$ and $c_i = e^{\varphi_i - \varphi_{i+1}}$.

The main goal of this paper is to construct a bi-Hamiltonian theory of systems (1.14) and (1.15). We show that the space of strictly lower-triangular difference operators L admits two different structures of a Poisson manifold and specify the corresponding Hamiltonians.

For $k = 1$, systems (1.14) and (1.15) have the simplest and most interesting form:

$$\partial_\xi \varphi_{i-1} - \partial_\xi \varphi_{i+1} = e^{\varphi_i - \varphi_{i-1}} - e^{\varphi_{i+1} - \varphi_i}, \tag{1.16}$$

$$\partial_\eta \varphi_i - \partial_\eta \varphi_{i-1} = e^{\varphi_{i-1} - \varphi_{i+1}} - e^{\varphi_{i-2} - \varphi_i}. \tag{1.17}$$

A posteriori, in these cases, one of our main results can easily be verified. Namely, it is easy to check that systems (1.16) and (1.17) are Hamiltonian with respect to the form $\omega = \sum_{i=1}^n d\varphi_i \wedge d\varphi_{i+1}$, $\varphi_i = \varphi_{i+n}$, and the corresponding Hamiltonians are

$$H^- = \sum_{i=1}^n e^{\varphi_i - \varphi_{i-1}}, \quad H^+ = \sum_{i=1}^n e^{\varphi_{i-2} - \varphi_i}, \qquad \varphi_i = \varphi_{i+n}, \tag{1.18}$$

respectively. But even in this simple case, the second Hamiltonian structure of Eqs. (1.16) and (1.17) is far from obvious. In the last section we prove that under the (one-to-one for odd n) change of variables $e^{\varphi_i - \varphi_{i-1}} = x_i - x_{i-2} + e_1$ Eqs. (1.16) take the form of Hamiltonian equations with respect to the form $\widetilde{\omega} = \sum_{i=1}^n dx_i \wedge dx_{i-1}, x_i = x_{i+n}$, with Hamiltonian

$$\widetilde{H}^- = \sum_{i=1}^n x_i^2 (x_{i-1} - x_{i+1}).$$

2. Preliminaries

In this section we give the necessary facts from the spectral theory of strictly lower-triangular operators and describe the construction of algebraic-geometrical solutions of the $2D$ Toda hierarchy.

2.1. The spectral theory of lower-triangular difference operators. In the modern approach to the spectral theory of periodic difference operators a central role is played by the notion of a *spectral curve* associated with an n-periodic difference operator L. By definition, the points of the spectral curve parameterize the Bloch solutions of the equation

$$L\psi = E\psi, \tag{2.1}$$

i.e., the solutions of (2.1) that are eigenfunctions for the monodromy operator

$$T^{-n}\psi = w\psi. \tag{2.2}$$

Let $\mathcal{L}(E)$ be the solution space of Eq. (2.1). This is a linear space of dimension equal to the order of L. The monodromy operator preserves $\mathcal{L}(E)$ and, hence, defines a finite-dimensional operator $T^{-n}(E)$ on this space. The pairs of complex numbers (w, E) for which there exists a common solution of Eqs. (2.1) and (2.2) are determined by the characteristic equation

$$R(w, E) = \det(w \cdot 1 - T^{-n}(E)) = 0.$$

The polynomial $R(w,E)$ can also be obtained as the characteristic polynomial of the finite-dimensional operator $L(w)$ being the restriction of L to the space $\mathcal{T}(w) := \{\psi \mid w\psi_{i+n} = \psi_i\}$:

$$R(w,E) = \det(E \cdot 1 - L(w)) = 0, \qquad L(w) := L|_{\mathcal{T}(w)}. \tag{2.3}$$

The family of algebraic curves that arise as spectral curves depends on the choice of a family of difference operators. It was shown in [5] that, in the case of strictly lower-triangular difference operators L, the characteristic polynomial has the form

$$R(w,E) = w^{k+1} - E^n + \sum_{i>0,\, j\geqslant 0,\, ni+(k+1)j<n(k+1)} r_{ij} w^i E^j = 0, \tag{2.4}$$

where $r_{1,0} = \prod_{i=1}^n a_i^1 \neq 0$ (by virtue of assumption (1.4)).

If n and $k+1$ are coprime, then the affine curve defined in $\mathbb{C}^2$ by (2.4) is compactified by one point p_-, at which the functions $w(p)$ and $E(p)$ naturally defined on Γ have poles of orders n and $k+1$, respectively. In other words, if one chooses a local coordinate z in a neighborhood of p_- so that $w = z^{-n}$, then the Laurent expansion of E has the form

$$E = z^{-k-1}\left(1 + \sum_{s=1}^{\infty} e_s z^s\right), \qquad w = z^{-n}. \tag{2.5}$$

As shown in [5], the specific form of Eq. (2.4) allows one to single out another marked point p_+ on Γ, namely, the preimage of $E = 0$ with $w = 0$. It turns out that at this point $E = E(p)$ has a simple zero, and the functions $w = w(p)$ have a zero of order n:

$$w = \frac{1}{r_{1,0}} E^n \left(1 + \sum_{s=1}^{\infty} w_s E^s\right). \tag{2.6}$$

Analytic properties of the Bloch solution in a neighborhood of the marked points are described by the following two statements.

Lemma 2.1 [5]. *Let L be an operator of the form* (1.3) *whose order and period are coprime. Then there is a unique formal series $E(z)$ of the form* (2.5) *such that the equation $L\psi = E\psi$ has a unique formal solution of the form*

$$\psi_i(z) = z^i\left(1 + \sum_{s=1}^{\infty} \xi_s^-(i) z^s\right) \tag{2.7}$$

with periodic coefficients $\xi_s^-(i) = \xi_s^-(i+n)$ normalized by the condition $\xi_s^-(0) = 0$.

For further use, we briefly outline the proof.

Proof. The substitution of (2.7) and (2.5) into the equation $L\psi = E\psi$ gives a system of difference equations for the unknown constants e_s and the unknown functions $\xi_s(i)$ of the discrete variable i. The first of them is the equation

$$e_1 + \xi_1^-(i) - \xi_1^-(i-k-1) = a_i^{(k)}. \tag{2.8}$$

The periodicity constraint on ξ_1^- uniquely determines

$$e_1 = n^{-1} \sum_{i=1}^{n} a_i^{(k)} \tag{2.9}$$

and reduces the difference equation (2.8) of order $k+1$ to the first-order difference equation

$$me_s + \xi_1^-(i) - \xi_1^-(i-1) = \sum_{j=0}^{m-1} a_{i-j(k+1)}^{(k)}, \tag{2.10}$$

where m is an integer such that $1 \leqslant m < n$ and $m(k+1) = 1 \pmod n$. Equation (2.10) and the initial condition $\xi_1^-(0) = 0$ uniquely determine $\xi_1^-(i)$.

For arbitrary s, the equation determining e_s and ξ_s^- has the form

$$e_s + \xi_s^-(i) - \xi_s^-(i-k-1) = Q_s(e_1, \dots, e_{s-1}; \xi_1, \dots, \xi_{s-1}, a_i^{(j)}), \tag{2.11}$$

where Q_s is a function linear in $e_{s'}$ and $\xi_{s'}$, $s' < s$, and polynomial in $a_i^{(j)}$. The same argument as above shows that it has a unique periodic solution, which proves the lemma. □

Lemma 2.2 [5]. *The equation $L\psi = E\psi$ has a unique formal solution of the form*

$$\psi_i(E) = e^{\varphi_i} E^{-i}\left(1 + \sum_{s=1}^{\infty} \xi_s^+(i) E^s\right), \qquad a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}}, \tag{2.12}$$

normalized by the condition $\xi_s^+(0) = 0$.

Proof. The substitution of (2.12) into (2.1) gives a system of nonhomogeneous first-order difference equations for the unknown coefficients ξ_s^-. For $s = 1$, we have

$$\xi_1^+(i) - \xi_1^+(i-1) = e^{\varphi_{i-2} - \varphi_i} a_i^{(2)}. \tag{2.13}$$

For any s, the equations have the similar form

$$\xi_s^+(i) - \xi_s^+(i-1) = e^{-\varphi_i} q_s(\xi_1^+, \dots, \xi_{s-1}^+, a_i^{(j)}); \tag{2.14}$$

together with the initial conditions, these equations recursively define the $\xi_s^+(i)$ for all i. □

The uniqueness of the formal solution (2.12) implies the following assertion.

Corollary 2.3. *The formal series* (2.12) *is a Bloch solution*, *i.e.*, *it satisfies* (2.2) *with*

$$w(E) = \psi_{-n}(E) = r_{1,0}^{-1} E^n \left(1 + \sum_{s=1}^{\infty} w_s E^s\right). \tag{2.15}$$

From Lemma 2.1 it follows that the components $\psi_i(p)$, $p := (w, E) \in \Gamma$, of the Bloch solution $\psi(p)$ considered as functions on the spectral curve have a *zero of order* i at the marked point p_-. Lemma 2.2 implies that $\psi_i(p)$ has a *pole of order* i at the marked point p_+.

It can be proved in a standard way that, in this case, ψ_i is a meromorphic function on Γ having (for generic operators) g poles $\gamma_1, \dots, \gamma_g$ *not depending* on i outside the marked points $p_\pm$ (see [1] for details). These analytic properties are determining for the discrete Baker–Akhiezer function introduced in [2].

The identification of the Bloch functions of periodic difference operators with the discrete Baker–Akhiezer function is key for establishing a connection between the spectral theory of lower-triangular operators, the theory of commuting difference operators (see [2]), and the theory of algebraic-geometric solutions of the $2D$ Toda hierarchy.

The correspondence

$$L \longmapsto \{\Gamma, D = \gamma_1 + \dots + \gamma_g\}, \tag{2.16}$$

where Γ is the spectral curve of the operator L and D is the pole divisor of the Bloch solution ψ, is usually referred to as the *direct spectral transform*.

This is a one-to-one correspondence between the open dense subsets of the space of operators and those of the space of algebraic-geometric spectral data. The construction of the *inverse* spectral transform is a particular case of the general construction of algebraic-geometric solutions of the $2D$ Toda hierarchy.

2.2. Algebraic-geometric solutions of the $2D$ Toda hierarchy. Let Γ be a smooth algebraic curve of genus g with fixed local coordinates $z_\pm$ in neighborhoods of the two marked points $p_\pm \in \Gamma$ such that $z_\pm(p_\pm) = 0$, and let $t = \{t_j^\pm,\ j = 1, 2, \dots\}$ be a set of complex parameters (it is assumed that only *finitely many* of them are nonzero). Then, as shown in [3], the following assertion is valid.

Lemma 2.4. *For a generic set of g points $\gamma_1, \dots, \gamma_g$, there is a unique meromorphic function $\Psi_i(t,p)$, $p \in \Gamma$, such that*

(i) *outside the marked points $p_\pm$ it has simple poles at γ_s (provided that the γ_s are distinct)*;

(ii) *in neighborhoods of the marked points it has the form*

$$\Psi_i(t, z_\pm) = z_\pm^{\mp i} e^{(\sum_m t_m^\pm z_\pm^{-m})} \left(\sum_{s=1}^{\infty} \xi_s^\pm(i,t)\, z_\pm^s \right), \qquad \xi_0^- = 1. \tag{2.17}$$

The function Ψ_i is a particular case of the so-called *multi-point Baker–Akhiezer function* (see, e.g., [10]).

The uniqueness of the function Ψ_i implies the following result.

Theorem 2.5 [3]**.** *Let $\Psi_i(t,p)$ be the Baker–Akhiezer function corresponding to any set of data $\{\Gamma, p_\pm, z_\pm; \gamma_1, \dots, \gamma_g\}$. Then there exist unique operators $L_m^\pm$ of the form* (1.9), (1.10) *with $\varphi_i(t) := \ln \xi_0^+(t)$ such that Eqs.* (1.8) *hold.*

Remark 2.6. By definition, the Baker–Akhiezer function depends on the choice of local coordinates $z_\pm$ in neighborhoods of the marked points $p_\pm$. A change of the local coordinate corresponds to a triangular transformation of the times $t_m^\pm$ (cf. the remark in the introduction).

The algebraic-geometric solutions of the $2D$ Toda hierarchy can be explicitly expressed in terms of the Riemann theta-function. Choosing a basis of cycles a_i and b_i, $i = 1, \dots, g$, on Γ with canonical intersection matrix, i.e., so that $a_i \circ a_j = b_i \circ b_j = 0$ and $a_i \circ b_j = \delta_{ij}$, we can define

(a) a basis of normalized holomorphic differentials ω_i for which $\oint_{a_j} \omega_i = \delta_{ij}$;

(b) the matrix B of their b-periods for which $B_{ij} = \oint_{b_j} \omega_i$ and the corresponding Riemann theta-function

$$\theta(z) = \theta(z|B) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (m,z) + \pi i (Bm,m)}, \qquad z = z_1, \dots, z_g;$$

(c) the Abel transform $A(p)$ under which the vector $A(p)$ has coordinates $A_k(p) = \int^p \omega_k$;

(d) the normalized Abelian differential $d\Omega_0$ of the third kind for which $\oint_{a_i} d\Omega_0 = 0$ having simple poles with residues ∓ 1 at $p_\pm$ and the normalized Abelian differential $d\Omega_{m,\pm}$ of the second kind having poles at $p_\pm$ of the form $d\Omega_{m,\pm} = d(z_\pm^{-m} + O(z_\pm))$ and normalized by the condition $\oint_{a_i} d\Omega_{m,\pm} = 0$.

Lemma 2.7 [3]**.** *The Baker–Akhiezer function is given by the formula*

$$\Psi_i(t,p) = \frac{\theta(A(p) + iU_0 + \sum U_{m,\pm} t_m^\pm + Z)\, \theta(A(p_-) + Z)}{\theta(A(p_-) + iU_0 + \sum U_{m,\pm} t_m^\pm + Z)\, \theta(A(p) + Z)}\, e^{i\Omega_0(p) + \sum t_m^\pm \Omega_{m,\pm}(p)}\,. \tag{2.18}$$

Here the summation is over all pairs of indices $(m, \pm)$ and

(a) *$\Omega_0(p)$ and $\Omega_{m,\pm}(p)$ are the Abelian integrals $\Omega_0(p) = \int^p d\Omega_0$ and $\Omega_{m,\pm}(p) = \int^p d\Omega_{m,\pm}$ corresponding to the differentials introduced above and normalized so that, in a neighborhood of p_-, they have the form*

$$\Omega_0(z_-) = \ln z_- + O(z_-), \quad \Omega_{m,-}(z_-) = z_-^{-m} + O(z_-), \quad \Omega_{m,+}(z_-) = O(z_-);$$

(b) *$2\pi i U_0$ and $2\pi i U_{\alpha,j}$ are the vectors of their b-periods, i.e., the vectors with coordinates*

$$U_0^k = \frac{1}{2\pi i} \oint_{b_k} d\Omega_0, \qquad U_{m,\pm}^k = \frac{1}{2\pi i} \oint_{b_k} d\Omega_{m,\pm}; \tag{2.19}$$

(c) *Z is an arbitrary vector corresponding to the pole divisor of the Baker–Akhiezer function.*

Note that from the bilinear Riemann relations it follows that $U_0 = A(p_-) - A(p_+)$, and the termwise comparison of the coefficients of the same powers on the left- and right-hand sides of (2.18) imply the following result.

Theorem 2.8 [3]. *The algebraic-geometrical solutions of the* 2*D Toda lattice are given by the formula*

$$\varphi_i(t) = \ln \frac{\theta((i-1)U_0 + \sum U_{m,\pm} t_m^{\pm} + \widetilde{Z})}{\theta(iU_0 + \sum U_{m,\pm} t_m^{\pm} + \widetilde{Z})} + ic_0 + \sum c_{m,\pm} t_m^{\pm}, \tag{2.20}$$

where $\widetilde{Z} = Z + A(p_-)$ *is an arbitrary vector, the vectors* U_0 *and* $U_{m,\pm}$ *are defined in* (2.19), *and the constants* c_0 *and* $c_{m,\pm}$ *are the leading coefficients of the expansions of the Abelian integrals in a neighborhood of* p_+:

$$\Omega_0(z_+) = -\ln z_+ + c_0 + O(z_+), \tag{2.21}$$
$$\Omega_{m,+}(z_+) = z_+^{-m} + c_{m,+} + O(z_+), \quad \Omega_{m,-}(z_+) = c_{m,-} + O(z_+).$$

From (2.20) it is easy to see that, in the general case, the algebraic-geometric solution is a *quasi-periodic* function of all variables, including i. It is n-periodic in the discrete variable i if the vector $nU_0 = n(A(p_+) - A(p_-))$ is a vector in the lattice defining the Jacobian of the corresponding curve Γ. The last statement is equivalent to the following assertion.

Lemma 2.9. *Let* Γ *be a smooth algebraic curve on which a meromorphic function* w *with a unique zero at some point* p_+ *and a unique pole at another point* p_- *of order* n *is defined. Then the Baker–Akhiezer function corresponding to the curve* Γ, *the points* $p_\pm$, *and any divisor* γ_s *satisfies Eq.* (2.2), *and therefore the corresponding solution of the* 2*D Toda hierarchy is* n*-periodic.*

To prove this statement, it is enough to check that the functions Ψ_{i-n} and $w\Psi_n$ have the same analytical properties and hence coincide.

2.3. The dual Baker–Akhiezer function. For further use, we recall the important notion of the *dual Baker–Akhiezer function* (a detailed discussion of the notion of dual functions is contained in [10]).

For a nonspecial divisor $D = \gamma_1 + \dots + \gamma_g$ of degree g on a smooth algebraic curve Γ of genus g with two marked points, one can define the *dual* effective divisor $D^+ = \gamma_1^+ + \dots + \gamma_g^+$ of degree g as follows: for the given D, there exists a unique meromorphic differential $d\Omega$ having simple poles with resides ± 1 at the marked points that is holomorphic everywhere except at these points and has zeros at γ_s ($d\Omega(\gamma_s) = 0$). The zero divisor of $d\Omega$ is of degree $2g$. Hence, in addition to zeros at γ_s, the differential $d\Omega$ has zeros at g other points γ_s^+ ($d\Omega(\gamma_s^+) = 0$). In other words, the divisor D^+ is defined by the equation $D + D^+ = \mathcal{K} + p_+ + p_- \in J(\Gamma)$, where $\mathcal{K}$ is the canonical class, i.e., the equivalence class of the zero divisor of the holomorphic differential on Γ.

The function $\Psi_i^+(t,p)$ dual to the Baker–Akhiezer function $\Psi_i(t,p)$ corresponding to a divisor D is determined by the following analytical properties: (i) outside the marked points $p_\pm$ it is meromorphic and has simple poles at γ_s^+ (if γ_s^+ are distinct); (ii) in neighborhoods of the marked points it has the form

$$\Psi_i^+(t, z_\pm) = z_\pm^{\pm i} e^{-(\sum_m t_m^{\pm} z_\pm^{-m})} \left(\sum_{s=1}^{\infty} \chi_s^{\pm}(i,t)\, z_\pm^s \right), \qquad \chi_0^- = 1. \tag{2.22}$$

It follows from this definition that the differential $\Psi_i^+ \Psi_j d\Omega$ is a meromorphic differential on Γ, which may have poles only at the marked points $p_\pm$. Moreover, for $i > j$ ($i < j$), it is holomorphic at p_+ (p_-). Since the sum of residues of a meromorphic differential equals zero, we have

$$\operatorname*{res}_{p_\pm} \Psi_i^+ \Psi_j \, d\Omega = \pm \delta_{i,j}, \tag{2.23}$$

which implies that Ψ^+ satisfies the equation

$$(\Psi^+ L)_i \equiv \Psi_{i+k+1}^+ + a_{i+k}^{(k)} \Psi_k^+ + \dots + a_{i+1}^{(1)} \Psi_{i+1}^+ = E \Psi_i^+ \tag{2.24}$$

adjoint to (2.1) and the equation

$$-\partial_{t_m^{\pm}} \Psi^+ = \Psi^+ L_m^{\pm}. \tag{2.25}$$

The theta-functional formula (2.20) for the dual Baker–Akhiezer function has the form

$$\Psi_i^+(t,p) = \frac{\theta(A(p) - iU_0 - \sum U_{m,\pm} t_m^\pm + Z^+)\,\theta(A(p_-) + Z^+)}{\theta(A(p_-) - iU_0 - \sum U_{m,\pm} t_m^\pm + Z^+)\,\theta(A(p) + Z^+)}\, e^{-i\Omega_0(p) - \sum t_m^\pm \Omega_{m,\pm}(p)}, \tag{2.26}$$

where $Z + Z^+ = \mathcal{K} + A(p_+) + A(p_-)$. The analytical properties of Ψ^+ easily imply the following assertion.

Lemma 2.10. *Under the assumptions of Lemma* 2.9 *the dual Baker–Akhiezer function satisfies the equation*

$$\Psi_i^+ = w\Psi_{i-n}^+. \tag{2.27}$$

Remark 2.11. As mentioned above, the construction of an inverse spectral transform can be regarded as a special case of the construction of the algebraic-geometric solutions of a $2D$ Toda hierarchy. Indeed, let Γ be the curve determined by an equation of the form (2.4); then a simple comparison of analytical properties shows that the Bloch function of the operator L coincides with the Baker–Akhiezer function depending on an infinite set of variables when all continuous times vanish: $\psi_i = \Psi_i(t_k^\pm = 0)$.

3. The Hamiltonian Theory of Reduced Systems

The systems of equations (1.14) and (1.15) were defined as special reductions of the $2D$ Toda hierarchy. Therefore, the solutions of the corresponding equations are given by (2.18), where the Riemann theta-function corresponds to any curve defined by Eq. (2.4).

In this section we develop a Hamiltonian theory of this reduced system, following the general scheme proposed in [7] and [8]. According to this scheme, on the *space of operators* L, which is identified with the phase space of the system, one can define a family of two-forms by

$$\omega^{(i)} = -\frac{1}{2}\sum_\alpha \operatorname*{res}_{p_\alpha} E^{-i}\langle \psi^+(w)\,\delta L \wedge \delta\psi(w)\rangle\, d\Omega, \tag{3.1}$$

where $\delta F(L)$ stands for the variation of a function F on the space of operators (the Baker–Akhiezer function with fixed eigenvalue w and fixed normalization is such a function) and the summation is over the set of those points p_α on the corresponding spectral curve at which the expression on the right-hand side *a priori* has poles, namely, the marked points $p_\pm$, at which the Baker–Akhiezer function and its dual have poles, and, for $i > 0$, of the zeros p_ℓ, $\ell = 1, \dots, k$, of the function $E = E(p)$ at which $w = w(p)$ does not vanish, i.e., $E(p_\ell) = 0$ and $w(p_\ell) \neq 0$.

3.1. The differential $d\Omega$. Our first goal is to derive a closed expression for the differential $d\Omega$ specified above via its analytic properties in terms of the Bloch eigenfunctions ψ and the dual functions ψ^+.

Suppose that the coefficients of the operator are n-periodic. Following the same line of reasoning as in [4], consider the differential $d\psi$ with respect to the spectral parameter. It satisfies the nonhomogeneous linear equation

$$(L - E)\,d\psi = dE\psi, \tag{3.2}$$

which is the differential of Eq. (2.1). Differentiating Eq. (2.2), we see that $d\psi$ satisfies the monodromy relation

$$w\,d\psi_i + dw\psi_i = d\psi_{i-n}. \tag{3.3}$$

Let us denote the mean of a function f_i on the interval $l+1 \leqslant i \leqslant l+n$ by $\langle f\rangle_l := \frac{1}{n}\sum_{i=l+1}^{l+n} f_i$; in the case of n-periodic functions, where this mean does not depend on l, we shall use the short notation $\langle f\rangle$. From (3.2) it follows that

$$E\langle\psi^+ d\psi\rangle_l + dE\,\langle\psi^+\psi\rangle = \langle\psi^+(L\,d\psi)\rangle_l = \frac{1}{n}\sum_{j=1}^{k+1}\sum_{i=l+1}^{l+n} a_i^{(j)}\psi_i^+ d\psi_{i-j}. \tag{3.4}$$

Equation (2.24) implies

$$E\langle\psi^+ d\psi\rangle_l = \frac{1}{n}\sum_{j=1}^{k+1}\sum_{i=l+1}^{l+n} a_{i+j}^{(j)}\psi_{i+j}^+ \, d\psi_i = \frac{1}{n}\sum_{j=1}^{k+1}\sum_{i=l+1+j}^{l+n+j} a_i^{(j)}\psi_i^+ d\psi_{i-j}. \tag{3.5}$$

Substituting (3.5) into (3.4) and using (3.3), we obtain

$$dE\,\langle\psi^+\psi\rangle = \frac{dw}{nw}\sum_{j=1}^{k+1}\sum_{i=l+1}^{l+j} a_i^{(j)}\psi_i^+\psi_{i-j}. \tag{3.6}$$

Note that the left-hand side of (3.6) does not depend on l. Hence the right-hand side of (3.6) is independent of l as well. Averaging over l, we obtain the equation

$$dE\,\langle\psi^+\psi\rangle = \frac{dw}{nw}\langle\psi^+(L^{(1)}\psi)\rangle, \tag{3.7}$$

where

$$L^{(1)} := \sum_{j=1}^{k+1} j a_i^{(j)} T^{-j} \tag{3.8}$$

is the difference analogue of the first *descendant* of a differential operator introduced in [4].

From (3.7) it follows that the zeroes of dw coincide with the zeroes of the meromorphic function $\langle\psi^+\psi\rangle$ and the zeros of dE coincide with the zeros of $\langle\psi^+(L^{(1)}\psi)\rangle$. Hence the following lemma is valid.

Lemma 3.1. *The differential*

$$d\Omega := \frac{dw}{nw\langle\psi^+\psi\rangle} = \frac{dE}{\langle\psi^+(L^{(1)}\psi)\rangle} \tag{3.9}$$

is holomorphic outside the marked points $p_\pm$, has zeros at the poles of ψ and ψ^+, and has simple poles with resides ± 1 at $p_\pm$.

Lemma 3.1 allows us to regard (3.9) as an explicit expression for the differential $d\Omega$, which we introduced by specifying its analytical properties in the definition of the dual Baker–Akhiezer function.

Examples. For $k = 1$,

$$d\Omega = \frac{dE}{\langle a_i^{(1)}\psi_i^+\psi_{i-1}^+ + 2\psi_i^+\psi_{i-2}\rangle} = \frac{dw}{nw\langle\psi^+\psi\rangle}, \tag{3.10}$$

and for $k = 2$,

$$d\Omega = \frac{dE}{\langle a_i^{(1)}\psi_i^+\psi_{i-1}^+ + 2a_i^{(2)}\psi_i^+\psi_{i-2} + 3\psi_i^+\psi_{i-3}\rangle} = \frac{dw}{nw\langle\psi^+\psi\rangle}\,. \tag{3.11}$$

3.2. Symplectic leaves and the Darboux coordinates. We emphasize that the form $\omega^{(i)}$ is not closed, and it is degenerate on the space of *all* operators L. It becomes closed after being restricted to certain subvarieties. As we shall see below, only the forms $\omega^{(0)}$ and $\omega^{(1)}$ are nondegenerate on the corresponding subvarieties. Thus, on the space of operators L, there exist two structures of a *Poisson* manifold. The existence of such structures reflects the bi-Hamiltonian nature of integrable systems.

In the framework of the approach of [7] and [8] the constrains defining the symplectic leaves in each of the Poisson structures are equivalent to the condition that the form $\omega^{(i)}$ does not depend on the choice of the normalization of the Bloch eigenvector ψ. The change of normalization is equivalent to the transformation $\psi_i \to \psi_i h$, $\psi_i^+ \to \psi_i^+ h^{-1}$, where $h = h(w)$ is a scalar function. Under this transformation the differential on the right-hand side of (3.1) is mapped to

$$E^{-i}\langle\psi^+(w)\,\delta L \wedge \delta\psi(w)\rangle\, d\Omega + E^{-i}\langle\psi^+(w)\,\delta L\psi(w)\rangle \wedge \delta\ln h\, d\Omega. \tag{3.12}$$

Hence the form $\omega^{(i)}$ is normalization independent when the last term in (3.12) is holomorphic near the points p_α. It follows from the equation

$$(L - E)\delta\psi(w) = -(\delta L - \delta E(w))\psi \tag{3.13}$$

and the definition of the adjoint operator that

$$\langle\psi^+((\delta L - \delta E)\psi)\rangle = \langle(\psi^+(E - L))\delta\psi\rangle = 0. \tag{3.14}$$

Using (3.9), we obtain the following statement.

Lemma 3.2. *The restriction of the form $\omega^{(i)}$ given by (3.1) to a subvariety of the space of all operators on which the differential $E^{-i}\delta E(w)d\ln w$ is holomorphic in neighborhoods of the points p_α is normalization independent.*

Example. For $i = 0$, the summation in (3.1) is over the marked points $p_\pm$. At the point p_+ (where $w = 0$) the function E has a zero. Therefore, the form $E d\ln w$ has a pole only at p_-, and hence it has zero residue at p_-. Thus, in (2.5) we have $e_{k+1} = 0$.

In a neighborhood of p_-, where the function E has a pole of order $k+1$, the form $\delta E(w)d\ln w$ has a pole of order $k+2$ with zero residue. Hence, for the subvariety Λ_0^c defined for any set $c = (c_1, \dots, c_k)$ of constants as

$$\Lambda_0^c := \{L \in \Lambda_0^c \mid e_s(L) = c_s,\ s = 1, \dots, k\}, \tag{3.15}$$

where the $e_s = e_s(L)$ are the coefficients of expansion (2.5), the following assertion is valid.

Corollary 3.3. *The form $\omega^{(0)}$ restricted to the subvariety Λ_0^c is normalization independent.*

Example. The form $E^{-1}\delta E(w)d\ln w$ is holomorphic in a neighborhood of the marked point p_-. Since the sum of its residues equals zero, it follows that this form is holomorphic at the point p_+ if it is holomorphic at the points p_ℓ, $\ell = 1, \dots, k$. Using the chain rule, we see that the variation of $E(w)$ with fixed w is related to the variation of $w(E)$ with fixed E by $\delta E(w)\,dw + \delta w(E)\,dE = 0$. Hence $\delta \ln E(w)\,d\ln w$ is holomorphic at the points p_ℓ (the preimages of $E = 0$ at which $w \neq 0$) if $\delta w(p_\ell) = 0$. The last condition holds on the subvariety

$$\Lambda_1^c := \{L \in \Lambda_1^c \mid r_{i,0}(L) = c_i,\ 1 = 1, \dots, k\}, \tag{3.16}$$

where $c = (c_1, \dots, c_k)$ is a k-tuple of constants and the $r_{i,0}(L) = r_{i,0}$ are the coefficients of the polynomial $\det L(w) = w^{k+1} + \sum_{i=1}^k r_{i,0} w^i$.

Corollary 3.4. *The form $\omega^{(1)}$ restricted to the subvariety Λ_1^c is normalization independent.*

Remark 3.5. For $i > 1$, the subvariety Λ_i^c, on which the restriction of $\omega^{(i)}$ is normalization independent, is described by a system of $i(k+1) - 1$ equations:

$$\Lambda_i^c := \{L \in \Lambda_i^c \mid w_{\ell,s} = c_{\ell,s},\ s = 1, \dots, i;\ w_s = c_s,\ s = 2, \dots, i\}, \tag{3.17}$$

where the $w_{\ell,s}$ are the coefficients of the expansion

$$w = \sum_{s=0}^{\infty} w_{\ell,s} E^s \tag{3.18}$$

of w at the preimages p_ℓ of $E = 0$ on Γ at which $w(p_\ell) \neq 0$, the w_s are the coefficients of the expansion (2.15) of w at p_+, and the $c_{i,s}$ and c_s are constants. Hence Λ_i^c is of dimension $(n-1)k - i + 1$. Recall that the dimension of a family of curves Γ defined by equations of the form (2.4) equals $k(n+1)/2$ (the number of the coefficients r_{ij}). For generic values of the coefficients r_{ij}, the curve Γ is smooth and has genus $g = k(n-1)/2$. Therefore, the correspondence (2.16) restricted to Λ_i^c identifies the latter with the total space of Jacobian bundles over the space of the corresponding spectral curves. For $i > 1$, the dimension of a fiber is *higher* than the dimension of the base. Hence the form $\omega^{(i)}$ restricted to Λ_i^c is degenerate for $i > 1$.

3.3. The Darboux coordinates. For completeness, we describe a construction of the Darboux coordinates for the restriction $\widehat{\omega}^{(i)}$ of $\omega^{(i)}$ to the subvariety Λ_i^c, i.e.,

$$\widehat{\omega}^{(i)} := \omega^{(i)}|_{\Lambda_i^c}. \tag{3.19}$$

Theorem 3.6. *Let γ_s be the poles of the Baker–Akhiezer function. Then*

$$\widehat{\omega}^{(i)} = \frac{1}{n}\sum_{s=1}^{g} E^{-i}(\gamma_s)\delta E(\gamma_s) \wedge \delta \ln w(\gamma_s). \tag{3.20}$$

Remark 3.7. The meaning of the right-hand side of this formula is as follows. By definition, on each spectral curve meromorphic functions E and w are given. The values $E(\gamma_s)$ and $w(\gamma_s)$ of these functions at the points γ_s define a set of functions on the space of operators L. The wedge product of their differentials is a two-form on our phase space.

Proof. The idea of the proof of formula (3.20) is very general and does not rely on the specific form of L. We follow the proof of Lemma 5.1 in [6] (see also [9]).

The differential whose residues determine $\omega^{(i)}$ according to (3.1) is a meromorphic differential on the spectral curve Γ. Therefore, the sum of its residues at the point p_α is equal to the negative sum of the other residues on Γ. The differential has poles of two types. The poles of the first type are the poles γ_s of ψ. They are simple in general position. Note that $\delta\psi$ has a pole of order 2 at γ_s. Taking into account the fact that $d\Omega$ has a zero at γ_s, we obtain

$$\operatorname*{res}_{\gamma_s} E^{-i}\langle\psi^+\,\delta L \wedge \delta\psi\rangle\, d\Omega = \frac{E^{-i}\langle\psi^+\delta L\psi\rangle}{n\langle\psi^+\psi\rangle}(\gamma_s) \wedge \delta\ln w(\gamma_s) = \frac{1}{n}E^{-i}(\gamma_s)\delta E(\gamma_s)\wedge\delta\ln w(\gamma_s). \tag{3.21}$$

The last equality follows from Eq. (3.14), which is merely the standard formula for the variation of an eigenvalue of an operator.

The poles of the second type of the differential on the right-hand side of (3.1) are the zeros q_j of the differential dw. Indeed, in a neighborhood of q_j the local coordinate on the spectral curve is $\sqrt{w - w(q_j)}$ (in general position, where the zero is simple). Varying the Taylor expansion of ψ in this coordinate, we obtain

$$\delta\psi = -\frac{d\psi}{dw}\,\delta w(q_j) + O(1). \tag{3.22}$$

Therefore, $\delta\psi$ has a simple pole at q_j. Similarly,

$$\delta E = -\frac{dE}{dw}\,\delta w(q_j). \tag{3.23}$$

Relations (3.22) and (3.23) imply

$$\operatorname*{res}_{q_j} E^{-i}\langle\psi^+\delta L\wedge\delta\psi\rangle\, d\Omega = \operatorname*{res}_{q_j}\frac{E^{-i}\langle\psi^+\delta L d\psi\rangle}{n\langle\psi^+\psi\rangle}\wedge\frac{\delta E d\ln w}{dE}\,. \tag{3.24}$$

Due to the skew-symmetry of wedge product, we can replace δL in (3.24) by $(\delta L - \delta E)$. Then, using the identities $\psi^*(\delta L - \delta E) = \delta\psi^*(E - L)$ and $(E - L)d\psi = -dE\psi$, we obtain

$$\operatorname*{res}_{q_j} E^{-i}\langle\psi^+\delta L\wedge\delta\psi\rangle\, d\Omega = -\operatorname*{res}_{q_j}\frac{E^{-i}\langle\delta\psi^+\psi\rangle}{n\langle\psi^+\psi\rangle}\wedge\delta E\, d\ln w = \operatorname*{res}_{q_j}\frac{E^{-i}\langle\psi^+\delta\psi\rangle}{n\langle\psi^+\psi\rangle}\wedge\delta E\, d\ln w; \tag{3.25}$$

to obtain the last equality, we used the identity $\langle\psi^+\psi\rangle(q_j) = 0$ (which follows, as mentioned above, from (3.7)). By the definition of the subvariety on which $\omega^{(i)}$ is normalization independent (see Lemma 3.2) the form on the right-hand side of (3.25) has no poles at the points p_α. It has poles only at q_i and at γ_s. Hence, after restriction to such a subvariety, we obtain

$$\sum_j \operatorname*{res}_{q_j} \frac{E^{-i}\langle\psi^+\delta\psi\rangle}{n\langle\psi^+\psi\rangle} \wedge \delta E\, d\ln w = -\sum_s \operatorname*{res}_{\gamma_s} \frac{E^{-i}\langle\psi^+\delta\psi\rangle}{n\langle\psi^+\psi\rangle} \wedge \delta E\, d\ln w$$
$$= \frac{1}{n}\sum_s E^{-i}(\gamma_s)\delta E(\gamma_s) \wedge \delta \ln w(\gamma_s). \qquad (3.26)$$

Relations (3.21), (3.25), and (3.26) directly imply (3.20). This completes the proof of the theorem. □

3.4. The Hamiltonians. The next step in the construction of a Hamiltonian theory for systems admitting the Lax representation is to show that the substitution of the vector field ∂_t defined by the Lax equation into the form $\omega^{(i)}$ restricted to the subvariety on which it is normalization independent yields an exact 1-form, i.e., $\widehat{\omega}^{(i)}(\partial_t, X) = \delta H^{(i)}(X)$. This means that on a subvariety on which the form $\widehat{\omega}^{(i)}$ is nondegenerate the vector field ∂_t is Hamiltonian with Hamiltonian H.

Below we apply the general scheme to Eqs. (1.14) and (1.15) and compute the corresponding Hamiltonians. Let ∂_t be the vector field defined by the Lax equations; then

$$\partial_t L = [M, L], \qquad \partial_t \psi = M\psi - \psi f, \qquad (3.27)$$

where f is a meromorphic function on the spectral curve.

Remark 3.8. The appearance of the term with f in the expression for $\partial_t\psi$ is due to the fact that in the definition of the form $\omega^{(i)}$ it is assumed that the normalization of the Bloch function ψ is *time independent*: $\psi_0 \equiv 1$. If the dependence of the operator L on t is determined by the Lax equation, then the time dependence of the pole divisor $D(t)$ of $\psi(t)$ becomes linear after the application of the Abel transform. This follows form the relation

$$\psi_i(t,p) = \Psi_i(t,p)\Psi_0^{-1}(t,p), \qquad (3.28)$$

where Ψ is the Baker–Akhiezer function given by (2.18). Equation (1.8) implies (3.27) with $f(t,p) = \partial_t \ln \Psi_0(t,p)$. The function f has poles at the marked points $p_\pm$ and can be represented in the form

$$f = \sum_{s=1}^{m_\pm} c_s^\pm z^{-s} + O(1), \qquad (3.29)$$

where $c_s^\pm$ are *constants*, which in fact parameterize the commuting flows of the hierarchy, and $m_\pm$ are the positive and negative orders of the operator M.

Theorem 3.9. *The restrictions of the vector-field $\partial_{t_m^\pm}$ defined by the Lax equation* (1.13) *to the subvarieties Λ_i^c, $i = 1, 2$, are Hamiltonian with respect to the forms $\widehat{\omega}^{(i)}$ with Hamiltonians*

$$H^{(0)}_{t_m^-} = \operatorname*{res}_{p_-} z^{-m} E(z)\, d\ln z = e_{m+k+1}, \qquad (3.30)$$

$$H^{(1)}_{t_m^-} = \operatorname*{res}_{p_-} z^{-m} \ln E(z)\, d\ln z, \qquad (3.31)$$

where $E(z)$ is the series (2.5) *with coefficients defined in Lemma* 2.1, *and*

$$H^{(i)}_{t_m^+} = \frac{1}{n}\operatorname*{res}_{p_+} E^{-m-i} \ln w(E)\, dE, \qquad i = 0, 1, \qquad (3.32)$$

where $w(E)$ is defined in (2.15).

Proof. The substitution of (3.27) and (3.9) into (3.1) gives

$$\omega^{(i)}(\partial_t, \cdot\,) = -\frac{1}{2}\sum_{p_\alpha} \operatorname*{res}_{p_\alpha}(\langle\psi^+[M,L]\delta\psi\rangle - \langle\psi^+\delta L(M\psi - \psi f)\rangle)\, \frac{d\ln w}{nE^i\langle\psi^+\psi\rangle}\,. \qquad (3.33)$$

Using the equation $(L-E)\delta\psi = -(\delta L - \delta E)\psi$, we see that the differential on the right-hand side of (3.33) is equal to

$$-\frac{1}{2}\left(\langle\psi^+(M\delta E + \delta L f)\psi\rangle - \langle\psi^+(\delta L\, M + M\,\delta L)\psi\rangle\right)\frac{d\ln w}{nE^i\langle\psi^+\psi\rangle}\,. \tag{3.34}$$

The second term has poles only at the points p_α. Hence the sum of its residues at these points is equal to zero. The first term is equal to

$$-\frac{1}{2}\langle\psi^+(2f + (M-f))\psi\rangle\delta E\,\frac{d\ln w}{nE^i\langle\psi^+\psi\rangle}\,. \tag{3.35}$$

From the definition of f in (3.27) it follows that $\langle\psi^+(M-f)\psi\rangle$ is holomorphic at p_α. Since the restriction of $E^{-i}\delta E d\ln w$ to Λ_i^c is holomorphic at the marked points p_α, it follows that the second term in (3.35) restricted to Λ_i^c has no residues at p_α. Recall that the function f has poles only at the points $p_\pm$. Using the identity $\delta E(w)d\ln w = -\delta\ln w(E)dE$ for the residue at p_+, we finally obtain the equation

$$\widehat{\omega}^{(i)}(\partial_t,\,\cdot\,) = \frac{1}{n}\operatorname*{res}_{p_+} f(E)\delta\ln w(E)E^{-i}\,dE - \frac{1}{n}\operatorname*{res}_{p_-} f(w)E^{-i}(w)\delta E(w)\,d\ln w. \tag{3.36}$$

Recall that the choice of the basis vector fields $\partial_{t_m^\pm}$ of the hierarchy depends on the choice of local coordinates in neighborhoods of the marked point $p_\pm$. As follows from the proofs of Lemmas 2.1 and 2.2, the most natural choice is $z = w^{-1/n}$ at p_- and $z = E$ at p_+. In this case, the functions $f_m^\pm$ corresponding to $t = t_m^\pm$ have poles at $p_\pm$ of the forms $f_m^+ = E^{-m} + O(E)$ and $f_m^- = z^{-m} + O(z), z = w^{-1/n}$, respectively. Therefore, (3.36) implies $\widehat{\omega}^{(i)}(\partial_{t_m^\pm},\,\cdot\,) = \delta H^{(i)}_{t_m^\pm}$. The theorem is proved. ☐

4. Special Coordinate Systems. Examples

We begin this section by introducing special systems of coordinates on the space of lower-triangular operators in which $\omega^{(\ell)}$, $\ell = 1, 2$, have *local densities*, i.e., coordinates $x_i^{(j)}$ in which $\omega^{(\ell)}$ have the form $\omega = \sum f_{i,i_1}^{(j,j_1)}\delta x_i^{(j)} \wedge \delta x_{i_1}^{(j_1)}$, where the summation is over the set of all pairs of indices i, i_1 such that $|i - i_1| < d_1$ for some integer d_1 not depending on the period n of the operator. It is also assumed that the coefficients $f_{i,i_1}^{(j,j_1)}$ are functions of parameters $x_{i_2}^{(j_2)}$ such that $|i - i_2| < d_2$ for some number d_2 not depending on n.

Remark 4.1. Note that in the natural coordinates on the space of lower-triangular operators, which coincide with the coefficients $a_i^{(j)}$ of these operators, the forms have no local densities.

4.1. The form $\boldsymbol{\omega^{(0)}}$. We identify coordinates in which the form $\omega^{(0)}$ has local densities with the set of the first k coefficients of the expansion (2.7) of the Bloch solution at the marked point p_-. Relations (2.8) and (2.11) for $s = 1, \ldots, k$ can be regarded as the definition of the map

$$\{\xi_s^-(i), e_s\} \longmapsto \{a_i^{(j)}\}, \tag{4.1}$$

where the functions $\xi_s^-(i)$ are defined up to a common shift $\xi_s^-(i) \to \xi_s^-(i) + c_i$. This shift can be fixed by the normalization condition $\xi_s^-(0) = 0$.

The form $\omega^{(0)}$ in definition (3.1) is the average over i of an expression depending on $\xi_s^-(i-j)$, $j = 0, \ldots, k$, and the first $k-1$ coefficients of the expansion at p_- of the function

$$\psi_i^* := \frac{\psi_i^+}{\langle\psi^+\psi\rangle}, \tag{4.2}$$

where ψ^+ is the dual Baker–Akhiezer function (2.22). The coefficients of ψ_i^* can be found recursively from the relations

$$\operatorname*{res}_{p_-}\psi_i^*\psi_{i-j}\,d\ln z = \delta_{0,j}, \tag{4.3}$$

which follow from (2.23) and (3.9). The expressions for these coefficients in terms of ξ_s^- are local. Therefore, the statement that $\omega^{(0)}$ has local densities in the new coordinates is an obvious corollary of the definition.

Example with $k = 1$. The natural coordinates on the space of n-periodic lower-triangular operators $L = a_i T^{-1} + T^{-2}$ of order 2 are their coefficients a_i. The special coordinates $x_i := \xi_1^-(i)$ are defined up to a common shift and a constant e_1. The expression for the natural coordinates in terms of the new ones is given by (2.8):

$$a_i = x_i - x_{i-2} + e_1. \tag{4.4}$$

The substitution of the expansion of ψ and ψ^+ into (3.1) gives the following expression for the restriction of $\omega^{(0)}$ to the symplectic leaf $e_1 = const$ at $k = 1$:

$$\widehat{\omega}^{(0)} = \frac{1}{2}\langle da_i \wedge dx_{i-1}\rangle = \langle dx_i \wedge dx_{i-1}\rangle, \tag{4.5}$$

where, as before, $\langle\cdot\rangle$ denotes the mean value of the periodic expression in brackets over the period.

Remark 4.2. Above we denoted the variation on the phase space (the space of parameters) by δ in order do distinguish it form the differential d, which is taken with respect to the spectral parameter. After taking the residues of the differential, here and in what follows, we use only the notation d, i.e., set $dx_i := \delta x_i$.

According to Theorem 3.9, Eqs. (1.16) restricted to the symplectic leaf $\langle a_i\rangle = \langle e^{\varphi_i - \varphi_{i-1}}\rangle = e_1 = \text{const}$ are Hamiltonian with respect to $\widehat{\omega}^{(0)}$ with Hamiltonian $H^{(0)}_{t_1^-} := e_3$. In order to write this expression explicitly in terms of the new coordinates, we use Eqs. (2.11). For $s = 2$ and $k = 1$, we have

$$\xi_2^-(i) - \xi_2^-(i-2) + e_1\xi_1(i)^- + e_2 = a_i\xi_1^-(i-1). \tag{4.6}$$

From (4.4) it follows that

$$\xi_2^-(i) - \xi_2^-(i-2) + e_2 = x_i x_{i-1} - x_{i-1}x_{i-2} + e_1(x_{i-1} - x_i). \tag{4.7}$$

Taking the mean of Eq. (4.7), we obtain $e_2 = 0$ (recall that in the proof of Lemma 3.2 it was shown that $e_{k+1} = 0$ for any k). For $s = 3$ and $k = 1$, Eq. (2.11) has the form

$$\xi_3^-(i) - \xi_3^-(i-2) + e_1\xi_2^-(i) + e_3 = a_i\xi_2^-(i-1) = (x_i - x_{i-2} + e_1)\xi_2^-(i-1). \tag{4.8}$$

Averaging (4.8), we obtain the following explicit expression for the Hamiltonian of Eq. (1.16) in terms of the new coordinates:

$$H^{(0)}_{\partial_{t_1^-}} = e_3 = \langle (x_i - x_{i-2})\xi_2^-(i-1)\rangle = \langle x_i(\xi_2^-(i-1) - \xi_2^-(i+1)\rangle = \langle x_i^2(x_{i-1} - x_{i+1})\rangle; \tag{4.9}$$

the last equality follows from (4.6).

Example with $k = 2$. The expressions for the coefficients of a lower-triangular operator of order 3 in terms of the coordinates $x_i := \xi_1^-(i)$ and $y_i := \xi_2^-(i)$ are given by (2.8) and (2.9):

$$a_i^{(2)} = x_i - x_{i-3} + e_1, \tag{4.10}$$

$$\begin{aligned} a_i^{(1)} &= y_i - y_{i-3} + e_1 x_i + e_2 - a_i^{(2)} x_{i-2} \\ &= y_i - y_{i-3} - (x_i - x_{i-3})x_{i-2} + e_1(x_i - x_{i-2}) + e_2. \end{aligned} \tag{4.11}$$

The substitution of the expansions of ψ and ψ^+ into (3.1) gives

$$\omega^{(0)} = \frac{1}{2}\langle\, da_i^{(1)} \wedge dx_{i-1} + da_i^{(2)} \wedge (\chi_1^-(i)\, dx_{i-2} + d\xi_2^-(i-2))\rangle, \tag{4.12}$$

where χ_1^- is the first coefficient of the expansion of ψ^+ at the marked point p_-. Equation (4.3) with $j = 1$ implies $\chi_1^-(i) = -x_{i-1}$. Straightforward computations yield the following expression for the form $\omega^{(0)}$ restricted to a leaf along which e_1 and e_2 are constant:

$$\widehat{\omega}^{(0)} = \langle dy_i \wedge (dx_{i-1} - dx_{i+2}) + d(x_{i-1}x_{i-2}) \wedge dx_i\rangle + e_1\langle dx_i \wedge dx_{i-1}\rangle. \tag{4.13}$$

Equation (1.16) with $k = 2$ restricted to a leaf where e_1 and e_2 are constant is Hamiltonian with respect to the form (4.13) with Hamiltonian $H^{(0)}_{t_1^-} = e_4$. Straightforward but lengthy computations give the following expression for the Hamiltonian $H := e_4$:

$$H = \langle y_{i-1}(y_i - y_{i-3})\rangle + \langle x_i x_{i-1} x_{i-2}(x_{i-1} - x_i)\rangle + e_1\langle (x_i^2(x_{i-1} - x_{i+1})\rangle + e_2\langle x_{i-1}(x_i - x_{i-1})\rangle + \langle y_i(x_{i+2}^2 - x_{i-1}^2 - x_{i+2}x_{i+1} + x_{i-2}x_{i-1}\rangle. \quad (4.14)$$

4.2. The form $\boldsymbol{\omega^{(1)}}$. The choice of a system of coordinates in which the form $\omega^{(1)}$ has local density is suggested by the very definition (3.1), which involves the values of ψ_i at the marked points $p_\ell \in \Gamma$ that are the preimages of $E = 0$, at which $w(p_\ell) \neq 0$.

Let $\Phi = \{\phi_i^\ell\}$ be a $k \times n$ matrix of rank k, i.e., $i = 1, \dots, n$ and $\ell = 1, \dots, k$. We say that two matrices are equivalent and write $\Phi \sim \Phi'$ if $\Phi' = \Phi\lambda$, where $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$. The space of equivalence classes $[\Phi] := (\Phi/\sim)$ can be seen as the space of (ordered) sets of k *distinct* points in $(n-1)$-dimensional projective space: $[\phi^\ell] \in \mathbb{P}^{n-1}$.

Consider the space of pairs $\{[\Phi], W\}$, where $W = \{w_1, \dots, w_k\}$ is a set of nonzero numbers ($w_\ell \neq 0$). The symmetric group S_k acts on the space of such pairs by simultaneous permutations of rows of the matrix Φ and coordinates of the vector W.

Now we are going to define a map from the corresponding quotient space to the space of n-periodic operators L of the form (1.3):

$$\{[\Phi], W\}/S_k \mapsto L. \quad (4.15)$$

First, note that, given a set $W = \{w_1, \dots, w_k\}$ of nonzero numbers, any $k \times n$ matrix Φ can be extended to a unique $k \times \infty$ matrix ϕ_i^ℓ, $i \in \mathbb{Z}$, such that $\phi_{i-n}^\ell = w_\ell \phi_i^\ell$. Such an extension uniquely determines an operator L of the form (1.3) such that, for any ℓ, the sequence $\phi^\ell = \{\phi_i^\ell\}$ is a solution of the equation

$$L\phi^\ell = 0 \iff \sum_{j=1}^{k} a_i^{(j)} \phi_{i-j}^\ell = -\phi_{i-k-1}^\ell. \quad (4.16)$$

Indeed, for fixed i, (4.16) is a system of k nonhomogeneous linear equations for the unknown coefficients of L. Applying Cramer's rule, we obtain

$$a_i^{(j)} = -\frac{|\phi_{i-1}, \dots, \phi_{i-j+1}, \phi_{i-k-1}, \phi_{i-j-1}, \dots, \phi_{i-k}|}{|\phi_{i-1}, \dots, \phi_{i-j+1}, \ \phi_{i-j} \ \phi_{i-j-1}, \dots, \phi_{i-k}|}. \quad (4.17)$$

Here and in what follows, we use the following notation: ϕ_i is the k-vector with coordinates $\phi_i := \{\phi_i^\ell\}$, and, for any set $V_1, \dots, V_k$ of k-vectors, $|V_1, \dots, V_k|$ stands for the determinant of the corresponding matrix, i.e., $|V_1, \dots, V_k| := \det(V_i^\ell)$.

Recall that above we parameterized the leading coefficient $a_i^{(1)}$ by variables φ_i such that $a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}}$. Relation (4.17) with $j = 1$ allows us to identify these variables with

$$e^{-\varphi_i} := (-1)^{ik} |\phi_{i-1}, \dots, \dots, \phi_{i-k}| \quad (4.18)$$

and represent Eq. (4.17) in the form

$$a_i^{(j)} = (-1)^{ik+1} e^{\varphi_i} |\phi_{i-1}, \dots, \phi_{i-j+1}, \phi_{i-k-1}, \phi_{i-j-1}, \dots, \phi_{i-k}|. \quad (4.19)$$

Theorem 4.3. *The map* (4.15) *defined by* (4.18) *and* (4.19) *is a one-to-one correspondence between open domains. Under this correspondence Eqs.* (1.14) *and* (1.15) *restricted to leaves with fixed w_ℓ are Hamiltonian with respect to the form*

$$\widehat{\omega}^{(1)} = \frac{1}{2}\left\langle d\varphi_{i-1} \wedge d\varphi_i - (-1)^{(i-1)k} e^{\varphi_{i-1}} \sum_{j=1}^{k} da_i^{(j)} \wedge |\phi_{i-2}, \dots, \phi_{i-k}, d\phi_{i-j}| \right\rangle \quad (4.20)$$

with Hamiltonians

$$H^- = \langle a_i^{(k)} \rangle \quad \textit{and} \quad H^+ = -\langle a_i^{(2)} e^{\varphi_{i-2} - \varphi_i} \rangle, \quad (4.21)$$

respectively.

Proof. The right-hand side of (4.17) is symmetric with respect to the simultaneous permutations of rows of the matrices in the numerator and denominator. Hence the map (4.15) is well defined on an open domain where the denominator does not vanish. The inverse map identifies w_ℓ with nonzero roots of the polynomial $R(w,0) = \det L(w)$ defined in (2.3). In other words, w_ℓ is the value of the function $w(p)$ on the spectral curve Γ of L at one of the preimages of $E = 0$, i.e., p_ℓ: $(w_\ell, 0) \in \Gamma$. It follows from this identification that ϕ_i is nothing but the value of the Baker–Akhiezer function at p_ℓ, i.e., $\phi_i^\ell = \psi_i(p_\ell)$. This proves the first statement of the theorem. ☐

Recall that, by definition, $\omega^{(1)}$ is equal to the sum of resides at $p_\pm$ and p_ℓ of the form

$$-\frac{1}{2n}\sum_{j=1}^{k}\delta a_i^{(j)} \wedge (\psi_i^* \delta\psi_{i-j})E^{-1} d\ln w \tag{4.22}$$

averaged over i. The Baker–Akhiezer function ψ_i and its dual ψ_i^+ have, respectively, a zero and a pole of order i at p_-. Since E has a pole of order $k+1$ at p_-, the form (4.22) is holomorphic at p_-. Hence it has no residue at p_-. At p_+ the function E has a simple zero. Therefore, the form $E^{-1} d\ln w$ has a pole of order 2 at p_+. At the same time, at p_+ the functions ψ_i^+ and ψ_i have, respectively, a zero and a pole of order i. Hence the terms with $j > 1$ in sum (4.22) are holomorphic at p_+. From (2.12) and (2.22) it follows that

$$-\frac{1}{2n}\operatorname*{res}_{p_+}\delta a_i^{(1)} \wedge (\psi_i^* \delta\psi_{i-1})E^{-1} d\ln w = -\frac{1}{2}\delta(e^{\varphi_i - \varphi_{i-1}}) \wedge e^{-\varphi_i}\delta(e^{\varphi_{i-1}}) = \frac{1}{2}\delta\varphi_{i-1} \wedge \delta\varphi_i\,. \tag{4.23}$$

Our next goal is to express $\psi_i^+(p_\ell)$ in terms of $\phi^\ell = \psi(p_\ell)$ in order to obtain a closed expression for $\omega^{(1)}$ in terms of ϕ^ℓ.

Lemma 4.4. *Let* $r_\ell := \operatorname{res}_{p_\ell} E^{-1} d\Omega$. *Then*

$$r_\ell \psi_i^+(p_\ell) = \frac{(-1)^{\ell+k-1}\det \widehat{\Phi}_i^{\ell,k}}{|\phi_{i-2}, \dots, \phi_{i-k-1}|}, \tag{4.24}$$

where $\widehat{\Phi}_i$ *is the* $k\times k$ *matrix with columns* $(\phi_{i-1}, \dots, \phi_{i-k})$ *and* $\widehat{\Phi}_i^{\ell,k}$ *is obtained from* $\widehat{\Phi}_i$ *by removing the* ℓ*th row and the last column.*

Proof. By the definition of $d\Omega$ the differential $\psi_i^+\psi_{i-j}E^{-1}d\Omega$ is holomorphic outside the marked points $p_\pm$ and the points p_ℓ at which E vanishes. For $2 \leqslant j \leqslant k$, it is holomorphic at $p_\pm$. Hence the sum of its residues at p_ℓ equals zero:

$$\sum_{\ell=1}^{k}\operatorname*{res}_{p_\ell}\psi_i^+\psi_{i-j}E^{-1}d\Omega = \sum_{\ell} r_\ell\psi_i^+(p_\ell)\phi_{i-j}^\ell = 0, \qquad j = 2, \dots, k. \tag{4.25}$$

The differential $\psi_i^+\psi_{i-j}E^{-1}d\Omega$ is holomorphic at p_- and has a simple pole at p_+ with residue -1. Hence

$$\sum_{l}\operatorname*{res}_{p_\ell}\psi_i^+\psi_{i-k-1}E^{-1}d\Omega = \sum_{\ell} r_\ell\psi_i^+(p_\ell)\phi_{i-k-q}^\ell = 1. \tag{4.26}$$

Equations (4.25) and (4.26) form a system of linear equations for the unknowns $r_\ell\psi_i(p_\ell)$. Cramer's rule implies (4.24). ☐

Note that, multiplying the right-hand side of (4.24) by $d\phi_{i-j}^\ell$ and then averaging over ℓ, we can identify the latter with the expansion of the determinant along the last column, i.e.,

$$-\frac{1}{2}\sum_{\ell=1}^{k} r_\ell\psi_i^+(p_\ell)\, d\phi_{i-j}^\ell = -\frac{1}{2}\frac{|\phi_{i-2}, \dots, \phi_{i-k}, d\phi_{i-j}|}{|\phi_{i-2}, \dots, \phi_{i-k-1}|} = \frac{(-1)^{k(i-1)+1}}{2}|\phi_{i-2}, \dots, \phi_{i-k}, d\phi_{i-j}|e^{\varphi_{i-1}}. \tag{4.27}$$

The right-hand side of (4.20) is equal to the sum of the right-hand side of (4.23) and the wedge product of (4.27) and $da_i^{(j)}$. This proves (4.20).

To complete the proof of the theorem, it remains to note that, according to Theorem 3.9, the Hamiltonians of Eqs. (1.14) and (1.15) are equal to

$$H^- := H_{\partial_{t_1^-}} = \operatorname*{res}_{z=0} \ln E(z) z^{-2}\, dz = e_1 = \langle a_i^{(k)} \rangle \tag{4.28}$$

and

$$H^+ := H_{\partial_{t_1^+}} = \frac{1}{n} \operatorname*{res}_{E=0} \ln w(E) E^{-2} dE = w_1, \tag{4.29}$$

where w_1 is the first coefficient of expansion (2.15). According to Corollary 2.3,

$$n^{-1} \ln w = n^{-1}(\ln \psi_{-n} - \ln \psi_0) = \langle \psi_{i-1} - \psi_i \rangle. \tag{4.30}$$

Therefore, from (2.12) and (2.13) we obtain

$$w_1 = \langle \xi_1^+(i-1) - \xi_1^+(i) \rangle = -\langle a_i^{(2)} e^{\varphi_{i-2} - \varphi_i} \rangle, \tag{4.31}$$

which completes the proof of the theorem.

Example. For $k = 1$, Eq. (4.18) takes the form $e^{-\varphi_i} = (-1)^i \phi_{i-1}$. In this case, we have

$$\omega^{(1)} = \frac{1}{2} \langle d\varphi_{i-1} \wedge d\varphi_i - (-1)^{i-1} e^{\varphi_{i-1}} d(e^{\varphi_i - \varphi_{i-1}}) \wedge d\phi_{i-1} \rangle = \langle d\varphi_{i-1} \wedge d\varphi_i \rangle. \tag{4.32}$$

Note that, for $k = 1$, the coefficient $a_i^{(2)}$ equals 1, and (4.21) takes the form (1.18).

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