Finite quotients of the fundamental groups of 3-manifolds (joint work in progress with Melanie Wood)

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3-manifolds and their fundamental groups

Let M be a 3-manifold. I will always take it to be closed (compact, no boundary) and orientable.

Examples: 3-sphere S^3, 3-torus $(S^1)^3$, 2-sphere times circle $S^2 \times S^1$, ...

The fundamental group $\pi_1(M)$ is the group of loops in M starting and ending at a point $*$ in M, up to homotopy, where composition is concatenation.

Examples: $\pi_1(S^3) = 1$, $\pi_1(S^2 \times S^1) = \mathbb{Z}$, $\pi_1((S^1)^3) = \mathbb{Z}^3$.

There are many deep questions about the fundamental groups of 3-manifolds. I will talk about their finite quotients.

Fundamental groups and their finite quotients

Given a 3-manifold M, which finite groups are quotients of $\pi_1(M)$?

- $M = S^3, \pi_1(M) = 1.$ Only the trivial group is a quotient.
- $M = S^2 \times S^1, \pi_1(M) = \mathbb{Z}.$ All cyclic groups are quotients.
- $M = (S^1)^3, \pi_1(M) = \mathbb{Z}^3.$ All finite abelian groups with at most 3 generators are quotients.
- $M = S^3/SL_2(\mathbb{F}_5) =$ a dodecahedron with opposite faces glued after rotation by $\pi/5$. $\pi_1(M) = SL_2(\mathbb{F}_5)$. Only $SL_2(\mathbb{F}_5), SL_2(\mathbb{F}_5)/\pm 1$, and the trivial group are quotients.
- $M =$ a dodecahedron with opposite faces glued after rotation by $3\pi/5$ (Weber-Seifert). $\pi_1(M)$ generated by p, q, r, s with relations $pqsrpr^{-2}s^{-1}qps = pqr^{-1}pr^{-2}s^{-1}qsr^{-1}s^{-q} = pq^2srspqsr^2p^{-1}s^{-1}q = pqsr^2p^{-1}rs^{-1}r^{-1}s^{-1}q^{-1}p^{-1}s^{-1}q = 1.$ Many complicated finite quotients.

Question: What question should we ask?

Is every finite group the quotient of the fundamental group of a 3-manifold?

- Yes. Take S^3, remove $2n$ balls, leaving $2n$ spheres as boundary. Glue the spheres together in n pairs. Resulting space M_n has $\pi_1(M)$ the free group on n generators, so every finite group with n generators is a quotient.

Let G and H be two finite groups. If G is a quotient of $\pi_1(M)$, then must H be a quotient of $\pi_1(M)$?

- This question is much more interesting!
- The answer is clearly no for many pairs of groups G, H.
- But the answer is sometimes yes.

Universal facts about finite quotients

Let G be the semidirect product $(\mathbb{Z}/5) \rtimes (\mathbb{Z}/4)$, where the generator of $\mathbb{Z}/4$ acts on $\mathbb{Z}/5$ by multiplication by 2.

Let H be the semidirect product $(\mathbb{Z}/5)^2 \rtimes (\mathbb{Z}/4)$, where the generator of $\mathbb{Z}/4$ acts on $(\mathbb{Z}/5)^2$ by multiplication by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Theorem

Let M be a 3-manifold. If G is a quotient of $\pi_1(M)$, then H is also a quotient of $\pi_1(M)$.

A more complicated statement involving simpler groups:

Theorem

Let M be a 3-manifold. If S_3 is a quotient of $\pi_1(M)$, then at least one of S_4, $S_3 \times S_2$, or $(\mathbb{Z}/3) \rtimes (\mathbb{Z}/4)$ is a quotient of $\pi_1(M)$.

Description of the main theorem

We prove a (computably enumerable) list of properties of the fundamental group of the 3-manifold of the form: For K the fundamental group of a 3-manifold, if G is a quotient of K, then at least one of $H_1, \dots, H_n$ is a quotient of K, for finite groups $G, H_1, \dots, H_n$.

(Will describe these properties explicitly, later, using group cohomology)

We prove these properties are universal in the following sense:

If K is a finitely generated group satisfying these properties, then for each natural number N there exists a 3-manifold M such that, for each finite group G of order $\leq N$, G is a quotient of $\pi_1(M)$ if and only if G is a quotient of K.

Example of an existential result

Let Q_8 be the eight-element quaternion group $\{1, -1, i, -i, j, -j, k, -k\}$. There are 3 nontrivial homomorphisms $Q_8 \to \{\pm 1\}$.

Let K be defined as a semidirect product $(\mathbb{Z}/3 \times \mathbb{Z}/5) \rtimes Q_8$ where Q_8 acts on $\mathbb{Z}/3$ by one of these homomorphimss $Q_8 \to \{\pm 1\}$ and acts on $\mathbb{Z}/5$ by a different homomorphism.

Then K satisfies all our properties, so there exists a 3-manifold M such that K is a quotient of $\pi_1(M)$ and every group of order $\leq 1,000,000$ that is a quotient of $\pi_1(M)$ is a quotient of G.

But K is not itself the fundamental group of a 3-manifold (Perelman).

Strategy of proof

Our proofs of the two parts of the main theorem are mostly independent.

The first part finds properties satisfied by the fundamental group of an arbitrary 3-manifold. This uses very standard topological methods (based on Poincaré duality, Euler characteristic, Heegard splittings, and cobordism).

The second part produces examples of 3-manifolds which have certain finite groups as a quotient but don't have other finite groups as a quotient. We do this using a probabilistic method – we show that a random 3-manifold has these properties with positive probability.

We use a notion of random 3-manifold defined by Dunfield and Thurston.

Random Heegard Splittings (Dunfield-Thurston)

A handlebody of genus g is a ball with g handles glued on. It is a 3-manifold with boundary - its boundary is a Riemann surface of genus g.

We can make a 3-manifold by gluing two handlebodies together using a homeomorphism of their boundaries. This is called a Heegard splitting.

We only care about homeomorphisms of the Riemann surface of genus g to itself up to isotopy. These form a group, the mapping class group, that is finitely generated. Choose a finite set of generators including the identity.

We define a random mapping class to be a random word in these generators of length L. We define a random Heegard splitting by gluing two genus g handlebodies along a random mapping class.

We calculate all probabilities by first taking the limit as L goes to ∞ and then taking the limit as g goes to ∞.

A typical probabilistic result

Let's consider the first homology group $H_1(M)$.

- This is equivalent to asking which finite *abelian* groups are a quotient of $\pi_1(M)$.

For most 3-manifolds (probability 1) this is a torsion abelian group. So it splits as a product of its p-parts for all primes p.

Theorem

For G a finite abelian p-group and a random 3-manifold M, the probability that the p-part of $H_1(M)$ is isomorphic to G is equal to $\frac{1}{|G||\mathrm{Aut}(G)|}$ times the number of symmetric nondegenerate pairings $G \times G \to \mathbb{Q}/\mathbb{Z}$ times $\prod_{j=1}^{\infty}(1+p^{-j})^{-1}$.

Why symmetric nondegenerate pairings? There is a "torsion linking pairing" $H_1(M)^{tors} \times H_1(M)^{tors} \to \mathbb{Q}/\mathbb{Z}$ which is symmetric and nondegenerate.

Each pairing is equally likely, so the probability of getting a group G together with a pairing ℓ is simply $\frac{1}{|G||\mathrm{Aut}(G,\ell)|} \prod_{j=1}^{\infty}(1+p^{-j})^{-1}$.

Prior Work

Dunfield and Thurston more than just define this model. They calculated, for a fixed finite group Q, the limit of the expected number of surjections $\pi_1(M) \to Q$ for a random 3-manifold M. It is

$$\max_{\tilde{Q} \to Q \text{ central extension}} \frac{|\tilde{Q}|}{|\tilde{Q}^{ab}|}.$$

However, they were only able to convert this information on the average number of surjections to the probability that there is at least one surjection in a few special cases (e.g. finite simple groups Q).

We calculate the probability of obtaining a given group as a quotient, and related probabilities, by using the information about the total number of surjections in a more extensive way (plus a little more topological information).

This is analogous to the classical problem in probability theory of showing a probability distribution is determined by its moments. It requires a mix of analysis and group theory.

Prior Work: Number theory

A classic analogy:

Topology	Number Theory
3-manifolds	Number fields
Fundamental group	Galois group of a maximal unramified extension
$\Uparrow$ (abelianizes) $\Uparrow$	$\Uparrow$ (abelianizes) $\Uparrow$
First homology	Class group

The class group of a random number field has been heavily studied for a while (the Cohen-Lenstra heuristics). The Galois group of a random number field was more recently studied as a generalization (Boston-Bush-Hajir, . . .).

Our analytic and algebraic techniques were developed to study the number theory analogue, building on prior work of Heath-Brown, Ellenberg-Venkatesh-Westerland, Wood, Boston-Wood, Wang-Wood, Lipnowski-S.-Tsimerman, Liu-Wood-Zurieck-Brown, S.

Group cohomology

Group cohomology is the cohomology of the classifying space of the goup G.

For V a representation of a group G, group cohomology $H^i(G, V)$ is calculated combinatorially, with cochains in degree i functions from G^i to V and coboundary maps some explicit formula. There is also a dual homology theory.

If G is a quotient of $\pi_1(M)$, then we obtain a map $H^3(G, A) \to H^3(M, A) \to A$ for any group A by integrating. Call this τ.

The properties we proved for 3-manifold groups are defined using group cohomology and the integration map τ.

Theorem Statement: Properties of 3-manfiold groups

Theorem (S.-Wood): G a group and $\tau : H^3(G, A) \to A$ satisfy the following if $G = \pi_1(M)$ and τ is integration:

(P1): Let V be a representation of G over $\mathbb{F}_p$. If V is irreducible, then $\dim H^1(G, V) = \dim H^1(G, V^\vee)$.

(P2) Let V be a representation of G over $\mathbb{F}_p$. For $0 \neq \alpha \in H^2(G, V)$, there exists $\beta \in H^1(G, V^\vee)$ such that $\tau(\alpha \cup \beta) \neq 0$.

(P3): Let V be a representation of G over $\mathbb{F}_q$ for q odd. If V is irreducible and symplectic (i.e. the action of G preserves a nondegenerate skew-symmetric bilinear form), then $\dim H^1(G, V)$ is even.

(P4) Let V be a representation of G defined over $\mathbb{F}_{2^n}$, which is irreducible and symplectic, such that the map $G \to Sp(V)$ lifts to a certain extension $1 \to V \to ASp(V) \to Sp(V) \to 1$ defined using the Heisenberg group. Then $\dim H^1(G, V)$ is congruent mod 2 to $\tau(c_V)$, where $c_V \in H^3(ASp(V), \mathbb{Z}/2)$ is a certain universal class.

Ideas from the proof of probabilistic result

The key thing is to understand the probability that G is a quotient of $\pi_1(M)$ and $H_1, \dots, H_n$ are not where H_i are slightly larger than G, i.e. H_i surjects onto G with kernel a minimal normal subgroup.

Formally represent this probability as a weighted combination of the number of expected surjections onto
$G, H_1, \dots, H_n, H_1 \times_G H_1, \dots H_1 \times_G H_n, \dots, H_i \times_G H_j, \dots, H_i \times_G H_j \times_G H_k, \dots$

Evaluate this formal sum using algebra (need to understand group cohomology of extensions like $H_i \times_G H_j \times_G H_k$, do this with a spectral sequence)

Check the formal calculation is correct using analysis (have to exchange sum with a limit, use dominated convergence, etc.)

More on the link to number theory

Cohen-Lenstra theory describes the distribution of the class group of a random number field. What is a random number field?

- To generate a random quadratic extension of $\mathbb{Q}$, take a uniformly random element from the set of all quadratic extensions of $\mathbb{Q}$ with discriminant $< X$, then let $X \to \infty$.
- More generally, to generate a random extension of $\mathbb{Q}$ with Galois group Γ, take a uniformly random element from the set of all extensions of $\mathbb{Q}$ with Galois group Γ where the product of all ramified primes is $< X$, then let $X \to \infty$.
- Even more generally, to generate a random extension of F with Galois group Γ, take a uniformly random element from the set of all extensions of F with Galois group Γ where the product of the norms of all ramified primes is $< X$, then let $X \to \infty$.

In the last two case, the class group of our random number field and the Galois group of its maximum unramified extension admit an action of Γ, and it's reasonable to consider the group together with this action.

Which fields need the new methods

The approach of Liu, Wood, and Zurieck-Brown could be applied to random Γ-extensions K of a base field F where F does not contain too many roots of unity.

- Can consider only the largest quotient of the Galois group of order prime to the number of roots of unity in F

Our new approach fixes this.

Why are number fields with many roots of unity like 3-manifolds?

Artin-Verdier duality gives a map $H^3(\mathcal{O}_K, \mu_n) \to \mathbb{Z}/n$ for a number field K and natural number n. If K contains the n'th roots of unity then we have a composition

$$H^3(\pi_1(\mathcal{O}_K), \mathbb{Z}/n) \to H^3(\mathcal{O}_K, \mathbb{Z}/n) \to H^3(\mathcal{O}_K, \mu_n) \to \mathbb{Z}/n$$

that behaves like τ.

Function fields: a partial bridge between geometry and topology

The key to guessing the distribution of the Galois group of the maximal unramified extension of Γ-extensions K of F is to guess the moments: The expected number of unramified extensions of K with Galois group H, for a fixed Γ-group H.

This is equivalent to finding the expected number of extensions of F with Galois group $H \rtimes \Gamma$, with certain restrictions on their ramification. This is hard to do rigorously except in a few special cases!

To guess non-rigorously, model the field F with a field $\mathbb{F}_q(t)$ of functions in one variable over a finite field $\mathbb{F}_q$. Then extensions with Galois group $H \rtimes \Gamma$ correspond to algebraic curves that are branched covers of the projective line with Galois group $H \rtimes \Gamma$.

Can count these algebraic curves by understanding the topology of moduli spaces of branched covers (Hurwitz spaces).