Wednesday, January 26, 2022 5:58 PM

Higgs-Coulomb correspondence in GLSM

GLSM is a curve counting theory (GW, quasimap, DT, ...)

Consider the following problem: let's count maps from

$$\begin{array}{ccc} \mathbb{C}^* \circlearrowright \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \circlearrowleft (\mathbb{C}^*)^2 \\ \uparrow & & \uparrow \\ (x:y) & & (X:Y) \end{array}$$

$(x:y) \to (q\,x:y)$

$(X:Y) \to (a_1 X : a_2 Y)$

Degree d map $(P(x,y) : Q(x,y))$, where P, Q are homog-s of degree d $\left(\begin{array}{c}\mathcal{O}(d)\\ \downarrow \\ \mathbb{P}^1\end{array}\right)$ such that they do not have common zeros.

Let's compactify space of such maps.

- $\to$ stable maps (GW theory)
- $\to$ quasimaps (Ciocan-Fontanine, Kim)

Quasimap $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is $(P(x,y) : Q(x,y))$ such that both P and Q are not zero polynomials.

Points (x,y) in the domain $\mathbb{P}^1$ s.t. that $P(x,y) = Q(x,y) = 0$ are called the base points

- Space of q-maps to $\mathbb{P}^1$ is $\mathbb{P}^{2(d+1)}$

Equivariant cohomology

$$H^\bullet(\mathbb{P}^1) = \frac{\mathbb{C}[p]}{\langle p^2 = 0\rangle} \qquad H^\bullet_{(\mathbb{C}^\times)^2_{a_1,a_2}}(\mathbb{P}^1) = \frac{\mathbb{C}[p]}{\langle (p-\varepsilon_1)(p-\varepsilon_2)\rangle}$$

equivariant vars corresponding to the group action

Equivariant localization

X - sm. var. G - reductive group action on X (for simplicity has only

<u>Equivariant localization</u>

equivariant vars corresponding to the group action

X - sm. proj, G - reductive group action on X (for simplicity has only fixed point)

$$H_G(X)_{loc} \xrightarrow{\ 2\ } \cong H_G(X^G)_{loc}$$

localized wrt equivariant vars.

$$\alpha \in H_G(X)$$

$$2(\alpha) = \sum_{p \in \text{fixed pts}} \frac{\alpha|_p}{Eu_G(T_pX)}$$

- Let's apply this to q maps on $\mathbb{P}_1$

What are the fixed pts?

It has to be $(P(x,y) : 0)$ or $(0 : Q(x,y))$.

$$P(x,y) = \sum_{i=0}^{d} c_i x^i y^{d-i} \longrightarrow c_i q^i x^i y^{d-i}$$

$\mathbb{C}^*_q$

It has to be $(x^i y^{d-i} : 0)$ or $(0 : x^i d^{-i})$

We will not allow base pts at $(0:1)$ of the domain $\mathbb{P}^1$

The only remaining fixed pts are $\underset{p_1}{\underset{\shortparallel}{(y^d : 0)}}$ and $\underset{p_2}{\underset{\shortparallel}{(0 : y^d)}}$

Let's compute $Eu_G(T_p\mathbb{P}^{2(d+1)})$

$\boxed{p_1}$ $(y^d + \underset{\underset{c_1 \cancel{y^d} + c_2 x y^{d-1} + \ldots + c_{d+1} x^d}{\shortparallel}}{p(x,y)} ; q(x,y))$ deformations of the q-map.

We compute $T_p\mathbb{P}^{2(d+1)}$ as a $\mathbb{C}^\times_q \times (\mathbb{C}^\times)^2_{a_1,a_2}$ -rep.

$$T_p\mathbb{P}^{2(d+1)} = \cancel{a_1}(\cancel{1} \oplus q \oplus \ldots \oplus q^d) \oplus a_2 a_1^{-1}(1 \oplus q \oplus \ldots \oplus q^d)$$

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Use the target space rescaling to set coeff in front of y^d to 1

$$Eu_G(T_p\mathbb{P}^{2(d+1)}) = \prod_{n=1}^{d} n\cdot z \cdot \prod_{n=0}^{d} (\varepsilon_2 - \varepsilon_1 + nz)$$

$$Eu_G(\oplus_i \mathcal{L}_i) = \prod_i c_1(\mathcal{L}_i)$$

Let's denote $c_1(a_i) = \varepsilon_i$, $c_1(q) = z$ equivariant vars

$$i(\alpha) = z^{-2d-1}\frac{\alpha|_{p_1}}{d!\cdot\prod_{n=0}^{d}\left(\frac{\varepsilon_2-\varepsilon_1}{z}+n\right)} + z^{-2d-1}\frac{\alpha|_{p_2}}{\prod_{n=0}^{d}\left(\frac{\varepsilon_1-\varepsilon_2}{z}+n\right)\cdot d!}$$

$$J^{qmap} = \sum_{d=0}^{\infty} Q^d \cdot\left[z^{-2d-1}\frac{1}{d!\cdot\boxed{\prod_{n=0}^{d}\left(\frac{\varepsilon_2-\varepsilon_1}{z}+n\right)}}[p_1] + z^{-2d-1}\frac{1}{\prod_{n=0}^{d}\left(\frac{\varepsilon_1-\varepsilon_2}{z}+n\right)\cdot d!}[p_2]\right]$$

$$\Gamma(x+1) = x\Gamma(x) \qquad " = \frac{\Gamma\left(\frac{\varepsilon_2-\varepsilon_1}{z}+d+1\right)}{\Gamma\left(\frac{\varepsilon_2-\varepsilon_1}{z}\right)}$$

explicit function

Remark

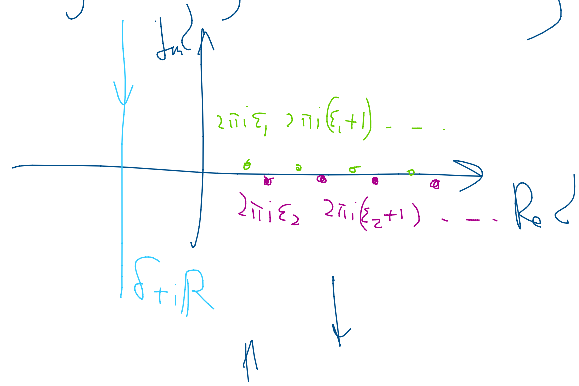
K-theory class on target $\mathbb{P}^1$

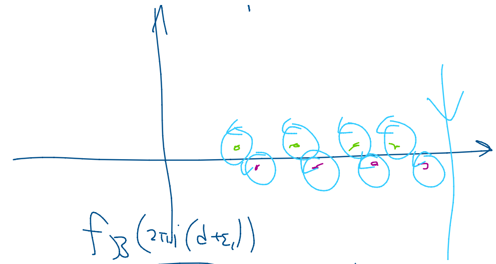
$$Z_{D^2}(\mathcal{B}) := \int_{\delta+i\mathbb{R}} d\zeta \cdot \Gamma\left(-\frac{\zeta}{2\pi i}+\varepsilon_1\right)\Gamma\left(-\frac{\zeta}{2\pi i}+\varepsilon_2\right)\cdot Q^{-\frac{\zeta}{2\pi i}}\cdot \boxed{f_{\mathcal{B}}(\zeta)}$$

"hemisphere partition function"

We can compute this integral by residues shifting the contour to $Re(\zeta) \to +\infty$

$$\operatorname{Res}_{z=-d} \Gamma(z) = \frac{(-1)^d}{d!}$$





$$Z_{D^2}(\mathcal{B}) = \sum_{d=0}^{\infty} Q^d \left[\frac{Q^{\varepsilon_1}}{d!\,\Gamma(\varepsilon_1-\varepsilon_2+d+1)} \frac{f_{\mathcal{B}}(2\pi i(d+\varepsilon_1))}{\sin \pi \varepsilon_1} + \frac{Q^{\varepsilon_2}}{\Gamma(\varepsilon_2-\varepsilon_1+d+1)\,d!} \frac{f_{\mathcal{B}}(2\pi i(d+\varepsilon_2))}{\sin \pi \varepsilon_2} \right]$$

$$\boxed{Z_{D^2}(\mathcal{B}) = \left\langle J^{q\text{-map}}\big|_{z=2\pi i} \cdot \hat{\Gamma}^{-1},\ ch(\mathcal{B}) \right\rangle}$$

Iritani's Γ-class

Poincare (with Todd class)

Higgs-Coulomb correspondence in our case.

Hori-Romo '13

Generalizations: target $\mathbb{P}^1 \rightsquigarrow$ toric smooth DM stack

$$[V /\!/_{\omega} G] = [V^{\omega\text{-ss}}/G] = \mathcal{X}$$

$V = \mathbb{C}^{n+k}$, $G = (\mathbb{C}^\times)^k$

GIT quotient w.r.t. to stability parameter $\omega \in \mathfrak{g}^*$

Variable Q generalizes to e^{ω}, where $\omega \in H^2(\mathcal{X};\mathbb{R})$

$Z_{D^2}(\mathcal{B})$ - analytic function in ω

Power series expansion recovers $J^{q\text{map}}$-function

$\langle J^{q\text{-map}}_{1},\ \hat{\Gamma}^{-1},\ ch(\mathcal{B})\rangle$

Power series expansion recovers J-function

$$\langle J^{q\text{-map}}\big|_{z=2\pi i} \cdot \hat{\Gamma}^{-1}, \, ch(\mathfrak{B}) \rangle$$

Expansions at diff. points recover different q-map theories

q-maps to $[V/\!/_\omega G]$ for different stab. ω

<u>Example</u>: $[\mathbb{C}^6 /\!/ \mathbb{C}^*]$

$(1\ 1\ 1\ 1\ 1\ -5)$

$\omega > 0$ $\qquad\qquad$ $\omega < 0$

$$K_{\mathbb{P}^4} \quad \rightsquigarrow \quad [\mathbb{C}^5 / \mathbb{Z}_5]$$

version of Ruan's crepant transformation

Further gener-n: GLSM

"quasimap counts into critical loci of functions on $[V/\!/_\omega G]$

$$\mathbb{C}^6_{x_1 \dots x_5, p} \qquad W(\bar{x}, p) = p \cdot \sum_{i=1}^{5} x_i^5$$

$$\mathrm{Crit}\, W \subset K_{\mathbb{P}^4} \qquad\qquad \mathrm{Crit}\, W \subset [\mathbb{C}^5/\mathbb{Z}_5]$$

$$\mathrm{Crit}\, W \cong \{p=0\} \cap \{\textstyle\sum x_i^5 = 0\} \cong \text{quintic 3-fold in } \mathbb{P}^4 \qquad\qquad \mathrm{Crit}\, W \cong 0$$