

BLOCH INVARIANTS OF HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. We define an invariant $\beta(M)$ of a finite volume hyperbolic 3-manifold M in the Bloch group $\mathcal{B}(\mathbb{C})$ and show it is determined by the simplex parameters of any degree one ideal triangulation of M . We show $\beta(M)$ lies in a subgroup of $\mathcal{B}(\mathbb{C})$ of finite \mathbb{Q} -rank determined by the invariant trace field of M . Moreover, the Chern-Simons invariant of M is determined modulo rationals by $\beta(M)$. This leads to a simplicial formula and rationality results for the Chern Simons invariant which appear elsewhere.

Generalizations of $\beta(M)$ are also described, as well as several interesting examples. An appendix describes a scissors congruence interpretation of $\mathcal{B}(\mathbb{C})$.

1. INTRODUCTION

Let $M = \mathbb{H}^3/\Gamma$ be an oriented hyperbolic manifold of finite volume (so Γ is a torsion free Kleinian group). It is known that M has a degree one ideal triangulation by ideal simplices $\Delta_1, \dots, \Delta_n$ (see sect. 2.1). Let $z_i \in \mathbb{C}$ be the parameter of the ideal simplex Δ_i for each i . These parameters define an element $\beta(M) = \sum_{i=1}^n [z_i]$ in the pre-Bloch group $\mathcal{P}(\mathbb{C})$ (as defined below and in [14], for example).

Theorem 1.1. *The above element $\beta(M)$ can be defined without reference to the ideal decomposition, so it depends only on M . Moreover, it lies in the Bloch group $\mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})$.*

The independence of $\beta(M)$ on ideal triangulation holds even though our concept of degree one ideal triangulation is rather more general than ideal triangulation concepts often considered.

We prove this theorem as follows. There is an exact sequence (mod 2-torsion) due to Bloch and Wigner (cf. [14])

$$0 \rightarrow \mu \rightarrow H_3(\mathrm{PGL}(2, \mathbb{C}); \mathbb{Z}) \rightarrow \mathcal{B}(\mathbb{C}) \rightarrow 0,$$

where $\mu \subset \mathbb{C}^*$ is the group of roots of unity. If M is compact then there is a “fundamental class” $[M] \in H_3(\mathrm{PGL}(2, \mathbb{C}); \mathbb{Z})$ and we show $\beta(M)$ is the image of $[M]$ in $\mathcal{B}(\mathbb{C})$. We do this by factoring through a certain relative homology group $H_3(\mathrm{PGL}(2, \mathbb{C}), \mathbb{C}\mathbb{P}^1; \mathbb{Z})$ for which the relationship between $[M]$ and $\beta(M)$ is easier to see (in fact Dupont and Sah [14] show this relative homology group maps isomorphically to $\mathcal{P}(\mathbb{C})$). In the non-compact case we also find a fundamental class $[M]$ in this relative homology group that maps to $\beta(M) \in \mathcal{P}(\mathbb{C})$, thus proving that $\beta(M)$ is independent of triangulation. The fact that it lies in $\mathcal{B}(\mathbb{C})$ is the relation $\sum z_i \wedge (1 - z_i) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*$ on the simplex parameters z_i . For a more restrictive type of ideal triangulation than those considered here this relation has

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been attributed to Thurston (unpublished) by Gross [19] (according to [44]). It also follows easily from [32] (see also [27]). We give a cohomological proof here.

Recall (e.g., [36, 28]) that the *invariant trace field* $k(M)$ is the field generated over \mathbb{Q} by squares of traces of elements of Γ . It is known that modulo torsion the Bloch group $\mathcal{B}(k)$ of a number field is isomorphic to \mathbb{Z}^{r_2} , where r_2 is the number of complex embeddings of k , so $\mathcal{B}(k) \otimes \mathbb{Q} \cong \mathbb{Q}^{r_2}$. Moreover, if k is given as a subfield of \mathbb{C} , then naturally $\mathcal{B}(k) \otimes \mathbb{Q} \subset \mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$.

Theorem 1.2. *As an element of $\mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$, the invariant $\beta(M)$ lies in the subgroup $\mathcal{B}(k(M)) \otimes \mathbb{Q}$.*

In the non-compact case we can define $\beta(M)$ directly as an element of $\mathcal{B}(k(M))$ which is independent of triangulation, so the “ $\otimes \mathbb{Q}$ ” of the previous theorem can be deleted, but we do not know if it can in the compact case. Theorem 6.1 is of interest in this regard.

The Chern-Simons invariant $\text{CS}(M)$ is determined modulo rational multiples of π^2 by $\beta(M)$. Chern and Simons defined what is now called the Chern Simons invariant in ([8]) for any compact $(4n - 1)$ -dimensional Riemannian manifold. Meyerhoff [23] extended the definition in the case of hyperbolic 3-manifolds to allow noncompact ones, that is hyperbolic 3-manifolds with cusps. The Chern-Simons invariant $\text{CS}(M)$ of such a hyperbolic 3-manifold M is an element in $\mathbb{R}/\pi^2\mathbb{Z}$. There is a map $\rho: \mathcal{B}(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Q}$ called the “Bloch regulator map” whose definition we recall in section 7.

Theorem 1.3. $\rho(\beta(M)) = \frac{i}{2\pi^2}(\text{vol}(M) + i \text{CS}(M)) \in \mathbb{C}/\mathbb{Q}$.

As is pointed out in [30], Theorems 1.2 and 1.3 put strong restrictions on $\text{CS}(M)$. For example, it follows that $\text{CS}(M)$ is rational (by which we mean that it is zero in $\mathbb{R}/\pi^2\mathbb{Q}$) if $k(M)$ is a quadratic extension of a totally real field, and — assuming the “Ramakrishnan Conjecture” — $\text{CS}(M)$ is irrational if $k(M) \cap \overline{k(M)} \subset \mathbb{R}$.

We also discuss a definition of our invariant for any homomorphism $f: \Gamma \rightarrow \text{PGL}(2, \mathbb{C})$. It then generally lies in $\mathcal{P}(\mathbb{C})$ rather than $\mathcal{B}(\mathbb{C})$, but it equals $\beta(M)$ for the discrete embedding of Γ . We generalize this also to higher dimensions, but it is not clear at this point what the significance of this is.

In section 9 we describe several interesting examples. The final section 10 is an appendix describing a scissors congruence interpretation of the Bloch group $\mathcal{B}(\mathbb{C})$.

Many of the results of this paper were announced in [31]. The question of defining invariants in K -groups for hyperbolic manifolds is also investigated by A. Goncharov. We thank him for sharing his preprint [17] with us.

2. PRELIMINARIES

2.1. Ideal simplices and degree one ideal triangulations. We shall denote the standard compactification of \mathbb{H}^3 by $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \mathbb{C}\mathbb{P}^1$. An ideal simplex Δ with vertices $z_1, z_2, z_3, z_4 \in \mathbb{C}\mathbb{P}^1$ is determined up to congruence by the cross ratio

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

This z lies in the upper half plane of \mathbb{C} if the orientation induced by the given ordering of the vertices agrees with the orientation of \mathbb{H}^3 . Permuting the vertices

by an even (i.e., orientation preserving) permutation replaces z by one of

$$z, \quad 1 - \frac{1}{z}, \quad \text{or} \quad \frac{1}{1 - z},$$

while an odd permutation replaces z by

$$\frac{1}{z}, \quad \frac{z}{z - 1}, \quad \text{or} \quad 1 - z.$$

We will also allow degenerate ideal simplices where the vertices lie in a plane, so the parameter z is real. However, we always require that the vertices are distinct. Thus the parameter z of the simplex lies in $\mathbb{C} - \{0, 1\}$ and every such z corresponds to an ideal simplex.

If one takes finitely many geometric 3-simplices and glues them together by identifying all the 2-faces in pairs then one obtains a cellular complex Y which is a manifold except possibly at isolated points. If the complement of these “bad” points is oriented we call Y a *geometric 3-cycle*. In this case the complement $Y - Y^{(0)}$ of the vertices is an oriented manifold.

Suppose $M^3 = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold¹. A *degree one ideal triangulation* of M consists of a geometric 3-cycle Y plus a map $f: Y - Y^{(0)} \rightarrow M$ satisfying

- f is degree one almost everywhere in M ;
- for each 3-simplex S of Y there is a map f_S of S to an ideal simplex in $\overline{\mathbb{H}}^3$, bijective on vertices, such that $f|_S - S^{(0)}: S - S^{(0)} \rightarrow M$ is the composition $\pi \circ (f_S|_S - S^{(0)})$, where $\pi: \mathbb{H}^3 \rightarrow M$ is the projection.

In [40] Thurston shows that any compact hyperbolic 3-manifold has degree one ideal triangulations with $|Y| \simeq M$. Ideal triangulations also arise “in practice” (e.g., in the program SNAPPEA for exploring hyperbolic manifolds — [42]) as follows. Epstein and Penner in [15] show that any non-compact M has a genuine ideal triangulation, that is, one for which f is arbitrarily closely deformable to a homeomorphism (they actually give an ideal polyhedral subdivision; to subdivide these polyhedra into ideal tetrahedra it is conceivable that one may need flat ideal tetrahedra to match triangulations of faces of polyhedra — see section 10 for more details). The ideal simplices can be deformed to give degree one ideal triangulations (based on the same geometric 3-cycle Y) on almost all manifolds obtained by Dehn filling cusps of M (see e.g., [32]).

2.2. Bloch group. There are several different definitions of the Bloch group in the literature. They differ at most by torsion and they agree with each other for algebraically closed fields. We shall use the following.

Definition 2.1. Let k be a field. The *pre-Bloch group* $\mathcal{P}(k)$ is the quotient of the free \mathbb{Z} -module $\mathbb{Z}(k - \{0, 1\})$ by all instances of the following relations:

$$(1) \quad [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - x}{1 - y}\right] = 0,$$

$$(2) \quad [x] = \left[1 - \frac{1}{x}\right] = \left[\frac{1}{1 - x}\right] = -\left[\frac{1}{x}\right] = -\left[\frac{x - 1}{x}\right] = -[1 - x].$$

¹Throughout this paper “hyperbolic manifold” means oriented hyperbolic 3-manifold of finite volume. Similarly Kleinian groups are assumed to have finite covolume

The first of these relations is usually called the *five term relation*. The *Bloch group* $\mathcal{B}(k)$ is the kernel of the map

$$\mathcal{P}(k) \rightarrow k^* \wedge_{\mathbb{Z}} k^*, \quad [z] \mapsto 2(z \wedge (1 - z)).$$

(The above five term relation is the one of Suslin [38]. It had misprints in its formulations in [30] and [31]. Dupont and Sah use $[x] - [y] + [y/x] - [(1-y)/(1-x)] + [(1-y^{-1})/(1-x^{-1})] = 0$, which is conjugate to Suslin's by the self-map $z \mapsto z^{-1}$ of $\mathbb{Z}(k - \{0, 1\})$. One easily deduces that this map induces an isomorphism between the Bloch groups resulting from the two choices of five-term relation. We will thus use Suslin's without further comment. The justification of the choice comes from Suslin's version of the Bloch group which we do not discuss here, see [38] or [30] for more details.)

Dupont and Sah's definition of the Bloch group does not use the relations (2). We shall need their versions later so we will denote their groups obtained by omitting relations (2) by $\mathcal{P}'(k)$ and $\mathcal{B}'(k)$ (this is the reverse of their convention).

Dupont and Sah show in [14] that $\mathcal{P}'(k)$ is more natural than $\mathcal{P}(k)$ from a homological point of view. They also show:

Lemma 2.2. *If the characteristic of k is not 2 then $\mathcal{P}(k)$ and $\mathcal{P}'(k)$ differ by at most torsion of order dividing 6. If, moreover, k is algebraically closed then $\mathcal{P}(k) = \mathcal{P}'(k)$. \square*

Thus, if we are willing to ignore torsion or if we are working over \mathbb{C} we can use either definition.

Definition 2.3. Another definition starts with a pre-Bloch group $\mathcal{P}(k)$ defined as the quotient of the free \mathbb{Z} -module $\mathbb{Z}(k \cup \{\infty\})$ by the following relations:

$$\begin{aligned} [0] &= [1] = [\infty] = 0, \\ [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] &= 0. \end{aligned}$$

It is not hard to see that this gives the same result as our definition.

Remark 2.4. For $k = \mathbb{C}$, the relations (2) express the fact that $\mathcal{P}(\mathbb{C})$ may be thought of as being generated by isometry classes of ideal hyperbolic 3-simplices. The five term relation (1) then expresses the fact that in this group we can replace an ideal simplex on four ideal points by the cone of its boundary to a fifth ideal point. As we show in an appendix (section 10), the effect is that $\mathcal{P}(\mathbb{C}) = \mathcal{P}'(\mathbb{C})$ is a group generated by ideal polyhedra with ideal triangular faces modulo the relations generated by cutting and pasting along such faces.

2.3. The Bloch invariant. Let $f: Y - Y^{(0)} \rightarrow M$ be a degree 1 ideal triangulation of the hyperbolic manifold M as in subsection 2.1 above. Each 3-simplex of Y maps to an ideal hyperbolic simplex in $\overline{\mathbb{H}}^3$. Let z_1, \dots, z_n be the cross ratio parameters of these ideal simplices.

Definition 2.5. The *Bloch invariant* $\beta(M)$ is the element $\sum_1^n [z_j] \in \mathcal{P}(\mathbb{C})$. If the z_j 's all belong to a subfield $K \subset \mathbb{C}$, we may consider $\beta(M)$ as an element of $\mathcal{P}(K)$.

We show in section 6 that if $\beta(M)$ can be defined as above in $\mathcal{P}(K)$ then it lies in $\mathcal{B}(K) \subset \mathcal{P}(K)$ and is independent of triangulation.

3. RELATIVE HOMOLOGY OF Γ

If G is a group and Ω is a G -set then $\mathbb{Z}\Omega$ is a $\mathbb{Z}G$ -module. Let $J\Omega$ be the kernel of the augmentation map $\epsilon: \mathbb{Z}\Omega \rightarrow \mathbb{Z}$. Then, following [10], we define

$$H_n(G, \Omega) = H_n(G, \Omega; \mathbb{Z}) = \text{Tor}_{n-1}^{\mathbb{Z}G}(J\Omega, \mathbb{Z}).$$

For our purposes it is convenient to capture the dimension shift in this definition as follows. Suppose

$$\cdots \rightarrow S_3 \rightarrow S_2 \rightarrow S_1 \rightarrow J\Omega \rightarrow 0$$

is a $\mathbb{Z}G$ -projective resolution of $J\Omega$. Then $H_n(G, \Omega)$ is the homology at index n of the chain complex

$$\cdots \rightarrow (S_3)_G \rightarrow (S_2)_G \rightarrow (S_1)_G \rightarrow 0,$$

where we are using the notation

$$M_G := M \otimes_{\mathbb{Z}G} \mathbb{Z}.$$

Let $S_n(\mathbb{CP}^1)$ denote the free abelian group generated by all ordered $(n+1)$ -tuples $\langle z_0, \dots, z_n \rangle$ of *distinct* points of \mathbb{CP}^1 modulo the relations

$$\langle z_0, \dots, z_n \rangle = \text{sgn } \tau \langle z_{\tau(0)}, \dots, z_{\tau(n)} \rangle$$

for any permutation τ of $\{0, \dots, n\}$. With the standard boundary map, they form a chain complex $S_\bullet(\mathbb{CP}^1)$. This is the cellular chain complex for the complete simplex on \mathbb{CP}^1 (considered as a discrete set), so it gives a resolution of \mathbb{Z} , i.e., the following sequence

$$\cdots \rightarrow S_2(\mathbb{CP}^1) \rightarrow S_1(\mathbb{CP}^1) \rightarrow S_0(\mathbb{CP}^1) \rightarrow \mathbb{Z} \rightarrow 0$$

is exact.

If $G \subset \text{PGL}(2, \mathbb{C})$ is any subgroup, then G acts on \mathbb{CP}^1 , so the above sequence is a sequence of $\mathbb{Z}G$ -modules. We can truncate this exact sequence to get an exact sequence

$$\cdots \rightarrow S_2(\mathbb{CP}^1) \rightarrow S_1(\mathbb{CP}^1) \rightarrow J\mathbb{CP}^1 \rightarrow 0.$$

If this resolution of $J\mathbb{CP}^1$ were a free $\mathbb{Z}G$ -resolution then $S_{\bullet \geq 1}(\mathbb{CP}^1)_G$ would compute the homology $H_n(G, \mathbb{CP}^1)$. Instead we only get a homomorphism

$$H_n(G, \mathbb{CP}^1) \rightarrow H_n(S_{\bullet \geq 1}(\mathbb{CP}^1)_G)$$

for each n , determined by mapping any free resolution to the above resolution.

For the rest of the section, suppose we have a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, so $\Gamma \subset \text{PGL}(2, \mathbb{C})$ is a torsion free Kleinian group. Then Γ acts on $\mathbb{CP}^1 = \partial\mathbb{H}^3$. In this case we shall see that the above homomorphism is an isomorphism in degree 3 (and higher).

Lemma 3.1. $S_n(\mathbb{CP}^1)$ is a free $\mathbb{Z}\Gamma$ -module for $n \geq 2$.

Proof. We will show that Γ acts freely on the basis of $S_n(\mathbb{CP}^1)$ for $n \geq 2$. Since each element $\gamma \in \Gamma$ is uniquely determined by its action on 3 distinct points of \mathbb{CP}^1 , the only way for γ to fix a basis element of $S_n(\mathbb{CP}^1)$ for $n \geq 2$ is if it acts as a permutation of $n+1$ distinct points in \mathbb{CP}^1 . But such a γ must be a torsion element which is impossible since Γ is torsion free. \square

Proposition 3.2. 1. $H_3(\Gamma, \mathbb{CP}^1) \cong \mathbb{Z}$.

2. The above map $H_3(\Gamma, \mathbb{CP}^1) \rightarrow H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ is an isomorphism.

3. Furthermore, if Γ is cocompact, then the natural map $H_3(\Gamma; \mathbb{Z}) \rightarrow H_3(\Gamma, \mathbb{CP}^1)$ is also an isomorphism.

Proof. If M is compact then part 1 follows from part 3, since $H_3(\Gamma; \mathbb{Z}) = H_3(M; \mathbb{Z}) = \mathbb{Z}$. Thus assume M has cusps. Let \mathcal{C} be the set of *cuspidal points*, that is, preimages of cusps of M in $\partial\overline{\mathbb{H}}^3 = \mathbb{CP}^1$, or, equivalently, fixed points of parabolic elements of Γ . Let M_0 be the result of removing open horoball neighbourhoods of the cusps of M , so M_0 is compact with toral boundary components. Then \mathcal{C} can also be identified with $\pi_0(\partial\widehat{M}_0)$, where \widehat{M}_0 is the universal cover. As described for instance in [10] (see also the proof of Lemma 4.2), one has $H_3(\Gamma, \mathcal{C}) \cong H_3(M_0, \partial M_0) \cong \mathbb{Z}$, generated by the fundamental class $[M_0]$. The short exact sequence

$$0 \rightarrow J\mathcal{C} \rightarrow J\mathbb{CP}^1 \rightarrow \mathbb{Z}(\mathbb{CP}^1 - \mathcal{C}) \rightarrow 0$$

gives rise to the long exact sequence

$$\cdots \rightarrow H_3(\Gamma, \mathbb{Z}(\mathbb{CP}^1 - \mathcal{C})) \rightarrow H_3(\Gamma, \mathcal{C}) \rightarrow H_3(\Gamma, \mathbb{CP}^1) \rightarrow H_2(\Gamma, \mathbb{Z}(\mathbb{CP}^1 - \mathcal{C})) \rightarrow \cdots$$

By Shapiro's Lemma ([7]) $H_i(\Gamma, \mathbb{Z}(\mathbb{CP}^1 - \mathcal{C}))$ is isomorphic to the direct sum over the orbits of Γ on $\mathbb{CP}^1 - \mathcal{C}$ of H_i of the isotropy groups of these orbits. Since these isotropy groups are all trivial or infinite cyclic, both $H_3(\Gamma, \mathbb{Z}(\mathbb{CP}^1 - \mathcal{C}))$ and $H_2(\Gamma, \mathbb{Z}(\mathbb{CP}^1 - \mathcal{C}))$ are trivial. It follows that

$$H_3(\Gamma, \mathcal{C}) \xrightarrow{\cong} H_3(\Gamma, \mathbb{CP}^1),$$

completing the proof of part 1.

We prove parts 2 and 3 together. Since $S_\bullet(\mathbb{CP}^1)$ is a resolution of \mathbb{Z} , the standard spectral sequence associated to $H_*(\Gamma, S_\bullet(\mathbb{CP}^1))$ converges to $H_*(\Gamma; \mathbb{Z})$. The $E_{p,q}^1$ term of the spectral sequence is $H_q(\Gamma, S_p(\mathbb{CP}^1))$. It follows from Lemma 3.1 and the fact that $H_i(G; M) = \{0\}$ for any free G -module M and $i \geq 1$, that $E_{p,q}^1 = \{0\}$ for $p \geq 2$. The isotropy group of the Γ action of a point in \mathbb{CP}^1 can only be the trivial group, an infinite cyclic group or a torus group (i.e., $\mathbb{Z} \oplus \mathbb{Z}$), which all have homology dimension at most 2. By Shapiro's Lemma $H_i(\Gamma, S_0(\mathbb{CP}^1))$ is therefore trivial for $i \geq 3$. The isotropy group in Γ of an unordered pair of distinct points of \mathbb{CP}^1 is either trivial or an infinite cyclic group. Again by Shapiro's Lemma, $H_i(\Gamma, S_1(\mathbb{CP}^1)) = 0$ for $i \geq 2$.

Putting everything together, we deduce that the E^1 -terms of the spectral sequence in total degree ≤ 4 have the following picture

$$\begin{array}{cccccc} & & 0 & & & \\ & & 0 & & 0 & \\ & & H_2(\Gamma, S_0(\mathbb{CP}^1)) & & 0 & \\ & & H_1(\Gamma, S_0(\mathbb{CP}^1)) & & H_1(\Gamma, S_1(\mathbb{CP}^1)) & & 0 & & 0 \\ & & S_0(\mathbb{CP}^1)_\Gamma & & S_1(\mathbb{CP}^1)_\Gamma & & S_2(\mathbb{CP}^1)_\Gamma & & S_3(\mathbb{CP}^1)_\Gamma & & S_4(\mathbb{CP}^1)_\Gamma \end{array}$$

In fact, all the omitted terms are zero except those in the bottom row. For the time being, assume the following

Lemma 3.3. *The d_1 differential $H_1(\Gamma, S_1(\mathbb{CP}^1)) \rightarrow H_1(\Gamma, S_0(\mathbb{CP}^1))$ is injective.*

It then follows that the d_2 differential maps $H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ to the trivial subgroup of $H_1(\Gamma, S_1(\mathbb{CP}^1))$. Now if Γ is cocompact, then it does not contain any parabolic elements. Hence for any point in \mathbb{CP}^1 the isotropy group in Γ is either trivial or infinite cyclic. It follows that $H_2(\Gamma, S_0(\mathbb{CP}^1))$ is trivial. So if Γ is cocompact, the

spectral sequence collapses at E^3 , and it follows that $H_3(\Gamma, \mathbb{Z}) \rightarrow H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ is an isomorphism.

To examine the relationship between $H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ and $H_3(\Gamma, \mathbb{CP}^1)$, we use the exact sequence

$$\cdots \rightarrow S_2(\mathbb{CP}^1) \rightarrow S_1(\mathbb{CP}^1) \rightarrow J\mathbb{CP}^1 \rightarrow 0$$

The spectral sequence associated to $H_*(\Gamma, S_{\bullet \geq 1}(\mathbb{CP}^1))$ computes $H_*(\Gamma, \mathbb{CP}^1)$. The E^1 -terms of this spectral sequence are simply the E^1 -terms of the spectral sequence of $H_*(\Gamma, S_\bullet(\mathbb{CP}^1))$ without the first column (with the appropriate degree shift). The only possible non-zero d_2 is the one from $H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ to $H_1(\Gamma, S_1(\mathbb{CP}^1))$ which has already been shown to be trivial above. It follows that this spectral sequence collapses at E^2 and therefore $H_3(\Gamma, \mathbb{CP}^1) \rightarrow H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ is an isomorphism. \square

The proof of Lemma 3.3 requires a more geometric argument.

Proof of Lemma 3.3. We view \mathbb{CP}^1 as the natural boundary of \mathbb{H}^3 . Consider the action of Γ on the set $(\mathbb{CP}^1)^2 - \Delta$ of ordered pairs (z_0, z_1) of distinct points of \mathbb{CP}^1 . If an element of Γ takes one such pair (z_0, z_1) to another (z'_0, z'_1) then it takes the \mathbb{H}^3 -geodesic joining z_0 and z_1 to the \mathbb{H}^3 -geodesic joining z'_0 and z'_1 . Since Γ has no torsion, no element of Γ can take (z_0, z_1) to (z_1, z_0) . Let X be a set of orbit representatives of the $\Gamma \times C_2$ action on $(\mathbb{CP}^1)^2 - \Delta$, where C_2 is the order 2 group that maps $(z_0, z_1) \mapsto (z_1, z_0)$. Then, as a $\mathbb{Z}\Gamma$ -module, $S_1(\mathbb{CP}^1)$ is the sum over $(z_0, z_1) \in X$ of $\mathbb{Z}\Gamma/\Gamma_{(z_0, z_1)}$. Let X_0 be the subset of X consisting of (z_0, z_1) whose isotropy groups $\Gamma_{(z_0, z_1)}$ are non-trivial (and hence infinite cyclic). Then by Shapiro's lemma

$$H_1(\Gamma, S_1(\mathbb{CP}^1)) \cong \bigoplus_{(z_0, z_1) \in X_0} H_1(\Gamma, \mathbb{Z}(\Gamma/\Gamma_{(z_0, z_1)})) = \bigoplus_{(z_0, z_1) \in X_0} \mathbb{Z} = \mathbb{Z}X_0.$$

Similarly, let X_1 denote the set of orbits in \mathbb{CP}^1/Γ whose isotropy groups are infinite cyclic. They are in fact the orbits of points that appear in X_0 . Again by Shapiro's Lemma, $H_1(\Gamma, S_0(\mathbb{CP}^1))$ is $\mathbb{Z}X_1$. The differential d_1 takes $(z_0, z_1) \in \mathbb{Z}X_0$ to $[z_1] - [z_0]$ in $\mathbb{Z}X_1$.

Now if $(z_0, z_1) \in X_0$ then z_0 and z_1 are in different Γ -orbits, for if not then the \mathbb{H}^3 -geodesic connecting z_0 and z_1 would map to a closed geodesic in $M = \mathbb{H}^3/\Gamma$ which is asymptotic to itself with reversed direction, which is clearly absurd. Similarly, if (z_0, z_1) and (z'_0, z'_1) are distinct elements of X_0 then z_0, z_1, z'_0, z'_1 represent four distinct Γ -orbits, for otherwise (z_0, z_1) and (z'_0, z'_1) would represent two closed geodesics in M which are asymptotic to each other. The injectivity of d_1 now follows immediately. \square

4. THE FUNDAMENTAL HOMOLOGY CLASS

Suppose $M = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold with a degree one ideal triangulation given by a geometric 3-cycle Y and map $f: Y - Y^{(0)} \rightarrow M$. Form the pull-back covering

$$\begin{array}{ccc} \widehat{Y - Y^{(0)}} & \longrightarrow & \mathbb{H}^3 \\ \downarrow & & \downarrow \\ Y - Y^{(0)} & \longrightarrow & M \end{array}$$

We can complete to get a simplicial complex \widehat{Y} with a Γ -action (which is free except maybe at the vertices) plus a Γ -equivariant map $\widehat{Y} \rightarrow \overline{\mathbb{H}^3}$. Since this map takes vertices to \mathbb{CP}^1 it induces a Γ -equivariant map of \widehat{Y} to the complete simplex on \mathbb{CP}^1 . Denote by $C_\bullet(\widehat{Y})$ the simplicial chain complex of \widehat{Y} . We get an induced map $C_\bullet(\widehat{Y}) \rightarrow S_\bullet(\mathbb{CP}^1)$ of chain complexes and hence a map in homology

$$\gamma: H_*(Y) = H_*(C_\bullet(\widehat{Y})_\Gamma) \rightarrow H_*(S_\bullet(\mathbb{CP}^1)_\Gamma),$$

since $C_\bullet(\widehat{Y})_\Gamma$ is the simplicial chain complex $C_\bullet(Y)$ of Y . There is also a natural homomorphism

$$\mu: H_3(S_\bullet(\mathbb{CP}^1)_\Gamma) \rightarrow \mathcal{P}(\mathbb{C}),$$

given by sending any 4-tuple of points in \mathbb{CP}^1 to their cross ratio.

Lemma 4.1. *If $[Y] \in H_3(Y)$ is the fundamental class then $\mu \circ \gamma[Y] \in \mathcal{P}(\mathbb{C})$ is the Bloch invariant $\beta(M)$.*

Proof. The fundamental class $[Y]$ is represented by the sum of the 3-simplices of Y . Under $\mu \circ \gamma$ this maps to the sum of the cross ratio parameters of the corresponding ideal simplices. This is $\beta(M)$ by definition. \square

Recall that in Proposition 3.2 we showed a natural isomorphism $H_3(\Gamma, \mathbb{CP}^1) \cong H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ and showed both groups are infinite cyclic with a natural generator which we will denote in both cases by $[M]$.

Lemma 4.2. $\gamma[Y] = [M] \in H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$.

Proof. We introduce a homomorphism $v: H_3(S_\bullet(\mathbb{CP}^1)_\Gamma) \rightarrow \mathbb{R}$ and show first that $v(\gamma[Y]) = \text{vol}(M)$, the volume of M . We will then show the same for the generator of $H_3(\Gamma, \mathbb{CP}^1)$ to complete the proof.

We will use $\langle z_0, \dots, z_n \rangle_\Gamma$ to denote the image of $\langle z_0, \dots, z_n \rangle$ in $S_n(\mathbb{CP}^1)_\Gamma$. We can think of $\langle z_0, \dots, z_3 \rangle_\Gamma$ as representing an ideal simplex in \mathbb{H}^3 that is well defined up to the action of Γ . We define $v(\langle z_0, \dots, z_3 \rangle)$ to be plus or minus the hyperbolic volume of this ideal simplex, with the sign chosen according as the orientation of the simplex agrees or not with the orientation of \mathbb{H}^3 (if the simplex is planar then the volume is zero and orientation is irrelevant). Given a 3-cycle $\alpha = \sum_i n_i \langle z_0^i, z_1^i, z_2^i, z_3^i \rangle_\Gamma$ in $S_3(\mathbb{CP}^1)_\Gamma$ we define $v(\alpha) = \sum_i n_i v(\langle z_0^i, z_1^i, z_2^i, z_3^i \rangle)$. If we have five distinct points $z_0, \dots, z_4 \in \mathbb{CP}^1$ then it is easy to see geometrically that $v(\partial \langle z_0, \dots, z_4 \rangle) = 0$. It follows that v induces a map, which we also call v :

$$v: H_3(S_\bullet(\mathbb{CP}^1)_\Gamma) \rightarrow \mathbb{R}.$$

The value of v on a class given by a degree one ideal triangulation is the sum of the signed volumes of the ideal tetrahedra of the triangulation, which is just the volume of M , by the degree one condition and Fubini's theorem.

Now let Z be the end compactification of M . Then $Z = X/\Gamma$ where $X = \mathbb{H}^3 \cup \mathcal{C}$ with \mathcal{C} as in the previous section. Note that Z is homeomorphic to the result of collapsing each boundary component of M_0 to a point, where M_0 is as in the proof of Proposition 3.2. In particular, $H_3(Z) = H_3(M_0, \partial M_0) = \mathbb{Z}$.

Consider a triangulation of Z and the lifted triangulation of X . Let $C_\bullet(X)$ denote the simplicial chain complex of X . We can think of $S_q(\mathbb{CP}^1)$ as being generated by arbitrary $(q+1)$ -tuples of points of \mathbb{CP}^1 modulo the relations

$$\langle z_0, \dots, z_q \rangle = \text{sgn}(\tau) \langle z_{\tau(0)}, \dots, z_{\tau(q)} \rangle$$

for any permutation τ of $\{0, \dots, q\}$ and

$$\langle z_0, \dots, z_q \rangle = 0 \quad \text{if the } z_i \text{ are not distinct.}$$

We can thus map $C_\bullet(X) \rightarrow S_\bullet(\mathbb{CP}^1)$ by taking any equivariant map of the vertices of the triangulation of X to \mathbb{CP}^1 . We claim that the induced map $C_\bullet(X)_\Gamma = C_\bullet(Z) \rightarrow S_\bullet(\mathbb{CP}^1)_\Gamma$ induces the isomorphism $H_3(\Gamma, \mathcal{C}) = H_3(M_0, \partial M_0) = H_3(Z) \rightarrow H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ of Proposition 3.2.

If M is compact then, since $X = \mathbb{H}^3$ is contractible, $C_\bullet(X)$ is a free $\mathbb{Z}\Gamma$ resolution of \mathbb{Z} , and the above map is indeed the map of Proposition 3.2.

Suppose M is non-compact, so \mathcal{C} is non-empty. Let $\tilde{C}_\bullet(X)$ be the reduced chain complex (so $\tilde{C}_q(X) = C_q(X)$ for $q > 0$ and $\tilde{C}_0(X)$ is the kernel of augmentation $\epsilon: C_0(X) \rightarrow \mathbb{Z}$). Then, since X is contractible,

$$\dots \rightarrow C_2(X) \rightarrow C_1(X) \rightarrow \tilde{C}_0(X) \rightarrow 0$$

is exact. It is clearly a free $\mathbb{Z}\Gamma$ -resolution of $\tilde{C}_0(X)$. Moreover, $\tilde{C}_0(X)$ is isomorphic to $J\mathcal{C} \oplus F$ in the notation of the previous section, where F is the submodule of $C_0(X)$ generated by *finite* vertices. Since F is clearly a free $\mathbb{Z}\Gamma$ -module, it follows that $C_{\geq 1}(X)_\Gamma = C_{\geq 1}(Z)$ computes $H_*(\Gamma, \mathcal{C})$. This gives the isomorphism $H_3(M_0, \partial M_0) = H_3(Z) \cong H_3(\Gamma, \mathcal{C})$ used in the proof of Proposition 3.2.

Now consider the above map $C_\bullet(X) \rightarrow S_\bullet(\mathbb{CP}^1)$. The induced map of reduced groups in degree zero, $\tilde{C}_0(X) \rightarrow J\mathbb{CP}^1$, is, up to a map of free summands, the inclusion $J\mathcal{C} \rightarrow J\mathbb{CP}^1$. The map $C_{\geq 1}(X)_\Gamma \rightarrow S_{\geq 1}(\mathbb{CP}^1)_\Gamma$ thus induces the map $H_3(\Gamma, \mathcal{C}) = H_3(Z) \rightarrow H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ of Proposition 3.2 as claimed.

To compute v of the generator of $H_3(S_\bullet(\mathbb{CP}^1)_\Gamma)$ we must thus map the vertices of X equivariantly to \mathbb{CP}^1 and then sum the volumes of the ideal simplices in \mathbb{H}^3 corresponding to a set of Γ -orbit representatives of the 3-simplices of X . Replace the simplices of the triangulation of X by geodesic simplices. Then the sum of the signed volumes of a set of orbit representatives of these simplices is $\text{vol}(M)$. In fact, if we consider the triangulation of Z to be given by a map from an abstract simplicial complex to Z , then the above sum of signed simplex volumes is just the integral of the pull-back of the volume form on Z to this simplicial complex. If we now move the vertices in \mathbb{H}^3 of the triangulation of X continuously and equivariantly within \mathbb{H}^3 , then we are just homotoping the map of the simplicial complex to Z , so the sum of signed simplex volumes stays equal to $\text{vol}(M)$. If we now let the vertices move continuously all the way out to \mathbb{CP}^1 , then the simplex volumes change continuously, so their sum is still $\text{vol}(M)$ when the vertices reach \mathbb{CP}^1 . \square

Combining the isomorphism of Proposition 3.2 with the above map μ gives a map

$$H_3(\Gamma, \mathbb{CP}^1) \rightarrow \mathcal{P}(\mathbb{C}).$$

The preceding lemmas imply immediately:

Proposition 4.3. *This homomorphism maps $[M] \in H_3(\Gamma, \mathbb{CP}^1)$ to $\beta(M)$. In particular $\beta(M)$ is independent of triangulation.* \square

Remark 4.4. Note that the above map $H_3(\Gamma, \mathbb{CP}^1) \rightarrow \mathcal{P}(\mathbb{C})$ is defined for *any* subgroup $\Gamma \subset \text{PGL}(2, \mathbb{C})$, including $\text{PGL}(2, \mathbb{C})$ itself. Now $\text{PGL}(2, \mathbb{C})$ acts transitively on the basis of $S_2(\mathbb{CP}^1)$, so the boundary map $S_3(\mathbb{CP}^1)_{\text{PGL}(2, \mathbb{C})} \rightarrow S_2(\mathbb{CP}^1)_{\text{PGL}(2, \mathbb{C})}$

is trivial. It follows that

$$H_3(S_\bullet(\mathbb{CP}^1)_{\mathrm{PGL}(2,\mathbb{C})}) \xrightarrow{\cong} \mathcal{P}(\mathbb{C}).$$

The map of the above proposition can thus be described as the map $H_3(S_\bullet(\mathbb{CP}^1)_\Gamma) \rightarrow H_3(S_\bullet(\mathbb{CP}^1)_{\mathrm{PGL}(2,\mathbb{C})}) = \mathcal{P}(\mathbb{C})$ induced by the inclusion $\Gamma \rightarrow \mathrm{PGL}(2,\mathbb{C})$.

5. COMPLETION OF PROOF OF THEOREM 1.1

For Theorem 1.1 it remains to show that $\beta(M)$ is in $\mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})$.

We first assume M is not compact. The inclusion $\mathcal{JC} \rightarrow \mathcal{ZC}$ induces (because of the dimension shift in the definition of $H_*(\Gamma, \mathcal{C})$) a map $H_3(\Gamma, \mathcal{C}) \rightarrow H_2(\Gamma, \mathcal{ZC})$. Similarly, the inclusion $\mathcal{JCP}^1 \rightarrow \mathcal{ZCP}^1$ induces a map $H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1) \rightarrow H_2(\mathrm{PGL}(2,\mathbb{C}), \mathbb{ZCP}^1)$. We thus have the following commutative diagram

$$\begin{array}{ccc} H_3(\Gamma, \mathcal{C}) & \longrightarrow & H_2(\Gamma, \mathcal{ZC}) \\ \downarrow & & \downarrow \phi \\ H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1) & \longrightarrow & H_2(\mathrm{PGL}(2,\mathbb{C}), \mathbb{ZCP}^1) \end{array}$$

Since $\mathrm{PGL}(2,\mathbb{C})$ acts transitively on \mathbb{CP}^1 with isotropy group the Borel subgroup B of upper triangular matrices, we have $H_2(\mathrm{PGL}(2,\mathbb{C}), \mathbb{ZCP}^1) \cong H_2(B; \mathbb{Z})$. Let T denote the maximal torus in $\mathrm{PGL}(2,\mathbb{C})$ and U the upper unipotent matrices. Then $H_2(B, \mathbb{Z}) \cong H_2(T, \mathbb{Z}) \cong \wedge^2 \mathbb{C}^*$, induced by $B \rightarrow B/U \cong T$, (cf. [14] (A11)). By [14], one has the following commutative diagram

$$\begin{array}{ccc} H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1) & \longrightarrow & H_2(\mathrm{PGL}(2,\mathbb{C}), \mathbb{ZCP}^1) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{P}(\mathbb{C}) & \xrightarrow{\lambda} & \wedge^2 \mathbb{C}^* \end{array}$$

and the map λ is simply given by $\lambda[z] = 2(z \wedge (1 - z))$.

So to show that the element $\beta(M)$ is in fact in $\mathcal{B}(\mathbb{C})$, it suffices to show that the homomorphism ϕ is trivial. By Shapiro's lemma, $H_2(\Gamma, \mathcal{ZC})$ is the sum of H_2 of isotropy groups of Γ on \mathcal{C} . These isotropy groups are unipotent subgroups of $\mathrm{PGL}(2,\mathbb{C})$, so H_2 of such an isotropy group when mapped to $H_2(B, \mathbb{Z})$ under ϕ factors through $H_2(U, \mathbb{Z})$. Since $H_2(B, \mathbb{Z}) \cong H_2(T, \mathbb{Z})$ is induced by $B \rightarrow B/U \cong T$, the homomorphism from $H_2(U, \mathbb{Z})$ to $H_2(B, \mathbb{Z})$ is trivial. It follows that ϕ is trivial, hence $\beta[M]$ is an element in $\mathcal{B}(\mathbb{C})$.

If M is compact then an argument was given in the Introduction. We expand on it since we need it again below. In this case $\beta(M)$ is in the image of $H_3(\Gamma) \rightarrow H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1) \rightarrow \mathcal{P}(\mathbb{C})$, which factors through $H_3(\mathrm{PGL}(2,\mathbb{C}))$. Since the sequence $H_3(\mathrm{PGL}(2,\mathbb{C})) \rightarrow H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1) \rightarrow H_2(\mathrm{PGL}(2,\mathbb{C}), \mathbb{ZCP}^1)$ is exact, the result again follows from the above diagram. This finishes the proof of Theorem 1.1. \square

6. RESTRICTING THE FIELD OF DEFINITION

There are two situations in which we can define the Bloch invariant $\beta(M)$ as an element of $\mathcal{P}(K)$ for a subfield $K \subset \mathbb{C}$ rather than as an element of $\mathcal{P}(\mathbb{C})$.

Case 1. If M is a hyperbolic manifold with a degree one ideal triangulation into ideal simplices $\Delta_1, \dots, \Delta_n$ we call the subfield of \mathbb{C} generated by the cross ratio

parameters z_1, \dots, z_n of these simplices the *tetrahedron field* associated with the triangulation. If this tetrahedron field is a subfield of K we can define

$$\beta(M) := \sum_{i=1}^n [z_i] \in \mathcal{P}(K).$$

Case 2. If $M = \mathbb{H}^3/\Gamma$ with $\Gamma \subset \mathrm{PGL}(2, K)$ then we can define $\beta(M)$ as the image of $[M]$ under the composition

$$H_3(\Gamma, \mathcal{C}) \xrightarrow{\cong} H_3(S_\bullet(\mathbb{CP}^1(K))_\Gamma) \rightarrow H_3(S_\bullet(\mathbb{CP}^1(K))_{\mathrm{PGL}(2, K)}) \xrightarrow{\cong} \mathcal{P}(K),$$

where $\mathbb{CP}^1(K) = K\mathbb{P}^1$ is the set of K -rational points of \mathbb{CP}^1 .

Theorem 6.1. *If $M = \mathbb{H}^3/\Gamma$ has a tetrahedron field contained in K , then Γ has a discrete embedding into $\mathrm{PGL}(2, \mathbb{C})$ with image in $\mathrm{PGL}(2, K)$. That is, if Case 1 holds then so does Case 2. Moreover, in either case $\beta(M)$ lies in $\mathcal{B}(K)$ and only depends on M and the field K .*

Proof. If $f: Y - Y^{(0)} \rightarrow M$ is a degree one ideal triangulation then, as above, we can lift to a map $\widehat{Y} \rightarrow \mathbb{H}^3$. Let $V \subset \mathbb{CP}^1 = \partial\mathbb{H}^3$ be the image of the set of vertices of \widehat{Y} . Then, as in [28], if we apply an isometry to put three of the points of V at 0, 1, and ∞ , then the fact that Y is connected implies that V will be a subset of $\mathbb{CP}^1(K)$, where K is the field generated by the cross ratio parameters of the simplices of K . Since V is a Γ -invariant set, it also follows as in [28] that Γ is then in $\mathrm{PGL}(2, K)$. We can now repeat the proof of Theorem 1.1 using $\mathbb{CP}^1(K)$ in place of \mathbb{CP}^1 . The proof that $\beta(M) \in \mathcal{P}(K)$ is independent of choices is just as before. The proof that it lies in $\mathcal{B}(K)$ needs slightly more care.

We first assume M is non-compact. We consider the versions of the commutative diagrams of section 5 with \mathbb{C} replaced by K . The first,

$$\begin{array}{ccc} H_3(\Gamma, \mathcal{C}) & \longrightarrow & H_2(\Gamma, \mathbb{Z}\mathcal{C}) \\ \downarrow & & \downarrow \phi \\ H_3(\mathrm{PGL}_2(K), \mathbb{CP}^1(K)) & \longrightarrow & H_2(\mathrm{PGL}_2(K), \mathbb{Z}\mathbb{CP}^1(K)), \end{array}$$

is unproblematic. For the second, one must follow the argument presented in the appendix of Dupont-Sah [14] carefully. It gives a commutative diagram

$$\begin{array}{ccc} H_3(\mathrm{PGL}_2(K), \mathbb{CP}^1(K)) & \longrightarrow & H_2(\mathrm{PGL}_2(K), \mathbb{Z}\mathbb{CP}^1(K)) \\ \downarrow T & & \downarrow \cong \\ \mathcal{P}'(K) & \xrightarrow{\lambda} & \wedge^2 K^* \end{array}$$

with the only difference to the diagram for $K = \mathbb{C}$ being that T is now only an isomorphism modulo 2-torsion and we are using the Dupont-Sah version $\mathcal{P}'(K)$ of the Bloch group rather than $\mathcal{P}(K)$ (see section 2.2). The map λ is still $[z] \mapsto 2(z \wedge (1 - z))$. Since this vanishes on the extra relations (2) that define $\mathcal{P}(K)$ from $\mathcal{P}'(K)$, we can replace $\mathcal{P}'(K)$ by $\mathcal{P}(K)$ in the above diagram, and the rest of the argument carries through as in the case $K = \mathbb{C}$. Similarly, the argument in the compact case also carries through. \square

Now if M has cusps, then by [15], M has a genuine decomposition into convex ideal polyhedra with vertices at the cusps. We can further subdivide these polyhedra into ideal tetrahedra. These subdivisions may not agree on common faces

of the ideal polyhedra, in which case we must add some flat ideal tetrahedra to mediate between the two triangulations of the faces in question. We obtain what we call a “genuine” ideal triangulation. In [28], it was shown that all the cross ratio parameters of these ideal tetrahedra lie in the invariant trace field $k = k(M)$. Hence $\beta(M)$ can in fact be defined in $\mathcal{P}(k)$.

Corollary 6.2. *If M is non-compact then $\beta(M)$ is well-defined in $\mathcal{B}(k)$, where $k = k(M)$ is the invariant trace field of M . \square*

Proof of Theorem 1.2. If Γ can be conjugated into $\mathrm{PGL}(2, K)$, then K is called a potential coefficient field of Γ (or M) in [22]. By theorem 6.1, $\beta(M)$ can be defined in $\mathcal{B}(K)$ for each potential coefficient field K . According to [22], one can find infinitely many such potential coefficient fields K , which are at most quadratic extensions of the trace field K_0 of Γ . Choose two of them, say K_1 and K_2 . Then $K_1 \cap K_2 = K_0$. Let L be any field which contains K_1 and K_2 . The inclusions $K_i \rightarrow L$ induce modulo torsion injections $\mathcal{B}(K_i) \rightarrow \mathcal{B}(L)$ for $i = 0, 1, 2$ (see e.g., [30]). Since we are willing to work modulo torsion, we shall identify each $\mathcal{B}(K_i)$ with its image in $\mathcal{B}(L)$. In [30] it is shown that with these identifications $\mathcal{B}(K_0) \subset \mathcal{B}(K_1) \cap \mathcal{B}(K_2)$ with torsion quotient. By Theorem 6.1 we know $\beta(M) \in \mathcal{B}(L)$ is in the subgroup $\mathcal{B}(K_1) \cap \mathcal{B}(K_2)$ and hence some positive multiple is in the subgroup $\mathcal{B}(K_0) \subset \mathcal{B}(L)$. In particular, $\beta(M)$ is well defined in $\mathcal{B}(\mathrm{TraceField}(M)) \otimes \mathbb{Q}$.

Now, if $\Gamma^{(2)}$ is the subgroup of Γ generated by squares of elements in Γ then $\Gamma/\Gamma^{(2)}$ is an elementary abelian group of order 2^s for some s . In [36] it is shown that the trace field of $\Gamma^{(2)}$ is the invariant trace field $k(M)$ of M . Let $M^{(2)} = \mathbb{H}^3/\Gamma^{(2)}$. Then clearly $\beta(M^{(2)}) = 2^s \beta(M)$ and since $\beta(M^{(2)}) \in \mathcal{B}(k(M))$ it follows that there is a well defined element $\beta_{k(M)}(M) \in \mathcal{B}(k(M)) \otimes \mathbb{Q}$ whose image is $\beta(M)$ in $\mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$. \square

7. CHERN-SIMONS INVARIANT

Theorem 1.3 involved the *Bloch regulator map*

$$\rho: \mathcal{B}(\mathbb{C}) \longrightarrow \mathbb{C}/\mathbb{Q},$$

which is defined as follows. For $z \in \mathbb{C} - \{0, 1\}$, define

$$\rho(z) = \frac{\log z}{2\pi i} \wedge \frac{\log(1-z)}{2\pi i} + 1 \wedge \frac{\mathcal{R}(z)}{2\pi^2},$$

where $\mathcal{R}(z)$ is the “Rogers dilogarithm function”

$$\mathcal{R}(z) = \frac{1}{2} \log(z) \log(1-z) - \int_0^z \frac{\log(1-t)}{t} dt.$$

See section 4 of [14] or [20] for details on how to interpret this formula. This ρ vanishes on the relations (1) and (2) which define $\mathcal{P}(\mathbb{C})$ and hence ρ induces a map

$$\rho: \mathcal{P}(\mathbb{C}) \longrightarrow \mathbb{C} \wedge_{\mathbb{Z}} \mathbb{C}.$$

This fits in a commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathbb{C}) & \xrightarrow{\mu} & \mathbb{C}^* \wedge \mathbb{C}^* \\ \downarrow \rho & & \downarrow = \\ \mathbb{C} \wedge \mathbb{C} & \xrightarrow{\epsilon} & \mathbb{C}^* \wedge \mathbb{C}^* \end{array}$$

where $\epsilon = 2(e \wedge e)$ with $e(z) = \exp(2\pi iz)$. The kernel of μ is $\mathcal{B}(\mathbb{C})$ and the kernel of ϵ is \mathbb{C}/\mathbb{Q} . Hence ρ restricts to give the desired map $\rho: \mathcal{B}(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Q}$.

Recall that Theorem 1.3 is the formula

$$\frac{2\pi^2}{i}\rho(\beta(M)) = \text{vol}(M) + i \text{CS}(M) \in \mathbb{C}/(i\pi^2\mathbb{Q}).$$

The volume part of this result is not hard (see e.g. [14], [27], [30]). In fact the imaginary part of $2\pi^2\rho$ can be extended to a map $\text{vol}: \mathcal{P}(\mathbb{C}) \rightarrow \mathbb{R}$ given on generators by $[z] \mapsto D_2(z)$ where D_2 is the *Bloch-Wigner dilogarithm* (cf. [5])

$$D_2(z) = \text{Im} \ln_2(z) + \log |z| \arg(1 - z), \quad z \in \mathbb{C} - \{0, 1\}.$$

The name vol is justified because $D_2(z)$ is the hyperbolic volume of an ideal tetrahedron Δ with cross ratio z .

The real part of the formula, giving Chern-Simons invariant, lies deeper. If $M = \mathbb{H}^3/\Gamma$ is compact then Dupont [11] proved² the above formula with $\beta(M)$ replaced by the image of the fundamental class $[M] \in H_3(\Gamma; \mathbb{Z})$ under $H_3(\Gamma; \mathbb{Z}) \rightarrow H_3(\text{PGL}(2, \mathbb{C})^\delta; \mathbb{Z}) \rightarrow \mathcal{B}(\mathbb{C})$. But this image is $\beta(M)$ by Propositions 3.2 and 4.3. Thus Theorem 1.3 follows in this case. If M is non-compact it follows from the formula for Chern-Simons invariant of [27]. We give some of the details for completeness.

Suppose M has an ideal triangulation which subdivides it into n ideal tetrahedra

$$M = \Delta_1 \cup \dots \cup \Delta_n.$$

Choose an ordering of the vertices of the j -th tetrahedron that is compatible with the orientation of M and let z_j^0 be the cross ratio parameter which then describes this tetrahedron. Let

$$\mathcal{Z}^0 = \begin{pmatrix} \log z_1^0 \\ \vdots \\ \log z_n^0 \\ \log(1 - z_1^0) \\ \vdots \\ \log(1 - z_n^0) \end{pmatrix}.$$

Recall from [32] (see also [27] which we are following here) that if M has h cusps then the z_j^0 are determined by so-called consistency and cusp relations which can be written in the form

$$U\mathcal{Z}^0 = \pi i \mathbf{d},$$

where U is a certain integral $(n + 2h) \times 2n$ -matrix and

$$\mathbf{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_{n+2h} \end{pmatrix}$$

is some integral vector. Geometrically, the consistency relations say that the tetrahedra fit together around each edge of the triangulation and the cusp relations say that generators of the cusp groups represent parabolic isometries.

The consistency relations are given by a $n \times 2n$ submatrix of U and the component containing $z^0 = (z_1^0, \dots, z_n^0)$ in the set of $z = (z_1, \dots, z_n)$ that satisfy these relations will be called *Dehn surgery space* and denoted \mathcal{D} (this is actually a 2^h -fold branched

²The proof in [11] is for the “flat” Chern-Simons invariant and the assumption implicit there that this agrees with the “riemmanian” Chern-Simons invariant for M is confirmed in [13].

cover of what is usually called Dehn surgery space, see [32], but the difference is irrelevant to the current discussion).

The equation

$$(3) \quad U\mathbf{c} = \mathbf{d}$$

has a solution $\mathbf{c} = \mathcal{Z}^0/\pi i \in \mathbb{C}^{2n}$. Since U is an integral matrix, equation (3) also has solutions

$$\mathbf{c} = \begin{pmatrix} c'_1 \\ \vdots \\ c'_n \\ c''_1 \\ \vdots \\ c''_n \end{pmatrix} \in \mathbb{Q}^{2n}.$$

In [27] it is shown that solutions \mathbf{c} can be found in \mathbb{Z}^{2n} , in fact even in a certain affine sublattice of \mathbb{Z}^{2n} .

Let M' be the result of a hyperbolic Dehn filling on M obtained by deforming the parameter $z^0 = (z_1^0, \dots, z_n^0)$ to a new value $z = (z_1, \dots, z_n)$ in Dehn surgery space \mathcal{D} (cf. e.g., [32]). Topologically M' differs from M in that a new closed geodesic γ_j has been added at the j -th cusp for some $j \in \{1, \dots, h\}$. Let λ_j be the complex number which has real part equal to the length of this geodesic and imaginary part equal to its torsion (the latter is only well-defined modulo 2π). If no geodesic has been added at the j -th cusp we put $\lambda_j = 0$.

Theorem 7.1 ([27]). *Given any solution $\mathbf{c} \in \mathbb{Q}^{2n}$ to equation (3), there exists a constant $\alpha = \alpha(\mathbf{c}) \in i\pi^2\mathbb{Q}$ such that if M' is any result of hyperbolic Dehn filling obtained by deforming M as above, then*

$$\text{vol}(M') + i \text{CS}(M') = \alpha - \frac{\pi}{2} \sum_{j=1}^h \lambda_j - i \sum_{\nu=1}^n \left(\mathcal{R}(z_\nu) - \frac{i\pi}{2} (c'_\nu \log(1-z_\nu) - c''_\nu \log(z_\nu)) \right).$$

Moreover, if $\mathbf{c} \in \mathbb{Z}^{2n}$ then conjecturally $\alpha \in \frac{i\pi^2}{12}\mathbb{Z}$, while if \mathbf{c} is in the sublattice mentioned above then conjecturally $\alpha \in \frac{i\pi^2}{6}\mathbb{Z}$.

The following summary of the proof in [27] also explains how Theorem 1.3 follows.

In [43] Yoshida proved a conjecture of [32] that a formula of the following form hold on the space of Dehn fillings of M :

$$\text{vol}(M') + i \text{CS}(M') = -\frac{\pi}{2} \sum_{j=1}^h \lambda_j + f(z),$$

where $f(z)$ is some analytic function of $z = (z_1, \dots, z_n) \in \mathcal{D}$. The formula of the above theorem has this form and it is not hard to verify that its real part is correct with $\alpha = 0$. We thus have two analytic functions whose real parts agree at the points $z \in \mathcal{D}$ that correspond to Dehn fillings. These points limit on z^0 , which is a smooth point of \mathcal{D} , from all tangent directions. It follows that the two analytic functions agree up to an imaginary constant on \mathcal{D} . This imaginary constant is α . On the other hand, it is shown in [27] that the right side of the formula of Theorem 7.1 without the constant α gives $\frac{2\pi^2}{i}\rho(\beta(M'))$ modulo $i\pi^2\mathbb{Q}$. We have already shown this equals $\text{vol}(M') + i \text{CS}(M')$ modulo $i\pi^2\mathbb{Q}$ if M' is compact. Thus $\alpha \in$

$i\pi^2\mathbb{Q}$, so Theorem 7.1 is proved. (Note that we have used that Theorem 1.3 holds in the compact case, which Dupont [11] proved for the image of the fundamental class rather than for $\beta(M)$. The result that the image of fundamental class equals $\beta(M)$ (Proposition 4.3) thus fills a gap in the proof of Theorem 7.1 in [27].)

We have just mentioned that [27] shows that the right side of the formula of Theorem 7.1 equals $\frac{2\pi^2}{i}\rho(\beta(M))$ modulo $i\pi^2\mathbb{Q}$. Applying this and Theorem 7.1 to M itself proves Theorem 1.3 in the non-compact case.

8. GENERALIZATIONS: HIGHER DIMENSIONS AND HOMOMORPHISMS

Definition 8.1. Let $S_q(\partial\overline{\mathbb{H}}^n)$ be the abelian group generated by arbitrary $(q+1)$ -tuples of points of $\partial\overline{\mathbb{H}}^n$ modulo the relations

$$\langle z_0, \dots, z_q \rangle = \text{sgn } \tau \langle z_{\tau(0)}, \dots, z_{\tau(q)} \rangle$$

for any permutation τ of $\{0, \dots, q\}$ and

$$\langle z_0, \dots, z_q \rangle = 0 \quad \text{if the } z_i \text{ are not distinct.}$$

Define a boundary map $S_q(\partial\overline{\mathbb{H}}^n) \rightarrow S_{q-1}(\partial\overline{\mathbb{H}}^n)$ by the usual formula $\partial\langle z_0, \dots, z_q \rangle = \sum_{i=0}^q (-1)^i \langle z_0, \dots, \widehat{z}_i, \dots, z_q \rangle$. Note that $S_\bullet(\partial\overline{\mathbb{H}}^n)_{\text{Isom}^+(\mathbb{H}^n)}$ is the result of adding the relations

$$\langle gz_0, \dots, gz_q \rangle = \langle z_0, \dots, z_q \rangle$$

for $g \in \text{Isom}^+(\mathbb{H}^n)$ to the above definition. We define

$$\mathcal{P}_n := H_n(S_\bullet(\partial\overline{\mathbb{H}}^n)_{\text{Isom}^+(\mathbb{H}^n)}).$$

In particular, if $n = 3$ then $S_\bullet(\partial\overline{\mathbb{H}}^3) = S_\bullet(\mathbb{C}\mathbb{P}^1)$ and $\mathcal{P}_3 = \mathcal{P}(\mathbb{C})$.

Now let M^n be a manifold which is homeomorphic to the interior of a compact manifold M_0 with (possibly empty) boundary such that the universal cover \widetilde{M}_0 and all its boundary components are contractible. For example, a complete hyperbolic n -manifold of finite volume has this property. Let $\Gamma = \pi_1(M) = \pi_1(M_0)$. We will define an invariant of a homomorphism $f: \Gamma \rightarrow \text{Isom}^+(\mathbb{H}^n)$ which generalizes the invariant $\beta(M)$ of previous sections. We shall need the homomorphism to satisfy a condition which we describe and discuss below.

Γ acts by covering transformations on \widetilde{M}_0 . Let X be the end compactification of \widetilde{M} and Z the end compactification of M (these can be obtained by collapsing each boundary component of \widetilde{M}_0 respectively M_0 to a point). Denote $\mathcal{C} = X - \widetilde{M}$. For each $c \in \mathcal{C}$ denote $P_c = \{g \in \Gamma : gc = c\}$. P_c is isomorphic to the fundamental group of the boundary component of M_0 corresponding to c .

Condition 8.2. We assume that $f: \Gamma \rightarrow \text{Isom}^+(\mathbb{H}^n)$ has the property that $f(P_c)$ fixes some point $x_c \in \partial\overline{\mathbb{H}}^n$ for each c . As we discuss at the end of this section, this condition can be relaxed and is then automatically satisfied in many cases, for instance if M is an odd-dimensional hyperbolic manifold.

Γ acts on $\partial\overline{\mathbb{H}}^n$ via the homomorphism f . We can choose the assignment $h: c \mapsto x_c$ to be Γ -equivariant, since if $f(P_c)$ fixes x_c then $f(P_{gc}) = f(gP_c g^{-1})$ fixes $f(g)x_c$. We then get an induced map $H_n(\Gamma, \mathcal{C}) \rightarrow H_n(\text{Isom}^+(\mathbb{H}^n), \partial\overline{\mathbb{H}}^n)$ and the image $\beta_h(f) \in H_n(\text{Isom}^+(\mathbb{H}^n), \partial\overline{\mathbb{H}}^n)$ of the fundamental class in $H_n(\Gamma, \mathcal{C}) = H_n(M_0, \partial M_0) = \mathbb{Z}$ is an invariant of the given situation. If M is compact we use

instead the image of the fundamental class under $H_n(\Gamma) \rightarrow H_n(\text{Isom}^+(\mathbb{H}^n)) \rightarrow H_n(\text{Isom}^+(\mathbb{H}^n), \partial\overline{\mathbb{H}^n})$.

In general $\beta_h(f)$ presumably depends on the choice of h . But trivially:

Proposition 8.3. *If h is unique, for example if M is compact, or M is a finite volume hyperbolic manifold and $f: \Gamma \rightarrow \text{Isom}^+(\mathbb{H}^n)$ the homomorphism that determines its hyperbolic structure, then $\beta_h(f) \in H_n(\text{Isom}^+(\mathbb{H}^n), \partial\overline{\mathbb{H}^n})$ is a well defined invariant of f that generalizes the $\beta(M)$ of previous sections. \square*

We also have a natural map

$$\mu: H_n(\text{Isom}^+(\mathbb{H}^n), \partial\overline{\mathbb{H}^n}) \rightarrow \mathcal{P}_n$$

generalizing the map $H_3(\text{PGL}(2, \mathbb{C}), \mathbb{C}\mathbb{P}^1) \xrightarrow{\cong} \mathcal{P}(\mathbb{C})$ of remark 4.4.

Theorem 8.4. *$\mu(\beta_h(f))$ does not depend on h and is thus an invariant just of M and $f: \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^n)$. We denote it simply $\beta(f) \in \mathcal{P}_n$.*

Proof. The compact case is trivial, so we assume M non-compact. We shall think of $S_n(\partial\overline{\mathbb{H}^n})_{\text{Isom}^+(\mathbb{H}^n)}$ as being generated by isometry classes of ideal n -simplices. Triangulate Z and lift the triangulation to a triangulation of X . Note that Z is homeomorphic to the result of adding a cone on each boundary component of M_0 . We may therefore assume that Z is triangulated by first triangulating M_0 and then coning the triangulation at each boundary component.

As in the proof of 4.2, the map $H_n(\Gamma, \mathcal{C}) \rightarrow \mathcal{P}_n$ can be identified with the map $H_n(Z) = H_n(\mathcal{C}_\bullet(X)_\Gamma) \rightarrow H_n(S_\bullet(\partial\overline{\mathbb{H}^n})_{\text{Isom}^+(\mathbb{H}^n)})$ induced by extending the map $h: \mathcal{C} \rightarrow \partial\overline{\mathbb{H}^n}$ to an equivariant map h' of all the vertices of the triangulation of X to $\partial\overline{\mathbb{H}^n}$. Given an n -simplex of Z , we can lift it to X and then h' maps it to an ideal $(n+1)$ -simplex $\langle z_0, \dots, z_n \rangle \in S_n(\partial\overline{\mathbb{H}^n})$ which is well-defined up to the action of Γ . In particular it is well-defined up to isometry. $\mu(\beta_h(f))$ is the sum of these ideal simplices corresponding to simplices of Z .

Since there are only finitely many Γ -orbits in \mathcal{C} , it suffices to see that $\mu(\beta_h(f))$ is unchanged if we change h on just one Γ -orbit of \mathcal{C} . Let v be the corresponding vertex of Z and let $[v, v_1, \dots, v_n]$ be a simplex of Z that involves v . Let $\langle z, z_1, \dots, z_n \rangle$ be the corresponding ideal simplex before its change and $\langle z', z_1, \dots, z_n \rangle$ the ideal simplex after the change. In $\mathcal{P}_n = H_n(S_\bullet(\partial\overline{\mathbb{H}^n})_{\text{Isom}^+(\mathbb{H}^n)})$ the difference of these two equals

$$\sum_{i=1}^n (-1)^i \langle z', z, z_1, \dots, \tilde{z}_i, \dots, z_n \rangle.$$

But the link of the vertex v is a manifold (it is the corresponding boundary component of M_0) and is triangulated by simplices like $[v_1, \dots, v_n]$, so the simplex $[v_2, \dots, v_n]$ appears in exactly two simplices of this link with opposite orientations. Thus the summand $-\langle z', z, z_2, \dots, z_n \rangle$ of the above sum is canceled by a corresponding summand from a neighboring simplex. This is true for all the summands, so the result follows. \square

In particular, since volume is well defined on \mathcal{P}_n , theorem 8.4 gives a way of defining the ‘‘volume’’ of a homomorphism f as above. The existence of such a volume in the 3-dimensional case was mentioned in [41]. In the three-dimensional case our proof easily gives a little more information.

Theorem 8.5. *Assume M is a hyperbolic 3-manifold. If f is the homomorphism corresponding to some Dehn filling M' of M then $\beta(f) = \beta(M')$. If each cusp subgroup of Γ has non-trivial elements γ with $f(\gamma)$ parabolic (or trivial) then $\beta(f) \in \mathcal{B}(\mathbb{C})$.*

Proof. If we are given an ideal triangulation of M then we can compute $\beta(f)$ as the sum of ideal simplices obtained by taking the vertices to fixed points of the corresponding cusp subgroups for the Γ -action given by f . If f corresponds to a Dehn filling then this gives a degree one ideal triangulation of M' (see e.g., [32]) so the sum of the resulting ideal simplices represents $\beta(M')$. The final sentence follows by the same argument as in section 5, except that the isotropy groups of Γ on \mathcal{C} may map to cyclic rather than unipotent subgroups of $\mathrm{PGL}(2, \mathbb{C})$ and we get a zero map in homology in the cyclic case because $H_2(\mathbb{Z}) = 0$. \square

We end this section by discussing to what degree Condition 8.2 is restrictive, and to what extent it can be relaxed. If we consider $\beta_h(f) \in H_n(\mathrm{Isom}^+(\mathbb{H}^n), \partial\overline{\mathbb{H}^n}) \otimes \mathbb{Q}$ and $\beta(f) \in \mathcal{P}_n \otimes \mathbb{Q}$ then we need only require that Condition 8.2 is satisfied for a subgroup of finite index in Γ , since we can then compute the invariant for this subgroup and divide by the degree of the covering. In particular, this relaxed condition holds for any f if n is odd and M is a hyperbolic manifold (it suffices that the cusp groups P_c are virtually polycyclic). However even without going to a subgroup of finite index, the condition is not very restrictive. For instance, if $n = 3$ and M is a hyperbolic 3-manifold then the cusp groups are isomorphic to \mathbb{Z}^2 and the only way the image of such a group can fail to have a fixed point in $\partial\overline{\mathbb{H}^3}$ is if its image is a Klein four-group fixing a point of \mathbb{H}^3 .

9. EXAMPLES

In this section, we will look at a few interesting examples that illustrate our results and conjectures of this paper and [30]. We are grateful to Alan Reid, Frank Calegari, and Craig Hodgson for noticing some of these examples. Many of the calculations in this section are done with the aid of the software packages Snappea, [42], Snap, [18] and Pari-GP [3]. The number theory justification of these calculations can be found in the book by Henri Cohen [9].

The result that underlies many of our calculations is a theorem of Borel (reinterpreted in the light of work of Bloch and Suslin — for more details, see e.g. [30]). Let F be a number field and let $\sigma_1, \bar{\sigma}_1, \dots, \sigma_{r_2}, \bar{\sigma}_{r_2}: F \rightarrow \mathbb{C}$ be a list of all complex embeddings of F . The “Borel regulator” is the map

$$c_2: \mathcal{B}(F) \longrightarrow \mathbb{R}^{r_2}$$

defined on generators by

$$c_2([z]) = (D_2(\sigma_1(z)), \dots, D_2(\sigma_{r_2}(z))),$$

where D_2 is the Bloch-Wigner dilogarithm defined in section 7.

Theorem 9.1. *The Borel regulator c_2 has kernel the torsion of $\mathcal{B}(F)$ and has image a maximal sublattice of \mathbb{R}^{r_2} .* \square

Thus we can verify computationally whether elements α and β in $\mathcal{B}(F)$ are equal mod torsion by computing whether $D_2(\sigma(\alpha)) = D_2(\sigma(\beta))$ for all possible embeddings $\sigma: F \subset \mathbb{C}$. Computing this numerically to sufficient precision gives absolute proof of the equality of α and β modulo torsion if the size of a smallest

element of the lattice $c_2(B(F)) \subset \mathbb{R}^{r_2}$ is known. The size of a smallest element can be bounded in terms of the covolume of the lattice once a rational basis has been found, and conjectures exist for this covolume which suggest that one or two digits of precision in calculations would normally be ample. These conjectures are open for all but very few cases, so our examples below cannot be considered to be proved. But since we compute to over 50 digits precision they can probably be considered to be correct beyond reasonable doubt.

Before we discuss the examples we recall the relationship of the Chern-Simons invariant with the η -invariant of [1]. The η -invariant is a real valued invariant defined for compact riemannian manifolds of dimension congruent to 3 modulo 4. Atiyah, Patodi, and Singer prove in [2]

Theorem 9.2. $(1/2\pi^2)CS(M) = (3/2)\eta(M) \pmod{1/2}$ for any compact riemannian 3-manifold M . \square

The formula of Theorem 7.1 has been improved to a formula for the eta-invariant $\eta(M)$ in [24] and [33], and we use this to give $(3/2)\eta(M)$ rather than $(1/2\pi^2)CS(M)$ in cases where we have done the computation. The normalization $(1/2\pi^2)CS$, which is a well defined invariant modulo 1/2, is a fairly standard normalization for CS because for compact 3-manifolds $(1/2\pi^2)CS$ is actually well defined modulo 1. Nevertheless, our computations of $(1/2\pi^2)CS$ are only valid modulo 1/2, even in the compact case. (The value of $(1/2\pi^2)CS(M)$ modulo 1 for a compact 3-manifold is determined by $\eta(M)$ and the homology of M ; see [2].)

Example 9.3. In Weeks' and Hodgson's census of closed manifolds, included in Snappea, the one with the smallest volume is $m003(-3, 1)$, commonly known as the Weeks manifold W . It is conjectured to be the hyperbolic 3-manifold of smallest volume. It is obtained via three $(-3, 2)$ Dehn surgeries on the alternating 6-crossing link which is a circular chain of three circles. It can also be obtained as $((5, 1), (5, 2))$ Dehn surgery on the Whitehead link, which is the description that Weeks originally found.

The invariant trace field of the Weeks manifold is known to be the cubic field k of discriminant -23 generated by the complex root with positive imaginary part of the polynomial $x^3 - x + 1$. We denote this root by θ . It satisfies

$$D_2(\theta) = 0.94270736277692772092129960309221164759032710576688316\dots,$$

which is the volume of W . By Borel's theorem $\mathcal{B}(k)$ is of rank 1. Since $\theta \wedge (1 - \theta) = \theta \wedge \theta^3 = 0$, $[\theta]$ generates $\mathcal{B}(k) \otimes \mathbb{Q}$. The computation of $D_2(\theta)$ shows that $\beta(W) = [\theta]$ in $\mathcal{B}(k) \otimes \mathbb{Q}$ beyond reasonable doubt.

Since the invariant trace field of W is of odd degree over \mathbb{Q} , the conjecture in the introduction of [30] says that $(1/2\pi^2)CS(W)$ should be irrational. Computation gives

$$\frac{3}{2}\eta(M) = 0.060043066678727155012132615144817756316780200913123686\dots,$$

so this is $(1/2\pi^2)CS(W)$ modulo 1/2. Using continued fractions it is easily checked that any rational expression for this would have to have over 30 digits in numerator and denominator. This provides numerical evidence for its irrationality.

Alan Reid found that the manifold M obtained via $(0, 1)$ surgery on the 8_9 -knot has zero Chern-Simons invariant (in fact, zero η -invariant since it admits an orientation reversing symmetry) and the volume of M appears numerically to be 6

times that of W . The Ramakrishnan conjecture (cf. [30]) suggests that the Bloch invariant given by $[M]$ be $3([\theta] - [\bar{\theta}])$ modulo torsion and that the invariant trace field of M therefore contain $\mathbb{Q}(\theta, \bar{\theta})$, which is the Galois closure of k .

Computations using Snap show that the invariant trace field of M is in fact exactly the Galois closure of k and that, at least numerically, $\beta(M) = 3([\theta] - [\bar{\theta}])$ modulo torsion.

Example 9.4. For a number field F with just one complex place there exist arithmetic 3-manifolds with this field as invariant trace field. Any such will give a Bloch invariant which generates $\mathcal{B}(F) \otimes \mathbb{Q}$. For fields with more than one complex place we do not know how much of $\mathcal{B}(F) \otimes \mathbb{Q}$ can be generated by Bloch invariants of hyperbolic 3-manifolds. The following example is of interest in this regard.

The polynomial $f(x) = x^4 + x^2 - x + 1$ is irreducible with roots $\tau_1^\pm = 0.54742\dots \pm 0.58565\dots i$ and $\tau_2^\pm = -0.54742\dots \pm 1.12087\dots i$. The field $F = \mathbb{Q}(x)/(f(x))$ is thus of degree 4 over \mathbb{Q} with two complex embeddings σ_1, σ_2 up to complex conjugation, one with image $\sigma_1(F) = \mathbb{Q}(\tau_1^-)$ and one with image $\sigma_2(F) = \mathbb{Q}(\tau_2^-)$ (the discriminant of F is 257, which is prime, so F has no proper subfield other than \mathbb{Q}). The Bloch group $\mathcal{B}(F)$ is thus of rank 2 modulo torsion.

It turns out that in the cusped and closed census lists of Snappea there are a total of five manifolds of volume

$$3.163963228883143983991014715973154484812787671518\dots,$$

two of them noncompact and three of them compact, and they all have invariant trace field equal to $\sigma_1(F)$ or its complex conjugate. (Reversing orientation of a manifold replaces invariant trace field by the complex conjugate field, so after adjusting orientations they all have invariant trace field $\sigma_1(F)$.) Moreover, there are ten manifolds of volume

$$3.821687586179977739110922224290385516821302495504\dots,$$

three noncompact and seven compact, and they all have invariant trace field equal to $\sigma_2(F)$ or its complex conjugate. One of the seven compact ones double covers a manifold of half this volume called $M10(-1, 3)$.

The noncompact manifolds of volume 3.1639... are $M032$ and $M033$ in the 5-simplex census and have $(1/2\pi^2)CS(M032) =$

$$0.155977016743515161236016645699315761220516234595001\dots$$

and $(1/2\pi^2)CS(M033) = (1/2\pi^2)CS(M032) - 1/4$, while the three compact ones have $(1/2\pi^2)(CS(M) - CS(M032))$ equal to $-1/3, -3/8$, and $11/60$ respectively. The noncompact manifolds of volume 3.82168... are $M159, M160$, and $M161$ and have $(1/2\pi^2)CS(M159) + 1/4 = (1/2\pi^2)CS(-M160) = (1/2\pi^2)CS(M161) =$

$$0.191492799941975387695803880629160641602608213619566\dots$$

The compact manifold which double covers $M10(-1, 3)$ also has this Chern Simons invariant. (In fact $(3/2)\eta(M10(-1, 3)) = (.1914927999\dots - 1)/2$.) The other six compact manifolds of this volume have Chern-Simons invariants less than $.1914927999\dots$ by $1/6, 1/3, 1/3, 1/3, 1/3, 5/12$ respectively.

Numerical computation shows that the manifolds in the first group all have the same rational bloch invariant $\beta_1 \in \mathcal{B}(F) \otimes \mathbb{Q}$ (more precisely, they all have Bloch invariant $\sigma_1(\beta_1) \in \mathcal{B}(\sigma_1 F) \otimes \mathbb{Q}$). Similarly the second group gives a class $\beta_2 \in \mathcal{B}(F) \otimes \mathbb{Q}$. In fact, in the non-compact case the triangulation gives an exact

Bloch invariant in $\mathcal{B}(F)$. For each of the two non-compact manifolds of the first class this invariant is

$$\beta_1 = 2\left[\frac{1}{2}(1 - \tau^2 - \tau^3)\right] + [1 - \tau] + \left[\frac{1}{2}(1 - \tau^2 + \tau^3)\right] \in \mathcal{B}(F)$$

Here τ denotes the class of x in $F = \mathbb{Q}(x)/(x^4 + x^2 - x + 1)$. (It is interesting to note that despite the exact equality of Bloch invariants the Chern-Simons invariants differ by $1/4$ modulo $1/2$.) Similarly the first two noncompact manifolds of the second group give the class

$$\beta_2 = 2[2 - \tau - \tau^3] + 2[\tau + \tau^2 + \tau^3] \in \mathcal{B}(F)$$

and the third gives

$$\beta'_2 = \left[\frac{1}{4}(3 + \tau^2)\right] + 2\left[\frac{1}{2}(\tau^2 + \tau^3)\right] + \left[\frac{1}{4}(-3 - 2\tau - 1\tau^2 + \tau^3)\right] + \left[\frac{1}{13}(8 - 5\tau - 2\tau^2 - 4\tau^3)\right] \in \mathcal{B}(F).$$

We do not know if the torsion class $\beta_2 - \beta'_2$ vanishes.

The Borel regulator map gives:

$$\begin{aligned} c_2(\beta_1) &= (3.1639632288831439839910147159731544848127876715181, \\ &\quad - 1.4151048972655633406895085877105020361346679596016) \\ c_2(\beta_2) &= (-0.69854408278444071973072661203684276397736670535490, \\ &\quad 3.8216875861799777391109222242903855168213024955043), \end{aligned}$$

proving that β_1 and β_2 generate $\mathcal{B}(F) \otimes \mathbb{Q}$.

In fact, as we now describe, the whole of $\mathcal{B}(\sigma_1(F))$ is generated by Bloch invariants of 3-manifolds with invariant trace field $\sigma_1(F)$.

Searching the closed manifold census for manifolds whose volumes are small linear combinations of $D_2(\sigma_1(\beta_1)), D_2(\sigma_1(\beta_2))$ results in five candidates, four of them with volume $4.396672801932495\dots$ and one with volume $5.629382374981847\dots$. Checking with Snap then confirms that they all have invariant trace field $\sigma_1(F)$ and their Bloch invariants in $\mathcal{B}(F) \otimes \mathbb{Q}$ are numerically $(3/2)\beta_1 + (1/2)\beta_2$ for the four of volume $4.396672801932495\dots$ and $2\beta_1 + \beta_2$ for the one of volume $5.629382374981847\dots$.

A similar search for compact manifolds with invariant trace field $\sigma_2(F)$ yielded no new examples.

The Galois closure \overline{F} of F is degree 24 over \mathbb{Q} . The element β_1 has four distinct Galois conjugates in $\mathcal{B}(\overline{F}) \otimes \mathbb{Q}$ (which is of rank 12), and hence in $\mathcal{B}(\mathbb{C})$. As elements of $\mathcal{B}(\mathbb{C})$ these are $\sigma_1(\beta_1), \overline{\sigma}_1(\beta_1), \sigma_2(\beta_1), \overline{\sigma}_2(\beta_1)$. Their sum is zero (it follows from Theorem 9.1 that the sum of all Galois conjugates of any element of $\mathcal{B}(\overline{\mathbb{Q}})$ is zero) and they generate a rank 3 subgroup of $\mathcal{B}(\mathbb{C})$. Similar remarks apply to β_2 , and one checks that the two rank 3 subgroups of $\mathcal{B}(\mathbb{C})$ generated by the Galois conjugates of β_1 and β_2 generate a rank 6 subgroup. The Bloch invariants of all the above manifolds and their orientation reversals generate a rank 5 subgroup of this; namely the subgroup generated by the six elements $\sigma_1(\beta_1), \overline{\sigma}_1(\beta_1), \sigma_1(\beta_2), \overline{\sigma}_1(\beta_2), \sigma_2(\beta_2), \overline{\sigma}_2(\beta_2)$, the sum of the last four of which is zero.

Example 9.5. It is of interest to know to what extent the Bloch invariant determines Chern Simons invariant in $\mathbb{R}/\pi^2\mathbb{Z}$ rather than in $\mathbb{R}/\pi^2\mathbb{Q}$. Let

$$z_1 = \frac{3 + i - \sqrt{4 + 2i}}{2}, \quad z_2 = 2z_1 - 2z_1^2 + z_1^3/2, \quad z_3 = \frac{1 + i}{2}.$$

Then Snappea shows that the manifolds $M6(1,3)$ and $M11(1,3)$ both have degree one ideal triangulations using the three simplices with the above parameters, so their Bloch invariants in $\mathcal{B}(\mathbb{Q}(z_1))$ are equal. However, their Chern Simons invariants are $11/48$ and $7/48$ which differ by $1/12$ modulo $1/2$. (In fact $(3/2)\eta(M6(1,3)) = -61/48$.) This example is especially interesting because these two manifolds are not commensurable, despite having congruent ideal triangulations. In fact, they are both arithmetic with invariant trace field $\mathbb{Q}(i)$, so by Reid (see e.g., [28]) their commensurability classes are determined by their invariant quaternion algebras. But computation using Snap shows that each of $M6(1,3)$ and $M11(1,3)$ has quaternion algebra ramified at just two primes, one of which is the prime dividing 2 but the other of which divides 5 or 13 respectively.

The volume of these manifolds is

$$1.831931188354438030109207029864768221548298748563344268534.$$

Snappea knows five compact orientable manifolds of this volume and they lead to several examples like the above. In fact, the five-vertex graph with edges according to whether the corresponding manifolds have congruent triangulations with parameters in some quadratic extension of $\mathbb{Q}(i)$ is a connected graph. In each case we thus see that the Bloch invariants of the corresponding manifolds are equal in some quadratic extension of $\mathbb{Q}(i)$. We do not know if the Bloch invariants can be defined in $\mathcal{B}(\mathbb{Q}(i))$, and if so, whether they are equal there (modulo torsion they are just the element $2[i] \in \mathcal{B}(\mathbb{Q}(i))$).

Example 9.6. The manifold $X = V3066$ in the seven-simplex cusped census has the surprising property that the Dehn filled manifolds $X(p, q)$ and $X(-p, q)$ appear to have equal volume for each (p, q) and to have Chern-Simons invariants which sum to the apparently irrational number

$$\alpha = 0.02172669391945231711932766534448768004430408 \dots$$

The manifold $X(1, 2)$ has volume

$$5.137941201873417769841348339474845035649675 \dots$$

and Chern-Simons invariant α . Moreover, its invariant trace field k is generated by a root of $x^3 + 2x - 1$ and has discriminant -59 . The Ramakrishnan conjecture would imply that the invariant trace field of $X(p, q)$ and conjugate invariant trace field of $X(-p, q)$ generate a field that contains the join of above cubic field and its complex conjugate, i.e., the Galois closure of this cubic field. Experiment suggests that this holds, in fact that the invariant trace field of $X(p, q)$ always contains the above cubic field. For example, the Ramakrishnan conjecture would imply that $X(-1, 2)$ (which has the same volume as $X(1, 2)$ but zero Chern-Simons invariant) must have invariant trace field containing the Galois closure K of the above cubic field, and in fact its invariant trace field is exactly this Galois closure. The Bloch invariant of $X(1, 2)$ is in fact $4(2[\theta] + [1 + \theta^2]) \in \mathcal{B}(k) \otimes \mathbb{Q}$, where θ is the complex root (with positive imaginary part) of $x^3 + 2x - 1$. And $X(-1, 1)$ and $X(1, 1)$ have volume exactly half the above volume, Chern-Simons invariant $\alpha/2 + 5/24$ and $\alpha/2 - 5/24$ respectively, and both also have the above cubic field as invariant trace field. The manifold X itself has volume $6.2328329776455 \dots$, Chern-Simons invariant $\alpha/2 - 1/4$, and a degree 6 invariant trace field of discriminant $-2^6 59^2$.

10. APPENDIX: SCISSORS CONGRUENCE

In this appendix, we will prove that the pre-Bloch group $\mathcal{P}(\mathbb{C})$ defined in definition 2.1 has a more geometric (scissors congruence) description. Throughout the appendix, by a face-triangulated polyhedron we mean a convex ideal polyhedron in \mathbb{H}^3 with ideally triangulated faces. As we describe below, we will allow degenerate (flat) polyhedra, though this is not essential.

One should think of a flat face-triangulated polyhedron as having infinitesimal thickness, so it is an ideal polygon in \mathbb{H}^3 with two ideal triangulations, one on each “side”. (A more formal definition might be to associate a triangulation of the polygon to each of its two normal directions.) Flat polyhedra occur as follows in triangulation. In [15], it was shown that a cusped hyperbolic 3-manifold M can be decomposed into convex ideal polyhedra. In order to get a triangulation of M , one needs to triangulate each resulting polyhedron. After this triangulation, a common face of two different polyhedra may now have different triangulations. In order to make this a true triangulation for M , one needs to insert a flat polyhedron for each such face, which one should consider to have two sides, with triangulations on each side to match the triangulations coming from the faces of the two polyhedra. One can then triangulate these flat polyhedra into flat tetrahedra to complete the triangulation of M . (Of course, by changing the triangulations of the polyhedra one may be able to avoid the need for flat tetrahedra — it is unknown whether this is always possible.)

In particular, a flat ideal tetrahedron, that is, one with a real cross ratio parameter r , is thus an ideal quadrilateral triangulated by drawing one diagonal on one “side” of it and the other diagonal on the other “side”. To understand which side gets which diagonal, thicken the flat tetrahedron slightly by deforming r to $r + i\epsilon$ with $\epsilon > 0$.

Define a group $\mathcal{Q}(\mathbb{C})$ generated by face-triangulated polyhedra subject to the following relations:

- for each face-triangulated polyhedron P and isometry $g \in \text{Isom}^+(\mathbb{H}^3)$, we have $[gP] = [P]$;
- if a face-triangulated polyhedron P is obtained by gluing two face-triangulated polyhedra P_1 and P_2 along a face then $[P] = [P_1] + [P_2]$. The face along which P_1 and P_2 are glued together not only should have the same physical shape, but should also have compatible triangulation.

Remark. There are a couple of points worth noting here:

1. The importance of requiring triangular faces in our definition was made clear to us by a remark of David Kazhdan. Take an ideal pyramid on an ideal quadrilateral base. The two ways of cutting the quadrilateral by a diagonal give two decompositions of the pyramid into two ideal tetrahedra. If we put these equal then we have made the flat tetrahedron given by the quadrilateral zero. However the cross ratios of flat tetrahedra gives $\mathcal{P}(\mathbb{R})$, the set of all the real elements in $\mathcal{P}(\mathbb{C})$. $\mathcal{P}(\mathbb{R})$ is not trivial in $\mathcal{P}(\mathbb{C})$. In fact, after passing to $\mathcal{B}(\mathbb{C})$, $\mathcal{B}(\mathbb{R})$ maps onto $\mathcal{B}(\mathbb{C})_+$. In short, without requiring triangulated faces, we would be looking at $\mathcal{P}(\mathbb{C})/\mathcal{P}(\mathbb{R})$, which is not what we want.

2. It is obvious from the 5-term relation in the definition of $\mathcal{P}(\mathbb{C})$ that any real cross ratio can be written as the alternating sum of complex cross ratios. Thus, if we define a group $\mathcal{P}_0(\mathbb{C})$ by replacing $\mathbb{Z}(\mathbb{C} - \{0, 1\})$ in Definition 2.1 by $\mathbb{Z}(\mathbb{C} - \mathbb{R})$,

then $\mathcal{P}_0(\mathbb{C})$ surjects to $\mathcal{P}(\mathbb{C})$. It is not hard to verify that this surjection is an isomorphism (the same holds if \mathbb{C} and \mathbb{R} are replaced by any field and proper subfield). This suggests that the group $\mathcal{Q}(\mathbb{C})$ remains the same if we use only non-degenerate polyhedra, which is indeed true and can be proved without much difficulty. We leave the details to the reader.

Proposition 10.1. *The homomorphism*

$$\Phi: \mathcal{P}(\mathbb{C}) \rightarrow \mathcal{Q}(\mathbb{C})$$

induced by sending $[z]$, for $z = x + iy \in \mathbb{C} - \{0, 1\}$ with $y \geq 0$, to any ideal tetrahedron with cross ratio z is well defined and is an isomorphism.

Proof. First we prove well definedness.

Given an ideal tetrahedron Δ and an ordering of its vertices, the orientation of Δ induced by that ordering may or may not agree with the orientation induced on Δ from \mathbb{H}^3 (this induced orientation makes sense even if Δ is flat, since we are giving flat simplices infinitesimal thickness). We will consider the tetrahedron Δ plus the ordering of its vertices to represent the element $\Delta \in \mathcal{Q}(\mathbb{C})$ or $-\Delta \in \mathcal{Q}(\mathbb{C})$ according as these two orientations do or do not agree. With this convention it is clear that Equation (2) in Definition 2.1 is preserved by the map Φ .

Geometrically, equation (1) of Definition 2.1 can be interpreted as follows: given a polyhedron P_0 with five vertices, there is an isometry of \mathbb{H}^3 which moves the vertices to positions $\infty, 0, 1, x, y$, so that the polyhedron can either be decomposed into three ideal tetrahedra with vertices $\langle \infty, 0, 1, x \rangle$, $\langle \infty, 0, x, y \rangle$ and $\langle 0, 1, x, y \rangle$, or it can be decomposed into two ideal tetrahedra $\langle \infty, 0, 1, y \rangle$ and $\langle \infty, 1, x, y \rangle$, and these vertex orderings give the correct orientations of these five simplices. Then the five term relation (1) of definition 2.1 expresses the equality of these two decompositions and is thus respected by Φ . It is not hard to check that if one now permutes the five vertices, the sign of a term in the five term relation is changed only if the orientation induced by the vertex ordering of the corresponding simplex has changed, so the five term relation still expresses the same geometric fact as before. Therefore the map Φ is well defined.

If we think of the five-term relation as allowing us to move from one triangulation of P_0 to another, it makes sense to call such a move a “cycle move”.

We will now show that Φ is an isomorphism. To do so we must show that any face-triangulated polyhedron P has an ideal triangulation, that is a subdivision into ideal tetrahedra compatible with the face triangulations, and moreover, that any two ideal triangulations of P are related by a sequence of cycle moves.

By a *triangle of P* we shall mean a triangle of the face-triangulation of a face of P . Choose one vertex v of P and then take the set of cones to v of triangles of P which do not contain v . These cones are clearly 3-simplices which triangulate P (there will be flat simplices only if there are triangles of P not containing v in faces that do contain v).

Given an arbitrary triangulation of P , for each 3-simplex Δ of the triangulation we can use a cycle move to replace it by the sum (with appropriate signs or orientations) of the cones to v of the faces of Δ . This relates this arbitrary triangulation by cycle moves to the triangulation just constructed and thus shows that any two triangulations of P are related by cycle moves, completing the proof. \square

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