

CHARACTERIZING JACOBIANS OF ALGEBRAIC CURVES WITH INVOLUTION

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ABSTRACT. We give two characterizations of Jacobians of curves with involution having fixed points in the framework of two particular cases of Welter's trisecant conjecture. The geometric form of each of these characterizations is the statement that such Jacobians are exactly those containing a shifted Abelian subvariety whose image under the Kummer map is orthogonal to an explicitly given vector.

1. INTRODUCTION

The problem of characterization of the Jacobians of curves among irreducible principally polarized Abelian varieties (ppav) is the famous Riemann-Schottky problem. Over years of its more than 130 years history, quite a few geometrical characterizations of the Jacobians have been found (see surveys [1, 2, 3]). None of them provides an explicit system of equations for the image of the Jacobian locus in the projective space under the level two theta imbedding.

The first effective solution of the Riemann-Schottky problem was obtained by T. Shiota [4], who proved the famous Novikov's conjecture: *the Jacobians of smooth algebraic curves are precisely those indecomposable principally polarized abelian varieties (ppavs) whose theta-functions provide solutions to the Kadomtsev-Petviashvili (KP) equation.*

The strongest known characterization of a Jacobian variety in this direction, the so-called *Welters' (trisecant) conjecture*, formulated in [5]: *an indecomposable ppav X is the Jacobian of a curve if and only if its Kummer variety $K(X)$ has a trisecant line* was proved by the author in [6, 7]. The approach to its solution, proposed in [6], is general enough to be applicable to a variety of Riemann-Schottky-type problems. In [8, 9] it was used for a characterization of principally polarized Prym varieties. The latter problem is almost as old and famous as the Riemann-Schottky problem but is much harder.

The goal of that work is to characterize the Jacobians of curves with involution. The curves with involution naturally appears as a

part of algebraic-geometrical data defining solutions to integrable system with symmetries. Numerous examples of such systems include the Kadomtsev-Petviashvili hierarchies of type B and C (BKP and CKP hierarchies, respectively) introduced in [10, 11] and the Novikov-Veselov hierarchy introduced in [12, 13].

The existence of an involution of a curve is key in proving that the constructed solutions have the necessary symmetry. The solutions corresponding to the same curve are usually parameterized by points of its Prym variety. In other words the existence of involution plus some extra constraints on the divisor of the Baker-Akhiezer function are *sufficient* conditions ensuring required symmetry. The problem of proving that these conditions are *necessary for two-dimensional* integrable hierarchies is much harder and that is a problem we address in that paper.

Remark 1. To the best of our knowledge from pure algebraic-geometrical perspective the characterization problem of curves with involution in terms of their Jacobians has never been considered in its full generality. The only known to the author works in this directions are [15, 16, 17]. In a certain sense the setup we will consider – the Jacobian and the Prym variety in it – resembles the setup arising in the famous Schottky-Jung relations [18].

Let B be an indecomposable symmetric matrix with positive definite imaginary part. It defines an indecomposable principally polarized abelian variety $X = \mathbb{C}^g/\Lambda$, where the lattice Λ is generated by the basis vectors $e_m \in \mathbb{C}^g$ and the column-vectors B_m of B . Throughout the paper $\pi : \mathbb{C}^g \rightarrow X$ denotes the corresponding projection.

The Riemann theta-function $\theta(z) = \theta(z|B)$ corresponding to B is given by the formula

$$(1.1) \quad \theta(z) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i(z,m) + \pi i(Bm,m)}, \quad (z, m) = m_1 z_1 + \dots + m_g z_g$$

The Kummer variety $K(X)$ is an image of the Kummer map

$$(1.2) \quad K : Z \in X \mapsto \{\Theta[\varepsilon_1, 0](Z) : \dots : \Theta[\varepsilon_{2g}, 0](Z)\} \in \mathbb{CP}^{2g-1}$$

where $\Theta[\varepsilon, 0](z) = \theta[\varepsilon, 0](2z|2B)$ are level two theta-functions with half-integer characteristics ε .

A trisecant of the Kummer variety is a projective line which meets $K(X)$ at least at three points. There are three particular cases of the characterization of the Jacobians by trisecants, corresponding to three possible configurations of the intersection points (a, b, c) of $K(X)$ and the trisecant:

- (i) all three points coincide ($a = b = c$),
- (ii) two of them coincide ($a = b \neq c$);
- (iii) all three intersection points are distinct ($a \neq b \neq c \neq a$).

Of course the first two cases can be regarded as degenerations of the general case (iii). However, when the existence of only one trisecant is assumed, all three cases are independent and require its own approach. The approaches used in [6, 7] were based on the theories of three main soliton hierarchies (see details in [19]): the KP hierarchy for (i), the 2D Toda hierarchy for (ii) and the Bilinear Discrete Hirota Equations (BDHE) for (iii). Recently, pure algebraic proof of the first two cases of the trisecant conjecture were obtained in [20].

The main goal of this work is to give two characterizations of the Jacobians of curves with involution, which distinguish such Jacobians within the framework of their characterizations given by cases (i) and (ii) above. Both of them are limited to the case of involutions having at least one fixed point, i.e. to two-sheeted *ramified* covers. The first, related to the KP theory, is limited by the obvious reason, since a curve with one marked point is used in constructing its solutions.

Theorem 1.1. *An indecomposable principally polarized abelian variety (X, θ) is the Jacobian variety of a smooth algebraic curve Γ of genus g with involution $\sigma : \Gamma \rightarrow \Gamma$ having at least one point fixed if and only if there exist g -dimensional vectors $U \neq 0, V, A, \zeta$ and constants Ω_1, Ω_2, b_1 such that:*

(A) *the equality*

$$(1.3) \quad (\partial_y - \partial_x^2 + u) \psi = 0,$$

where

$$(1.4) \quad u = -2\partial_x^2 \ln \theta(Ux + Vy + Z) + b_1, \quad \psi = \frac{\theta(A + Ux + Vy + Z)}{\theta(Ux + Vy + Z)} e^{\Omega_1 x + \Omega_2 y},$$

holds, for an arbitrary vector Z ;

and

(B) *the intersection of the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$ with a shifted abelian subvariety $Y \subset X$ which is the Zariski closure of $\pi(Ux + \zeta) \subset X$ is reduced and the equation*

$$(1.5) \quad \partial_V \theta|_{\Theta \cap Y} = 0$$

holds.

Moreover, the locus Π of points $\zeta \in X$ for which the equation (1.5) holds is the locus of points for which the equation $\zeta + \sigma(\zeta) = 2P + K \in X$, where K is the canonical class, holds.

Remark 2. Note that if $V \neq 0$ then (1.5) implies that $\dim Y < g$, since for any nonzero vector V the restriction $\partial_V \theta|_{\Theta} \neq 0$. The case $V = 0$ when (1.5) is automatically satisfied is the case of hyperelliptic curves.

The condition (A) is one of three equivalent forms of the characterization of the Jacobians among ppav proved in [6]. The direct substitution of the expression (1.4) into equation (1.3) and the use of the addition formula for the Riemann theta-functions

$$(1.6) \quad \theta(z+w)\theta(z-w) = \sum_{\varepsilon \in ((1/2)\mathbb{Z}/\mathbb{Z})^g} \Theta[\varepsilon, 0](z)\Theta[\varepsilon, 0](w).$$

shows the equivalence of (A) to the condition that for all theta characteristics $\varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ the equations

$$(1.7) \quad (\partial_V - \partial_U^2 - 2\Omega_1 \partial_U + (\Omega_2 - \Omega_1^2)) \Theta[\varepsilon, 0](A/2) = 0$$

hold. (Here and below ∂_U, ∂_V are the derivatives along the vectors U and V , respectively). Equation (1.7) means that the image of the point $A/2$ under the Kummer map is an inflection point (case (i) of Welters' conjecture).

There are two other equivalent forms of the condition (B), which, in particular, give its *geometric form*. The first one is:

(C) there is a vector W and a constants Ω_3, b_3 such that the equality

$$(1.8) \quad \left(\partial_t - \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x + b_3 \right) \psi = 0,$$

where

$$(1.9) \quad u = -2\partial_x^2 \ln \theta(Ux + Wt + \zeta) + b_1, \quad \psi = \frac{\theta(A + Ux + Wt + \zeta)}{\theta(Ux + Wt + \zeta)} e^{\Omega_1 x + \Omega_3 t},$$

holds.

The fact that the condition (C) holds for curves with involution is known. In the CKP theory [10, 11] equation (1.8) plays the same role as equation (1.3) in the KP theory. Namely, both equations define the first flows of the corresponding hierarchies (see details in [22]).

In [21, 22] equation (1.8) was obtained and used in another but equivalent form:

(C') there is a constant b_2 such that the equality

$$(1.10) \quad \partial_U \partial_V \ln \theta|_{\widehat{Y}} = b_2$$

holds on \widehat{Y} which is Zariski closure of $\pi(Ux + Wt + \zeta) \subset X$.

Remark 3. As will be seen in what follows, if there is a set of vectors and constants for which the above conditions are satisfied, then there is a family of such sets. One of the consequences of this is that without loss of generality, any (but only one) of the constants b_i above can be set equal to 0.

The addition theorem (1.6) implies that (1.10) is equivalent to the condition that the vector $(\partial_U \partial_V K(0) - b_2 K(0))$ is *orthogonal to the image under the Kummer map $K(\Pi)$ of the shifted abelian subvariety \widehat{Y}* :

$$(1.11) \quad \sum_{\varepsilon \in ((1/2)\mathbb{Z}/\mathbb{Z})^g} (\partial_U \partial_V \Theta[\varepsilon, 0](0) - b_2 \Theta[\varepsilon, 0](0)) \Theta[\varepsilon, 0](z) = 0, \quad z \in \widehat{Y}$$

whence follows the condition *of a kind of flatness* of the image under the Kummer map of the shifted Prym subsubvariety $\Pi \subset X$, that is, $K(\Pi)$ lies in a proper (projective) linear subspace.

Equations (1.3) and (1.10) are used for the proof of (B), which is the strongest form of the characterization the Jacobians of curves with involution. The implication (B) \Rightarrow (C) is not by all means direct and comes only as a result of the proof the theorem. Namely, (B) is what we *really* use in the proof that the corresponding curve is a curve with involution. The latter, as it was mentioned above implies (C).

The explicit meaning (B) is as follows. As shown in [4, 23] the affine line $Ux + Z$ is not contained in Θ for any vector Z . Hence, the function $\tau(x, y) := \theta(Ux + Vy + z), z \in Y$ is a *nontrivial* entire function of x . The assumption that $\Theta \cap Y$ is reduced means that zeros $q(y)$ of τ considered as a function of x (depending on y) are generically simple, $\tau(q(y), y) = 0, \tau_x(q(y), y) \neq 0$. Then (1.10) is the equation $\partial_y q|_{y=0} = 0$.

The second characterization of the Jacobians of curves with involution is related to the 2D Toda theory. A priori, unlike the KP case, there is no obvious reason why it is not applicable to all types of involution including unramified covers. It turned out that there is an obstacle for the case unramified covers and our second theorem also gives a characterization of the Jacobians of curves with involution *with* fixed points.

Theorem 1.2. *An indecomposable, principally polarized abelian variety (X, θ) is the Jacobian of a smooth curve of genus g with involution having fixed points if and only if there exist non-zero g -dimensional vectors $U \neq A \pmod{\Lambda}$, V, ζ , constants Ω_0, Ω_1, b_1 such that the following two conditions are satisfied.*

(A) *The differential-functional equation*

$$(1.12) \quad (\partial_y - T - u)\psi = 0, \quad T = e^{\partial_x}$$

where

$$(1.13) \quad u = b_1 + (T - 1)\partial_y \ln \theta(xU + yV + Z)$$

and

$$(1.14) \quad \psi = \frac{\theta(A + xU + yV + Z)}{\theta(xU + yV + Z)} e^{x\Omega_0 + y\Omega_1}$$

holds for an arbitrary vector Z .

(B) (i) *The intersection of the theta-divisor with the shifted Abelian variety Y , which is a closure of $\pi(Ux + \zeta)$, is reduced and is not invariant under the shift by U , $\Theta \cap Y \neq (\Theta + U) \cap Y$, and (ii) the equation*

$$(1.15) \quad ((\partial_V \theta(z))^2 + b_2 \theta(z + U)\theta(z - U))|_{z \in \Theta \cap Y} = 0,$$

where $b_2 \neq 0$ is a constant, holds.

Moreover, the locus of the points $\zeta \in X$ for which the equation (1.15) holds is the locus of point for which the equation $\zeta + \zeta^\sigma = K + P_1 + P_2 \in J(\Gamma)$, where (P_1, P_2) are points of the curves permuted by σ and such that $U = A(P_2) - A(P_1)$, is satisfied.

Remark 4. *Under the assumption that U spans an elliptic curve Theorem 1.2 was proved in [30]*

The condition (A) is one of three equivalent forms of the characterization of the Jacobians proved in [7]. It is equivalent to the condition (case (ii) of the Welter's conjecture):

the equations

$$(1.16) \quad \partial_V \Theta[\varepsilon, 0] ((A - U)/2) - e^p \Theta[\varepsilon, 0] ((A + U)/2) + E \Theta[\varepsilon, 0] ((A - U)/2) = 0,$$

are satisfied for all $\varepsilon \in Z_2^g$.

In the course of proving that condition (B) holds for ramified double covers, we first prove that:

there is a vector W and constants Ω_2, b_2 such that the differential-functional equation

$$(1.17) \quad (\partial_t - T - w_1 - wT^{-1})\psi = 0$$

where

$$(1.18) \quad w_1 = b_1 + \frac{1}{2}(T-1)\partial_t \ln \theta, \quad w = b_2 \frac{T\theta T^{-1}\theta}{\theta^2}$$

$$\theta = \theta(xU + tW + \zeta)$$

and

$$(1.19) \quad \psi = \frac{\theta(A + xU + tW + \zeta)}{\theta(xU + tW + \zeta)} e^{x\Omega_0 + t\Omega_2},$$

holds.

The equivalent form of the statement above is the statement that for the ramified double covers

there is a constant b_3 such that the equality

$$(1.20) \quad \theta^2(z)\partial_V^2 \ln \theta(z) - b_2 \theta(z+U)\theta(z-U)|_{z \in \widehat{Y}} = b_3,$$

where \widehat{Y} which is Zariski closure of $\pi(Ux + Wt + \zeta)$, holds.

The addition theorem (1.6) implies that (1.20) is equivalent to the condition that the vector $(2\partial_V^2 K(0) - b_2 K(U) - b_3 K(0))$ is orthogonal to the image under the Kummer map of the abelian subvariety \widehat{Y} , i.e. the equation

$$(1.21) \quad \sum_{\varepsilon \in ((1/2)\mathbb{Z}/\mathbb{Z})^g} (2\partial_V^2 \Theta[\varepsilon, 0](0) - b_2 \Theta[\varepsilon, 0](U) - b_3 \Theta[\varepsilon, 0](0)) \Theta[\varepsilon, 0](z) = 0$$

with $z \in \widehat{Y}$, holds.

Equations (1.17) and (1.20) are analogs of the conditions (C) and (C') in the flex case. But unlike the latter, we do not claim that they are equivalent to (B), because we came short in proving that they imply (B), and will use additional arguments for the last step in the proof. The condition (B) is what we really use for the proof of "if" part of the theorem.

Remark 5. As we shall see below, the case $2U \in \Lambda$ and $W = 0$ in (1.18) and (1.19) corresponds to hyperelliptic curves, which are curves with involution.

We conclude the introduction by saying that it is tempting to see if the third characterization associated with the fully discrete BDHE hierarchy might be applicable to the case of unramified covers.

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2. PRELIMINARIES

The KP and CKP hierarchies. The Kadomtsev-Petviashvili (KP) hierarchy is one of the most fundamental in the modern theory of integrable systems. It has at least three well-known definitions/representations usually called: the Zakharov-Shabat form [24], the Sato form [25] and the Hirota bilinear equation form [11, 26].

In the Sato form it is defined as a system of commuting flows on the space of sequences $(u_1(x), u_2(x), \dots)$ of functions of one variable x , which can be identified with the space of pseudo-differential operators of the form

$$(2.1) \quad \mathcal{L} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \dots$$

The flows are defined by the Lax equations

$$(2.2) \quad \partial_{t_k} \mathcal{L} = [B_k, \mathcal{L}], \quad B_k := \mathcal{L}_+^k \quad k = 1, 2, 3, \dots$$

where $(\cdot)_+$ stands for the differential part of a pseudo-differential operator. The commutativity of flows implies that the equations

$$(2.3) \quad \partial_{t_l} B_k - \partial_{t_k} B_l + [B_k, B_l] = 0$$

are satisfied for all pairs k, l . For $k = 2, l = 3$ operator equation (2.3) where

$$(2.4) \quad B_2 = \partial_x^2 - u, \quad u = -2u_1$$

$$(2.5) \quad B_3 = \partial_x^3 - \frac{3}{2}u\partial_x - w, \quad w = -\frac{3}{2}u_x - 3u_2$$

after change of notation $t_1 = x, t_2 = y, t_3 = t$, is equivalent to the system of two equations

$$(2.6) \quad 4w_x = 3u_{xx} + 3u_y, \quad 4w_y = (4u_t + 6uu_x - u_{xxx})_x + 3u_{xy}$$

Eliminating w from the system one gets the original KP equation for the remaining function u

$$(2.7) \quad 3u_{yy} = (4u_t + 6uu_x - u_{xxx})_x$$

In [10] an infinite integrable hierarchy of partial differential was introduced and called the Kadomtsev-Petviashvili hierarchy of type C (CKP). It is a hierarchy of commuting flows that are the restriction of the flows of the KP hierarchy corresponding to “odd” times $\mathbf{t}_o = \{t_1, t_3, t_5, \dots\}$ onto the space of *anti self-adjoint* pseudo-differential operators \mathcal{L} of the form (2.1), i.e. such that

$$(2.8) \quad \mathcal{L}^* = -\mathcal{L},$$

where $(\cdot)^*$ means the formal adjoint defined by the rule $(f(x) \circ \partial_x^m)^* = (-\partial_x)^m \circ f(x)$.

Remark 6. Note, that if (2.8) is satisfied then the operator B_3 in (2.5) is of the form (1.8).

In [22] the CKP hierarchy was characterized in terms of the KP tau-function which is a function of KP "times", $\tau(t_1, t_2, \dots)$. More precisely, each solution of the CKP hierarchy has a unique extension to the solution of the full KP hierarchy via the flows (2.2) with even k (which obviously do not preserve constraint (2.8)). Such solution is naturally to call *KP extension of the solution to the CKP hierarchy*. In [22] it was proved that the KP tau-function is the tau-function of such a solution if and only if the equation

$$(2.9) \quad \partial_x \partial_y \log \tau \Big|_{\mathbf{t}_e=0} = 0$$

holds for all \mathbf{t}_o , where all "even" times $\mathbf{t}_e = (t_2 = y, t_4, \dots)$ are set equal to zero.

Theorem 1.1 is a stronger version of that result in the algebraic-geometric setting when τ function coincides with the theta-function up to a factor which is the exponent of a quadratic in times form. Namely it states, that it is enough to require that equation (2.9) holds only for the first two times $t_1 = x, t_3 = t$ of the CKP hierarchy (compare with (1.10)).

The Baker-Akhiezer function. Recall that a smooth genus g algebraic curve Γ with fixed local coordinate $k^{-1}(p)$ in the neighborhood of a point $P \in \Gamma$, $k^{-1}(P) = 0$ and a generic effective degree g divisor $D = \gamma_1 + \dots + \gamma_g$ defines the Baker-Akhiezer function $\psi(\mathbf{t}, p)$ which is a function of complex variables $\mathbf{t} = (t_1, t_2, t_3, \dots)$ (it is always assumed that only finite number of $t_i \neq 0$) and $p \in \Gamma$. For fixed \mathbf{t} it is defined as a unique function of $p \in \Gamma$ with the following analytic properties:

1⁰. Outside P the singularities of ψ are poles at points of the divisor D of order not greater then the multiplicity of the point in D , i.e., $(\psi) + D \geq 0$.

2⁰. In the neighborhood of P the function ψ has the form

$$(2.10) \quad \psi(\mathbf{t}, k) = e^{\sum_i k^i t_i} \left(1 + \sum_{s=1}^{\infty} \xi_s(\mathbf{t}) k^{-s} \right), \quad k = k(p),$$

In order to present an explicit formula for ψ in terms of the Riemann theta function we first choose a symplectic basis of a - and b -cycles on Γ . Then define a basis of normalized holomorphic differentials ω_k , $k =$

$1, \dots, g$, and the matrix B of their b -periods

$$\oint_{a_k} \omega_\ell = \delta_{k\ell}, \quad B_{k\ell} = \oint_{b_\ell} \omega_k = B_{\ell k},$$

and the Riemann-theta function by formula (1.1)

Denote by $A(p)$ the vector (depending on a path of integration) with coordinates $A_k(p) = \int_P^p \omega_k$ and by $\Omega_i(p), i = 1, 2, 3, \dots$, the Abelian integral $\Omega_i = \int_P^p d\Omega_i$ where $d\Omega_i$ is the normalized (i.e. having zero a -periods) meromorphic differential whose the only singularity is at P of the form $d\Omega_i = dk^i(1 + O(k^{-i-2}))$.

The definition of Ω_i needs a clarification since $d\Omega_i$ has the pole at P . In the definition of Ω_i it is assumed that its branch in the neighborhood of P is fixed such that there is no constant term in the expansion

$$(2.11) \quad \Omega_i = k^i + \sum_{s=1}^{\infty} a_s^{(i)} k^{-s} + O(k^{-2})$$

and then extended analytically along the path. It is assumed that the paths in the definition of $A(p)$ and $\Omega_i(p)$ are the same.

Lemma 2.1 ([27, 28]). *(i) The Baker-Akhiezer function defined above equals*

$$(2.12) \quad \psi(\mathbf{t}, p) = \frac{\theta(A(p) + \sum_i t_i U_i + Z) \theta(Z)}{\theta(\sum_i t_i U_i + Z) \theta(A(p) + Z)} e^{\sum_i t_i \Omega_i(p)},$$

where U_i are vectors with the coordinates

$$(2.13) \quad U_i^k = \frac{1}{2\pi i} \oint_{b_k} d\Omega_i,$$

and

$$(2.14) \quad Z = - \sum_s A(\gamma_s) + \mathcal{K},$$

where \mathcal{K} is the vector of Riemann constants.

(ii) The BA function ψ satisfies the equations

$$(2.15) \quad (\partial_{t_k} - B_k) \psi = 0, \quad k = 1, 2, \dots$$

where B_k is a monic differential operator in x of order k .

Remark 7. The compatibility conditions of equations (2.15) is the KP hierarchy in the Zakharov-Shabat form. From the definition of B_k in [27] it is easy to show that $B_k = \mathcal{L}_+^k$ where \mathcal{L} is a unique pseudo-differential operator such that the equation

$$(2.16) \quad \mathcal{L}\psi = k\psi$$

holds. The compatibility conditions of (2.15) and (2.16) are the Lax equations (2.2).

The substitution of (2.10) into (2.15) with B_2 of the form (2.4) gives the equations

$$(2.17) \quad \partial_y \xi_s - 2\partial_x \xi_{s+1} - \partial_x^2 \xi_s + u \xi_s = 0, \quad s = 0, \dots$$

The first of them with $s = 0$ gives an expression of u in terms of ξ_1 . Namely, $u = 2\partial_x \xi_1$.

Similarly, the substitution (2.10) into (2.15) with B_3 of the form (2.5) gives the equations:

$$(2.18) \quad \partial_t \xi_s - 3\partial_x \xi_{s+2} - 3\partial_x^2 \xi_{s+1} - \partial_x^3 \xi_s + \frac{3}{2}u \xi_{s+1} + \frac{3}{2}u \partial_x \xi_s + w \xi_s = 0.$$

The first of them with $s = 0$ gives the expression for w .

The explicit theta-functional formulas for u and w are obtained by expansion of (2.12) near P . From the bilinear Riemann identities for periods of Abelian differentials it follows that the expansion of the Abel map $A(p)$ near P has the form

$$(2.19) \quad A(k) = - \sum_{i=1}^{\infty} \frac{1}{i} U_i k^{-i}, \quad k = k(p)$$

where the vectors U_i are given by formula (2.13). From (2.12) and (2.20) it follows that

$$(2.20) \quad u = 2\partial_x \xi_1 = -2\partial_x^2 \ln \theta \left(\sum_i t_i U_i + Z \right) + 2a_1^{(1)}$$

where the constant $a_1^{(1)}$ is defined in (2.11).

Computing the next coefficient of expansion of (2.12) one gets

$$(2.21) \quad w = \frac{3}{4}u_x - \frac{3}{2}\partial_x \partial_y \ln \theta \left(\sum_i t_i U_i + Z \right) + b_3$$

Although for completeness we defined above the Baker-Akhiezer function, depending on the full set of times of the KP hierarchy, in the future we will restrict ourselves to its dependence only on the first three times, setting $t_i = 0, i > 3$. Recall that above we have already denoted these times by $x = t_1, y = t_2, t = t_3$. Below we will use notation U, V, W for the vectors U_1, U_2, U_3 , respectively.

Bloch properties of the Baker-Akhiezer function. For further use we elaborate more on monodromy properties of the Baker-Akhiezer function along the subvariety $Y \subset J(\Gamma)$. Recall that Y is Zariski closure of $\pi(xU + \zeta)$ where $\pi : \mathbb{C}^g \mapsto J(\Gamma) = \mathbb{C}^g/\Lambda$ is the projection.

The preimage $\tilde{Y} = \pi^{-1}(Y)$ is an affine subspace. A choice of a vector $\zeta \in \tilde{Y}$ identifies it with the image of an imbedding map

$$(2.22) \quad \iota : \mathbb{C}^d \hookrightarrow \mathbb{C}^g$$

Let $\Lambda_U \subset \Lambda$ be the sublattice of the Jacobian lattice such that under this identification

$$(2.23) \quad Y = \mathbb{C}^d/\Lambda_U$$

For further use define the vector $\eta \in \mathbb{C}^d$ by the equation

$$(2.24) \quad \iota(\eta) = U$$

Consider the function

$$(2.25) \quad \phi(z, k) := \frac{\theta(A(k) + \iota(z)) \theta(\zeta)}{\theta(\iota(z)) \theta(A(k) + \zeta)} e^{\ell(k), z}, \quad z \in \mathbb{C}^d,$$

where $A(k)$ is defined in (2.19) and $\ell(k)$ is a formal series

$$(2.26) \quad \ell(k) = \sum_{s=1}^{\infty} \ell_s k^{-s}, \quad \ell_s \in \mathbb{C}^d,$$

such that the equation

$$(2.27) \quad (\ell(k), \eta) = \Omega_1(k) - k,$$

holds.

A simple comparison of (2.12) and (2.25) shows that the coefficients $\xi_s(x, 0)$ of the Baker-Akhiezer function $\psi(x, 0, k)$ expansion at P are of the form

$$(2.28) \quad \xi_s(x, 0) = \varphi_s(x\eta)$$

where $\varphi_s(z)$ are coefficients of the expansion of ϕ ,

$$(2.29) \quad \phi(z, k) = 1 + \sum_{s=1}^{\infty} \varphi_s(z) k^{-s}$$

The latter are of the form

$$(2.30) \quad \varphi_s(z) = \frac{\tau_s(z)}{\tau(z)}, \quad \tau(z) := \theta(\iota(z))$$

where $\tau_s(z)$ is a holomorphic function of z .

The monodromy properties of the theta-function imply that under the shift of the arguments by a vector $\lambda \in \Lambda_U$ the function ϕ gets transformed to

$$(2.31) \quad \phi(z + \lambda, k) = \phi(z, k)\rho_\lambda(k), \quad \rho_\lambda(k) = 1 + \sum_{s=1}^{\infty} \rho_{\lambda,s}k^{-s}$$

Equation (2.31) is equivalent to the equations

$$(2.32) \quad \varphi_s(z + \lambda) = \sum_{j=0}^s \varphi_{s-j}(z)\rho_{\lambda,j}$$

Lemma 2.2. *Let $\lambda_1, \dots, \lambda_d$ be a set of linear independent vectors in Λ_U . Then there is a unique coordinate $k^{-1}(p)$ in the neighborhood of P and a unique linear form $\ell(k)$ satisfying (2.27) such that $\phi(z, k)$ is invariant under the shift by these vectors, i.e.,*

$$(2.33) \quad \phi(z + \lambda_i, k) = \phi(z, k) \Leftrightarrow \rho_{\lambda_i}(k) = 1, \quad i = 1, \dots, d.$$

For the proof it is enough to note, that the differential $d\Omega_1$ depends only on the first jet of the local coordinate, i.e., it does not change under the change of the coordinate of the form $k' = k + O(k^{-1})$, but the coefficients of his expansion (2.11) *do depend* on a choice of the local coordinate. Conversely, for any given set of coefficients $a_s^{(1)}$ equation (2.11) with $i = 1$ can be regarded as the definition of the corresponding formal local coordinate.

The vectors ℓ_s and constants $a_s^{(1)}$ are defined recurrently. On each step we defined first ℓ_s by a system of nonhomogeneous linear equations which are needed for the equation $\rho_{\lambda_i,s} = 0$ to be satisfied, and then define $a_s^{(1)}$ by the equation $a_s^{(1)} = (\ell_s, \eta)$.

The dual Baker-Akhiezer function. A notion of duality is defined first for a generic effective degree g divisor $D = \gamma_1 + \dots + \gamma_g$. For any such divisor there is a unique meromorphic differential $d\Omega$ having second order pole at the marked point P ,

$$(2.34) \quad d\Omega = dk \left(1 + \sum_{s=2}^{\infty} c_s k^{-s} \right)$$

vanishing at the points of the divisor D with the multiplicity at least equal to the multiplicity of the point in D . The zero divisor of $d\Omega$ is of degree $2g$. Let D^* be the effective degree g complimentary divisor, i.e. $D + D^* = K + 2P \in J(\Gamma)$, where K is the canonical class.

The dual BA function is defined as a unique function $\psi^*(x, y, p)$ with the following analytic properties with respect to $p \in \Gamma$:

1⁰. Outside P the singularities of ψ^* are poles at the divisor D^* , $(\psi^*) + D^* \geq 0$;

2⁰. In the neighborhood of P the function ψ has the form

$$(2.35) \quad \psi^*(x, y, k) = e^{-kx - k^2y} \left(1 + \sum_{s=1}^{\infty} \xi_s^*(x, y) k^{-s} \right), \quad k = k(p),$$

The explicit formula for ψ^* in terms of the Riemann theta-function is:

$$(2.36) \quad \psi^*(x, y, k) = \frac{\theta(A(p) - xU - yV - Z) \theta(Z)}{\theta(xU + yV + Z) \theta(A(p) - Z)} e^{-x\Omega_2(p) - y\Omega_3(p)},$$

From the definition of the dual BA function it is easy to see that the equations

$$(2.37) \quad \text{res}_P \left(\partial_x^i \psi^*(x, 0, p) \partial_x^m \psi(x, 0, p) \right) d\Omega = 0, \quad m, i = 0, 1, \dots$$

hold. Indeed, the differential in the left hand side of the equation is holomorphic away of P since the poles of ψ and ψ^* cancel with zeros of $d\Omega$. The essential singularities of ψ and ψ^* at P cancel each other. Hence, the differential in l.h.s. is a meromorphic differential with the only pole at P .

Computing the residue with $i = 0$ in terms of the coefficients of the expansions (2.10) and (2.35) we get the system of equations

$$(2.38) \quad \xi_{m+1} + \xi_{m+1}^* + c_{m+1} = h_{m+1}(\xi_1, \xi_1^*, \dots, \xi_m, \xi_m^*; c_2, \dots, c_m)$$

where h_{n+1} are some explicit differential polynomials in ξ_s , linear in ξ_s^* , $s \leq n$, and in the coefficients of the expansion (2.34) of $d\Omega$. Equations (2.38) recurrently express coefficients of the dual function expansion in terms of the coefficients ξ_s and c_s .

The dual BA function satisfies the equation that is formally adjoint to (1.3):

$$(2.39) \quad (-\partial_y - \partial_x^2 + u(x, y))\psi^*(x, y, p) = 0$$

To prove (2.39) it is enough to note that the same arguments as in the proof of (1.3) show that ψ^* satisfies (2.39) with potential equal to $v = -2\partial_x \xi_1^*$. Equation (2.38) for $m = 1$, i.e. the equation

$$(2.40) \quad \xi_1 + \xi_1^* = 0$$

implies $v = u$ and (2.39) is proved.

The substitution of (2.35) into (2.39) gives a system of equations

$$(2.41) \quad -\partial_y \xi_s^* + 2\partial_x \xi_{s+1}^* - \partial_x^2 \xi_s^* + u \xi_s^* = 0, \quad s = 0, \dots$$

Introduce the function dual to ϕ :

$$(2.42) \quad \phi^*(z, k) := \frac{\theta(A(k) - \iota(z)) \theta(\zeta)}{\theta(\iota(z)) \theta(A(k) - \zeta)} e^{-\ell(k), z},$$

and the coefficients $\varphi_s^*(z)$ of its expansion. From (2.36) it follows that

$$(2.43) \quad \xi_s^*(x, 0) = \varphi_s^*(x\eta)$$

The functions φ_s^* have the form

$$(2.44) \quad \varphi_s^*(z) = \frac{\tau_s^*(z)}{\tau(z)},$$

where τ_s^* is a holomorphic function of $z \in \mathbb{C}^g$. They satisfy the following monodromy properties

$$(2.45) \quad \varphi_s^*(z + \lambda) = \sum_{j=0}^s \varphi_{s-j}^*(z) \rho_{\lambda, j}^*$$

where $\rho_{\lambda, s}^*$ are the coefficients of the expansion of $\rho_\lambda^*(k) = \rho_\lambda^{-1}(k)$.

3. PROOF OF THEOREM 1.1

To begin with let us show that if a curve Γ admits an involution under which P is fixed and k is odd, $k(p) = -k(\sigma(p))$, then the conditions (C) and (C') of the theorem are satisfied for $\zeta \in J(\Gamma)$ such that

$$(3.1) \quad \zeta + \zeta^\sigma = K + 2P \in J(\Gamma)$$

In other words for ζ that is the image under the Abel transform of a divisor D such that $D + \sigma(D)$ is the zero-divisor of a meromorphic differential $d\tilde{\Omega}$ with the only pole of second order at P .

Consider the differential $(\psi(x, 0, t, \sigma(p)) \partial_x \psi(x, 0, t, p)) d\tilde{\Omega}$. It is a meromorphic differential on Γ with the only pole at P . Hence, it has no residue at P . Computing this residue in terms of the coefficients of the expansion (2.10) we get

$$(3.2) \quad 2\xi_2(x, 0, t) - \xi_1^2(x, 0, t) + \partial_x \xi_1(x, 0, t) + c_2 = 0$$

where c_2 is a constant defined by the Laurent expansion of $d\tilde{\Omega}$ at P .

Taking the x -derivative of (3.2) and using (2.17) with $s = 1$ we get the equation

$$(3.3) \quad 0 = \partial_x (2\xi_2(x, 0, t) - \xi_1^2(x, 0, t) + \partial_x \xi_1(x, 0, t)) = \partial_y \xi_1|_{y=0}$$

Recall, that $\xi_1 = -\partial_x \ln \theta + a_1^{(1)} x + a_1^{(2)} y$. Therefore, equation (1.10) with $b_2 = a_1^{(2)}$ is satisfied for all ζ satisfying (3.1). Then from (2.21) it follows that $w(x, 0, t) = \frac{3}{4} u_x + d$, i.e equation (1.8) holds.

The formulae for ψ in (1.4) and (1.9) are just the evaluation of the formula (2.12) for the Baker Akhiezer function at any $p \in \Gamma$.

Our next goal is to prove $(C) \Rightarrow (B)$. As shown in [4, 23] the theta divisor Θ in the Jacobian $J(\Gamma)$ of an a smooth algebraic curve Γ does not contain a ∂_U invariant locus. In other words: for any vector $Z \in J(\Gamma)$ the function $\tau(x; y, t) := \theta(Ux + Vy + Wt + Z)$ is a *non-zero* entire function of the variable x depending on the variables y and t and Z . For brevity we omit explicit indication on Z -dependence. The zeros q of τ in x (depending on y, t, Z) correspond to intersection points of $Ux + Vy + Wt + Z$ with the theta-divisor Θ . Since Θ is reduced and U -direction is transverse to Θ on an Zariski open set, generic zeros of τ are simple. It might be not the case when we consider $z \in \widehat{Y}$.

The multiplicity of a zero of an entire function in x depending on parameters is an upper-continuous function of the parameters. Hence, there is a constant N such that for z in an open everywhere dense set $(\Theta \cap Y)^0 \subset \Theta \cap \widehat{Y}$ the multiplicity of the zero $q(t)$ in the neighborhood of $x = 0$ of the function $\theta(Ux + Wt + z)$, $z \in (\Theta \cap Y)^0$ for sufficiently small t is exactly N . Then, by the implicit function theorem $q(t)$ is a smooth function of t . Then the Laurent expansion of $u = -2\partial_x^2 \ln \tau$ is of the form

$$(3.4) \quad u(x, 0, t) = \frac{2N}{(x - q(t))^2} + v(t) + w(t)(x - q(t)) + \dots$$

whose coefficients v, w, \dots are smooth functions of t .

The function $\psi(x, 0, t)$ given by formula (1.4) is a meromorphic function of x with a pole at $x = q(t)$ of order $n \leq N$ (the numerator might have a zero at $q(t)$), i.e. its Laurent expansion of ψ at q has the form

$$(3.5) \quad \psi(x, 0, t) = \frac{\alpha(t)}{(x - q(t))^n} + O((x - q(t))^{-n+1}), \quad \alpha \neq 0$$

Substitution of (3.5) into equation (1.8) and calculation of the coefficient at the leading term of the Laurent expansion of the left-hand side, which is of order $-n - 3$ gives the equation

$$(3.6) \quad 3N = n(n + 2)$$

Now we are going to use the equivalent form of the condition (C) , i.e., equation (1.10). Multiplying it by θ we get the equation

$$(3.7) \quad \partial_x T(x, z) - T(x, z) \partial_x \ln \theta(Ux + z) = 0, \quad z \in \widehat{Y}$$

where $T(x, z) := \partial_y \theta(Ux + Vy + z)|_{y=0}$. Since at $x = q(0)$ the function $\partial_x \ln \theta(Ux + z)$ has simple pole with residue N , equation (3.7) implies that T at $x = q(0)$ has zero of the same order as $\theta(Ux + z)$. Therefore,

the function $\partial_V \ln \theta|_{\Theta \cap \widehat{Y}}$ is a regular function. The latter implies that $\partial_y \psi(x, y)|_{y=0}$ for ψ defined in (1.4) has pole $x = q(0)$ of order at most n . Two other terms on the left hand side of (1.3) have poles of order $n + 2$. The comparison of their leading coefficients gives the equation

$$(3.8) \quad 2N = n(n + 1)$$

From (3.6) and (3.8) it follows that $N = n = 1$. The implication $(C) \Rightarrow (B)$ is proved. That completes the proof of the "only if" part of the theorem statement.

We begin the proof of the "if" part of the Theorem statement by proving

Lemma 3.1. *Suppose that equation (1.5) is satisfied and let k^{-1} be the local coordinate defined in Lemma 2.2. Then, for the expansions (2.10), (2.35) of the Baker-Akhiezer function and its dual in this coordinate, the equation*

$$(3.9) \quad \psi^*(x, 0, k) = \psi(x, 0, -k)$$

holds.

Proof. Equation (3.9) is equivalent to the equations

$$(3.10) \quad \xi_s^*(x, 0) = (-1)^s \xi_s(x, 0), \quad s = 0, 1, \dots$$

The latter, by (2.28) and (2.43), are corollaries of the equations

$$(3.11) \quad \varphi_s(z) = (-1)^s \varphi_s^*(z)$$

We are going to prove (3.11) by induction. The initial step of the induction is the equation (2.40).

Suppose that equations (3.11) hold for $s \leq n$. Then for $s \leq n$ equations (3.10) hold. The latter and equations (2.17), (2.41) with $s = n$ imply the equation

$$(3.12) \quad \partial_y \xi_n(x, 0) + (-1)^n \partial_y \xi_n^*(x, 0) = 2\partial_x(\xi_{n+1}(x, 0) + (-1)^n \xi_{n+1}^*(x, 0)).$$

A priori the left hand side of (3.12) has poles of second order at poles of ξ_n and ξ_n^* (i.e. at zeros of $\tau(x\eta)$) but equation (1.5) ensures that it has simple poles, only. At the same time the right hand side has no x -residues at its poles. Hence, the function $2\partial_x(\xi_{n+1}(x, 0) + (-1)^n \xi_{n+1}^*(x, 0))$ is a *holomorphic* function of x . By (2.28) and (2.43) it equals to $f_{n+1}(x\eta)$, where

$$(3.13) \quad f_{n+1}(z) := 2\partial_\eta(\varphi_{n+1}(z) + (-1)^n \varphi_{n+1}^*(z))$$

Consider the monodromies of f_{n+1} . From the induction assumption and equations (2.32),(2.45) it follows that

$$(3.14) \quad \rho_{\lambda,s} = (-1)^s \rho_{\lambda,s}^*, \quad s = 1, \dots, n.$$

The latter imply that the function f_{n+1} has trivial monodromies

$$(3.15) \quad f_{n+1}(z + \lambda) = f_{n+1}(z), \quad \lambda \in \Lambda_U$$

i.e. it descends to an abelian function on Y . Since upon restriction on the line $x\eta$ it is holomorphic, and the projection of the line is dense in Y we get that $f_{n+1}(z) = f_{n+1}$ is a constant. Then the equation $\xi_s(0,0) = \xi_s^*(0,0) = 0$ imply

$$(3.16) \quad F_{n+1}(x\eta) = x f_{n+1}, \quad F_{n+1}(z) := \varphi_{n+1}(z) + (-1)^n \varphi_{n+1}^*(z)$$

From (3.14), (2.32) and (2.45) it follows that

$$(3.17) \quad F_{n+1}(z + \lambda) - F_{2n}(z) = \rho_{\lambda,n+1} + (-1)^n \rho_{\lambda,n+1}^*$$

The latter imply that $F_n(z)$ is a linear form, i.e.

$$(3.18) \quad F_{n+1}(z) = (\tilde{\ell}_{n+1}, z), \quad (\tilde{\ell}_{n+1}, \eta) = f_{n+1}$$

Recall, that by our choice of the local coordinate we have that $\rho_{\lambda_i,m} = \rho_{\lambda_i,m}^* = 0$ for any $m > 0$, i.e. the linear form F_{n+1} vanishes at d linear independent vectors. Hence, $F_{n+1} = 0$ and the induction step is completed.

Corollary 3.1. *Let k^{-1} be the local coordinate defined in Lemma 2.33. Then for the expansion in this coordinate of the differential $d\Omega$ defining the dual Baker-Akhiezer function (i.e. the differential vanishing at the poles of the Baker-Akhiezer function), the equation*

$$(3.19) \quad d\Omega(k) = -d\Omega(-k)$$

holds.

Proof. Equation (3.19) is equivalent to the equations

$$(3.20) \quad c_{2s-1} = 0$$

where c_s are the coefficients in (2.34).

Let $c_{2n+1} \neq 0$ be the non-zero coefficient of $d\Omega$ expansion with the smallest odd index. Let $d\Omega_{2n}$ be the sum of the first $2n$ terms of $d\Omega(k)$ expansion, i.e.,

$$d\Omega(k) = d\Omega_{2n}(k) + dk (c_{2n+1} k^{-2n+1} + O(k^{-n-2}))$$

The differential $d\Omega_{2n}$ is odd, $d\Omega_{2n}(k) = -d\Omega_{2n}(-k)$. Equation (3.9) implies that $(\partial_x^n \psi \partial_x^n \psi_{y=0}^*) d\Omega_{2n}$ is also odd. The residue of an odd differential is zero. Then using (2.37) with $i = m = n$ we get

$$(3.21) \quad 0 = \text{res}_{k=\infty} (\partial_x^n \psi \partial_x^n \psi_{y=0}^* |_{y=0}) d\Omega = (-1)^{n+1} c_{2n+1}$$

Contradiction. The corollary is proved.

Introduce new local coordinate $k' = k + O(k^{-1})$ by the equation

$$(3.22) \quad dk' = d\Omega(k)$$

From (3.19) it follows that this change preserves the equation (3.9), i.e.

$$(3.23) \quad \psi^*(x, 0, k') = \psi(x, 0, -k')$$

For further use note that in the new coordinate the residue equation for the BA function and its dual has the standard form

$$(3.24) \quad \text{res}_{k=\infty} \psi^*(x, 0, k') (\partial_x^i \psi(x, 0, k')) dk' = 0$$

Corollary 3.2. *Let \mathcal{L} be a unique pseudo-differential operator of the form (2.1) such that the equation*

$$(3.25) \quad \mathcal{L}\psi(x, 0, k') = k'\psi(x, 0, k')$$

holds. Then the equation (2.8), i.e. $\mathcal{L}^ = -\mathcal{L}$, holds.*

Proof. The proof is standard in the KP theory. Introduce, the so-called wave pseudo-differential operator $\Phi = 1 + O(\partial^{-1})$ by the equation

$$(3.26) \quad \psi(x, 0, k') = \Phi e^{k'x}$$

Then

$$(3.27) \quad \mathcal{L} = \Phi \cdot \partial_x \cdot \Phi^{-1}$$

It is known that (3.24) implies

$$(3.28) \quad \psi^*(x, 0, k') = (\Phi^{-1})^* e^{-k'x}$$

(see in [6]). Using (3.23) we get $\Phi^* = \Phi^{-1}$. The latter and (3.27)) implies (2.8). The corollary is proved.

Now we are going to use few more standard facts from the KP theory and the related theory of commuting ordinary differential operators. Let $B_n = \mathcal{L}_+^n$ be, as before, the differential part of the operator \mathcal{L}^n . From (2.8) it follows that

$$(3.29) \quad B_n^* = (-1)^n B_n$$

The commuting flows of the KP hierachy (2.2) correspond to linear flows on the Jacobian. Hence, among the two sets of flows corresponding to even or odd n there are only finite number of linear independent. Hence, for all but a finite number of integers n there exists a linear operator

$$(3.30) \quad L_n = B_n + \sum_{m=1}^{[n/2]} c_{n,m} B_{n-2m}$$

such that

$$(3.31) \quad L_n \psi(x, 0, p) = a_n(p) \psi(x, 0, p)$$

where $a_n(p)$ is a meromorphic function on Γ with the only pole at the marked point P . The operator L_n is self-adjoint for even n and anti self-adjoint for odd n . In both cases we have

$$(3.32) \quad L_n \psi^*(x, 0, p) = a_n^*(p) \psi^*(x, 0, p)$$

where $a_n^*(p)$ is a meromorphic function of Γ with the only pole at P of order n . Since $\psi^*(x, 0, k') = \psi(x, 0, -k')$, the expansions of a_n and a_n^* near P in k' are related by the equation

$$a_n^*(k') = (-1)^n a_n(-k')$$

The involution $a_n \rightarrow a_n^*$ of the ring $\mathcal{A}(\Gamma, P)$ of meromorphic functions on Γ with the only pole at P defines the involution $\sigma : \Gamma \rightarrow \Gamma$ under which the local parameter $k^{-1}(p)$ is odd, $k'(\sigma(p)) = -k'(p)$. The theorem is proved.

4. PROOF OF THEOREM 1.2

To begin with, recall necessary facts from the algebraic-geometrical construction of solutions of the $2D$ Toda hierarchy.

Two-point Baker-Akhiezer function. Let Γ be a smooth genus g algebraic curve with fixed coordinates $k_\alpha^{-1}(p)$ in the neighborhoods of two marked points P_α , $\alpha = 1, 2$. The two-point Baker-Akhiezer function ψ depends on the variables $(x, \mathbf{t} = (t_{\alpha, i > 0}))$ (as always, it is assumed that only a finite number of them are not zero). For fixed (x, \mathbf{t}) and a non-special effective degree g divisor D it is defined as a function of $p \in \Gamma$ with the following analytical properties.

On $\Gamma \setminus \{P_1, P_2\}$ the function ψ is a multi-valued meromorphic function with poles at D whose order at a point of D is not greater than the multiplicity of the point in D , i.e. $(\psi) + D \geq 0$. The multi-valuedness of ψ is due to the only not trivial monodromies around the marked points P_1 and P_2 which are equal to $e^{2\pi i x}$ and $e^{-2\pi i x}$, respectively (for integer x the function ψ is single-valued).

In the neighborhood of P_1 it has the form:

$$(4.1) \quad \psi(x, \mathbf{t}, p) = k_1^x \exp\left(\sum_{i=1}^{\infty} t_{1,i} k_1^i\right) \left(1 + \sum_{s=1}^{\infty} \xi_{1,s}(x, \mathbf{t}) k_1^{-s}\right),$$

and in the neighborhood of P_2

$$(4.2) \quad \psi(x, \mathbf{t}, p) = k_2^{-x} e^{f(x, \mathbf{t})} \exp\left(\sum_{i=1}^{\infty} t_{2,i} k_2^i\right) \left(1 + \sum_{s=1}^{\infty} \xi_{2,s}(x, \mathbf{t}) k_1^{-s}\right),$$

As shown in [29] the Baker-Akhiezer function satisfies the linear equations

$$(4.3) \quad (\partial_{1,1} - T - u)\psi = 0, \quad (\partial_{2,1} - wT^{-1})\psi = 0,$$

where

$$(4.4) \quad u = \partial_{1,1} f = (1 - T) \xi_{1,1}, \quad w = \exp((1 - T^{-1})f),$$

The compatibility condition (4.3) is equivalent to the $2D$ Toda equation

$$(4.5) \quad \partial_{\xi} \partial_{\eta} f_n = e^{f_n - f_{n-1}} - e^{f_{n+1} - f_n},$$

with $x = n$, $\xi = t_{1,1}$ and $\eta = t_{2,1}$.

The explicit formula for f follows from the theta-functional formula for the Baker-Akhiezer functions:

$$(4.6) \quad \psi = \frac{\theta(A(P) + xU + \sum U_{\alpha,i} t_{\alpha,i} + Z) \theta(Z)}{\theta(xU + \sum U_{\alpha,i} t_{\alpha,i} + Z) (\theta(A(P) + Z))} e^{(x\Omega_0 + \sum t_{\alpha,i} \Omega_{\alpha,i}(P))},$$

Here Ω_0 is an Abelian integral of the normalized meromorphic differential with simple poles at P_1, P_2 and residues -1 and 1 respectively. From the bilinear Riemann identities it follows that

$$(4.7) \quad U = A(P_2) - A(P_1)$$

where $2\pi iU$ is the vector of b -periods of $d\Omega_0$. The Abelian integrals $\Omega_{\alpha,i}$ and the corresponding vectors $U_{\alpha,i}$ are defined similarly to that in (2.12): $d\Omega_{\alpha,i}$ is the normalized meromorphic differential with the pole at P_{α} of the form $d(k_{\alpha}^i + O(k_{\alpha}^{-1}))$. The corresponding Abelian integrals are normalized such that their expansions near P_1 are of the form:

$$(4.8) \quad \Omega_{1,i} = k^i + \sum_{s=1}^{\infty} a_{1,s}^{(1,i)} k_1^{-s}, \quad \Omega_{2,i} = \sum_{s=1}^{\infty} a_{1,s}^{(2,i)} k_1^{-s}$$

and near P_2

$$(4.9) \quad \Omega_{1,i} = \sum_{s=0}^{\infty} a_{2,s}^{(1,i)} k_2^{-s}, \quad \Omega_{2,i} = k_2^i \sum_{s=1}^{\infty} a_{2,s}^{(2,i)} k_2^{-s}$$

The expansions of Ω_0 are of the form:

$$(4.10) \quad \Omega_0 = \ln k_1 + \sum_{s=1}^{\infty} a_{1,s}^{(0)} k_1^{-s}, \quad \Omega_0 = -\ln k_2 + \sum_{s=1}^{\infty} a_{2,s}^{(0)} k_2^{-s}$$

In the definition of the Abel transform we chose the normalization $A(P_1) = 0$. Then using (4.7) we get

$$(4.11) \quad f = (T - 1) \ln \theta + x a_{2,0}^{(0)} + \sum t_{\alpha,i} a_{2,0}^{(\alpha,i)},$$

$$\theta := \theta(xU + \sum U_{\alpha,i} t_{\alpha,i} + Z)$$

For further use notice that (4.6) and the expansion (2.19) of the Abel transform near P_1 imply

$$(4.12) \quad \xi_{1,1} = -\partial_{1,1} \ln \theta + x a_{1,1}^{(0)} + \sum t_{\alpha,i} a_{1,1}^{(\alpha,i)}$$

In what follows we restrict ourselves to the case the first three variables $x, t_{1,1}, t_{2,1}$ by setting $t_{\alpha,i>1} = 0$. Introduce the variables y and t such that $t_{11} = y + t$, $t_{2,1} = -t$.

In the new variables equations (4.3) take the form:

Lemma 4.1. *The Baker-Akhiezer function $\psi(x, y, t, p)$ given by formula (4.6) satisfies the equation*

$$(4.13) \quad (\partial_y - T - u) \psi = 0, \quad T = e^{\partial_x}$$

with

$$(4.14) \quad u = b_1 + (T - 1) \partial_y \ln \theta, \quad b_1 := a_{1,1}^{(0)}$$

and the equation

$$(4.15) \quad (\partial_t - T - u - wT^{-1}) \psi = 0$$

with

$$(4.16) \quad w = b_2 \frac{T\theta T^{-1}\theta}{\theta^2}, \quad b_2 := e^{a_{2,0}^{(0)}}$$

The substitution of (4.1) into (4.13) gives a system of the equations

$$(4.17) \quad \partial_y \xi_{1,s} = (T - 1) \xi_{1,s+1} + u \xi_{1,s}, \quad s = 0, 1, 2, \dots$$

The first of them gives an expression of u in terms of $\xi_{1,1}$

$$(4.18) \quad (1 - T) \xi_{1,1} = u = b_1 + (T - 1) \partial_y \ln \theta$$

Similarly, the substitution of (4.2) into (4.13) gives:

$$(4.19) \quad \partial_y \xi_{2,s} = e^{(T-1)f} (T \xi_{2,s-1}) = b_2 \frac{T^2 \theta \cdot \theta}{(T\theta)^2} (T \xi_{2,s-1}), \quad s = 0, 1, 2, \dots$$

where for the last equation we use (4.11).

For further use introduce the function

$$(4.20) \quad \phi(z, k_2) := \frac{\theta(A(k_2) + \iota(z)) \theta(Z)}{\theta(\iota(z)) \theta(A(k_2) + Z)} e^{\ell(k,z)}, \quad z \in \mathbb{C}^d,$$

where $A(k_2)$ is the expansion of the Abel transform near P_2 and $\ell(k_2)$ is the formal series

$$(4.21) \quad \ell(k_2) = \sum_{s=0}^{\infty} \ell_s k_2^{-s}, \quad \ell_s \in \mathbb{C}^d,$$

such that the equation

$$(4.22) \quad (\ell(k_2), \eta) = \Omega_0(k_2) + \ln k_2,$$

where $\Omega(k_2)$ is the expansion of the Abelian integral Ω_0 at P_2 , holds.

A simple comparison of (4.6) and (4.20) shows that the equation

$$(4.23) \quad \xi_s(x, 0) = \varphi_s(x\eta)$$

where φ_s are the coefficients of the expansion of ϕ . The latter are of the form

$$(4.24) \quad \varphi_s(z) = \frac{\tau_{2,s}(z)}{\tau(z + \eta)}, \quad \tau(z) := \theta(\iota(z))$$

where $\tau_{2,s}(z)$ is a holomorphic function of z .

The monodromy properties of the theta-function imply that under the shift of the arguments by a vector $\lambda \in \Lambda_U$ the function ϕ gets transformed to

$$(4.25) \quad \phi(z + \lambda, k_2) = \phi(z, k_2) \rho_\lambda(k_2), \quad \rho_\lambda(k_2) = 1 + \sum_{s=1}^{\infty} \rho_{\lambda,s} k_2^{-s}$$

Equation (2.31) is equivalent to the equations

$$(4.26) \quad \varphi_s(z + \lambda) = \sum_{j=0}^s \varphi_{s-j}(z) \rho_{\lambda,j}$$

The same arguments as in the proof of Lemma 2.2 prove:

Lemma 4.2. *Let $\lambda_1, \dots, \lambda_d$ be a set of linear independent vectors in Λ_U . Then there is a unique coordinate $k_2^{-1}(p)$ in the neighborhood of P_2 and a unique linear form $\ell(k_2)$ satisfying (4.22) such that $\phi(z, k_2)$ is invariant under the shift by these vectors, i.e.,*

$$(4.27) \quad \phi(z + \lambda_i, k_2) = \phi(z, k_2) \Leftrightarrow \rho_{\lambda_i}(k_2) = 1, \quad i = 1, \dots, d.$$

Dual Baker-Akhiezer function. As in the one-point case the definition of the dual two-point Baker-Akhiezer function begins with the definition of the dual divisor. Let D be an effective degree g divisor. In the two point case the dual divisor D^* is defined as the divisor such that $D + D^*$ is the zero divisor of the meromorphic differential $d\Omega$ with simple poles at the marked points, i.e.,

$$(4.28) \quad D + D^* = K + P_1 + P_2 \in J(\Gamma)$$

In what follows it will be assumed that $d\Omega$ is normalized such that it has residue -1 at P_1 .

The dual Baker-Akhiezer function $\psi^*(x, \mathbf{t}, p)$ as a function of $p \in \Gamma$ has the following analytic properties: away from the marked points it is a multi-valued meromorphic function with the pole divisor D^* . Its only non-trivial monodromy factors are $e^{-2\pi i x}$ around P_1 and $e^{2\pi i x}$ around P_2 .

In the neighborhood of P_1 it has the form:

$$(4.29) \quad \psi^*(x, \mathbf{t}, p) = k_1^{-x} \exp\left(\sum_{i=1}^{\infty} -t_{1,i} k_1^i\right) \left(1 + \sum_{s=1}^{\infty} \xi_{1,s}^*(x, \mathbf{t}) k_1^{-s}\right),$$

and in the neighborhood of P_2

$$(4.30) \quad \psi^*(x, \mathbf{t}, p) = k_2^x e^{f^*(x, \mathbf{t})} \exp\left(\sum_{i=1}^{\infty} -t_{2,i} k_2^i\right) \left(1 + \sum_{s=1}^{\infty} \xi_{2,s}^*(x, \mathbf{t}) k_2^{-s}\right),$$

The explicit theta-functional formula for the

$$(4.31) \quad \psi^* = \frac{\theta(A(P) - (x+1)U - \sum U_{\alpha,i} t_{\alpha,i} - Z) \theta(Z - U)}{\theta((x+1)U + \sum U_{\alpha,i} t_{\alpha,i} + Z) (\theta(A(P) - Z - U))} e^{-x\Omega_0 - \sum t_{\alpha,i} \Omega_{\alpha,i}(P)}$$

where we use the equality $Z^* = -Z - U$ which follows from (4.28) and (4.7).

From the definition of the dual divisor it is easy to see that the equations

$$(4.32) \quad \text{res}_{P_1} (\psi^* T^i \psi) d\Omega = -\delta_{i,0}$$

hold. Indeed the differential in the r.h.s. of (4.32) is a meromorphic differential on Γ which is holomorphic away from the marked points. For $i > 0$ it is holomorphic at P_2 . Hence, it has no residue at P_1 . Equation (4.32) for $i = 0$ follows from (4.1) and (4.29) and the normalization of $d\Omega$ defined above.

Notice, that (4.32) with $i = 0$ implies the equation

$$(4.33) \quad f(x, \mathbf{t}) = -f^*(x, \mathbf{t})$$

and with $i = 1$ implies the equation

$$(4.34) \quad \xi_{1,1}(x+1, \mathbf{t}) + \xi_{1,1}^*(x, \mathbf{t}) = 0$$

Equation (4.33) implies that the dual BA function satisfies the equation formally adjoint to (4.13)

$$(4.35) \quad (\partial_y + T^{-1} + u)\psi^* = 0$$

The substitution of (4.29) into (4.35) gives the equations

$$(4.36) \quad -\partial_y \xi_{1,s}^* = (T^{-1} - 1)\xi_{1,s+1}^* + u\xi_{1,s}^*, \quad s = 0, 1, \dots$$

For further use introduce the function

$$(4.37) \quad \phi^*(z, k_1) := \frac{\theta(A(k_1) - \iota(z + \eta)) \theta(Z + U)}{\theta(\iota(z + \eta)) \theta(A(k_1) - Z - U)} e^{-(\ell_1(k_1), z)}, \quad z \in \mathbb{C}^d,$$

where $A(k_1)$ is the expansion of the Abel transform near P_1 and $\ell_1(k_1)$ is the formal series

$$(4.38) \quad \ell(k_1) = \sum_{s=1}^{\infty} \ell_{1,s} k_1^{-s}, \quad \ell_{1,s} \in \mathbb{C}^d,$$

such that the equation

$$(4.39) \quad (\ell_1(k_1), \eta) = \Omega_0(k_1) - \ln k_1,$$

where $\Omega_0(k_1)$ is the expansion of the Abelian integral Ω_0 at P_1 , holds.

A simple comparison of (4.31) and (4.37) shows that the coefficients of the dual Baker-Akhiezer function expansion at P_1 are equal to

$$(4.40) \quad \xi_{1,s}^*(x, 0) = \varphi_s^*(x\eta)$$

where φ_s^* are the coefficients of the expansion of ϕ^* . The latter are of the form

$$(4.41) \quad \varphi_s^*(z) = \frac{\tau_{1,s}^*(z)}{\tau(z + \eta)}, \quad \tau(z) := \theta(\iota(z))$$

where $\tau_{2,s}^*(z)$ is a holomorphic function of z .

The monodromy properties of the theta-function imply that under the shift of the arguments by a vector $\lambda \in \Lambda_U$ the function ϕ^* gets transformed to

$$(4.42) \quad \phi^*(z + \lambda, k_1) = \phi(z, k_2) \rho_\lambda^*(k_1), \quad \rho_\lambda^*(k_1) = 1 + \sum_{s=1}^{\infty} \rho_{\lambda,s}^* k_1^{-s}$$

Equation (4.42) is equivalent to the equations

$$(4.43) \quad \varphi_s^*(z + \lambda) = \sum_{j=0}^s \varphi_{s-j}^*(z) \rho_{\lambda,j}^*$$

The same arguments as in the proof of Lemma 2.2 prove:

Lemma 4.3. *Let $\lambda_1, \dots, \lambda_d$ be a set of linear independent vectors in Λ_U . Then there is a unique coordinate $k_1^{-1}(p)$ in the neighborhood of*

P_1 and a unique linear form $\ell_1(k_1)$ satisfying (4.39) such that ϕ^* is invariant under the shift by these vectors, i.e.,

$$(4.44) \quad \phi^*(z + \lambda_i, k_1) = \phi^*(z, k_1) \quad \Leftrightarrow \quad \rho_{\lambda_i}^*(k_1) = 1, \quad i = 1, \dots, d.$$

Proof of the theorem. We begin the proof of the theorem by pointing on the obstacle which makes our approach for the characterization of the Jacobian of curves with involution non-applicable for the case of non-ramified covers.

Lemma 4.4. *Let Γ be a curve with involution having fixed points. Then for the differential $d\Omega_0$ defined above, i.e., the Abelian differential with simple poles at the marked points P_1, P_2 , which are permuted by involution, and normalized by the conditions $\oint_{\alpha_k} d\Omega_0 = 0$, the equation*

$$(4.45) \quad d\Omega_0(\sigma(p)) = -d\Omega_0(p), \quad p \in \Gamma$$

holds.

Remark 8. The author have not been able to find a reference to this statement in the literature, but he has no doubt that it is well-known. In any case we present a proof of it, since its arguments will be useful below.

Proof. Following the line of the proof of Lemma 3.3 in [8] it is easy to show that there are integers l_k such that the equation

$$(4.46) \quad d\Omega_0(\sigma(p)) = -d\Omega_0(p) + 2\pi i l_k \omega_k, \quad l_k \in \mathbb{Z},$$

where ω_k are normalized holomorphic differentials, holds.

On any compact $\bar{\Gamma} \in \{\Gamma \setminus P_1, P_2\}$ the differential $d\Omega_0$ depends continuously on the marked points $P_1, P_2 = \sigma(P_1)$. Hence, integers l_k do not depend on P_1 . In the limit $P_1 \rightarrow P$, where P is a fixed point of involution, the residues of $d\Omega_0$ cancel each other. The latter implies that on $\bar{\Gamma}$ the differential $d\Omega_0$ tends to zero uniformly. Then comparing the limits of two sides of (4.46) we get $l_k = 0$. The lemma is proved.

Remark 9. The statement of the lemma above fails to be true for the case of unramified covers.

Lemma 4.5. *Suppose that there is an involution $\sigma : \Gamma \rightarrow \Gamma$ with fixed points under which the marked points are permuted, $\sigma(P_1) = P_2$ and the local coordinates in the neighborhoods of marked points are chosen such that $k_1(p) = k_2(\sigma(p))$. Then the equation*

$$(4.47) \quad (\theta^2 \partial_y^2 \ln \theta - b_2 T \theta T^{-1} \theta) \Big|_{y=0} = b_3$$

where b_3 is a constant and $\theta = \theta(xU + yV + tW + \zeta)$ with ζ such that $\zeta + \zeta^\sigma = K + P_1 + P_2$, where K is the canonical class, holds.

Proof. Let ψ be the BA function defined by a divisor D such that $D + D^\sigma$ is the zero divisor of a meromorphic differential $d\tilde{\Omega}$ with simple poles at P_1 and P_2 . The vector ζ above is equal to $\zeta = -A(D) + \mathcal{K}$, where \mathcal{K} is the vector of Riemann constants.

Consider the differential $d\hat{\Omega} := (T\psi)\psi^\sigma|_{y=0}d\tilde{\Omega}$. It is holomorphic outside the marked points, since poles of ψ and $T\psi^\sigma$ cancel out with zeros of $d\tilde{\Omega}$. At the marked points essential singularities of $T\psi$ and ψ^σ cancel each other. The multi-valuedness of the BA function requires additional consideration. Notice that the differential $d\Omega_{1,1} - d\Omega_{2,1}$ is odd with respect to the involution. Then from (4.6) and (4.45) it follows that $d\hat{\Omega}$ is *single-valued*, i.e. it is a meromorphic differential on Γ with the only pole at P_1 . Hence, it has no residue at that point. Computing the residue in terms of the coefficients of expansions (4.1) and (4.2) we get the equation

$$(4.48) \quad \xi_{1,1}(x+1, 0, t) + \xi_{2,1}(x, 0, t) = 0$$

Then from equation (4.19) with $s = 1$ and equation (4.12) it follows that equation (4.47) holds. The lemma is proved.

Equation (4.47) implies equation(1.20), since by definition $\pi(Ux + Wt + \zeta)$ is dense in \hat{Y} . Note also, that equation (4.47) is equivalent to equation (1.15). Indeed, substituting (4.2) into (4.15) and comparing the leading coefficients of the expansions at P_2 we get

$$(4.49) \quad u = \partial_t f + (T^{-1} - 1)\xi_{2,1} = \partial_t f + (T - 1)\xi_{1,1} = \partial_t f - u$$

where in the last two equalities we use (4.48) and (4.4), respectively. From (4.49) it follows that

$$(4.50) \quad 2u = 2\partial_y f|_{y=0} = \partial_t f$$

Equation (1.15) is proved.

Our next goal is to prove the condition (B) of the theorem. The explicit meaning of part (i) of (B) is as follows: generic zero $x = q$ of $\tau(x\eta + z)$ with $z \in \pi^{-1}(Y)$ is simple, and $\tau((q+1)\eta + z)\tau((q-1)\eta + z) \neq 0$. Here and below $\tau(z, y, t) := \theta(\iota(z) + Vy + Wt)$, and for brevity we will write $\tau(z, y) := \tau(z, y, 0)$ and $\tau(z, t) := \tau(x, 0, t)$, and $\tau(z) = \tau(z, 0, 0)$.

Remark 10. As it was mentioned in Introduction the author was unable to complete a proof that equations (1.15) and (1.20) imply (B) although believes that it is true. What we were not able to exclude is the possibility that $\Theta \cap Y$ is a union of two components $(\Theta \cap Y)_\pm$ on

which zeros of τ are of multiplicity 2, and $(\Theta \cap Y)_- + U = (\Theta \cap Y)_+$. The proof below uses an extension of arguments in the proof of Lemma 4.4 and the result of the previous Section.

Consider in more details the differential $d\Omega_0$ when P_1 is close to a fixed point P of the involution. Let z be an odd with respect to σ coordinate in the neighborhood of P , $z(P) = 0$. Let $z_0 = z(P_1)$ then in the neighborhood of P the differential $d\Omega_0$ has the form

$$(4.51) \quad d\Omega_0 = \frac{dz}{z + z_0} - \frac{dz}{z - z_0} + R(z, z_0)dz$$

where $R(z, z_0)$ is even holomorphic function of z (in the neighborhood of $z = 0$) and odd function of z_0 (uniformly bounded by z_0). Equation (4.51) obviously shows that $d\Omega_0 \rightarrow 0$ as $z_0 \rightarrow 0$. More importantly, it shows that $(2z_0)^{-1}d\Omega_0 \rightarrow d\Omega_1$, where $d\Omega_1$ is the normalized differential with the pole of order 2 at P which we used in the previous section.

Change of notations. For the duration of these arguments we use notation $2\pi i U_0$ and $2\pi i U_1$ for the vectors of b -periods of the differentials $d\Omega_0$ and $d\Omega_1$ in order to avoid a confusion caused by use the same notation U for them in Theorems 1.1 and 1.2. Similarly, we will use the notations Y_0 and Y_1 for the closers of $\pi(U_0x + \zeta_0)$ and $\pi(U_1x + \zeta_1)$ in $J(\Gamma)$, respectively (which both were denoted above by Y).

The convergence $(2z_0)^{-1}d\Omega_0 \rightarrow d\Omega_1$ is uniform on any compact $\bar{\Gamma} \subset (\Gamma \setminus P)$. The latter implies the convergence of the vectors $(2z_0)^{-1}U_0 \rightarrow U_1$ and as a corollary the convergence $Y_0 \rightarrow Y_1$.

As it was shown in the proof of Theorem 1.1 the intersection $\Theta \cap Y_1$ is reduced. The latter condition is open. Hence, for P_1 sufficiently closed to P the $\Theta \cap Y_0$ is also reduced. Then, using again openness of the latter condition, we conclude that $\Theta \cap Y_0$ is reduced for all but at most finite number of points $P_1 \in \Gamma$. Upper continuity of multiplicity of zeros of entire function depending on parameters implies that $\Theta \cap \widehat{Y}_0$, where \widehat{Y}_0 is the closure of $\pi(U_0x + Wt + \zeta_0)$, is also reduced.

If $\tau(x, 0, t)$ has simple zero at $x = q(t)$, then from equations (4.47) and (4.50) it follows the equations

$$(4.52) \quad (\partial_t q(t))^2 = -2b_2 \frac{\tau(q+1, 0, t)\tau(q-1, 0, t)}{\tau_x^2(q, 0, t)}$$

Without loss of generality (see Remark 5) we may assume that the vector $W \neq 0$. Hence, generically the left hand side of (4.52) is not zero. Hence, generically, the r.h.s. of the equation does not vanish. That is the statement (B) of the theorem. The "only if" part of the theorem is proved.

The following lemma is central in the proof of the "if" part of the theorem.

Lemma 4.6. *Suppose that equation (1.15) is satisfied. Then for the coefficients of the expansions (4.2), (4.29) of the Baker-Akhiezer function and its dual in the coordinates k_1^{-1} and k_2^{-1} in the neighborhoods of the marked points defined in Lemmas 4.2 and 4.3 for the same set of d independent vectors $\lambda_i \in \Lambda_U$ the equations*

$$(4.53) \quad \xi_{1,s}^*(x, 0) = \xi_{2,s}(x, 0), \quad s = 0, 1, \dots$$

hold.

Proof. By (4.23) and (4.40) equations (4.53) are corollaries of the equations

$$(4.54) \quad \varphi_s(z) = \varphi_s^*(z), \quad s = 0, 1, \dots$$

We are going to prove (4.54) by induction. The initial step of the induction is the equation $\varphi_0 = \varphi_0^* = 1$. Suppose that equation (4.54) is satisfied for $s \leq n$.

Compute the residue r_{n+1} of $\xi_{2,n+1}(x, 0)$ at the zero q of $\tau(x + 1)$. Comparing the coefficient at $(x - q)^{-2}$ of the left and right sides of equation (4.19) we get

$$(4.55) \quad -((\partial_y q)|_{y=0})r_{n+1} = b_2 \frac{\tau(q+2)\tau(q)}{\tau_x^2(q+1)} \xi_{2,n}(q+1, 0)$$

Compute now the residue r_{n+1}^* of $\xi_{1,n+1}^*$. Consider the equation (4.36). The l.h.s. of the equation and the second term of the r.h.s. are regular at $x = q + 1$. Two remaining terms have simple poles at $x = q + 1$. Comparing their residues at $x = q + 1$ we get the equation

$$(4.56) \quad r_{1,n+1}^* = ((\partial_y q)|_{y=0})\xi_{1,n}^*(q+1, 0)$$

By induction assumption $\xi_{2,n}(q+1, 0) = \xi_{1,n}^*(q+1, 0)$. Therefore, equations (4.55), (4.56) and (4.52) imply

$$(4.57) \quad r_{2,n+1} = r_{1,n+1}^*$$

From (4.57) it follows that the function $F_{n+1}(z) := \varphi_{n+1}(z) - \varphi_{n+1}^*(z)$ restricted to the line $z = x\eta$ is holomorphic. Then the same arguments as in the previous section prove that $F_{n+1} = 0$. The induction step is completed. The lemma is proved.

Note that if we identify the local coordinates k_1^{-1} and k_2^{-1} then equations (4.53) take the form

$$(4.58) \quad \psi_2(x, 0, k) = e^{f(x,0)}\psi_1^*(x, 0, k)$$

The final steps in the proof of Theorem 1.2 are literally identical to those in the proof of Theorem 1.1, after changing the facts from KP theory on their analogues in $2D$ Toda theory. Let us briefly outline them.

First, introduce the pseudo-difference operators

$$(4.59) \quad \mathcal{L}_1 = \sum_{i=-1}^{\infty} u_{1,i} T^{-i}, \quad \mathcal{L}_2 = \sum_{i=-1}^{\infty} u_{2,i} T^i$$

such that the equations

$$(4.60) \quad \begin{aligned} \mathcal{L}_1 (e^{-f/2} \psi(x, 0, k_1)) &= k_1 e^{-f/2} \psi(x, 0, k_1), \\ \mathcal{L}_2 (e^{-f/2} \psi(x, 0, k_2)) &= k_2 e^{-f/2} \psi(x, 0, k_2) \end{aligned}$$

hold. Then from (4.53) or equivalently (4.58) it follows that the equation

$$(4.61) \quad \mathcal{L}_1 = \mathcal{L}_2^*$$

holds. Recall, that in the difference case the formal adjoint operator is defined by the rule $(w \cdot T^i)^* = T^{-i} \cdot w$ and then extended by linearity.

Define difference operators

$$(4.62) \quad B_{1,m} = ((\mathcal{L}_1^m)_{>0}) + \frac{1}{2} ((\mathcal{L}_1^m)_0), \quad B_{2,m} = (\mathcal{L}_2^m)_{<0} + \frac{1}{2} ((\mathcal{L}_2^m)_0).$$

where $(\cdot)_{>0}$, $(\cdot)_{<0}$ denote the positive and negative parts of the pseudo-differential operator, respectively, and $(\cdot)_0$ denotes its zero-order term.

The equations of $2D$ Toda hierarchy (in the so-called symmetric gauge, see [30] for details)

$$(4.63) \quad \partial_{\alpha,m} \mathcal{L}_\beta = [B_{\alpha,m}, \mathcal{L}_\beta], \quad \alpha, \beta = 1, 2, \quad m = 1, 2, \dots$$

define commuting flows on the space of pairs of pseudo-difference operators of the form (4.59). These flows correspond to linear flows on the Jacobian. Hence, among them there is only finite number of linear independent ones. Therefore, for all but a finite number of integers n there are constants $c_{n,m}^\pm$ such that the corresponding linear combination of these flows defined by the operators

$$(4.64) \quad L_n^\pm = \sum_{i=1}^m c_{n,m}^\pm (B_{1,m} \pm B_{2,m})$$

is trivial. From (4.61) it follows that $(L_n^\pm)^* = \pm L_n^\pm$. The latter implies the equations

$$(4.65) \quad L_n^\pm \psi(x, 0, p) = a_n^\pm(p) \psi(x, 0, p), \quad L_n^\pm \psi^*(x, 0, p) = \pm a_n^\pm(p) \psi^*(x, 0, p),$$

where $a_n^\pm(p)$ are meromorphic functions on Γ with the only poles of order n at the marked points P_1, P_2 , hold. Points of the curve Γ parameterise common eigenfunctions of the commuting difference operators L_n^\pm (see [31, 32]). Hence, the correspondence $\psi \rightarrow \psi^*$ defines an involution σ of Γ under which the equation

$$(4.66) \quad \psi(x, 0, p) = \psi^*(x, 0, \sigma(p))$$

hold. The theorem is proved.

REFERENCES

- [1] E. Arbarello, *Survey of Work on the Schottky Problem up to 1996*. Added section to the 2nd edition of Mumford's Red Book, pp. 287–291, 301–304, Lecture Notes in Math. 1358, Springer, 1999.
- [2] O. Debarre, *The Schottky problem: an update*. In: Current topics in complex algebraic geometry (Berkeley, CA, 1992/93); pp. 57–64. (H. Clemens and J. Kollár, eds.) MSRI Publ. 28, Cambridge Univ. Press, Cambridge, 1995.
- [3] I. Taimanov, *Secant of abelian varieties, theta-functions and soliton equations*. Russian Math. Surveys, 52 (1997), no. 1, 149–224.
- [4] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*. Invent. Math., 83(2), 333–382, 1986.
- [5] G.E. Welters, *A criterion for Jacobi varieties*. Ann. of Math., 120 (1984), no. 3, 497–504.
- [6] I. Krichever, *Integrable linear equations and the Riemann-Schottky problem*. In: Algebraic Geometry and Number Theory, Birkhäuser, Boston, 2006.
- [7] I. Krichever, *Characterizing Jacobians via trisecants of the Kummer Variety*. Ann. of Math. **172** (2010), 485–516.
- [8] S. Grushevsky, I. Krichever, *Integrable discrete Schrödinger equations and a characterization of Prym varieties by a pair of quadrisecants*. Duke Mathematical Journal, 152 (2010), no 2, 318–371.
- [9] I. Krichever, *A characterization of Prym varieties*. Int. Math. Res. Not. 2006, Art. ID 81476, 36 pp.
- [10] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *KP hierarchy of orthogonal and symplectic type – Transformation groups for soliton equations VI*, J. Phys. Soc. Japan **50** (1981) 3813–3818.
- [11] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in “Nonlinear Integrable Systems – Classical Theory and Quantum theory”, M. Jimbo and T. Miwa (eds.), World Sci., Singapore, 1983, pp. 39–119.
- [12] A.P. Veselov, S.P. Novikov, *Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations*, Dokl. Akad. Nauk SSSR, 279:1 (1984), 20–24 .
- [13] A.P. Veselov, S.P. Novikov, *Finite-zone, two-dimensional Schrödinger operators. Potential operators*, Dokl. Akad. Nauk SSSR, 279:4 (1984), 784–788 .
- [14] E. Arbarello, I. Krichever, G. Marini, *Characterizing Jacobians via flexes of the Kummer Variety*, Math. Res. Lett. **13** (2006), 1, 109–123.
- [15] A. Beauville, *Vanishing thetanulls on curves with involutions*, Rend. Circ. Mat. Palermo (2) **62** (2013), no. 1, 61–66.

- [16] C. Poor, *The hyperelliptic locus*, Duke Math. J. 76 (1994), no. 3, 809–884.
- [17] H. Farkas, S. Grushevsky, R. Salvati Manni, *An explicit solution to the weak Schottky problem*, Algebr. Geom. 8 (2021), no. 3, 358–373.
- [18] F. Schottky, H. Jung, *Neue Sätze über Symmetriefunktionen und die Abel'schen Funktionen der Riemann'schen Theorie*. S.-B. Preuss. Akad. Wiss. Berlin; Phys. Math. Kl. 1 (1909) 282–297.
- [19] I. Krichever, T. Shiota, *Soliton equations and the Riemann-Schottky problem* - in Advanced lectures in Mathematics, v. 26, Handbook of Moduli, v.II, (eds. G.Farcas, I. Morrison), International Press, 2013
- [20] Enrico Arbarello, Giulio Codogni, Giuseppe Pareschi, *Characterizing Jacobians via the KP equation and via flexes and degenerate trisecants to the Kummer variety: an algebro-geometric approach*, arXiv:2009.14324
- [21] I.Krichever, N.Nekrasov, *The Novikov-Veselov hierarchy and $O(N)$ sigma model*, in preparation
- [22] I.Krichever, A.Zabrodin, *Kadomtsev-Petviashvili turning points and CKP hierarchy*, arXiv:2012.04482
- [23] J.D. Fay, *Theta functions on Riemann surfaces*, Lect. Notes Math., Vol. 352, Berlin, Heidelberg, New York: Springer, 1973.
- [24] V.E. Zakharov and A.B. Shabat, *A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem I*, Funk. Anal. i ego Pril. **8:3** (1974) 43–53.
- [25] M. Sato, *Soliton equations and the universal Grassmann manifold*, Math. Notes. Series **18** (1984), Sophia University, Tokyo.
- [26] M. Jimbo and T. Miwa, *Soliton equations and infinite dimensional Lie algebras*, Publ. RIMS, Kyoto University **19** (1983) 943–1001.
- [27] I. M. Krichever, *Integration of non-linear equations by methods of algebraic geometry*, Funct. Anal. Appl., 11 (1977), n. 1, 12–26.
- [28] I. M. Krichever, *Methods of algebraic geometry in the theory of non-linear equations*, Russian Math. Surveys, 32 (1977), n. 6, 185–213.
- [29] I. Krichever, *The periodic nonabelian Toda lattice and two-dimensional generalization*. appendix to: B. Dubrovin, *Theta-functions and nonlinear equations*, Uspekhi Mat. Nauk **36**, no 2 (1981) 72–77.
- [30] I. Krichever, A. Zabrodin, *Constrained Toda hierarchy and turning points of the Ruijsenaars-Schneider model*, arXiv:2109.05240.
- [31] D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related non-linear equations* – Proceedings Int.Symp. Algebraic Geometry, Kyoto, 1977, 115–153, Kinokuniya Book Store, Tokyo, 1978.
- [32] I.Krichever, *Algebraic curves and non-linear difference equation*, Uspekhi Mat. Nauk 33 (1978), n 4, 215-216.

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